Lecture 17: Integration

As recalled in Lecture 12, the course is structured in two parts. The first part, organised under the slogan "space as a stage for things" emphasised the following concepts:

- metric space, topological space, symmetry groups, continuity, constructing new spaces from old ones, compactness, Hausdorff spaces.

The second part, "space as a stage for motion", has been organised around the concept of a function space. With Picard's theorem in Lecture 15 we got a glimpse of the fundamental role such spaces play in the study of dynamics. But something is missing. Recall from Lecture 16 that by the Stone–Weierstrass theorem the trigonometric polynomials are dense in $Cts(S^1, \mathbb{R})$. But this result is not constructive (although, see Ex. L16-5) in the sense that it does not tell us, given $f: S^1 \longrightarrow \mathbb{R}$ continuous, a "preferred" coefficient a_n of $\cos(n\theta)$ in an approximating polynomial for f.

Compare this to the situation for a vector v in a vector space V with basis $\{u_1, \ldots, u_m\}$. There is a unique expression of v as $\sum_{i=1}^m a_i u_i$ for $a_i \in \mathbb{R}$, and the coefficients are "read off" by the linear functionals $u_i^* \in V^*$ which send v to $u_i^*(v) = a_i$. These functionals tell us "how much" of v is in the direction u_i.

We know $Cts(S^1, \mathbb{R})$ is an $\mathbb{R}$-vector space, and it is not difficult to show that $\{\cos(n\theta), \sin(n\theta)\}_{n \geq 1} \cup \{1\}$ is a linearly independent set in this vector space (see Ex. L16-2 and L17-1 below). So it is natural to ask: is this a basis for the (infinite-dimensional) vector space $Cts(S^1, \mathbb{R})$? One might then think a dual basis $\cos(n\theta)^*$ could produce the desired coefficient a_n.

However, this is far too naive: the trigonometric polynomials do <u>not</u> span $\mathrm{Cts}(S^1, \mathbb{R})$ and even if they did, in the infinite-dimensional case we do <u>not</u> have a dual basis. Too bad! We seem to lack some basic conceptual framework for working constructively in infinite-dimensional vector spaces of this kind. The appropriate framework, whose study will occupy us for the remainder of the semester, is <u>Hilbert space</u>, and the Hilbert space structure on $\mathrm{Cts}(S^1, \mathbb{R})$ (or rather a suitable replacement denoted $L^2(S^1)$) is derived from the <u>integral</u>.

In today's lecture we develop integrals in the context of function spaces, which will lead us to L^2 spaces, whose structure we will axiomitise next lecture using the notion of a Hilbert space.

<u>Exercise L17-1</u> With $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ prove the set $\{\cos(n\theta), \sin(n\theta)\}_{n \geq 1} \cup \{1\}$ is linearly independent in $\mathrm{Cts}(S^1, \mathbb{R})$ (so the expressions in eqn (7.1) of Lecture 16 are unique). In particular this shows that $\mathrm{Cts}(S^1, \mathbb{R})$ is infinite-dimensional.

<u>Exercise L17-2</u> Prove $e^{\cos(\theta)} \in \mathrm{Cts}(S^1, \mathbb{R})$ is not in the linear span of the linearly independent set considered in the previous exercise.

<u>Defn</u> An <u>integral pair</u> $(X, \int)$ is a compact Hausdorff space X together with a function $\int : \mathrm{Cts}(X, \mathbb{R}) \longrightarrow \mathbb{R}$ which is linear and continuous, and satisfies for all $f \in \mathrm{Cts}(X, \mathbb{R})$:

(i) If $f \geq 0$ then $\int f \geq 0$, and [where $f \geq 0$ means for all $x \in X$ $f(x) \geq 0$]

(ii) if $f \geq 0$ then $\int f = 0$ if and only if $f = 0$.

Lemma L17-0 For $a<b$ the Riemann integral $\int_{[a,b]} : Cts([a,b], \mathbb{R}) \longrightarrow \mathbb{R}$ gives an integral pair $([a,b], \int_{[a,b]})$.

Proof As usual $Cts([a,b], \mathbb{R})$ has the compact-open topology (and metric d_∞), see Tutorial 8 for a reminder on definitions. For linearity see T. Tao "Analysis I" Theorem 11.4.1 (a), (b). Condition (i) is immediate from the definitions.

For (ii) suppose $\int_{[a,b]} f = 0$ and that $f(x_0) > 0$. Then $f^{-1}((\frac{1}{2}f(x_0), \infty))$ is an open neighborhood of x_0, which contains a closed interval, say

$$x_0 \in [c,d] \subseteq f^{-1}((\tfrac{1}{2}f(x_0), \infty)) \subseteq [a,b] \qquad (c \neq d)$$

Then $P = \{[a,c)\ [c,d], (d,b]\}$ is a partition and the function

$$g(x) = \begin{cases} 0 & x \in [a,c) \\ \frac{1}{2}f(x_0) & x \in [c,d] \\ 0 & x \in (d,b] \end{cases}$$

is piece-wise constant with respect to P, hence since $f \geq g$ we have

$$\int_{[a,b]} f = \underline{\int}_{[a,b]} f \geq p.c.\int_{[a,b]} g = (d-c)\tfrac{1}{2}f(x_0) > 0$$

which is a contradiction. Hence $f \equiv 0$, proving (ii).

Lemma L17-1 If $(X, \int)$ is an integral pair then for $f, g \in Cts(X, \mathbb{R})$

(i) $f \leq g$ implies $\int f \leq \int g$ ⌜$f \leq g$ means $f(x) \leq g(x)$ for all $x \in X$⌟
(ii) $|\int f| \leq \int |f|$
(iii) $\int : Cts(X, \mathbb{R}) \to \mathbb{R}$ is <u>uniformly</u> continuous.

<u>Proof</u> (i) If $f \leq g$ then $g - f \geqslant 0$ so $\int g - \int f = \int (g-f) \geqslant 0$.

(ii) Let $\lambda \in \{1, -1\}$ be such that $\lambda \int f \geqslant 0$. Then $\lambda f \leq |f|$ so

$$\left| \int f \right| = \lambda \int f = \int \lambda f \leq \int |f|.$$

(iii) Immediate from

$$\left| \int f - \int g \right| = \left| \int (f-g) \right| \leq \int |f-g| \leq \int d_\infty(f,g) = V \cdot d_\infty(f,g)$$

where $V = \int 1$. $\square$

<u>Exercise L17-3</u> Give a continuous linear $\int$ not equal to the $\int_{[a,b]}$ of Lemma L17-0 for which $([a,b], \int)$ is also an integral pair.

Recall that by the adjunction property (Theorem L12-4) for X, Y locally compact Hausdorff we have a homeomorphism (Ex. L12-13)

$$\begin{array}{ccc} \mathrm{Cts}(X \times Y, Z) & \xrightarrow[\cong]{\Psi_{X,Y,Z}} & \mathrm{Cts}(X, \mathrm{Cts}(Y, Z)). \\ F & \longmapsto & \{ x \mapsto F(x,-) \} \end{array} \tag{4.1}$$

We can use this to define the <u>product</u> of integral pairs. However, in order to prove that the definition is independent of the order of X, Y we will have to use Ex. L16-11 which in turn depends on Urysohn's lemma (which we have not proven). I will provide a proof of this lemma at the end of the semester, but for the moment I will just continue to flag explicitly which results depend on it. In any case, the independence is also a consequence of Fubini's theorem when X, Y are of the form given in Lemma L17-0.

<u>Def^n</u> A <u>topological $\mathbb{R}$-vector space</u> is a vector space V together with a topology on the underlying set of V, such that the structural maps

$$\begin{aligned} V \times V &\longrightarrow V, & \mathbb{R} \times V &\longrightarrow V \\ (v,w) &\longmapsto v+w & (\lambda, v) &\longmapsto \lambda v \end{aligned}$$

are all continuous. A topological vector space is in particular a topological group.

<u>Exercise L17-4</u> Prove that if X is locally compact Hausdorff and V is a topological vector space that $\mathrm{Cts}(X,V)$ with the compact-open topology and the pointwise operations (as on p. 6, 7 of Lecture 16) is a topological $\mathbb{R}$-vector space. (Hint: just copy the proof of Lemma L16-6).

<u>Lemma L17-1</u> Let $(X, \int_X), (Y, \int_Y)$ be integral pairs. Then $(X \times Y, \int_{X\times Y})$ is an integral pair, called the <u>product integral pair</u> where $\int_{X \times Y}$ is defined so as to make the diagram below commute:

$$\begin{tikzcd} \mathrm{Cts}(X\times Y, \mathbb{R}) \arrow[d, "\Psi_{X,Y,\mathbb{R}}"', "\cong"] \arrow[r, dashed, "\int_{X\times Y}"] & \mathbb{R} \\ \mathrm{Cts}(X, \mathrm{Cts}(Y,\mathbb{R})) \arrow[r, "\int_Y \circ -"'] & \mathrm{Cts}(X,\mathbb{R}) \arrow[u, "\int_X"'] \end{tikzcd} \tag{5.1}$$

Assuming the Urysohn lemma the following diagram also commutes:

$\zeta: Y\times X \to X \times Y$ is the swap map $(y,x) \mapsto (x,y)$

$$\begin{tikzcd} \mathrm{Cts}(X\times Y, \mathbb{R}) \arrow[d, "(-)\circ\zeta"', "\cong"] \arrow[r, "\int_{X\times Y}"] & \mathbb{R} \\ \mathrm{Cts}(Y\times X, \mathbb{R}) \arrow[d, "\Psi_{Y,X,\mathbb{R}}"', "\cong"] & \\ \mathrm{Cts}(Y, \mathrm{Cts}(X,\mathbb{R})) \arrow[r, "\int_X \circ -"'] & \mathrm{Cts}(Y,\mathbb{R}) \arrow[uu, "\int_Y"'] \end{tikzcd} \tag{5.2}$$

Proof The space $X \times Y$ is compact Hausdorff by Lemma L10-2, Lemma L11-3. Let us first unpack the definition of $\int_{X\times Y}$. Given $F : X\times Y \longrightarrow \mathbb{R}$

$$
\begin{array}{lcc}
\mathrm{Cts}(X\times Y, \mathbb{R}) & F & \underline{\text{informally}} \\
\Big\downarrow \Psi & & \\
\mathrm{Cts}(X, \mathrm{Cts}(Y,\mathbb{R})) & \Psi(F) & x \longmapsto F(x,-) \\
\Big\downarrow \int_Y \circ (-) & & \\
\mathrm{Cts}(X,\mathbb{R}) & \int_Y \circ \Psi(F) & x \longmapsto \int_Y F(x,y)\,dy \\
\Big\downarrow \int_X & & \\
\mathbb{R} & \int_X\left(\int_Y \circ \Psi(F)\right) & \int_X \int_Y F(x,y)\,dy\,dx
\end{array}
$$

As a composite of continuous maps (using Lemma 12-1 for $\int_Y \circ (-)$) it is clear $\int_{X\times Y}$ is continuous. By Exercise L17-4 all spaces involved are topological vectorspaces. The map $\Psi_{X,Y,\mathbb{R}}$ is linear since

$$
\begin{aligned}
\Psi_{X,Y,\mathbb{R}}(F+G)(x)(y) &= (F+G)(x,y) \\
&= F(x,y) + G(x,y) \\
&= \Psi_{X,Y,\mathbb{R}}(F)(x)(y) + \Psi_{X,Y,\mathbb{R}}(G)(x)(y) \\
&= [\Psi_{X,Y,\mathbb{R}}(F)(x) + \Psi_{X,Y,\mathbb{R}}(G)(x)](y) \\
&= [\{\Psi_{X,Y,\mathbb{R}}(F) + \Psi_{X,Y,\mathbb{R}}(G)\}(x)](y)
\end{aligned}
$$

which proves $\Psi_{X,Y,\mathbb{R}}(F+G) = \Psi_{X,Y,\mathbb{R}}(F) + \Psi_{X,Y,\mathbb{R}}(G)$. Similarly one checks that $\Psi_{X,Y,\mathbb{R}}(\lambda F) = \lambda \Psi_{X,Y,\mathbb{R}}(F)$. The map $\int_Y \circ (-)$ is also linear, since $[\int_Y \circ (f+g)](x) = \int_Y (f(x)+g(x)) = \int_Y (f(x)) + \int_Y (g(x))$ and similarly $\int_Y \circ (\lambda f) = \lambda \int_Y \circ f$. So as a composite of linear maps, $\int_{X\times Y}$ is linear.

It remains to check the axioms for an integral pair:

(i) If $F \geqslant 0$ then for $x \in X$ the function $F(x,-) : Y \to \mathbb{R}$ is non-negative, so since $\int_Y$ is an integral pair $\int_Y F(x,-) \geqslant 0$. Hence $x \mapsto \int_Y F(x,-)$ is a non-negative function, which has non-negative integral $\int_{X\times Y} F$.

(ii) Suppose $F \geqslant 0$ and $\int_{X\times Y} F = 0$. That means that the function $F_Y : X \longrightarrow \mathbb{R}$ defined by $F_Y(x) = \int_Y F(x,-)$ has $\int_X F_Y = 0$. By the axioms for $\int_X$ we deduce $F_Y \equiv 0$. But then for $x \in X$ $\int_Y F(x,-) = 0$ yields $F(x,-) \equiv 0$ and hence $F \equiv 0$.

Assuming Urysohn, we have to show the two ways around (5.2) are equal as continuous linear maps

$$\mathrm{Cts}(X\times Y, \mathbb{R}) \longrightarrow \mathbb{R}$$

But by Lemma L17-2 below it suffices to show they agree on a dense subset A of $\mathrm{Cts}(X\times Y, \mathbb{R})$. But as a consequence of Stone-Weierstrass (Ex. L16-11) we know a convenient dense subset, namely the set of all finite sums of products of functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, i.e.

$$A = \left\{ \textstyle\sum_i f_i g_i \;\middle|\; f_i \in \mathrm{Cts}(X,\mathbb{R}),\, g_i \in \mathrm{Cts}(Y,\mathbb{R}) \right\}.$$

Since the two ways around (5.2) are linear, to show they agree on A it suffices to show they agree on a single fg with $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$. But then both ways around are easily checked to send $F = fg$ to the product $(\int_X f) \cdot (\int_Y g)$ so we are done. $\square$

<u>Lemma L17-2</u> If $f, g : X \longrightarrow Y$ are continuous maps of topological spaces, with Y Hausdorff, and $A \subseteq X$ is dense then $f|_A = g|_A$ implies $f = g$.

<u>Proof</u> Consider the continuous map $(\Delta(x) = (x,x))$

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$$

since Y is Hausdorff the diagonal $\Delta = \{(y,y) \mid y \in Y\} \subseteq Y \times Y$ is closed, and its preimage under the above map $\{x \in X \mid f(x) = g(x)\}$ is therefore closed in X. Hence if $A \subseteq X$ is dense and $f|_A = g|_A$ then $A \subseteq \{x \mid f(x) = g(x)\}$ and therefore $\{x \mid f(x) = g(x)\} = X$. $\square$

The outcome of Lemma L17-1 is essentially Fubini's theorem: we may interchange the order of integrals, so roughly speaking (roughly because "dx", "dy" have not entered our notation)

$$\int_Y \int_X F(x,y)dxdy = \int_{X \times Y} F(x,y) = \int_X \int_Y F(x,y)dydx$$

<u>Example L17-1</u> Combining Lemma L17-0, L17-1 we have an integral pair

$$\left([a_1,b_1] \times \cdots \times [a_n,b_n], \int_{[a_1,b_1]} \int_{[a_2,b_2]} \cdots \int_{[a_n,b_n]} \right)$$

for any collection of intervals.

We learned in Lecture 7 a few other ways of constructing spaces: disjoint unions and quotients (and pushouts, which were a combination of the two). It is natural to extend these operations to integral pairs.

Exercise L17-5 (i) Prove that if V, V' are topological $\mathbb{R}$-vector spaces so is $V \times V'$ with the product topology and the usual operations.

(ii) Let X, Y be locally compact Hausdorff and V a topological $\mathbb{R}$-vector space. Prove that the bijection

$$\begin{aligned} \mathrm{Cts}(X \amalg Y, V) &\xrightarrow{\cong} \mathrm{Cts}(X, V) \times \mathrm{Cts}(Y, V) \\ F &\longmapsto (F|_X, F|_Y) \end{aligned}$$

of Ex L7-6 is a linear homeomorphism (that is, an isomorphism of topological vector spaces). Here you are using Ex. L17-4, and (i). (see also Ex. L11-5 and Lemma L10-5). (Hint: you might like to first prove $(X \amalg Y) \times Z \cong (X \times Z) \amalg (Y \times Z)$.)

Lemma L17-3 Let $(X, \int_X), (Y, \int_Y)$ be integral pairs. Then $(X \amalg Y, \int_{X \amalg Y})$ is an integral pair where $\int_{X \amalg Y}$ is defined so as to make the diagram below commute:

$$\begin{tikzcd} \mathrm{Cts}(X \amalg Y, \mathbb{R}) \arrow[rr, dashed, "\int_{X \amalg Y}"] \arrow[d, "\cong", "\text{Ex.17-5}"'] & & \mathbb{R} \\ \mathrm{Cts}(X, \mathbb{R}) \times \mathrm{Cts}(Y, \mathbb{R}) \arrow[rr, "\int_X \times \int_Y"'] & & \mathbb{R} \times \mathbb{R} \arrow[u, "+"'] \end{tikzcd}$$

Proof The space $X \amalg Y$ is compact Hausdorff by Lemma L10-5, Lemma L11-5. The map is continuous and linear as a composite of continuous linear maps (using Ex. L17-5). The axioms of an integral pair are immediate since if $F : X \amalg Y \to \mathbb{R}$ then $F \geq 0$ iff. $F|_X \geq 0$ and $F|_Y \geq 0$. $\square$

<u>Lemma L17-4</u> Let $(X, \int_X)$ be an integral pair and $\sim$ an equivalence relation on X such that $X/\sim$ is Hausdorff. Then $(X/\sim, \int_{X/\sim})$ is an integral pair where $\int_{X/\sim}$ is the composite (ρ is the quotient map)

$$\mathrm{Cts}(X/\sim, \mathbb{R}) \xrightarrow{(-)\circ\rho} \mathrm{Cts}(X, \mathbb{R}) \xrightarrow{\int_X} \mathbb{R}$$

<u>Proof</u> The composite is continuous and linear (and X is compact by Lemma L10-1). If $f \geq 0$ then $f \circ \rho \geq 0$ and hence $\int_{X/\sim} f = \int_X (f \circ \rho) \geq 0$. If $f \geq 0$ and $0 = \int_{X/\sim} f = \int_X (f \circ \rho)$ then $f \circ \rho = 0$ and hence $f = 0$. □

<u>Example L17-2</u> We define $(S^1, \int_{S^1}) := ([0, 2\pi]/\sim, \int_{[0,2\pi]})$, where $0 \sim 2\pi$.

Note that as Exercise L17-3 shows, a space can be equipped with many integrals, and for instance using the definition $[0,1]/\sim$ would induce a different integral on S^1. We choose $[0, 2\pi]$ so that

$$\int_{S^1} 1 = 2\pi.$$

Of course we are free to use a different model of S^1, say $\mathbb{R}/2\pi\mathbb{Z}$, but while these spaces are homeomorphic if we want to "move" $\int_{S^1}$ to be defined on $\mathbb{R}/2\pi\mathbb{Z}$ we have to <u>specify</u> <u>which</u> homeomorphism $\phi : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$ we mean and then we would obtain an integral pair from

$$\mathrm{Cts}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}) \xrightarrow{(-)\circ\phi} \mathrm{Cts}(S^1, \mathbb{R}) \xrightarrow{\int_{S^1}} \mathbb{R}.$$

Anyway, the point is that while we can switch around between $\mathbb{R}/2\pi\mathbb{Z}$, $[0,2\pi]/\sim$, $[0,1]/\sim$, $\{(x,y) \mid x^2+y^2=1\}$ as <u>spaces</u> we must be more careful as <u>integral pairs</u>.

<u>Example L17-3</u> Let X be a finite CW-complex with presentation $X_0, X_1, \dots, X_{n-1}, X_n = X$. For $j \geq 1$ choose a homeomorphism

$$\psi_j : [-1,1]^j \longrightarrow D^j \quad (\text{the } j\text{-disk})$$

This is not the usual Riemann integral on the disk!

and we make $(D^j, \int_{D^j})$ an integral pair using ψ_j and $\int_{[0,1]^j}$.
We make $X_0 = \{1, \dots, r\}$ an integral pair with (cf. Ex L12-5)

$$Cts(X_0, \mathbb{R}) \longrightarrow \mathbb{R}, \quad f \longmapsto \sum_{i=1}^{r} f(i)$$

Then by induction and Lemmas L17-3, L17-4 we obtain a continuous linear map $\int_X$ s.t. $(X, \int_X)$ is an integral pair. This will depend on the choice of presentation and of the ψ_j, but we can at least choose $\psi_j = \mathrm{id}$ canonically.

<u>Exercise L17-6</u> Let G be a finite unoriented graph and $X(G)$ the associated CW-complex (Ex. L7-4). Compute $\int_{X(G)} 1$.

<u>Lemma L17-5</u> If $(X, \int)$ is an integral pair then $d_1^{\int}(f,g) = \int |f-g|$ defines a metric on $Cts(X, \mathbb{R})$.

<u>Proof</u> (M1) Since $|f-g| \geq 0$ we have $d_1^{\int}(f,g) \geq 0$.

(M2) If $d_1^{\int}(f,g) = 0$ then by axiom (ii) of an integral pair $|f-g| = 0$ and hence $f = g$.

(M3) Clearly $d_1^{\int}$ is symmetric.

(M4) Since $|f-g| + |g-h| \geq |f-h|$ by the triangle inequality in $\mathbb{R}$, we have by Lemma L17-1(i) that $\int |f-g| + \int |g-h| \geq \int |f-h|$ and hence the triangle inequality holds.

Solutions to selected exercises

$\boxed{L17-1}$ Suppose $a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta) \right) = 0$ as functions.
Then differentiating yields

$$\sum_{n=1}^{N} \left(-n a_n \sin(n\theta) + n b_n \cos(n\theta) \right) = 0.$$

$$\sum_{n=1}^{N} \left(-n^2 a_n \cos(n\theta) - n^2 b_n \sin(n\theta) \right) = 0$$

$$\vdots$$

Setting $\theta = 0$ in all these equations yields

$$\sum_{n=1}^{N} n b_n = 0$$
$$\sum_{n=1}^{N} n^2 a_n = 0$$
$$\sum_{n=1}^{N} n^3 b_n = 0$$
$$\vdots$$

Collating every second equation gives the matrix eq^N

$$\begin{pmatrix} 1 & 2 & \cdots & N \\ 1^2 & 2^2 & \cdots & N^2 \\ \vdots & & & \\ 1^N & 2^N & \cdots & N^N \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = 0$$

But this is a Vandermonde matrix whose determinant is nonzero, so we conclude $b_i = 0$ for $1 \le i \le N$. Similarly $a_i = 0$ for $1 \le i \le N$, and then also $a_0 = 0$. $\square$