Fusion in LG models III

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In the previous two lectures we have defined defects in topological LG models via matrix factorisations, described the fusion of defects via the tensor product of matrix factorisations, and organised all this into a bicategory $\mathcal{LG}$, which we have seen is a very useful setting for things like generalised orbifolding.

Throughout we have emphasised that this defect fusion is <u>computable</u> and we have demonstrated code written for this purpose. In this final lecture we explain <u>how the code works</u>, i.e. the mathematics behind it.

Outline

1. The cut operation
2. Homological perturbation
3. An equivalence of bicategories

1 The cut operation

<u>The problem</u>: the composition in $\mathcal{LG}$ is "infinitary"

$$\mathcal{LG}(V,U) \times \mathcal{LG}(W,V) \longrightarrow \mathcal{LG}(W,U)$$

$$\begin{tikzcd} (Y,X) \arrow[r, mapsto] & Y \otimes X \arrow[d, phantom, "\cong" rotate=90] \\ & Y * X \end{tikzcd}$$

infinite matrix

What is the algorithm to find this?

<u>Answer</u> The algorithm goes via an intermediate object

$$(Y,X) \longmapsto Y|X \longmapsto Y * X$$

f.rank MF of $U-W$ with action of a Clifford algebra "the <u>cut</u>"

split an idempotent, i.e. project onto ground state.

- This can be promoted to an equivalence of bicategories $\mathcal{C} \cong \mathcal{LG}$ where composition in $\mathcal{C}$ is by the <u>cut</u>.

<u>Setup</u> · $W(x)$, $V(y)$, $U(z)$ are potentials, $m = |y|$.

· $Y \in hmf(\mathbb{C}[y,z], U - V)$
$X \in hmf(\mathbb{C}[x,y], V - W)$

· $J_V = \mathbb{C}[y] / (\partial_{y_1} V, \dots, \partial_{y_m} V)$ ← f.d. since V is a potential

<u>Lemma</u> The tensor product

$$Y | X := Y \underset{\mathbb{C}[y]}{\otimes} J_V \underset{\mathbb{C}[y]}{\otimes} X$$

with differential $d_Y \otimes 1 \otimes 1 + 1 \otimes 1 \otimes d_X$ is a f.rank MF of $U - W$ over $\mathbb{C}[x,z]$.

<u>Proof</u> Since $\mathbb{C}[y,z] \underset{\mathbb{C}[y]}{\otimes} J_V \underset{\mathbb{C}[y]}{\otimes} \mathbb{C}[x,y] \cong \mathbb{C}[z] \underset{\mathbb{C}}{\otimes} J_V \underset{\mathbb{C}}{\otimes} \mathbb{C}[x] \cong \mathbb{C}[x,z]^{\oplus \dim J_V}$. □

<u>Next</u> We define odd closed $\mathbb{C}[x,z]$-linear operators

$$\gamma_i, \gamma_i^\dagger \circlearrowright Y|X \qquad 1 \leq i \leq m$$

satisfying Clifford relations:

(*)
$$\boxed{\begin{gathered} \gamma_i \gamma_j + \gamma_j \gamma_i \simeq 0, \quad \gamma_i^\dagger \gamma_j^\dagger + \gamma_j^\dagger \gamma_i^\dagger \simeq 0 \\ \gamma_i \gamma_j^\dagger + \gamma_j^\dagger \gamma_i \simeq \delta_{ij}. \end{gathered}}$$

The main ingredient is <u>Atiyah classes</u>.

<u>Def^N</u> Set $t_i = \partial_{y_i} V$ so $t_1, \dots, t_m$ is a regular sequence.
There is a $\mathbb{C}$-linear flat connection

$$\nabla : \mathbb{C}[[y]] \longrightarrow \mathbb{C}[[y]] \otimes_{\mathbb{C}[t]} \Omega^1_{\mathbb{C}[t]/\mathbb{C}}$$

with components $\partial_{t_i} \circlearrowright \mathbb{C}[[y]]$.

<u>Example</u> $V = \frac{1}{N+1} y^{N+1} \in \mathbb{C}[y]$ so $t = \partial_y V = y^N$ and

$$\mathbb{C}[[y]] \cong J_V \otimes_{\mathbb{C}} \mathbb{C}[[t]] \cong \bigoplus_{i=0}^{N-1} \mathbb{C}[[t]] \cdot y^i$$

and given

$$f = \sum_{i=0}^{N-1} f_i \cdot y^i \qquad f_i \in \mathbb{C}[t]$$

$$\partial_t f = \sum_{i=0}^{N-1} \partial_t(f_i) \cdot y^i$$

<u>Lemma</u> The operator

$$[d_{Y \otimes X}, \partial_{t_i}] \circlearrowright Y \otimes_{\mathbb{C}[y]} X$$

actually for this to make sense we need to extend scalars to $\mathbb{C}[[y]]$, but this is a technicality.

is $\mathbb{C}[t]$-linear and therefore passes to a $\mathbb{C}[x,z]$-linear operator on

$$Y | X \cong (Y \otimes_{\mathbb{C}[y]} X) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t).$$

Proof By the Leibniz rule, for $h \in \mathbb{C}[y]$

$$\partial_{t_i}(t_j h) = \partial_{t_i}(t_j)h + t_j \partial_{t_i}(h)$$
$$= \delta_{ij} h + t_j \partial_{t_i}(h)$$

$$\therefore \ [\partial_{t_i}, t_j] = \delta_{ij}$$

Hence

$$[[d_Y \otimes x, \partial_{t_i}], t_j] = -[[t_j, d_Y \otimes x], \partial_{t_i}] - [[\partial_{t_i}, t_j], d_Y \otimes x]$$

$$= -[\delta_{ij}, d_Y \otimes x] = 0. \quad \square$$

Def^N The Atiyah classes of Y/X are the odd closed $\mathbb{C}[x,z]$-linear operators

$$At_i := [d_Y \otimes x, \partial_{t_i}] \circlearrowright Y/X.$$

Clifford operators $\quad \gamma_1, \dots, \gamma_n, \gamma_1^\dagger, \dots, \gamma_n^\dagger \circlearrowright Y/X$

$$\gamma_i = At_i \qquad \text{(annihilation)}$$

$$\gamma_i^\dagger = -\partial_{y_i}(dx) - \tfrac{1}{2}\sum_q \partial_{y_i y_q}(V)\, At_q$$

(creation)

Prop These operators satisfy Clifford relations. (up to homotopy)

<u>Def^n</u> The <u>cut</u> Y/X is the Clifford representation

$$(Y/X, \{\gamma_i, \gamma_i^\dagger\}_{1\le i\le m})$$

in the homotopy category $hmf(\mathbb{C}[x,z], V-W)$.

<u>Theorem</u> (Dyckerhoff–M[1], M[2]) The idempotent

$$e = \gamma_1 \cdots \gamma_n \gamma_n^\dagger \cdots \gamma_1^\dagger \qquad \text{(vacuum projector)}$$

on Y/X splits to $Y\otimes X$, i.e. there is a diagram of matrix factorisations

$$\begin{tikzcd} Y/X \arrow[loop left, "e"] \arrow[r, shift right, "g"'] & Y\otimes X \arrow[l, shift right, "f"'] \end{tikzcd}$$

with

$$fg \simeq e, \quad gf \simeq 1.$$

<u>The Singular code</u>

1. Computes e as a finite matrix of polynomials in $\mathbb{C}[x,z]$.
2. Strictifies $ee \simeq e$ to $EE = E$.
3. Computes the projector $Im(E) \cong Im(e) \cong Y\otimes X$.
4. Then this f.rank $Im(E)$ is the <u>fusion</u> $Y * X$.

② Homological perturbation

In the rest of this lecture we discuss the proof of the above theorem. For this we will need

$$S_m := \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \qquad |\theta_i| = 1$$

viewed as a $\mathbb{Z}_2$-graded k-module. Let C_m be the associative unital k-algebra generated by $\gamma_1, \ldots, \gamma_n, \gamma_1^+, \ldots, \gamma_n^+$ subject to the Clifford relations $(*)$. Then there is an iso

$$C_m \xrightarrow{\cong} \mathrm{End}_k(S_m)$$

$$\gamma_i \longmapsto \theta_i \wedge (-)$$
$$\gamma_i^+ \longmapsto \theta_i^* \lrcorner (-)$$

So C_m is the <u>Clifford algebra</u> and S_m its <u>spinor rep</u>.

The full version of the theorem is

<u>Thm</u> (M[2]) There is an isomorphism of C_m-representations

$$Y|X \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} S_m \otimes_{\mathbb{C}} (Y \otimes_{\mathbb{C}[y]} X)$$

with C_m acting canonically on the right.

<u>Def^n</u> A <u>strong deformation retract</u> (SDR) is

$$(M, d) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (A, d), \; \phi$$

where M, A are $\mathbb{Z}_2$-graded R-modules (R a ring) and both $d^2 = W$ for some $W \in R$, such that

- $\pi\sigma = 1$,
- $\sigma\pi = 1 - [d, \phi]$,
- $\phi^2 = 0$,
- $\phi\sigma = 0$,
- $\pi\phi = 0$.

Now let $J : A \longrightarrow A$ be an odd operator such that

$$(d + J)^2 = V \cdot 1_A \qquad (V \text{ possibly} \neq W)$$

<u>Lemma</u> (Homological perturbation) If ϕJ has finite order, there is a strong deformation retract

$$(M, d_\infty) \underset{\sigma_\infty}{\overset{\pi_\infty}{\rightleftarrows}} (A, d + J), \; \phi_\infty$$

Proof of Thm There is an SDR $(t_i = \partial_{y_i} V)$

$$(J_V, 0) \rightleftarrows \Big(\overset{\Lambda(k\theta_1\oplus\cdots\oplus k\theta_n)}{S_m},\ \overset{d_K}{\sum_i t_i\theta_i^*}\Big)$$

which gives rise to an SDR

$$(\underbrace{Y\otimes J_V\otimes X}_{Y|X}, 0) \rightleftarrows (S_m\otimes(Y\otimes X), d_K)$$

Then using $d_{Y\otimes X}$ as a perturbation, the perturbation lemma above yields an SDR

$$(Y|X, \overline{d_{Y\otimes X}}) \rightleftarrows (S_m\otimes(Y\otimes X), d_K + d_{Y\otimes X})$$

$$\cong$$

$$(S_m\otimes(Y\otimes X), d_{Y\otimes X}).$$

This because each t_i acts null-homotopically on $Y\otimes X$.

$\square$

3 An equivalence of bicategories

The homotopy equivalence

$$Y|X \rightleftarrows Sm \underset{\mathbb{C}}{\otimes} (Y \otimes X)$$

can be promoted to an equivalence of bicategories

$$\mathcal{C} \cong \mathcal{LG}$$

where $\mathcal{C}$ has

- objects same as $\mathcal{LG}$ (i.e. potentials)
- 1-morphisms $W \to V$ are f.rank MFs of $V - W$ equipped with the action of a Clifford alg.
- composition is by the cut operation.

All the structures and operations of $\mathcal{C}$ are finitary, and described as polynomial functions of their inputs. In this sense $\mathcal{C}$ is a "finite model" of $\mathcal{LG}$.

Conclusion
- 2D defect TFT = bicategories
- Bicategory $\mathcal{LG}$, fusion = composition
- Generalised orbifolding
- Can compute!

<u>Example</u> Let E, F be MFs of $W(x) \in \mathbb{C}[x]$. Then $E^\vee$ is a MF of $-W(x)$, and we have in $\mathcal{LG}$

$$0 \xrightarrow{F} W \xrightarrow{E^\vee} 0$$

- $E^\vee \otimes_{\mathbb{C}[x]} F \cong \mathrm{Hom}_{\mathbb{C}[x]}(E, F)$ is the Hom-complex

- The cut $E^\vee | F$ is a $\mathbb{Z}_2$-graded cpx of f.d. vector spaces and there is an idempotent $e \circlearrowright E^\vee | F$ such that $\mathrm{Im}(e) = E^\vee * F$, i.e. there is a homotopy equivalence

$$\mathrm{Im}(e) \cong E^\vee \otimes F$$

f.d. $\mathbb{Z}_2$-graded vector space

∞-rank vector space (f.d. cohomology)

References

[1] Dyckerhoff, Murfet "Pushing forward matrix factorisations" 1102.2957.

[2] Murfet "Cut systems and matrix factorisations" arXiv: 1402.4541.