Jet Schemes and LG/CFT 1

The LG/CFT correspondence suggests a relationship between matrix factorisations of ADE singularities and modules over vertex algebras. The most promising geometric intermediary between these two worlds are jet schemes of Slodowy slices [AM, AKM]. In this series of talks we pose some questions to guide further work:

1. An introduction to jet schemes (Part 1)
2. From matrix factorisations to jet schemes (Qs)

1. An introduction to jet schemes (Part 1)

The following historical comments are from [Y].

The study of jet (or arc) schemes was initiated by Nash in the 60's [N]. Nash was interested in whether the singularities of a variety X could be reflected in the arc space of X. We know from the Jacobian criterion how the existence of singularities is detected by tangents, and arcs are higher order (in the sense of higher order terms in a Taylor series) analogues of tangent vectors, so this seems reasonable. This turned out to be true:

appearing modulo birational equiv.
on every desingularisation

- There is an injective map from the set of "essential" exceptional divisors of a resolution of singularities of X to the set of irreducible components of $\pi^{-1}(X_{sing}) \subseteq J_\infty(X)$ where $J_\infty(X)$ is the jet scheme and $\pi: J_\infty(X) \longrightarrow X$ the canonical projection. Nash conjectured that this is bijective. While false in general it is true for A-type surface singularities [N, R].

- Mustaţă showed a local complete intersection has rational singularities if and only if all its jet schemes are irreducible [M].

So what is a jet? To explain we first recall how $k[t]/t^2$ represents tangent vectors in algebraic geometry. Let k be an algebraically closed field and V a finite-dimensional k-vector space, $\mathrm{Sym}(V^*)$ the symmetric algebra. Then there are bijections

$$
\begin{array}{cccc}
\{\mathfrak{m} \subseteq \mathrm{Sym}(V^*) \mid \mathfrak{m} \text{ is maximal}\} & \mathfrak{m} & \varphi^{-1}(\{0\}) & \\
\cong & \downarrow & \uparrow & \\
\mathrm{Hom}_{k\mathrm{Alg}}(\mathrm{Sym}(V^*), k) & \mathrm{Sym}(V^*) \to \frac{\mathrm{Sym}(V^*)}{\mathfrak{m}} \xleftarrow{\cong} k & \varphi & f \mapsto f(w) \\
\cong & & \downarrow & \uparrow \\
\mathrm{Hom}_k(V^*, k) & & \varphi|_{V^*} & ev_w \\
\cong & & & \uparrow \\
V & & & w
\end{array}
\qquad (2.1)
$$

which is how we identify the maximal spectrum *as a set* $\mathrm{Specm}(\mathrm{Sym}(V^*))$ with V. Note that if $e_1, \dots, e_n$ is a k-basis of V then a point $P = \sum_{i=1}^n P_i e_i$ of V corresponds to the k-algebra map $\varphi_P : \mathrm{Sym}(V^*) \longrightarrow k$ with $\varphi_P(f) = f(P_1, \dots, P_n)$ where $f = f(x_1, \dots, x_n)$, $x_i = e_i^*$.

Replacing k with other zero-dimensional (Krull dimension) k-algebras T we see more of the geometry of the affine space $\mathbb{A}^n := \mathrm{Spec}(\mathrm{Sym}(V^*))$, where $n = \dim V$. For example let us classify k-algebra morphisms

$$\psi : \mathrm{Sym}(V^*) \longrightarrow k[\varepsilon]/\varepsilon^2. \qquad (\cong k \oplus k\varepsilon) \qquad (2.2)$$

There is a canonical quotient map $\pi : k[\varepsilon]/\varepsilon^2 \longrightarrow k$ sending $\pi(\varepsilon) = 0$ and so $\pi \circ \psi$ "is" a point of V, that is $\pi \circ \psi = \varphi_P$ for some $P = (P_1, \dots, P_n) \in k^n$. Note

$$P_i = \varphi_P(x_i) = \pi(\psi(e_i^*)) = \text{const. term of } \psi(e_i^*) \qquad (2.3)$$

The k-algebra morphism ψ is determined by its restriction $\psi|_{V^*}$ and thus by the sequence $\psi(e_1^*), \dots, \psi(e_n^*) \in k \oplus k\varepsilon$ (and moreover such sequences are in <u>bijection</u> with k-algebra maps (2.2)). If we write

$$\psi(e_i^*) = P_i + Q_i \varepsilon \tag{3.1}$$

then we know $P = (P_1, \dots, P_n)$ is to be thought of as a point of V (because it "transforms" as a point, more on this in a moment). But what about $Q = (Q_1, \dots, Q_n)$? Notice that if we write down the Taylor series of $f \in \mathrm{Sym}(V^*) \cong k[x_1, \dots, x_n]$ at P

$$f = \sum_{i_1 \geq 0 \cdots i_n \geq 0} \frac{1}{i_1! \, i_2! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}}(f)\Big|_P (x_1 - P_1)^{i_1} \cdots (x_n - P_n)^{i_n} \tag{3.2}$$

and apply to it ψ

$$\begin{aligned}
\psi(f) &= \sum_{i_1 \geq 0 \cdots i_n \geq 0} \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}}(f)\Big|_P (\psi(x_1) - P_1)^{i_1} \cdots (\psi(x_n) - P_n)^{i_n} \\
&= \sum_{i_1 \geq 0 \cdots i_n \geq 0} \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}}(f)\Big|_P Q_1^{i_1} \cdots Q_n^{i_n} \underbrace{\varepsilon^{i_1 + \cdots + i_n}}_{\text{zero if } |i| > 1} \\
&= f(P) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(f)\Big|_P Q_j \varepsilon \qquad \text{1-jet}
\end{aligned} \tag{3.3}$$

So we recognise Q as specifying the k-linear <u>derivation</u>

$$f \longmapsto \sum_{j=1}^{n} Q_j \frac{\partial}{\partial x_j}\Big|_P$$

We know that such derivations are (by def^n) <u>tangent vectors</u> so the information in a k-algebra homomorphism $\psi : \mathrm{Sym}(V^*) \longrightarrow k[\varepsilon]/\varepsilon^2$ is a pair consisting of a <u>point</u> and a <u>tangent vector at that point</u>. Inspecting (3.3) leads us to believe that k-algebra homomorphisms $\mathrm{Sym}(V^*) \longrightarrow k[\varepsilon]/\varepsilon^{s+1}$ extract <u>s-jets</u> of functions.

<u>Def</u>

Def^n The <u>s-jet</u> of a function f at a point P is the polynomial

$$J^s_P f := \sum_{j=0}^{s} \sum_{|\underline{i}|\le s} \frac{1}{i_1!\cdots i_n!} \frac{\partial^{\underline{i}}}{\partial \underline{x}^{\underline{i}}}(f)\Big|_P \, z^s$$

Note that this is a polynomial in a <u>single</u> variable z, so these are <u>not</u> what differential geometers call jets if $s > 1$. This is why sometimes $J^s_P f$ is called an arc rather than a jet, but Arakawa calls these jet schemes and it seems too late to complain about the name.

1.1 Coordinate transforms

Before we consider "larger" Artinian k-algebras such as $k[\varepsilon]/\varepsilon^3$ (larger in k-dimension) let us take a moment to observe the coordinate dependence in our derivation of Q from Ψ in (3.1). Suppose there is a linear map $A : V \longrightarrow V$ hence

$Ae_i = \sum_j a_{ji} e_j$

$$A^* : V^* \xrightarrow{\cong} V^*$$
$$\mathrm{Sym}(A^*) : \mathrm{Sym}(A^*) \xrightarrow{\cong} \mathrm{Sym}(A^*)$$

$$e_i^* \longmapsto e_i^* \circ A = \sum_{j=1}^n a_{ij} e_j^* = \sum_{j=1}^n a_{ij} x_j$$

Then the following diagrams commute

$$\begin{tikzcd}
e_i^*\circ A & \mathrm{Sym}(V^*) \arrow[dr, "\varphi_P"] & \\
& & k \\
x_i = e_i^* & \mathrm{Sym}(V^*) \arrow[uu, "\mathrm{Sym}(A^*)"', "\cong"] \arrow[ur, "\varphi_{AP}"'] &
\end{tikzcd}$$

$\varphi_P(e_i^*\circ A) = \sum_{j=1}^n a_{ij} P_j = (AP)_i$

"points transform as points" (4.2)

$$\begin{tikzcd}
\mathrm{Sym}(V^*) \arrow[dr, "\Psi_{P,Q}"] & \\
& k[\varepsilon]/\varepsilon^2 \\
\mathrm{Sym}(V^*) \arrow[uu, "\mathrm{Sym}(A^*)"] \arrow[ur, "\Psi_{AP,\,?}"'] &
\end{tikzcd}$$

$\Psi_{P,Q}(f) = f(P) + \sum_j Q_j \frac{\partial f}{\partial x_j}\Big|_P \varepsilon$

what goes here? Surely AQ?

$$\Psi_{P,Q}\big(\mathrm{Sym}(A^*)(f)\big) = f(AP) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(\underbrace{\mathrm{Sym}(A^*)(f)}_{f\left(\sum_{j=1}^n a_{1j} x_j,\, \dots,\, \sum_{j=1}^n a_{nj} x_j\right)}\Big)\Big|_P Q_j\, \varepsilon$$

$$\overset{\text{chain rule}}{=} f(AP) + \sum_{j=1}^{n}\Big[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_{ij}\Big]\Big|_{AP} Q_j\, \varepsilon$$

$$= f(AP) + \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} a_{ij} Q_j}_{(AQ)_i} \frac{\partial f}{\partial x_i}\Big|_{AP} \varepsilon \qquad (5.1)$$

So indeed $\Psi_{AP,AQ}$ makes (4.2) commute. But suppose now we replace $A : V \to V$ by an <u>arbitrary</u> polynomial map A, that is, we assume polynomials $A_1, \dots, A_n$ such that

$$A(P_1, \dots, P_n) = \big(A_1(P_1, \dots, P_n), \dots, A_n(P_1, \dots, P_n)\big) \qquad (5.2)$$

This corresponds to the k-algebra morphism $\alpha : \mathrm{Sym}(V^*) \longrightarrow \mathrm{Sym}(V^*)$, uniquely determined by $\alpha(x_i) = A_i$ in the sense that

$$\begin{tikzcd}
\mathrm{Hom}_{k\mathrm{Alg}}(\mathrm{Sym}(V^*), k) \arrow[ddd, "(-)\circ\alpha"'] & & & V \arrow[lll, "\cong"'] \arrow[ddd, "A"] \\
& \varphi_P & P & \\
& \varphi_P \circ \alpha & A(P) & \\
\mathrm{Hom}_{k\mathrm{Alg}}(\mathrm{Sym}(V^*), k) & & & V \arrow[lll, "\cong"]
\end{tikzcd} \qquad (5.3)$$

commutes. To check this note that

$$(\varphi_P \circ \alpha)(x_i) = \varphi_P(A_i) = A_i(P_1, \dots, P_n)$$

<u>Lemma</u> For a polynomial function $A : V \longrightarrow V$ as above the diagram

$$\begin{tikzcd} \mathrm{Hom}_{k\mathrm{Alg}}(\mathrm{Sym}(V^*), k[\varepsilon]/\varepsilon^2) \arrow[d, "(-)\circ\alpha"'] & V \times V \arrow[l, "\cong"'] \arrow[d, "A \times JA(P)"] \\ \mathrm{Hom}_{k\mathrm{Alg}}(\mathrm{Sym}(V^*), k[\varepsilon]/\varepsilon^2) & V \times V \arrow[l, "\cong"'] \end{tikzcd} \qquad (6.1)$$

commutes, where the horizontal maps send (P, Q) to $\Psi_{P,Q}$ as in (3.3) and $JA(P)$ is the Jacobian $\left(\frac{\partial A_i}{\partial x_j} \Big|_P \right)_{1 \le i, j \le n}$ viewed as a linear transformation on V.

<u>Proof</u> Given (P, Q) we compute

$$\begin{aligned} (\Psi_{P,Q} \circ \alpha)(f) &= f(AP) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\alpha(f)) \Big|_P Q_j \varepsilon \\ &= f(AP) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f(A_1(x_1, \dots, x_n), \dots, A_n(x_1, \dots, x_n)) \Big|_P Q_j \varepsilon \\ &= f(AP) + \sum_{j=1}^{n} \Big[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_{AP} \frac{\partial A_i}{\partial x_j}\Big|_P \Big] Q_j \varepsilon \qquad (6.2) \\ &= f(AP) + \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} \frac{\partial A_i}{\partial x_j}\Big|_P Q_j \Big] \frac{\partial f}{\partial x_i}\Big|_{AP} \varepsilon \end{aligned}$$

as claimed. $\square$

The upshot is that the functor $\mathrm{Hom}_{k\mathrm{Alg}}(-, k[\varepsilon]/\varepsilon^2)$ sends a vector space V (viewed as a k-algebra $\mathrm{Sym}(V^*)$) to $V \times V$. We said earlier that $k[\varepsilon]/\varepsilon^2$ "represents" tangent vectors but this is not <u>quite</u> true, as the notion of a tangent vector is of a tangent vector <u>plus</u> the point it is "attached". This information is represented not by the single algebra $k[\varepsilon]/\varepsilon^2$ but by the <u>diagram</u>

$$k[\varepsilon]/\varepsilon^2 \xrightarrow{\ \pi\ } k \qquad \pi(\varepsilon) = 0 \qquad (6.3)$$

since the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{k\mathrm{Alg}}\left(\mathrm{Sym}(V^*), k[\varepsilon]/\varepsilon^2\right) \arrow[r, "\pi\circ(-)"] & \mathrm{Hom}_{k\mathrm{Alg}}\left(\mathrm{Sym}(V^*), k\right) \\
TV = V\times V \arrow[u, "\cong"] \arrow[r, "\text{projection}"'] & V \arrow[u, "\cong"']
\end{tikzcd} \qquad (7.1)$$

$\psi_{P,Q}$ $\qquad$ $\pi\circ\psi_{P,Q}$

(P,Q) $\qquad$ P

It is therefore natural to consider applying $\mathrm{Hom}_{k\mathrm{Alg}}(R,-)$ to the inverse system

$$\cdots \longrightarrow k[\varepsilon]/\varepsilon^3 \longrightarrow k[\varepsilon]/\varepsilon^2 \longrightarrow k$$

and defining the inverse limit to be "arcs" on $X = \mathrm{Spec}(R)$. We will keep going in this direction in the next talk. Roughly speaking $J_\infty X$ is this space of arcs.

2. From MFs to jet schemes

Let X be a singular scheme, e.g. $X = \mathrm{Spec}\left(\mathbb{C}[x,y,z]/(x^2+y^n-z^2)\right)$. Let X_{reg} denote the regular part (e.g. $X\setminus\{0\}$ in the example) and X_{sing} its complement, the singular locus. We expect $J_\infty(X)$ to look like some dense open set $J_\infty(X_{reg})$ which is boring, meeting some interesting geometry at the "boundary" $\pi^{-1}(X_{sing})$ where

$$\pi : J_\infty(X) \longrightarrow X$$

maps an arc to the point P it passes through. Indeed the theorem of Nash cited earlier says we have an injective map

$$\left\{\begin{matrix}\text{essential components of}\\ \text{the exceptional divisor}\\ \text{of a resolution } Y\to X\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{irreducible components}\\ \text{of } \pi^{-1}(X_{sing})\end{matrix}\right\}$$

In the case of ADE surface singularities $X = \mathrm{Spec}\left(\mathbb{C}[x,y,z]/f\right)$ we know

$$\begin{Bmatrix} \text{essential components of} \\ \text{the exceptional divisor} \\ \text{of a resolution } Y \to X \end{Bmatrix} \hookrightarrow \begin{Bmatrix} \text{irreducible components} \\ \text{of } \pi^{-1}(X_{sing}) \end{Bmatrix}$$

$$\cong$$

$$\{ \text{indecomposable MFs of } f \} \quad 1 \longrightarrow 2$$

which means we can associate to any MF $\mathcal{E}$ an irreducible component $C_\mathcal{E}$ of $\pi^{-1}(X_{sing})$ in $J_\infty(X)$. The first question

$\boxed{Q1}$ Give a direct geometric construction of $C_\mathcal{E}$

Following [AM, AKM] it seems we can construct from $C_\mathcal{E}$ modules over W-algebras.

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