DEFINITION Let S be a graded ring and let M be a graded S-module, $M = \bigoplus_{d\in\mathbb{Z}} M_d$. We define the sheaf associated to M on $\mathrm{Proj}\, S$, denoted by $\widetilde{M}$, as follows. For each $\mathfrak{p} \in \mathrm{Proj}\, S$, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in the localisation $T^{-1}M$, where T is the multiplicative system of homogenous elements of S not in $\mathfrak{p}$.

*of course $M_{(\mathfrak{p})}$ is a well-defined $S_{(\mathfrak{p})}$-module for any hom. prime $\mathfrak{p}$,

$$M_{(\mathfrak{p})} = \{\tfrac{m}{s} \mid m \in M_d \text{ and } s \in S_d \text{ for some } d \geq 0\} \qquad (1)$$

Note that not every pair (m,s) representing an element of $M_{(\mathfrak{p})} \subseteq T^{-1}M$ has the form given in (1). Then $M_{(\mathfrak{p})}$ becomes an $S_{(\mathfrak{p})}$-module in the obvious way. For any open subset $U \subseteq \mathrm{Proj}\, S$ we define $\widetilde{M}(U)$ to be the set of functions $s: U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ which are locally fractions. This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and homogenous elements $m \in M$ and $f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$. It is easy to check that $\widetilde{M}$ is a sheaf of groups, with the obvious restriction maps. Then $\widetilde{M}$ is a $\mathcal{O}_X$-module ($X = \mathrm{Proj}\, S$) via $(r\cdot s)(\mathfrak{p}) = r(\mathfrak{p})\cdot s(\mathfrak{p})$.

PROPOSITION 5.11 Let S be a graded ring, and M a graded S-module. Let $X = \mathrm{Proj}\, S$. Then

(a) For any $\mathfrak{p} \in X$, the stalk $\widetilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$.

(b) For any homogenous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ via the isomorphism of $D_+(f)$ with $\mathrm{Spec}\, S_{(f)}$, where $M_{(f)}$ denotes the group of elements of degree zero in M_f.

(c) $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M is finitely generated, then $\widetilde{M}$ is coherent. ?

PROOF Define $\varphi: \widetilde{M}_{\mathfrak{p}} \longrightarrow M_{(\mathfrak{p})}$ by $\varphi(V,s) = s(\mathfrak{p})$. The fact that φ is an isomorphism follows as in (2.5)a. It is easy to check that φ maps the action of $\mathcal{O}_{X,\mathfrak{p}}$ to the action of $S_{(\mathfrak{p})}$ in a way compatible with $\mathcal{O}_{X,\mathfrak{p}} \Longrightarrow S_{(\mathfrak{p})}$. The isomorphism $\widetilde{M}_{\mathfrak{p}} \Longrightarrow M_{(\mathfrak{p})}$ is natural in M.

(b) If $D_+(f) = \emptyset$ this is trivial, so we may assume f is not nilpotent. Recall the isomorphism $D_+(f) \longrightarrow \mathrm{Spec}\, S_{(f)}$ is defined by

$$\varphi: D_+(f) \longrightarrow \mathrm{Spec}\, S_{(f)}$$
$$\varphi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$$
$$\varphi^{\#}_V(g)(\mathfrak{p}) = \psi_{\mathfrak{p}}(g(\varphi(\mathfrak{p})))$$

where $\psi_{\mathfrak{p}}: (S_{(f)})_{\varphi(\mathfrak{p})} \longrightarrow S_{(\mathfrak{p})}$

$$\psi_{\mathfrak{p}}(a/f^n, b/f^m) = af^m/bf^n$$

To define an isomorphism of sheaves of $\mathcal{O}_{\mathrm{Spec}\, S_{(f)}}$-modules $\varphi_* \widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ we first define for $\mathfrak{p} \in D_+(f)$

$$\Upsilon_{\mathfrak{p}}: (M_{(f)})_{\varphi(\mathfrak{p})} \longrightarrow M_{(\mathfrak{p})}$$

where $M_{(f)}$ is the group of degree 0 elements in M_f, which becomes an $S_{(f)}$-module in the obvious way. We define

$$\Upsilon_{\mathfrak{p}}(a/f^n, b/f^m) = \frac{f^m a}{f^n b}$$

If $\deg f = e > 0$ then $\deg a = n \cdot \deg f$, $a \in M$
$\deg b = m \cdot \deg f$, $b \in S$ and $b \notin \mathfrak{p}$

To be careful, choose a representative pair (m,r) for an element of $(M_{(f)})_{\varphi(\mathfrak{p})}$. Then $m \in M_{(f)} \subseteq M_f$ has a representative of the form (a, f^m) with the stated properties, as does $r \in S_{(f)} \subseteq S_f$, $r \notin \varphi(\mathfrak{p})$. One checks the definition of $\Upsilon_{\mathfrak{p}}$ is independent of both choices. It is also easy to check that $\Upsilon_{\mathfrak{p}}$ is a morphism of groups, and that for $m \in (M_{(f)})_{\varphi(\mathfrak{p})}$ and $r \in (S_{(f)})_{\varphi(\mathfrak{p})}$

$$\Upsilon_{\mathfrak{p}}(r\cdot m) = \psi_{\mathfrak{p}}(r)\cdot \Upsilon_{\mathfrak{p}}(m)$$

One shows that $\Upsilon_{\mathfrak{p}}$ is an isomorphism as in (2.5).

We define $\omega : \widetilde{M_{(f)}} \longrightarrow \varphi_* \widetilde{M}|_{D_+(f)}$ by

$$\omega_V : \widetilde{M_{(f)}}(V) \longrightarrow \widetilde{M}(\varphi^{-1}V)$$
$$\omega_V(s)(\mathfrak{p}) = \gamma_{\mathfrak{p}}(s(\varphi(\mathfrak{p})))$$

As in (2.5) it is clear that $\omega_V(s) \in \widetilde{M}(\varphi^{-1}V)$, and that ω_V is a morphism of groups. Clearly ω is a morphism of sheaves of abelian groups, and is in fact an isomorphism since $\gamma_{\mathfrak{p}}$ is an isomorphism $\forall \mathfrak{p} \in D_+(f)$, and the following diagram commutes

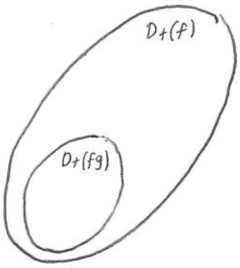
$$\begin{tikzcd}
\widetilde{M_{(f)}}_{\varphi(\mathfrak{p})} \arrow[r] \arrow[d, Rightarrow] & \widetilde{M}|_{D_+(f)\,\mathfrak{p}} \arrow[d, Rightarrow] \\
(M_{(f)})_{\varphi(\mathfrak{p})} \arrow[r, "\gamma_{\mathfrak{p}}"'] & M_{(\mathfrak{p})}
\end{tikzcd}$$

It only remains to show that ω is a morphism of $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$-modules. But if $r \in \mathcal{O}_{\operatorname{Spec} S_{(f)}}(V)$ then

$$\begin{aligned}
\omega_V(r \cdot s)(\mathfrak{p}) &= \gamma_{\mathfrak{p}}(r(\mathfrak{p}) \cdot s(\mathfrak{p})) \\
&= \gamma_{\mathfrak{p}}(r(\mathfrak{p})) \cdot \gamma_{\mathfrak{p}}(s(\mathfrak{p})) \\
&= \varphi^{\#}_V(r)(\mathfrak{p}) \cdot \omega_V(s)(\mathfrak{p}) \\
&= (r \cdot \omega_V(s))(\mathfrak{p})
\end{aligned}$$

(c) Since the $D_+(f)$ cover $\operatorname{Proj} S$, the fact that $\widetilde{M}$ is quasi-coherent follows from (b). If S is noetherian and M finitely generated Then? 84 Vena

<u>NOTE</u> Let S be a graded ring, $X = \operatorname{Proj} S$, M a graded S-module. Let f, g be homogenous elements of S of respective degrees $d, e \geq 1$. There is a bijection $M_{(f)} \Longrightarrow \widetilde{M}(D_+(f))$ defined by mapping m/f^n to $\mathfrak{p} \longmapsto m/f^n \in M_{(\mathfrak{p})}$. (Similarly $M_{(g)} \Longrightarrow \widetilde{M}(D_+(g))$). There is a morphism of groups $\varphi : M_{(f)} \longrightarrow M_{(fg)}$ defined by $\varphi(m/f^n) = g^n \cdot m/(fg)^n$. It is easily checked that the following diagram commutes:



$$\begin{tikzcd}
\widetilde{M}(D_+(f)) \arrow[d] & M_{(f)} \arrow[l, Leftarrow] \arrow[d, "\varphi"] \\
\widetilde{M}(D_+(fg)) & M_{(fg)} \arrow[l, Leftarrow]
\end{tikzcd}$$

At this point you should consult our typed notes “Modules over Projective Schemes”.

<u>DEFINITION</u> Let S be a graded ring, $X = \operatorname{Proj} S$. For any $n \in \mathbb{Z}$ we define the sheaf $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$, where $S(n)$ is the graded S-module

$$S(n)_d = S_{n+d}$$

So the S-module is S itself, just with a modified grading. We call $\mathcal{O}_X(1)$ the <u>twisting sheaf</u> of Serre. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\mathcal{F}(n)$ the <u>twisted sheaf</u> $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. If $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism of $\mathcal{O}_X$-modules, $\phi(n)$ denotes $\phi \otimes \mathcal{O}_X(n) : \mathcal{F}(n) \longrightarrow \mathcal{F}'(n)$.

<u>PROPOSITION 5.12</u> Let S be a graded ring and $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra.

(a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X, $n \in \mathbb{Z}$.
(b) For any graded S-module M, $\widetilde{M}(n) \cong (M(n))^{\sim}$. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.
(c) Let T be another graded ring, generated by T_1 as a T_0-algebra, and let $\varphi : S \longrightarrow T$ be a homomorphism preserving degrees, and let $U \subseteq Y = \operatorname{Proj} T$ and $f : U \longrightarrow X$ be the morphism determined by φ (Ex 2.14). Then $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$ and $f_*(\mathcal{O}_Y(n)|_U) \cong (f_* \mathcal{O}_U)(n)$.

<u>PROOF</u> (a) Let $f \in S_1$ and consider the restriction $\mathcal{O}_X(n)|_{D_+(f)}$. By the previous proposition this is isomorphic to $\widetilde{S(n)_{(f)}}$ on $\operatorname{Spec} S_{(f)}$. We will show that this restriction is free of rank 1. It suffices to show that $S(n)_{(f)} \cong S_{(f)}$ as $S_{(f)}$-modules. Define

$$\varphi : S_{(f)} \longrightarrow S(n)_{(f)}$$
$$\varphi(a/f^m) = f^n a / f^m \qquad \text{[Using the fact that } f \text{ is a unit if } n<0\text{]}$$

It is easy to check that this is a well-defined, injective morphism of $S_{(f)}$-modules. To see that it is surjective, let k/f^m be given, so $k \in S_{m+n}$. Then $k/f^{n+m} \in S_{(f)}$ and maps to k/f^m, as required. Since S is generated by S_1 as an S_0-algebra, X is covered by the open sets $D_+(f)$ for $f \in S_1$. Hence $\mathcal{O}(n)$ is invertible. (If $\mathfrak{p} \supseteq S_1$ then $\mathfrak{p} \supseteq S_+$ since any $f \in S_+$ is a poly in S_1 with coeffs in S_0 (no constant term since $\deg f > 0$).

(b) Let M be a graded S-module. Then for $n \in \mathbb{Z}$ there is an obvious isomorphism $(M \otimes_S S(n)) \Rightarrow M(n)$ of S-modules, which one easily sees to be an isomorphism of graded S-modules. Hence

$$\begin{aligned} \widetilde{M}(n) &= \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \\ &\cong \widetilde{M \otimes_S S(n)} \cong \widetilde{M(n)} \end{aligned}$$

since by assumption S is generated by S_1. This isomorphism is natural in M, in the sense that if $\phi : M \longrightarrow M'$ is a morphism of graded S-modules, $\phi(n) : M(n) \longrightarrow M'(n)$ induced by ϕ, then the following commutes:

$$\begin{tikzcd} \widetilde{M}(n) \arrow[r, Rightarrow] \arrow[d, "\tilde{\phi}(n)"'] & \widetilde{M(n)} \arrow[d, "\widetilde{\phi(n)}"] \\ \widetilde{M'}(n) \arrow[r, Rightarrow] & \widetilde{M'(n)} \end{tikzcd}$$

This follows from the naturality of $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong (M \otimes_S N)^{\sim}$ in M. In particular we see that $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$, via $\widetilde{S(n)} \otimes \widetilde{S(m)} \Rightarrow \widetilde{S(n) \otimes S(m)} \Rightarrow \widetilde{S(n+m)}$.

(c) In fact we do not need to assume T is generated by T_1. The morphism $S(n) \otimes_S T \longrightarrow T(n)$, $s \otimes t \longmapsto \varphi(s)t$ is an isomorphism of graded T-modules, so

$$\begin{aligned} f^*(\mathcal{O}_X(n)) &= f^*(\widetilde{S(n)}) \cong (S(n) \otimes_S T)^{\sim}|_U \\ &\cong \widetilde{T(n)}|_U = \mathcal{O}_Y(n)|_U \end{aligned}$$

For the last claim we do not even require that S be generated by S_1. Then

$$
\begin{aligned}
f_*(\mathcal{O}_Y(n)|_U) &\cong f_*(\widetilde{T(n)}|_U) \\
&\cong ({}_sT(n))^\sim \\
&= (({}_sT)(n))^\sim \\
&\cong ({}_sT)^\sim(n) \\
&\cong f_*(\widetilde{T}|_U)(n) \\
&= f_*(\mathcal{O}_U)(n)
\end{aligned}
$$

as required. $\square$

Assume S is generated by S_1 as an S_0-algebra.
The twisting operation allows us to define a graded S-module associated to any sheaf of modules on $X = \operatorname{Proj} S$. But first we need to properly understand the isomorphism $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$, $n, m \in \mathbb{Z}$.

$$
\begin{tikzcd}
T: \mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \arrow[rr, Rightarrow] & & \mathcal{O}_X(n+m) \\
\widetilde{S(n)} \otimes \widetilde{S(m)} \arrow[dr, Rightarrow] & & \widetilde{S(n+m)} \\
& \widetilde{S(n) \otimes S(m)} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Since $S(n)_{(\mathfrak{p})}$, $S(m)_{(\mathfrak{p})}$ and $S(n+m)_{(\mathfrak{p})}$ are all subgroups of $T^{-1}S$ (T = hom. elts not in $\mathfrak{p}$), we combine our previous work to see that for $U \subseteq X$ and $\mathfrak{p} \in U$

$$T_U(s)(\mathfrak{p}) = \textstyle\sum_i a_i(\mathfrak{p})\, b_i(\mathfrak{p})$$

$$s(\mathfrak{p}) = (V, \textstyle\sum_i a_i \otimes b_i)$$
$$a_i \in \widetilde{S(n)}(V),\ b_i \in \widetilde{S(m)}(V)$$

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on X then there is an isomorphism of $\mathcal{O}_X$-modules

$$
\begin{aligned}
K : \mathcal{F}(n)(d) = \mathcal{F}(n) \otimes \mathcal{O}_X(d) &= (\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(d) \\
&\cong \mathcal{F} \otimes (\mathcal{O}_X(n) \otimes \mathcal{O}_X(d)) \\
&\cong \mathcal{F} \otimes \mathcal{O}_X(n+d) \\
&= \mathcal{F}(n+d)
\end{aligned}
$$

defined by

$$K_U(s)(\mathfrak{p}) = \textstyle\sum_{i,j} (V \cap W,\ c_{ij}|_{V\cap W} \otimes T_{V\cap W}(d_{ij}|_{V\cap W} \dot{\otimes} b_i|_{V\cap W}))$$

where $s(\mathfrak{p}) = (V, \sum_i a_i \otimes b_i)$ $\quad a_i \in (\mathcal{F} \otimes \mathcal{O}_X(n))(V),\ b_i \in \widetilde{S(d)}(V)$

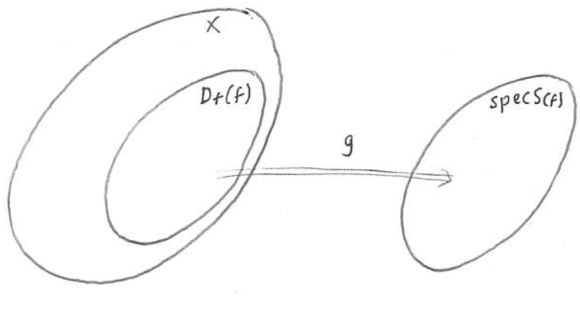
$a_i(\mathfrak{p}) = (W, \sum_j c_{ij} \otimes d_{ij})$ $\quad c_{ij} \in \mathcal{F}(W),\ d_{ij} \in \widetilde{S(n)}(W)$

If $s \in S_d$, $t \in S_e$ then there are canonical global sections $\dot{s} \in \Gamma(X, \mathcal{O}_X(d))$, $\dot{t} \in \Gamma(X, \mathcal{O}_X(e))$ defined by $\dot{s}(\mathfrak{p}) = s/1$. Then

$$T_X(\dot{s} \otimes \dot{t}) = \dot{st}$$

Also if $U \subseteq X$ and $a \in \mathcal{O}_X(n)(U)$, $b \in \mathcal{O}_X(U)$ then $T_U : (\mathcal{O}_X(n) \otimes \mathcal{O}_X(0))(U) \Rightarrow \mathcal{O}_X(n)(U)$ maps $a \dot{\otimes} b$ to $b \cdot a$.
This implies that $K : \mathcal{F}(n) \otimes \mathcal{O}_X(0) \longrightarrow \mathcal{F}(n)$ maps $m \dot{\otimes} b$ to $b \cdot m$.
[sim if $n \longleftrightarrow 0$]

NOTE We elaborate a little on (5.12a). Assume S is generated by S_1 as an S_0-algebra. We showed that if $f \in S_1$ then $\mathcal{O}_X(n)|_{D_+(f)} \cong \mathcal{O}_X|_{D_+(f)}$ as modules.



We defined an isomorphism $\varphi: S_{(f)} \longrightarrow S(n)_{(f)}$ of $S_{(f)}$-modules, giving $\tilde{\varphi}: \mathcal{O}_{\operatorname{spec} S_{(f)}} \longrightarrow \widetilde{S(n)_{(f)}}$. By (5.11b) there is an isomorphism $g_* \mathcal{O}_X(n) \cong \widetilde{S(n)_{(f)}}$. Let $f = g^{-1}$. Then applying f_* to the composite $\mathcal{O}_{\operatorname{spec} S_{(f)}} \cong g_* \mathcal{O}_X(n)$ we obtain $\mathcal{O}_X|_{D_+(f)} \cong f_* \mathcal{O}_{\operatorname{spec} S_{(f)}} \cong \mathcal{O}_X(n)|_{D_+(f)}$. For $V \subseteq D_+(f)$ we have

$$\mathcal{O}_X(n)(V) \longrightarrow \widetilde{S(n)_{(f)}}(g(V)) \longrightarrow \mathcal{O}_{\operatorname{spec} S_{(f)}}(g(V)) \longrightarrow \mathcal{O}_X(V)$$

If $s \in \mathcal{O}_X(n)(V)$ then the corresponding $s' \in \mathcal{O}_X(V)$ is defined by $s'(\mathfrak{p}) = v_{\mathfrak{p}}(s(\mathfrak{p}))$ where $v_{\mathfrak{p}}: S(n)_{(\mathfrak{p})} \longrightarrow S_{(\mathfrak{p})}$ is the isomorphism defined by $v_{\mathfrak{p}}(a/q) = a/f^n q$. So the isomorphisms expressing invertibility of $\mathcal{O}_X(n)$ are

$$\alpha_f: \mathcal{O}_X(n)|_{D_+(f)} \Longrightarrow \mathcal{O}_X|_{D_+(f)}$$

$$(\alpha_f)_V(s)(\mathfrak{p}) = v_{\mathfrak{p}}(s(\mathfrak{p}))$$

NOTE Twisting is Exact (Obsolete. Any locally free sheaf is flat)

Let S be a graded ring generated by S_1 as a S_0-algebra. We claim that if $X = \operatorname{Proj} S$ then for $n \in \mathbb{Z}$

$$- \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$$

is an exact functor. Let $0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules. Since S is generated by S_1 the $D_+(f)$, $f \in S_1$ cover X and to show $0 \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{G}(n) \longrightarrow \mathcal{H}(n) \longrightarrow 0$ is exact it suffices to show that

$$0 \longrightarrow (\mathcal{F} \otimes \mathcal{O}_X(n))|_{D_+(f)} \longrightarrow (\mathcal{G} \otimes \mathcal{O}_X(n))|_{D_+(f)} \longrightarrow (\mathcal{H} \otimes \mathcal{O}_X(n))|_{D_+(f)} \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_X|_{D_+(f)}$-modules. But the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{F} \otimes \mathcal{O}_X(n))|_{D_+(f)} \arrow[r] \arrow[d, Rightarrow] & (\mathcal{G} \otimes \mathcal{O}_X(n))|_{D_+(f)} \arrow[r] \arrow[d, Rightarrow] & (\mathcal{H} \otimes \mathcal{O}_X(n))|_{D_+(f)} \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & (\mathcal{F}|_{D_+(f)}) \otimes \mathcal{O}_X(n)|_{D_+(f)} \arrow[r] & (\mathcal{G}|_{D_+(f)}) \otimes \mathcal{O}_X(n)|_{D_+(f)} \arrow[r] & (\mathcal{H}|_{D_+(f)}) \otimes \mathcal{O}_X(n)|_{D_+(f)} \arrow[r] & 0
\end{tikzcd}$$

So it suffices to show that the bottom row is exact.
Let $\varphi : D_+(f) \longrightarrow \operatorname{Spec} S_{(f)}$ be the canonical isomorphism of schemes. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varphi_*(\mathcal{F}|_{D_+(f)} \otimes \mathcal{O}_X(n)|_{D_+(f)}) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(\mathcal{G}|_{D_+(f)} \otimes \mathcal{O}_X(n)|_{D_+(f)}) \arrow[r] \arrow[d, Rightarrow] & \varphi_*(\mathcal{H}|_{D_+(f)} \otimes \mathcal{O}_X(n)|_{D_+(f)}) \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & \varphi_*(\mathcal{F}|_{D_+(f)}) \otimes \widetilde{S(n)_{(f)}} \arrow[r] & \varphi_*(\mathcal{G}|_{D_+(f)}) \otimes \widetilde{S(n)_{(f)}} \arrow[r] & \varphi_*(\mathcal{H}|_{D_+(f)}) \otimes \widetilde{S(n)_{(f)}} \arrow[r] & 0
\end{tikzcd}$$

since $\varphi_*(\widetilde{S(n)}|_{D_+(f)}) \cong \widetilde{S(n)_{(f)}}$. So we have reduced to showing that $\widetilde{S(n)_{(f)}}$ is a flat $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$-module. But we have already shown that $S_{(f)} \cong S(n)_{(f)}$ as $S_{(f)}$-modules (since $f \in S_1$), so this is obvious, since tensoring with $\mathcal{O}_{\operatorname{Spec} S_{(f)}} = \widetilde{S_{(f)}}$ is naturally equivalent to the identity.

NOTE Pentagon for $\mathcal{O}_X$ tensoring

Let $\mathcal{O}_X$ be a sheaf of rings, and for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$ let $\lambda_{\mathcal{F},\mathcal{G},\mathcal{H}} : (\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H} \longrightarrow \mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H})$ be the isomorphism established earlier. Then for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{L}$ we claim the following diagram commutes:

$$\begin{tikzcd}
 & ((\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H})\otimes\mathcal{L} \arrow[dl, Rightarrow, "\lambda_{\mathcal{F},\mathcal{G},\mathcal{H}}\otimes 1"'] \arrow[dr, Rightarrow, "\lambda_{\mathcal{F}\otimes\mathcal{G},\mathcal{H},\mathcal{L}}"] & \\
(\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H}))\otimes\mathcal{L} \arrow[d, Rightarrow, "\lambda_{\mathcal{F},\mathcal{G}\otimes\mathcal{H},\mathcal{L}}"'] & & (\mathcal{F}\otimes\mathcal{G})\otimes(\mathcal{H}\otimes\mathcal{L}) \arrow[d, Rightarrow, "\lambda_{\mathcal{F},\mathcal{G},\mathcal{H}\otimes\mathcal{L}}"] \\
\mathcal{F}\otimes((\mathcal{G}\otimes\mathcal{H})\otimes\mathcal{L}) \arrow[rr, Rightarrow, "1\otimes\lambda_{\mathcal{G},\mathcal{H},\mathcal{L}}"'] & & \mathcal{F}\otimes(\mathcal{G}\otimes(\mathcal{H}\otimes\mathcal{L}))
\end{tikzcd} \qquad (1)$$

It suffices to show commutativity on stalks. But for $x\in X$ the following diagram commutes (save for the inner pentagon):

$$\begin{tikzcd}[column sep=small]
 & ((\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H})_x\otimes\mathcal{L}_x \arrow[rr, Rightarrow] \arrow[dl, Rightarrow] \arrow[r, phantom, "a"] & & ((\mathcal{F}\otimes\mathcal{G})_x\otimes\mathcal{H}_x)\otimes\mathcal{L}_x \arrow[dr, Rightarrow] & \\
(\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H}))_x\otimes\mathcal{L}_x \arrow[d, Rightarrow] \arrow[r, phantom, "\mathrm{I}"] & & \{((\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H})\otimes\mathcal{L}\}_x \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[r, phantom, "\mathrm{II}"] & & (\mathcal{F}\otimes\mathcal{G})_x\otimes(\mathcal{H}_x\otimes\mathcal{L}_x) \arrow[d, Rightarrow] \\
(\mathcal{F}_x\otimes(\mathcal{G}\otimes\mathcal{H})_x)\otimes\mathcal{L}_x \arrow[d, Rightarrow] \arrow[r, phantom, "b"] & \{(\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H}))\otimes\mathcal{L}\}_x \arrow[l, Rightarrow] \arrow[ul, Rightarrow] \arrow[d, Rightarrow] \arrow[dl, phantom, "\mathrm{III}"] & & \{(\mathcal{F}\otimes\mathcal{G})\otimes(\mathcal{H}\otimes\mathcal{L})\}_x \arrow[r, Rightarrow] \arrow[ur, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\mathrm{IV}"] & (\mathcal{F}_x\otimes\mathcal{G}_x)\otimes(\mathcal{H}\otimes\mathcal{L})_x \arrow[d, Rightarrow] \arrow[ul, phantom, "c"] \\
\mathcal{F}_x\otimes((\mathcal{G}\otimes\mathcal{H})_x\otimes\mathcal{L}_x) \arrow[dr, Rightarrow] \arrow[r, phantom, "d"] & \{\mathcal{F}\otimes((\mathcal{G}\otimes\mathcal{H})\otimes\mathcal{L})\}_x \arrow[l, Rightarrow] \arrow[rr, Rightarrow] \arrow[d, Rightarrow] \arrow[r, phantom, "\mathrm{V}"] & & \{\mathcal{F}\otimes(\mathcal{G}\otimes(\mathcal{H}\otimes\mathcal{L}))\}_x \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[r, phantom, "e"] & \mathcal{F}_x\otimes(\mathcal{G}_x\otimes(\mathcal{H}\otimes\mathcal{L})_x) \arrow[dl, Rightarrow] \\
 & \mathcal{F}_x\otimes((\mathcal{G}\otimes\mathcal{H})\otimes\mathcal{L})_x \arrow[rr, Rightarrow] & & \mathcal{F}_x\otimes(\mathcal{G}\otimes(\mathcal{H}\otimes\mathcal{L}))_x &
\end{tikzcd}$$

[That is, when we turn everything into tensor of modules, the induced map is just the module associator]

Squares I, III and IV commute by naturality of the associator w.r.t. stalks (proven earlier). Squares I, V commute by naturality of $(-\otimes-)_x \Rightarrow -_x\otimes-_x$. (a) commutes by definition, (b) for the same reason, also (d), (e). The triangle (c) commutes since $(1\otimes\phi)(\psi\otimes 1) = (\psi\otimes 1)(1\otimes\phi)$. So we have reduced to showing that both ways around the outside are the same. Consider the following diagram:

$$
\begin{tikzcd}[column sep=small]
 & ((\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H})_x\otimes\mathcal{L}_x \arrow[rr, Rightarrow] \arrow[dl, Rightarrow] & & ((\mathcal{F}\otimes\mathcal{G})_x\otimes\mathcal{H}_x)\otimes\mathcal{L}_x \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
(\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H}))_x\otimes\mathcal{L}_x \arrow[dd, Rightarrow] & \mathrm{I} & ((\mathcal{F}_x\otimes\mathcal{G}_x)\otimes\mathcal{H}_x)\otimes\mathcal{L}_x \arrow[ddl, Rightarrow] \arrow[ddr, Rightarrow] & \mathrm{IV} & (\mathcal{F}\otimes\mathcal{G})_x\otimes(\mathcal{H}_x\otimes\mathcal{L}_x) \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & & & a & (\mathcal{F}_x\otimes\mathcal{G}_x)\otimes(\mathcal{H}\otimes\mathcal{L})_x \arrow[dll, Rightarrow] \arrow[dd, Rightarrow] \\
(\mathcal{F}_x\otimes(\mathcal{G}\otimes\mathcal{H})_x)\otimes\mathcal{L}_x \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\mathrm{III}"] & (\mathcal{F}_x\otimes(\mathcal{G}_x\otimes\mathcal{H}_x))\otimes\mathcal{L}_x \arrow[d, Rightarrow] \arrow[drr, phantom, "\mathrm{VI}"] & & (\mathcal{F}_x\otimes\mathcal{G}_x)\otimes(\mathcal{H}_x\otimes\mathcal{L}_x) \arrow[d, Rightarrow] \arrow[drr, phantom, "\mathrm{V}"] & & \\
\mathcal{F}_x\otimes((\mathcal{G}\otimes\mathcal{H})_x\otimes\mathcal{L}_x) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \mathcal{F}_x\otimes((\mathcal{G}_x\otimes\mathcal{H}_x)\otimes\mathcal{L}_x) \arrow[rr, Rightarrow] \arrow[drrr, phantom, "\mathrm{II}"] & & \mathcal{F}_x\otimes(\mathcal{G}_x\otimes(\mathcal{H}_x\otimes\mathcal{L}_x)) & & \mathcal{F}_x\otimes(\mathcal{G}_x\otimes(\mathcal{H}\otimes\mathcal{L})_x) \arrow[ll, Rightarrow] \\
 & \mathcal{F}_x\otimes((\mathcal{G}\otimes\mathcal{H})\otimes\mathcal{L})_x \arrow[rrr, Rightarrow] & & & \mathcal{F}_x\otimes(\mathcal{G}\otimes(\mathcal{H}\otimes\mathcal{L}))_x \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Squares I, II commute since they are diagrams given by the tensor product of $\mathcal{L}_x$ (resp. $\mathcal{F}_x$) with commutative diagrams (associator on stalks). (a) commutes trivially. (VI) commutes due to the pentagon for the associator of $\mathcal{O}_{X,x}$-modules, IV, III, V respectively due to the naturality of the module associator in the first, second and third variable. That is, for a ring R and R-modules A, B, C, D the following commute:

$$
\begin{tikzcd}
 & ((A\otimes B)\otimes C)\otimes D \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, Rightarrow] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, Rightarrow] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, Rightarrow] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}
$$

$\phi: A \longrightarrow A'$

$$
\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, Rightarrow] \arrow[d, "(\phi\otimes 1)\otimes 1"'] & A\otimes(B\otimes C) \arrow[d, "\phi\otimes 1"] \\
(A'\otimes B)\otimes C \arrow[r, Rightarrow] & A'\otimes(B\otimes C)
\end{tikzcd}
$$

$\phi: B \longrightarrow B'$

$$
\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, Rightarrow] \arrow[d, "(1\otimes\phi)\otimes 1"'] & A\otimes(B\otimes C) \arrow[d, "1\otimes(\phi\otimes 1)"] \\
(A\otimes B')\otimes C \arrow[r, Rightarrow] & A\otimes(B'\otimes C)
\end{tikzcd}
$$

$\phi: C \longrightarrow C'$

$$
\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, Rightarrow] \arrow[d, "1\otimes\phi"] & A\otimes(B\otimes C) \arrow[d, "1\otimes(1\otimes\phi)"'] \\
(A\otimes B)\otimes C' \arrow[r, Rightarrow] & A\otimes(B\otimes C')
\end{tikzcd}
$$

This completes the proof that (1) commutes.

NOTE Compatibility of the Associator and $\sim$

Let S be a graded ring, (generated by S_1 as an S_0-algebra) M, N, T graded S-modules. The canonical isomorphism $M \otimes_S (N \otimes_S T) \Longrightarrow (M \otimes_S N) \otimes_S T$ of S-modules is clearly an isomorphism of graded S-modules. We claim that if $X = \operatorname{Proj} S$ then the associator for $\mathcal{O}_X$-modules is compatible with $\sim$ and the associator for graded S-modules, in the sense that the following commutes (all morphisms canonical)

$$\begin{tikzcd}
(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \otimes_{\mathcal{O}_X} \widetilde{T} \arrow[r, Rightarrow, "\lambda_X"] \arrow[d, Rightarrow, "\alpha_{M,N} \otimes 1"'] & \widetilde{M} \otimes_{\mathcal{O}_X} (\widetilde{N} \otimes_{\mathcal{O}_X} \widetilde{T}) \arrow[d, Rightarrow, "1 \otimes \alpha_{N,T}"] \\
\widetilde{M \otimes_S N} \otimes_{\mathcal{O}_X} \widetilde{T} \arrow[d, Rightarrow, "\alpha_{M \otimes N, T}"'] & \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{(N \otimes_S T)} \arrow[d, Rightarrow, "\alpha_{M, N \otimes T}"] \\
\widetilde{(M \otimes_S N) \otimes_S T} \arrow[r, Rightarrow, "\widetilde{\lambda_S}"'] & \widetilde{M \otimes_S (N \otimes_S T)}
\end{tikzcd} \qquad (1)$$

But given $U \subseteq X$, $s \in ((\widetilde{M} \otimes \widetilde{N}) \otimes \widetilde{T})(U)$, $s(\mathfrak{p}) = (V, \sum_i a_i \otimes b_i)$, $a_i \in (\widetilde{M} \otimes \widetilde{N})(V)$, $b_i \in \widetilde{T}(V)$, with say $a_i(\mathfrak{p}) = (V, \sum_j c_{ij} \otimes d_{ij})$, $c_{ij} \in \widetilde{M}(V)$, $d_{ij} \in \widetilde{N}(V)$, $c_{ij}(\mathfrak{p}) = m_{ij}/s_{ij} \in M_{(\mathfrak{p})}$ $d_{ij}(\mathfrak{p}) = n_{ij}/t_{ij} \in N_{(\mathfrak{p})}$ and $b_i(\mathfrak{p}) = k_i/e_i \in T_{(\mathfrak{p})}$. Then

$$\begin{aligned}
(\widetilde{\lambda}_S \alpha_{M \otimes N, T}(\alpha_{M,N} \otimes 1))_U(s)(\mathfrak{p}) &= \sum_{i,j} (\lambda_S)_{(\mathfrak{p})} \left(\frac{(m_{ij} \otimes n_{ij}) \otimes k_i}{s_{ij} t_{ij} e_i} \right) \\
&= \sum_{i,j} \frac{m_{ij} \otimes (n_{ij} \otimes k_i)}{s_{ij} t_{ij} e_i} \\
&= (\alpha_{M, N \otimes T}(1 \otimes \alpha_{N,T}) \lambda_X)_U(s)(\mathfrak{p})
\end{aligned}$$

as required. As an example, we show that for $n, e, d \in \mathbb{Z}$ the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{S(n)} \otimes \widetilde{S(e)}) \otimes \widetilde{S(d)} \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \widetilde{S(n)} \otimes (\widetilde{S(e)} \otimes \widetilde{S(d)}) \arrow[d, Rightarrow] \\
\widetilde{S(n+e)} \otimes \widetilde{S(d)} \arrow[dr, Rightarrow] & & \widetilde{S(n)} \otimes \widetilde{S(e+d)} \arrow[dl, Rightarrow] \\
& \widetilde{S(n+e+d)} &
\end{tikzcd} \qquad (2)$$

When we put in more detail, we get:

$$\begin{tikzcd}
& (\widetilde{S(n)} \otimes \widetilde{S(e)}) \otimes \widetilde{S(d)} \arrow[rr, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dl, phantom, shift left=4, "\bullet"] & & \widetilde{S(n)} \otimes (\widetilde{S(e)} \otimes \widetilde{S(d)}) \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[dr, phantom, shift right=4, "\bullet"] & \\
\widetilde{S(n) \otimes S(e)} \otimes \widetilde{S(d)} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\mathrm{I}"] & \widetilde{S(n+e)} \otimes \widetilde{S(d)} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[dr, phantom, shift right=4, "\bullet"] & & \widetilde{S(n)} \otimes \widetilde{S(e+d)} \arrow[d, Rightarrow] \arrow[dl, Rightarrow] \arrow[dl, phantom, shift left=4, "\bullet"] & \widetilde{S(n)} \otimes \widetilde{S(e) \otimes S(d)} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \arrow[dl, phantom, "\mathrm{II}"] \\
\widetilde{(S(n) \otimes S(e)) \otimes S(d)} \arrow[r, Rightarrow] \arrow[rrrr, Rightarrow, bend right, "\mathrm{III}"'] & \widetilde{S(n+e) \otimes S(d)} \arrow[r, Rightarrow] & \widetilde{S(n+e+d)} & \widetilde{S(n) \otimes S(e+d)} \arrow[l, Rightarrow] & \widetilde{S(n) \otimes (S(e) \otimes S(d))} \arrow[l, Rightarrow]
\end{tikzcd}$$

The triangles marked $\bullet$ commute by definition, the squares I, II by naturality of $\widetilde{-} \otimes \widetilde{-} \Rightarrow \widetilde{- \otimes -}$ and III since it is the image under $\sim$ of a commutative diagram. Hence since the outside commutes by (1), (2) commutes.

More generally, if $\mathcal{F}$ is any $\mathcal{O}_X$-module and $\kappa^{n,d} : \mathcal{F}(n) \otimes \mathcal{O}_X(d) \Longrightarrow \mathcal{F}(n+d)$ denotes the canonical isomorphism defined earlier, the following diagram commutes, for any $n, e, d \in \mathbb{Z}$:

$$
\begin{tikzcd}
(\mathcal{F}(n) \otimes \mathcal{O}_X(e)) \otimes \mathcal{O}_X(d) \arrow[rr, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\kappa^{n,e} \otimes 1"'] & & \mathcal{F}(n) \otimes (\mathcal{O}_X(e) \otimes \mathcal{O}_X(d)) \arrow[d, Rightarrow] \\
\mathcal{F}(n+e) \otimes \mathcal{O}_X(d) \arrow[dr, Rightarrow, "\kappa^{n+e,d}"'] & & \mathcal{F}(n) \otimes \mathcal{O}_X(e+d) \arrow[dl, Rightarrow, "\kappa^{n,e+d}"] \\
& \mathcal{F}(e+n+d) &
\end{tikzcd}
\tag{3}
$$

We redraw this diagram as follows:

$$
\begin{tikzcd}[column sep=small]
& & ((\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(e)) \otimes \mathcal{O}_X(d) \arrow[dll, Rightarrow] \arrow[drr, Rightarrow] & & \\
(\mathcal{F} \otimes (\mathcal{O}_X(n) \otimes \mathcal{O}_X(e))) \otimes \mathcal{O}_X(d) \arrow[rrrr, phantom, "P"] \arrow[ddd, Rightarrow] \arrow[dr, Rightarrow] & & & & (\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes (\mathcal{O}_X(e) \otimes \mathcal{O}_X(d)) \arrow[dl, Rightarrow] \arrow[ddd, Rightarrow] \\
& \mathcal{F} \otimes ((\mathcal{O}_X(n) \otimes \mathcal{O}_X(e)) \otimes \mathcal{O}_X(d)) \arrow[rr, Rightarrow] \arrow[ddd, Rightarrow] \arrow[ddrr, phantom, "\mathrm{O}"] & & \mathcal{F} \otimes (\mathcal{O}_X(n) \otimes (\mathcal{O}_X(e) \otimes \mathcal{O}_X(d))) \arrow[ddd, Rightarrow] & \\
& & & & \\
(\mathcal{F} \otimes \mathcal{O}_X(n+e)) \otimes \mathcal{O}_X(d) \arrow[uur, phantom, "\mathrm{I}"] \arrow[dr, Rightarrow] & & & & (\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(e+d) \arrow[uul, phantom, "\mathrm{II}"] \arrow[dl, Rightarrow] \\
& \mathcal{F} \otimes (\mathcal{O}_X(n+e) \otimes \mathcal{O}_X(d)) \arrow[dr, Rightarrow] & & \mathcal{F} \otimes (\mathcal{O}_X(n) \otimes \mathcal{O}_X(e+d)) \arrow[dl, Rightarrow] & \\
& & \mathcal{F} \otimes \mathcal{O}_X(n+e+d) & &
\end{tikzcd}
$$

The pentagon (P) is the associator pentagon, which commutes by earlier notes. The pentagon (O) is the functor $\mathcal{F} \otimes -$ applied to diagram (2), hence commutes. The squares I, II commute by naturality of the associator in the second and third variable. Hence (3) commutes.

generated by S_1 as an S_0-algebra.

DEFINITION Let S be a graded ring, let $X = \operatorname{Proj} S$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the graded S-module associated to $\mathcal{F}$ as a group to be

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \qquad (1)$$

We give it a structure of graded S-module as follows. If $s \in S_d$ then s determines a global section $\dot{s} \in \Gamma(X, \mathcal{O}_X(d))$ defined by $\dot{s}(\mathfrak{p}) = s/1$. Then for any $t \in \Gamma(X, \mathcal{F}(n))$ we define

$$s \cdot t = K_X(t \otimes \dot{s}) \in \Gamma(X, \mathcal{F}(n+d)) \qquad (2)$$

where K is the isomorphism defined on the previous page. If $s \in S$ and $m \in \Gamma_*(\mathcal{F})$ then let $m_i \in \Gamma(X, \mathcal{F}(i))$ denote the element of the sequence at position i. We define

$$s \cdot m = \sum_{\substack{d\geq 0\\ i\in\mathbb{Z}}} u_{d+i}(s_d \cdot m_i) \qquad \ulcorner u \text{ – injections in (1)} \lrcorner$$

or equivalently

$$(s\cdot m)_i = \sum_{\substack{d\geq 0\\ j\in\mathbb{Z}\\ d+j=i}} s_d \cdot m_j \qquad (3)$$

First one checks in (2) that for homogenous $s^{(\prime)}, t^{(\prime)}$ $s\cdot(t+t') = s\cdot t + s\cdot t'$, $(s+s')\cdot t = s\cdot t + s'\cdot t$ and for homogenous $s \in S_d$, $r \in S_e$, $t \in \Gamma(X, \mathcal{F}(n))$ $s\cdot(r\cdot t) = (sr)\cdot t$. The last is the only one that is difficult. For $n, d \in \mathbb{Z}$ let $K^{n,d} : \mathcal{F}(n)(d) \Rightarrow \mathcal{F}(n+d)$ be the map of the previous pages. We must show that

$$K_X^{e+n,d}\left(K_X^{n,e}(t \otimes \dot{r}) \otimes \dot{s}\right) = K_X^{n,e+d}(t \otimes \dot{(sr)})$$

But this follows from commutativity of (3) in "Compatibility of Assoc. and ~" on the global section $((t \otimes \dot{r}) \otimes \dot{s})$ of $(\mathcal{F}(n) \otimes \mathcal{O}_X(e)) \otimes \mathcal{O}_X(d)$.

Just by playing with indices in (3) it now follows for arbitrary $s, s' \in S$ and $m, m' \in \Gamma_*(\mathcal{F})$ that $s\cdot(m+m') = s\cdot m + s\cdot m'$, $(s+s')\cdot m = s\cdot m + s'\cdot m$ and $(ss')\cdot m = s\cdot(s'\cdot m)$. Clearly $1\cdot m = m$. Hence $\Gamma_*(\mathcal{F})$ is a graded S-module. Clearly (3) reduces to (2) for s, m homogenous.

If $X = \emptyset$ then $\Gamma_*(\mathcal{F}) = 0$. Also note that if $f \in S_e$ $(e > 0)$ and S_f is given the canonical $\mathbb{Z}$-grading, then for $n \in \mathbb{Z}$ we have $S(n)_{(f)} = (S_f)_n$ as subgroups of S_f.

PROPOSITION 5.13 Let A be a ring, $S = A[x_0, \dots, x_r]$ $r \geq 1$, and let $X = \operatorname{Proj} S$. (This is projective r-space over A). Then $\Gamma_*(\mathcal{O}_X) \cong S$, as graded S-modules.

PROOF If $A = 0$ then $S = 0$ and trivially $\Gamma_*(\mathcal{O}_X) \cong S$. Otherwise $S \neq 0$ and $X = \emptyset$ since x_0 is not nilpotent, hence $D_+(x_0) \neq \emptyset$. First we define a graded S-module Ω as follows:

$$\Omega = \{(t_0, \dots, t_r) \mid \forall i \ t_i \in S_{x_i} \text{ and } \forall i,j \text{ the images of } t_i \text{ and } t_j \text{ in } S_{x_i x_j} \text{ are the same}\}$$

For i, j the map $\phi : S_{x_i} \longrightarrow S_{x_i x_j}$ is defined by $\phi(a/x_i^n) = x_j^n a/(x_i x_j)^n$. This is a morphism of $\mathbb{Z}$-graded rings (canonical $\mathbb{Z}$-gradings) and is clearly a morphism of graded S-modules. Define the S-module structure on Ω by $s\cdot(t_0, \dots, t_r) = (st_0, \dots, st_r)$ with pointwise addition. One checks these are all well-defined. Define the grading on Ω by $(n \in \mathbb{Z})$

$$\Omega_n = \{(t_0, \dots, t_r) \mid \forall i \ t_i \in (S_{x_i})_n \text{ and } \forall i,j \text{ the images of } t_i, t_j \text{ agree in } S_{x_i x_j}\}$$

Each Ω_n is clearly a subgroup of Ω, and given $(t_0,\dots,t_r) \in \Omega$ we have $((t_0)_n,\dots,(t_r)_n) \in \Omega_n$ for all $n \in \mathbb{Z}$, and clearly $(t_0,\dots,t_r) = \sum_n ((t_0)_n,\dots,(t_r)_n)$. It is clear that this expression is unique, so Ω is indeed a graded S-module.

For all i,j we have the following commutative diagram for $n \in \mathbb{Z}$ (Note the deceptive notation: $\Gamma_*(\mathcal{O}_X)$ means $\bigoplus_{n\in\mathbb{Z}} \Gamma(X, (\mathcal{O}_X \otimes \mathcal{O}_X(n)))$, not $\mathcal{O}_X(n)$, although the two are isomorphic)

$$\begin{tikzcd}
(\mathcal{O}_X \otimes \mathcal{O}_X(n))(D_+(x_i)) \arrow[d] & S(n)_{(x_i)} \arrow[l, Rightarrow, "\alpha_i^n"'] \arrow[d] \\
(\mathcal{O}_X \otimes \mathcal{O}_X(n))(D_+(x_ix_j)) & S(n)_{(x_ix_j)} \arrow[l, Rightarrow, "\tilde{\alpha_j}"]
\end{tikzcd}$$

$a/x_i^n \mapsto x_j^n a/(x_ix_j)^n$

$\ulcorner \alpha_i^n$ is $S(n)_{(x_i)} \cong \widetilde{S(n)_{(x_i)}}(\mathrm{Spec}) \cong \widetilde{S(n)}(D_+(x_i)) \cong (\mathcal{O}_X \otimes \widetilde{S(n)})(D_+(x_i))\lrcorner$

$\ulcorner \hat{\mathcal{O}}_X(n) = \mathcal{O}_X \otimes \mathcal{O}_X(n)\lrcorner$

Since the $D_+(x_i)$ cover X, to give a global section $t \in \Gamma(X, \hat{\mathcal{O}}_X(n))$ is equivalent to giving sections $t_i \in \hat{\mathcal{O}}_X(n)(D_+(x_i))$ agreeing on $D_+(x_ix_j)$ $\forall i,j$. But this is equivalent to giving tuples $(t_0,\dots,t_r)$ with each $t_i \in S(n)_{(x_i)} = (S_{x_i})_n$ with t_i, t_j agreeing in $S_{x_ix_j}$. So there is a bijection $\Omega_n \xrightarrow{\alpha^n} \Gamma(X, \hat{\mathcal{O}}_X(n))$ defined by mapping $(t_0,\dots,t_r)$ to the unique t with $t|_{D_+(x_i)} = \alpha_i^n(t_i)$ $\forall i$. That is,

$$\alpha^n : \Omega_n \longrightarrow \Gamma(X, \hat{\mathcal{O}}_X(n))$$
$$\alpha^n(t_0,\dots,t_r)|_{D_+(x_i)} = \alpha_i^n(t_i) \qquad \forall\, 0 \le i \le r$$

One checks easily that this is an isomorphism of abelian groups, since the α_i^n are. Thus we obtain an isomorphism of abelian groups

$$\beta : \Omega \longrightarrow \Gamma_*(\mathcal{O}_X)$$
$$\beta(t_0,\dots,t_r)_n \in \Gamma(X, \hat{\mathcal{O}}_X(n))$$
$$\beta(t_0,\dots,t_r)_n|_{D_+(x_i)} = \alpha_i^n((t_i)_n)$$

The map β clearly respects grade. So to show $\Omega \cong \Gamma_*(\mathcal{O}_X)$ as graded S-modules, it only remains to show that for $s \in S$ and $(t_0,\dots,t_r) \in \Omega$, $\beta(st_0,\dots,st_r) = s \cdot \beta(t_0,\dots,t_r)$. It suffices to prove this in the case where $s \in S_d$ and $(t_0,\dots,t_r) \in \Omega_n$ are homogenous. Then both sides are homogenous of degree $d+n$, with

$$\beta(st_0,\dots,st_r)_{d+n}|_{D_+(x_i)} = \alpha_i^{d+n}((st_i)_{d+n}) = \alpha_i^{d+n}(st_i)$$

$$(s \cdot \beta(t_0,\dots,t_r))_{d+n} = s \cdot \beta(t_0,\dots,t_r)_n = K_X(\beta(t_0,\dots,t_r)_n \mathbin{\dot\otimes} \dot{s})$$

Hence $(s \cdot \beta(t_0,\dots,t_r))_{d+n}|_{D_+(x_i)} = K_{D_+(x_i)}(\beta(t_0,\dots,t_r)_n|_{D_+(x_i)} \mathbin{\dot\otimes} \dot{s}|_{D_+(x_i)}) = K_{D_+(x_i)}(\alpha_i^n((t_i)_n) \mathbin{\dot\otimes} \dot{s}|_{D_+(x_i)})$
And we have reduced to showing that

$$\alpha_i^{d+n}(st_i) = K_{D_+(x_i)}(\alpha_i^n(t_i) \mathbin{\dot\otimes} \dot{s}|_{D_+(x_i)})$$

Suppose that $t_i = a/x_i^r \in S_{x_i}$ (so $a \in S_{r+n}$), and let $p \in D_+(x_i)$ be given. Then

$$\begin{aligned} K_{D_+(x_i)}(\alpha_i^n(t_i) \mathbin{\dot\otimes} \dot{s}|_{D_+(x_i)})(p) &= (D_+(x_i), 1 \otimes T_{D_+(x_i)}(a/x_i^r \otimes s/1)) \\ &= (D_+(x_i), 1 \otimes as/x_i^r) \\ &= \alpha_i^{d+n}(st_i)(p) \end{aligned}$$

since $\alpha_i^n(t_i)(p) = (D_+(x_i), 1 \otimes a/x_i^r)$. This completes the proof that β is an isomorphism of graded S-modules.

The x_i are not zero divisors in S, so we get injective maps $S \longrightarrow S_{x_i}$, $S_{x_i} \longrightarrow S_{x_i x_j}$ and

$$\begin{aligned} \Upsilon_i : S_{x_i} &\longrightarrow S_{x_0 \cdots x_r} \\ a/x_i^n &\longmapsto \frac{a \prod_{j \neq i} x_j^n}{(x_0 \cdots x_r)^n} \end{aligned}$$

$\ulcorner S \to S_{x_i}$ epi$\lrcorner$

It is easy to check that $\forall i, j$ $\; S_{x_i} \longrightarrow S_{x_i x_j} \longrightarrow S_{x_0 \cdots x_r} = S_{x_i} \longrightarrow S_{x_0 \cdots x_r}$. It follows that

$$\begin{aligned} \Upsilon : \Omega &\longrightarrow S_{x_0 \cdots x_r} \\ \Upsilon(t_0, \ldots, t_r) &= \Upsilon_0(t_0) \end{aligned}$$

is an injective morphism (note $\Upsilon_i(t_i) = \Upsilon_0(t_0)$ by the condition on tuples in Ω). of S-modules. One checks that with the canonical $\mathbb{Z}$-grading on $S_{x_0 \cdots x_r}$, Υ is actually a morphism of graded S-modules. Considering the S_{x_i} as subrings of $S_{x_0 \cdots x_r}$ it is clear that the image of Υ is the intersection $\bigcap S_{x_i}$. Suppose that $a/x_i^n \in S_{x_i} \cap S_{x_j}$, so there is $b/x_j^e \in S_{x_j}$ with $(i \neq j)$

$$\frac{a \prod_{s \neq i} x_s^n}{(x_0 \cdots x_r)^n} = \frac{b \prod_{s \neq j} x_s^e}{(x_0 \cdots x_r)^e}$$

$$\therefore \quad a x_i^e \prod_{s \neq i} x_s^{n+e} = b x_j^n \prod_{s \neq j} x_s^{n+e}$$

So either the monomials are distinct and $a = b = 0$ or they are equal, implying $e = e + n$ and $n = n + e$, so $n = e = 0$. Hence as subrings of $S_{x_0 \cdots x_r}$ we have $S = S_{x_i} \cap S_{x_j}$ for any $i \neq j$. So there is an isomorphism of S-modules

$$\begin{aligned} S &\longrightarrow \Omega \\ a &\mapsto (a/1, \ldots, a/1) \end{aligned}$$

This is clearly an isomorphism of graded S-modules. Putting these together we have an isomorphism of graded S-modules

$$\begin{aligned} \eta : S &\longrightarrow \Gamma_*(\mathcal{O}_X) \\ \eta(a)_n|_{D_+(x_i)} &= \alpha_i^n(a_n/1) \end{aligned}$$

$$\therefore \quad \eta(a)_n = 1 \dot{\otimes} \dot{a}_n$$

$n \in \mathbb{Z}$
$0 \leq i \leq r$
$\alpha_i^n : S(n)_{(x_i)} \Rightarrow (\mathcal{O}_X \otimes \mathcal{O}_X(n))(D_+(x_i))$
canonical.

This completes the proof. $\square$

<u>CAUTION</u> If S is a graded ring which is not a polynomial ring, then it is not true in general that $\Gamma_*(\mathcal{O}_X) = S$.

At this point you should consult our typed notes “The Exponential Tensor Product”.

(EGA I, 9.3.1)

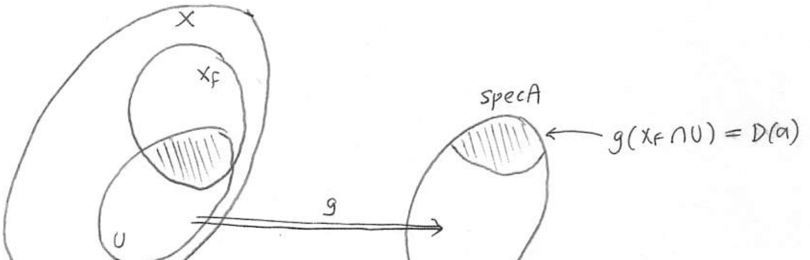
LEMMA 5.14 Let X be a scheme, let $\mathscr{L}$ be an invertible sheaf on X, let $f \in \Gamma(X, \mathscr{L})$, let X_f be the open set of points where $f_x \notin \mathfrak{m}_x \mathscr{L}_x$ and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then

(a) Suppose that X is quasi-compact, and let $s \in \Gamma(X, \mathscr{F})$ be a global section of $\mathscr{F}$ whose restriction to X_f is 0. Then for some $n > 0$ we have $f^n s = 0$, where $f^n s$ denotes the global section $s \otimes (f \otimes \cdots \otimes f)$ of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.

(b) Suppose furthermore that X has a finite covering by open affine subsets U_i such that $\mathscr{L}|_{U_i}$ is free for each i, and such that $U_i \cap U_j$ is quasi-compact for each i, j. Given a section $t \in \Gamma(X_f, \mathscr{F})$, then for some $n > 0$, the section $f^n t \in \Gamma(X_f, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ extends to a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.

PROOF To be clear on notation, $\mathscr{L}^{\otimes n} = \mathscr{L} \otimes (\mathscr{L} \otimes \cdots (\mathscr{L} \otimes \mathscr{L}) \cdots)$ with n-copies. If $U \subseteq X$ is open and $s \in \mathscr{F}(U)$, $f \in \mathscr{L}(U)$ then $f \dot\otimes f \in (\mathscr{L} \otimes \mathscr{L})(U)$, and $f \dot\otimes (f \dot\otimes f) \in \mathscr{L} \otimes (\mathscr{L} \otimes \mathscr{L})$. Inductively we define a section of $\mathscr{L}^{\otimes n}(U)$ which we denote f^n, or $f \otimes \cdots \otimes f$. We denote $s \dot\otimes f^n \in \mathscr{F} \otimes \mathscr{L}^{\otimes n}(U)$ by $f^n s$.

(a) Since X is quasi-compact, we can cover X with a finite number of open affines $U \overset{g}{\Rightarrow} \operatorname{Spec} A$ such that $\mathscr{L}|_U$ is free. Let $\Psi : \mathscr{L}|_U \Rightarrow \mathcal{O}_X|_U$ be an isomorphism expressing the freeness of $\mathscr{L}|_U$. Since $\mathscr{F}$ is quasi-coherent by (5.4) there is an A-module M with $g_* \mathscr{F}|_U \cong \widetilde{M}$. Our section $s \in \Gamma(X, \mathscr{F})$ restricts to give an element $s \in M$. On the other hand $f \in \Gamma(X, \mathscr{L})$ restricts to give a section of $\mathscr{L}|_U$, which in turn gives rise to an element $h = \Psi_U(f|_U) \in \mathcal{O}_X(U)$ and a corresponding element $a \in A$.



We have shown earlier that $g(X_f \cap U) = D(a)$. The fact that $s|_{X_f} = 0$ means that $a^n s = 0$ for some $n > 0$. Since there are a finite number of U, we can assume n is large enough to work for them all. But $a^n s = 0$ implies that $h^n s|_U = 0$. Consider the isomorphisms

$$(\mathscr{F} \otimes \mathscr{L}^{\otimes n})|_U \cong \mathscr{F}|_U \otimes (\mathscr{L}^{\otimes n})|_U \cong \mathscr{F}|_U \otimes (\mathscr{L}|_U)^{\otimes n}$$
$$\cong \mathscr{F}|_U \otimes (\mathcal{O}_X|_U)^{\otimes n} \cong \mathscr{F}|_U$$

which map $s|_U \dot\otimes (f|_U)^{\otimes n} \in (\mathscr{F} \otimes \mathscr{L}^{\otimes n})(U)$ to $s|_U \dot\otimes \Psi_U(f|_U)^{\otimes n} \longmapsto \Psi_U(f|_U)^n \cdot s|_U$. Of course $\Psi_U(f|_U) = h$, so $s|_U \dot\otimes (f|_U)^{\otimes n} = 0$. But $(s \dot\otimes f^{\otimes n})|_U = s|_U \dot\otimes (f|_U)^{\otimes n}$ and n works for every U, so we have $s \dot\otimes f^{\otimes n} = 0$ in $(\mathscr{F} \otimes \mathscr{L}^{\otimes n})(X)$.

(b) Let $t \in \Gamma(X_f, \mathscr{F})$ be given and isomorphisms $\Psi_i : \mathscr{L}|_{U_i} \Rightarrow \mathcal{O}_X|_{U_i}$, $g_i : U_i \Rightarrow \operatorname{Spec} A_i$. Let $h_i = (\Psi_i)_{U_i}(f|_{U_i}) \in \mathcal{O}_X(U_i)$ and $a_i \in A_i$ correspond to h_i. Since $\mathscr{F}$ is quasi-coherent there are A_i-modules M_i for each i s.t. $g_{i*} \mathscr{F}|_{U_i} \cong \widetilde{M_i}$. Let $t_i \in \widetilde{M_i}(D(a_i))$ correspond to $t|_{X_f \cap U_i} \in \mathscr{F}(X_f \cap U_i)$. Since $\widetilde{M_i}(D(a_i)) \cong (M_i)_{a_i}$ there is $n_i > 0$ and $q_i \in M$ s.t. $a_i^{n_i}|_{D(a_i)} \cdot t_i = q_i|_{D(a_i)}$. Hence if $m_i \in \mathscr{F}(U_i)$ corresponds to q_i, we have

$$h_i|_{X_f \cap U_i}{}^{n_i} \cdot t|_{U_i \cap X_f} = m_i|_{X_f \cap U_i} \in \mathscr{F}(X_f \cap U_i) \qquad (1)$$

As usual we can modify the m_i so that this works for a single $N > 0$ independent of i. By assumption the scheme $X_{ij} = U_i \cap U_j$ is quasi-compact, and $\mathscr{L}_{ij} = \mathscr{L}|_{U_i \cap U_j}$ is invertible, $\mathscr{F}_{ij} = \mathscr{F}|_{U_i \cap U_j}$ quasi-coherent. If $f_{ij} = f|_{U_i \cap U_j} \in \Gamma(X_{ij}, \mathscr{L}_{ij})$ then $X_{f_{ij}} = X_f \cap U_i \cap U_j$ and the element $m_i|_{U_i \cap U_j} - m_j|_{U_i \cap U_j}$ of $\mathscr{F}_{ij}(X_{ij})$ restricts to 0 on $X_{f_{ij}}$. This requires some work. Under the isomorphism

$$(\mathscr{F} \otimes \mathscr{L}^N)|_{U_i} \cong \mathscr{F}|_{U_i} \otimes (\mathscr{L}^{\otimes N})|_{U_i} \cong \mathscr{F}|_{U_i} \otimes (\mathcal{O}_X|_{U_i})^{\otimes N} \cong \mathscr{F}|_{U_i}$$

The element $t \dot\otimes f|_{X_f}^{\otimes N} \in (\mathscr{F} \otimes \mathscr{L}^{\otimes N})(X_f)$ restricts to $X_f \cap U_i$ and then maps to $h_i|_{X_f \cap U_i}{}^N \cdot t|_{U_i \cap X_f}$ $= m_i|_{X_f \cap U_i}$. NO. See printout.

quasi-compact

COROLLARY Let X be a scheme, $\mathcal{L}$ an invertible sheaf and $f \in \Gamma(X,\mathcal{L})$. Then $X_f = \phi$ iff. $f^{\otimes n} = 0$ in $\mathcal{L}^{\otimes n}$ for some $n > 0$.

PROOF Put $\mathcal{F} = \mathcal{O}_X$ and $s = 1$. Then (5.14)(a) shows that $X_f = \phi \Rightarrow f^{\otimes n} = 0$ some $n > 0$. Now suppose $f^{\otimes n} = 0$ in $\mathcal{L}^{\otimes n}$ for some $n > 0$ and suppose $X_f \neq \phi$. If $x \in X_f$ there is $x \in U \subseteq X$ with $f: U \cong \operatorname{Spec} A$ such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. Then if $h \in \mathcal{O}_X(U)$ corresponds to $f \in \mathcal{L}(U)$ we have $h^n = 0$. But we showed in (5.14)(a) that $f(X_f \cap U) = D(h)$, which is empty, contradicting $x \in X_f$. $\square$

REMARK 5.14.1 The hypotheses on X made in (a) and (b) above are satisfied either if X is noetherian (in which case every open set is quasi-compact) or if X is quasi-compact and separated (in which case the intersection of two affine open sets is affine, hence quasi-compact). In other words, if X is a scheme which is either noetherian or quasi-compact and separated, $\mathcal{L}$ an invertible sheaf on X, $f \in \Gamma(X,\mathcal{L})$ and $\mathcal{F}$ quasi-coherent, then

(a) If $s \in \Gamma(X,\mathcal{F})$ with $s|_{X_f} = 0$ then $s \otimes f^{\otimes n} = 0$ in $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for some $n > 0$.

(b) If $t \in \Gamma(X_f, \mathcal{F})$ then $t \otimes f^{\otimes n}$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for some $n > 0$.

PROPOSITION 5.15 Let S be a graded ring which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$ and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

PROOF see the following note. $\square$

At this point you should consult our typed notes "An Adjunction For Modules over $ProjS$".

LEMMA Let S be a graded ring, $\mathfrak{a} \subseteq S$ a homogenous ideal. Since $\sim : S\text{-}\underline{\mathrm{GrMod}} \longrightarrow \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$ is exact ($X = \operatorname{Proj} S$) the morphism $\tilde{\mathfrak{a}} \longrightarrow \mathcal{O}_X$ is a monomorphism. Let $f : \operatorname{Proj} S/\mathfrak{a} \longrightarrow X$ be the closed immersion associated to $\mathfrak{a}$. The ideal sheaf $\mathcal{J}$ of f is equivalent to $\tilde{\mathfrak{a}}$ as subobjects of $\mathcal{O}_X$.

PROOF There is an exact sequence of graded S-modules $0 \longrightarrow \mathfrak{a} \longrightarrow S \longrightarrow S/\mathfrak{a} \longrightarrow 0$. Hence we have an exact sequence $0 \longrightarrow \tilde{\mathfrak{a}} \longrightarrow \mathcal{O}_X \longrightarrow (S/\mathfrak{a})^{\sim} \longrightarrow 0$. Since f is a closed immersion, the morphism $f^{\#} : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ ($Y = \operatorname{Proj} S/\mathfrak{a}$) is an epimorphism in $\mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$. So it suffices to show $\mathcal{O}_X \longrightarrow (S/\mathfrak{a})^{\sim}$ and $f^{\#} : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ are equivalent quotients of $\mathcal{O}_X$. First we recall that the homogenous primes of $S/\mathfrak{a}$ are in bijection with the homogenous primes of S containing $\mathfrak{a}$, and

$$\varphi_{\mathfrak{p}} : (S/\mathfrak{a})_{(\mathfrak{p})} \longrightarrow (S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}$$
$$\varphi_{\mathfrak{p}}\left(\frac{s+\mathfrak{a}}{t}\right) = \frac{s+\mathfrak{a}}{t+\mathfrak{a}}$$

is an isomorphism of $S_{(\mathfrak{p})}$-modules for any homogenous prime $\mathfrak{p} \supseteq \mathfrak{a}$. The ring $(S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}$ is the graded ring $S/\mathfrak{a}$ localised at $\mathfrak{p}/\mathfrak{a}$, while $(S/\mathfrak{a})_{(\mathfrak{p})}$ denotes the graded S-module S localised at $\mathfrak{p}$. Here $(S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}$ is a $S_{(\mathfrak{p})}$-module via $S_{(\mathfrak{p})} \longrightarrow (S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}$. We define a morphism of $\mathcal{O}_X$-modules $\Phi : (S/\mathfrak{a})^{\sim} \longrightarrow f_* \mathcal{O}_Y$ by

$$\Phi_V(s)(\mathfrak{p}/\mathfrak{a}) = \varphi_{\mathfrak{p}}(s(\mathfrak{p}))$$

It is easy to see that $\Phi_V(s)$ actually belongs to $\mathcal{O}_Y(f^{-1}V)$. It is not difficult to check that Φ is a morphism of modules making the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr] \arrow[dr] & & (S/\mathfrak{a})^{\sim} \arrow[dl, "\Phi"] \\
& f_* \mathcal{O}_Y &
\end{tikzcd}$$

So it only remains to show that Φ is an isomorphism, for which it suffices to show that $\Phi_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in X$. The closed image of $f : \operatorname{Proj} S/\mathfrak{a} \longrightarrow \operatorname{Proj} S$ is $V(\mathfrak{a})$, and for $\mathfrak{p} \notin V(\mathfrak{a})$ we have $(f_* \mathcal{O}_Y)_{\mathfrak{p}} = 0$. Now $(S/\mathfrak{a})^{\sim}_{\mathfrak{p}} \cong (S/\mathfrak{a})_{(\mathfrak{p})}$ which is zero iff. $\mathfrak{a} \nsubseteq \mathfrak{p}$ so for $\mathfrak{p} \notin V(\mathfrak{a})$ we have $(S/\mathfrak{a})^{\sim}_{\mathfrak{p}} = 0$ as well. So for $\mathfrak{p} \notin V(\mathfrak{a})$, $\Phi_{\mathfrak{p}}$ is an isomorphism. For all $\mathfrak{p} \in V(\mathfrak{a})$ we have a commutative diagram

$$\begin{tikzcd}
(S/\mathfrak{a})^{\sim}_{\mathfrak{p}} \arrow[rr, "\Phi_{\mathfrak{p}}"] \arrow[d, Rightarrow] & & (f_* \mathcal{O}_Y)_{\mathfrak{p}} \arrow[d, Rightarrow] \\
(S/\mathfrak{a})_{(\mathfrak{p})} \arrow[rr, Rightarrow, "\varphi_{\mathfrak{p}}"'] & & (S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}
\end{tikzcd}$$

Which implies that $\Phi_{\mathfrak{p}}$ is an isomorphism, so Φ is an isomorphism and the proof is complete. $\square$

COROLLARY 5.16 Let A be a ring. Then

(a) If Y is a closed subscheme of $\mathbb{P}^r_A$ $(r \geq 1)$ then there is a homogeneous ideal $I \subseteq S = A[x_0, \dots, x_r]$ such that Y is the closed subscheme determined by I.

(b) A scheme Y over $\operatorname{Spec} A$ is projective if and only if it is isomorphic as a scheme over $\operatorname{Spec} A$ to $\operatorname{Proj} S$ for some graded A-algebra S with $S_0 = A$, and S finitely generated by S_1 as an S_0-algebra.

PROOF (a) Let $f : Y \longrightarrow X = \mathbb{P}^r_A$ be a closed immersion, with ideal sheaf $\mathscr{I}_Y$. Since $T_* : \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \longrightarrow S\,\underline{\mathrm{Gr\,Mod}}$ has a left adjoint it preserves monomorphisms, so $T_*(\mathscr{I}_Y) \longrightarrow T_*(\mathcal{O}_X)$ is monic. But by (5.13) $T_*(\mathcal{O}_X) \cong S$ as graded S-modules, so there is a homogeneous ideal $\mathfrak{a} \subseteq S$ and a commutative diagram

$$\begin{tikzcd} T_*(\mathscr{I}_Y) \arrow[r] & T_*(\mathcal{O}_X) \\ \mathfrak{a} \arrow[r] \arrow[u] & S \arrow[u, "\eta"'] \end{tikzcd}$$

Since $\mathscr{I}_Y$ and $\mathcal{O}_X$ are both quasi-coherent, by (5.15) the counits $T_*(\mathscr{I}_Y)^\sim \longrightarrow \mathscr{I}_Y$ and $T_*(\mathcal{O}_X)^\sim \longrightarrow \mathcal{O}_X$ are isomorphisms, and since the counit is natural we end up with a commutative diagram

$$\begin{tikzcd} \mathscr{I}_Y \arrow[r] & \mathcal{O}_X \\ T_*(\mathscr{I}_Y)^\sim \arrow[r] \arrow[u] & T_*(\mathcal{O}_X)^\sim \arrow[u, "\varepsilon"'] \\ \tilde{\mathfrak{a}} \arrow[r] \arrow[u] & \tilde{S} = \mathcal{O}_X \arrow[u, "\tilde{\eta}"'] \end{tikzcd} \qquad (1)$$

By the Lemma on the previous page $\tilde{\mathfrak{a}} \longrightarrow \mathcal{O}_X$ determines the same subobject as the ideal sheaf of $\operatorname{Proj} S/\mathfrak{a} \longrightarrow X$, so by (5.9) to show Y is the same closed subscheme as $\operatorname{Proj} S/\mathfrak{a}$ it suffices to show $\tilde{\mathfrak{a}}, \mathscr{I}_Y$ are the same subobject of $\mathcal{O}_X$. So we reduce to showing that $\varepsilon \tilde{\eta} = 1$ in (1). For this it suffices to show $\varepsilon_{\mathfrak{p}} \tilde{\eta}_{\mathfrak{p}} = 1 \ \forall \mathfrak{p} \in X$, and since $\tilde{S}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$ it suffices to show that $(\varepsilon \tilde{\eta})_{D_+(s)}(a/s) = a/s$ for homogeneous $a, s \in S$ of the same degree. First we notice that for $\mathfrak{p} \in D_+(x_i) \cap D_+(s)$ we have (a, s degree n)

$$\begin{aligned} \eta(a)_n(\mathfrak{p}) &= \alpha_i^n(a/1)(\mathfrak{p}) \\ &= (D_+(x_i), 1 \otimes q) \end{aligned}$$

where $q \in \widetilde{S(n)}(D_+(x_i))$ is $\mathfrak{p} \longmapsto a/1$

Hence for $\mathfrak{p} \in D_+(s)$, find some $0 \leq i \leq r$ with $\mathfrak{p} \in D_+(x_i)$. Then

$$\begin{aligned} \operatorname{germ}_{\mathfrak{p}} \varepsilon_{D_+(s)}(\tilde{\eta}_{D_+(s)}(a/s)) &= K_{\mathfrak{p}}(\eta(a)/s) \\ &= (D_+(s), \nu(1/s \cdot \eta(a)_n|_{D_+(s)})) \end{aligned}$$

By definition $1/s \cdot \eta(a)_n|_{D_+(s)} = K^{n,-n}_{D_+(s)}(\eta(a)_n|_{D_+(s)} \dot{\otimes} 1/s)$ and one checks that for $\mathfrak{p} \in D_+(x_i)$

$$\begin{aligned} K^{n,-n}_{D_+(s)}(\eta(a)_n|_{D_+(s)} \dot{\otimes} 1/s)(\mathfrak{p}) &= (D_+(s) \cap D_+(x_i), 1 \otimes T_{D_+(s) \cap D_+(x_i)}(q \dot{\otimes} 1/s)) \\ &= (D_+(s x_i), 1 \otimes a/s) \end{aligned}$$

So $1/s \cdot \eta(a)_n|_{D_+(s)} = 1 \dot{\otimes} a/s \in (\mathcal{O}_X \otimes \mathcal{O}_X)(D_+(s))$ so finally

$$\operatorname{germ}_{\mathfrak{p}} \varepsilon_{D_+(s)}(\tilde{\eta}_{D_+(s)}(a/s)) = \operatorname{germ}_{\mathfrak{p}} a/s$$

Since $\mathfrak{p} \in D_+(s)$ is arbitrary, $(\varepsilon \tilde{\eta})_{D_+(s)}(a/s) = a/s$ as required.

(b) Recall a graded A-algebra is a graded ring together with a ring morphism $A \longrightarrow S_0$, and our hypotheses are that (i) $A \longrightarrow S_0$ is an isomorphism (ii) S f.g. by S_1 as an S_0-algebra, or equivalently, S f.g. by S_1 as an A-algebra.

By Example 4.8.1 under these hypotheses $\operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ is projective. So if $Y \longrightarrow \operatorname{Spec} A$ is isomorphic to $\operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ as schemes over $\operatorname{Spec} A$, $Y \longrightarrow \operatorname{Spec} A$ will be projective. Conversely, suppose $Y \longrightarrow \operatorname{Spec} A$ is projective. Then for some $r \geq 1$, $Y \longrightarrow \operatorname{Spec} A$ factors as a closed immersion $Y \longrightarrow \mathbb{P}^r_A$ followed by $\mathbb{P}^r_A \longrightarrow \operatorname{Spec} A$. By (a) there is a homogenous ideal $I \subseteq S = A[x_0, \dots, x_r]$ such that $Y \longrightarrow \mathbb{P}^r_A$ is the same closed subscheme as $\operatorname{Proj} S/I \longrightarrow \mathbb{P}^r_A$, and thus the same closed subscheme as $\operatorname{Proj} S/I' \longrightarrow \mathbb{P}^r_A$ by Ex 3.12, where $I' = \bigoplus_{d\geq 0} I_d \subseteq S_+$. The graded A-algebra S/I' satisfies the necessary properties and $\operatorname{Proj} S/I' \longrightarrow \mathbb{P}^r_A \longrightarrow \operatorname{Spec} A = \operatorname{Proj} S/I' \longrightarrow \operatorname{Spec} A$, so Y and $\operatorname{Proj} S/I'$ are isomorphic as schemes over A. $\square$

NOTE By Example 4.8.1 if S is a graded A-algebra as in (b) above, S is isomorphic as a graded A-algebra to $A[x_0, \dots, x_n]/\mathfrak{a}$ for some $n \geq 0$ and homogenous ideal $\mathfrak{a}$. Hence $\operatorname{Proj} S$ is isomorphic as a scheme over $\operatorname{Spec} A$ to $\operatorname{Proj}(A[x_0, \dots, x_n]/\mathfrak{a})$, so any projective scheme over A has this form. (up to iso of schemes over A).

DEFINITION For any scheme Y, we define the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$ where $r \geq 0$ and $g : \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$ is part of $\mathbb{P}^r_Y$ (by $\mathbb{P}^r_Y$ we mean any pullback $\mathbb{P}^r_Y \longrightarrow Y$, $\mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$ of Y, $\mathbb{P}^r_{\mathbb{Z}}$ over $\operatorname{Spec} \mathbb{Z}$). Since the inverse image of invertible sheaves are invertible, the twisting sheaf on $\mathbb{P}^n_Y$ is invertible $n \geq 0$.

Suppose $\mathbb{P}^r_Y$ and $\overline{\mathbb{P}}^r_Y$ are two projective r-spaces over Y $(r \geq 0)$ with associated morphisms $g : \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$ and $\bar{g} : \overline{\mathbb{P}}^r_Y \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$. Let $f : \overline{\mathbb{P}}^r_Y \longrightarrow \mathbb{P}^r_Y$ be the canonical isomorphism. Then $gf = \bar{g}$ so

$$\bar{g}^*\mathcal{O}(1) = (gf)^*\mathcal{O}(1) \cong f^*g^*\mathcal{O}(1)$$

So the canonical isomorphisms identify twisting sheaves (up to isomorphism). If A is a ring and $r \geq 0$ let $\mathbb{P}^r_A$ be the canonical projective r-space over A, $g : \mathbb{P}^r_A \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$ canonical. By (5.12c) we have $g^*\mathcal{O}(1) \cong \mathcal{O}_X(1)$ $(X = \mathbb{P}^r_A)$, so in this case, up to isomorphism, the above definition agrees with the old one.

DEFINITION Fix a scheme Y and a projective r-space $\mathbb{P}^r_Y$ over Y $(r \geq 0)$. If $\mathcal{F}$ is a $\mathcal{O}_X$-module $(X = \mathbb{P}^r_Y)$ we denote by $\mathcal{O}(n)$ the module $\mathcal{O}(1)^{\otimes n}$ $(n > 0)$ and $\mathcal{F}(n)$ the module $\mathcal{F} \otimes \mathcal{O}(n)$. If $Y = \operatorname{Spec} A$ and $\mathbb{P}^r_A$ canonical, $\mathcal{O}(n) \cong \mathcal{O}_X(n)$ and so up to isomorphism $\mathcal{F}(n)$ agrees with the old definition.

DEFINITION A morphism $i : X \longrightarrow Z$ is an immersion if it can be factored as an open immersion $X \longrightarrow Y$ followed by a closed immersion $Y \longrightarrow Z$. The property of being an immersion is stable under isomorphisms on either end. Open and closed immersions are immersions, as are isomorphisms.

DEFINITION If $f : X \longrightarrow Y$ is a scheme over Y, an invertible sheaf $\mathcal{L}$ on X is very ample relative to Y if there is an immersion $i : X \longrightarrow \mathbb{P}^r_Y$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}$, (for some particular proj. r-space $\mathbb{P}^r_Y$, $r \geq 0$) and such that the following commutes

$$\begin{tikzcd} & \mathbb{P}^r_Y \arrow[dr] & \\ X \arrow[ur, "i"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Clearly if Y is a scheme and $r \geq 0$ the twisting sheaf on $\mathbb{P}^r_Y$ is a very ample invertible sheaf.

NOTE Let A be a ring, $m \geq 1$. There are m closed immersions $\Phi_i : \mathbb{P}^{m-1}_A \longrightarrow \mathbb{P}^m_A$ $0 \leq i \leq m-1$ each induced by a ring morphism $A[x_0, \dots, x_m] \longrightarrow A[x_0, \dots, x_{m-1}]$. By (5.12c) we have $\Phi_i^*(\mathcal{O}(1)) \cong \mathcal{O}(1)$. Also, the canonical isomorphism $k : \mathbb{P}^0_A \longrightarrow \operatorname{Spec} A$ has $k^*\mathcal{O}_{\operatorname{Spec} A} \cong \mathcal{O}(1)$ since $\mathbb{P}^0_A = \operatorname{Proj} A[x]$ so $D_+(x)$ covers all of $\mathbb{P}^0_A$, and so $\mathcal{O}(1) = \mathcal{O}(1)|_{D_+(x)} \cong \mathcal{O}_{\mathbb{P}^0_A}|_{D_+(x)} = \mathcal{O}_{\mathbb{P}^0_A}$, and always $k^*\mathcal{O}_{\operatorname{Spec} A} \cong \mathcal{O}_{\mathbb{P}^0_A}$. Hence

LEMMA For any scheme Y there are $n+1$ canonical closed immersions $j : \mathbb{P}^{n-1}_Y \longrightarrow \mathbb{P}^n_Y$ of schemes over Y for any $n \geq 1$, and $j^*\mathcal{O}(1) = \mathcal{O}(1)$, for all of these $n+1$ closed immersions.

PROOF Let $\mathbb{P}^{n-1}_Y, \mathbb{P}^n_Y$ be any projective spaces over Y. See our notes on projective space for the proof of the existence of j. Moreover, j fits into a commutative diagram

$$\begin{tikzcd} \mathbb{P}^{n-1}_Y \arrow[r, "g^{n-1}"] \arrow[d, "j"'] & \mathbb{P}^{n-1}_Z \arrow[d, "i"] \\ \mathbb{P}^n_Y \arrow[r, "g^n"'] & \mathbb{P}^n_Z \end{tikzcd}$$

By definition the twisting sheaves of $\mathbb{P}^{n-1}_Y, \mathbb{P}^n_Y$ are $g^{n-1*}(\mathcal{O}(1)), g^{n*}(\mathcal{O}(1))$ resp. (g^n is part of the def^N of $\mathbb{P}^n_Y$).
Thus

$$\begin{aligned} j^*\mathcal{O}(1) &= j^* g^{n*}\mathcal{O}(1) \\ &\cong (g^n j)^*\mathcal{O}(1) \\ &\cong g^{(n-1)*} i^*\mathcal{O}(1) \\ &\cong g^{n-1*}\mathcal{O}(1) = \mathcal{O}(1) \end{aligned}$$

as required. $\square$

So if X is a scheme over Y, $\mathcal{L}$ very ample relative to Y, say $i: X \longrightarrow \mathbb{P}^r_Y$ is an immersion with $i^*\mathcal{O}(1) \cong \mathcal{L}$, then the composite $i': X \longrightarrow \mathbb{P}^r_Y \xrightarrow{j} \mathbb{P}^{r+1}_Y$ is an immersion and $i'^*\mathcal{O}(1) \cong i^* j^* \mathcal{O}(1) \cong i^*\mathcal{O}(1) \cong \mathcal{L}$. So in the definition of very ample, we can restrict ourselves to $r \geq 1$.

LEMMA An immersion with closed image is a closed immersion.

PROOF Let $f: X \longrightarrow Y$ be an immersion factored as an open immersion $g: X \longrightarrow Z$ followed by a closed immersion $h: Z \longrightarrow Y$. If $f(X)$ is closed then clearly f is a homeomorphism of X with $f(X)$, so it suffices to show $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is surjective $\forall x \in X$. But for $x \in X$ the map $\mathcal{O}_{Z,g(x)} \longrightarrow \mathcal{O}_{X,x}$ is bijective since g is an open immersion, and the following commutes:

$$\begin{tikzcd} \mathcal{O}_{Y,f(x)} \arrow[rr] \arrow[dr] & & \mathcal{O}_{X,x} \\ & \mathcal{O}_{Z,g(x)} \arrow[ur] & \end{tikzcd}$$

Since h is a closed immersion $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{Z,g(x)}$ is surjective, so the composite $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is surjective, as required. $\square$

LEMMA Let Y be a noetherian scheme. Then a scheme X over Y is projective if and only if it is proper and there is a very ample sheaf on X relative to Y.

PROOF If X is projective over Y, then X is proper by (4.9). Also there is a closed immersion $i : X \longrightarrow \mathbb{P}^n_Y$ for some $n \geq 1$, so $i^*\mathcal{O}(1)$ is a very ample invertible sheaf on X, since the inverse image of invertible sheaves is invertible. Conversely, if X is proper over Y and $\mathcal{L}$ is a very ample invertible sheaf then $\mathcal{L} \cong i^*(\mathcal{O}(1))$ for some immersion $i : X \longrightarrow \mathbb{P}^r_Y$ of schemes over Y (we may assume $r \geq 1$). By (4.9) $\mathbb{P}^r_Y \longrightarrow Y$ is proper since Y is noetherian. So X, $\mathbb{P}^r_Y$ are both separated + finite type over Y, and by Ex 4.4 it follows that $X \longrightarrow \mathbb{P}^r_Y$ is proper and hence is an immersion with closed image, which must be a closed immersion by the previous Lemma. Hence X is projective over Y. $\square$

EXAMPLE Let A be a ring, S a graded A-algebra with $S_0 \cong A$ and S f.g. by S_1 over A. By Example 4.8.1 $\operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ is projective via the factorisation

$$\begin{tikzcd} & \mathbb{P}^n_A \arrow[dr] & \\ \operatorname{Proj} S \arrow[ur, "f"] \arrow[rr] & & \operatorname{Spec} A \end{tikzcd}$$

where $\operatorname{Proj} S \longrightarrow \operatorname{Proj} A[x_0, \dots, x_n]$ arises from a surjective morphism of graded rings $\varphi : A[x_0, \dots, x_n] \longrightarrow S$. Hence by (5.12c) $f^*\mathcal{O}(1) \cong \mathcal{O}_{\operatorname{Proj} S}(1)$. So the canonical twisting sheaf on $\operatorname{Proj} S$ is in fact a very ample sheaf, relative to $\operatorname{Spec} A$.

DEFINITION Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is <u>generated by global sections</u> if there is a family of global sections $\{s_i\}_{i \in I}$ $s_i \in \Gamma(X, \mathcal{F})$ such that for each $x \in X$ the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. (assume $I \neq \emptyset$) (By the next Lemma if $\mathcal{F}$ has this property and $\mathcal{F} \cong \mathcal{G}$ then so does $\mathcal{G}$)

LEMMA Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is generated by global sections iff. there is an epimorphism $\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$ for some I (possibly $\emptyset$).

PROOF First, for any ringed space $\mathcal{O}_X$ there is a bijection $\operatorname{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}(X)$ defined by $\phi \longmapsto \phi_X(1)$ and $m \longmapsto \phi_m$ where $(\phi_m)_U(1) = m|_U$. If $\{s_i\} \subseteq \mathcal{F}(X)$ are s.t. the stalk is generated by $\{germ_x s_i\}_{i \in I}$ let $\alpha_i : \mathcal{O}_X \longrightarrow \mathcal{F}$ correspond to s_i. The induced morphism $\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$ is an epimorphism since it is surjective on stalks. Conversely if $\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$ is epi, either $I = \emptyset$ (so $\mathcal{F} = 0$ and take $s = 0$ generates all the stalks) or $I \neq \emptyset$. Let $s_i \in \mathcal{F}(X)$ correspond to $\mathcal{O}_X \longrightarrow \bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$. Surjectivity on stalks shows $\{s_i\}$ has the required property, since taking stalks preserves coproducts, so $(\bigoplus_i \mathcal{O}_X)_x \cong \bigoplus_i \mathcal{O}_{X,x}$. $\square$

EXAMPLE 5.16.2 Any quasi-coherent sheaf on an affine scheme is generated by global sections. Indeed, if $\mathcal{F} = \widetilde{M}$ on $\operatorname{Spec} A$, any set of generators for M as an A-module will do.

EXAMPLE 5.16.3 Let $X = \operatorname{Proj} S$, where S is a graded ring which is generated by S_1 as an S_0-algebra. Then the elements of S_1 give global sections of $\mathcal{O}_X(1)$ which generate it. To see this it suffices to show $f/1$, $f \in S_1$ generate $S(1)_{(\mathfrak{p})}$ as a $S_{(\mathfrak{p})}$-module. If $a/s \in S(1)_{(\mathfrak{p})}$ say $s \in S_d \setminus \mathfrak{p}$ and $a \in S(1)_d = S_{d+1}$, then a can be written as a sum of terms of the form $s_0 f_1^{\alpha_1} \cdots f_n^{\alpha_n}$ where $s_0 \in S_0$, $f_i \in S_1$ and the α_i sum to $d+1$. But, assuming wlog $\alpha_1 \neq 0$,

$$\frac{s_0 f_1^{\alpha_1} \cdots f_n^{\alpha_n}}{s} = \frac{s_0 f_1^{\alpha_1 - 1} \cdots f_n^{\alpha_n}}{s} \cdot \frac{f_1}{1}$$

So $S(1)_{(\mathfrak{p})}$ is generated by the $f/1$, $f \in S_1$, as required. The above argument actually shows that if S is generated as an S_0-algebra by $f_1, \dots, f_n \in S_1$ then $S(1)_{(\mathfrak{p})}$ is generated as a $S_{(\mathfrak{p})}$-module by $f_1/1, \dots, f_n/1$. In particular the global sections $f_i \in \Gamma(X, \mathcal{O}_X(1))$ generate $\mathcal{O}_X(1)$.

NOTE If $f : X \longrightarrow Y$ is a morphism of schemes and a $\mathcal{O}_Y$-module $\mathcal{F}$ is generated by global sections, then so is $f^*\mathcal{F}$. see our Section 5.1. Notes.

THEOREM 5.17 (Serre) Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X relative to A, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$ can be generated by a finite number of global sections.

$\ulcorner n_0 > 0$,

PROOF For some $r \geq 1$ there is an immersion $i : X \longrightarrow \mathbb{P}^r_A$ such that $X \longrightarrow \mathbb{P}^r_A \longrightarrow \operatorname{Spec} A = X \longrightarrow \operatorname{Spec} A$ and $i^* \mathcal{O}_Y(1) \cong \mathcal{O}(1)$. (Here $Y = \operatorname{Proj} A[x_0, \dots, x_r]$ and $\mathcal{O}_Y(1) = A[x_0, \dots, x_r](1)^\sim$, $\mathcal{O}(1)$ is any very ample invertible sheaf on X). As in Remark 5.16.1 it follows that X, $\mathbb{P}^r_A$ are noetherian and so i is proper by Ex 4.4, so i is a closed immersion. So by Ex 5.5 $i_* \mathcal{F}$ is coherent. For $n > 0$

$$\begin{aligned} i_*(\mathcal{F}(n)) &= i_*(\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}) \\ &\cong i_*(\mathcal{F} \otimes \{i^* \mathcal{O}_Y(1)\}^{\otimes n}) \\ &\cong i_*(\mathcal{F} \otimes i^*(\mathcal{O}_Y(n))) && \ulcorner i^*(- \otimes -) = i^*(-) \otimes i^*(-)\lrcorner \\ &\cong (i_* \mathcal{F}) \otimes \mathcal{O}_Y(n) && \ulcorner \text{By Ex 5.1 d)} \lrcorner \\ &= (i_* \mathcal{F})(n) \end{aligned}$$

For $x \in X$ there is an isomorphism of abelian groups $i_*(\mathcal{F}(n))_{i(x)} \longrightarrow \mathcal{F}(n)_x$ compatible with $i^\#_x : \mathcal{O}_{Y, i(x)} \longrightarrow \mathcal{O}_{X,x}$. So if $i_*(\mathcal{F}(n))$ is generated by a finite number of global sections, so is $\mathcal{F}(n)$ $(n > 0)$. So we reduce to the case $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \dots, x_n]$ for $r \geq 1$, with $\mathcal{O}(1) = \mathcal{O}_X(1)$.

Now cover X with the open sets $D_+(x_i)$ $i = 0, \dots, r$. Since $\mathcal{F}$ is coherent, for each i there is a finitely generated module M_i over $B_i = A[x_0/x_i, \dots, x_n/x_i]$ such that $g_{i*} \mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$ where $g_i : D_+(x_i) \Rightarrow \operatorname{Spec} B_i$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate this module. These give rise to sections of $\mathcal{F}(D_+(x_i))$ which we also denote by s_{ij}. Since $D_+(x_i) = X_{\dot{x}_i}$ where $\dot{x}_i$ denotes the canonical global section of $\mathcal{O}_X(1)$ by Lemma 5.14 b) for some $n > 0$ the section $s_{ij} \dot{\otimes} \dot{x}_i|_{D_+(x_i)}^{\otimes n}$ of $(\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n})$ extends to a global section. Since $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n} \cong \mathcal{F}(n)$ there is a global section t_{ij} of $\mathcal{F}(n)$ with $t_{ij}|_{D_+(x_i)} = s_{ij} \dot{\otimes} x_i^n|_{D_+(x_i)}$. Using $\mathcal{F} \otimes \mathcal{O}_X(N) \cong (\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(N - n)$ for $N > n$ we may assume n is independent of i, j. There is a morphism of $\mathcal{O}_X$-modules $x_i^n : \mathcal{F} \longrightarrow \mathcal{F}(n)$ defined by $(x_i^n)_U(m) = m \dot{\otimes} \dot{x}_i^n|_U$. In particular $(x_i^n)_{D_+(x_i)}(s_{ij}) = s_{ij} \dot{\otimes} x_i^n|_{D_+(x_i)}$. Since $\mathcal{F}(n)$ is quasi-coherent there is a B_i-module M_i' such that $g_{i*} \mathcal{F}(n)|_{D_+(x_i)} \cong \widetilde{M_i'}$ for each i.

Let $\mathcal{O}_X(n)|_{D_+(x_i)} \xrightarrow{\gamma} \mathcal{O}_X|_{D_+(x_i)}$ be the isomorphism expressing the fact that $\mathcal{O}_X(n)$ is invertible. For $U \subseteq D_+(x_i)$ we have $\gamma_U(x_i^n|_U) = 1$. So the morphism $(x_i^n)|_{D_+(x_i)} : \mathcal{F}|_{D_+(x_i)} \longrightarrow \mathcal{F}(n)|_{D_+(x_i)}$ is actually the isomorphism

$$\begin{aligned} \mathcal{F}(n)|_{D_+(x_i)} &\cong \mathcal{F}|_{D_+(x_i)} \otimes \mathcal{O}_X(n)|_{D_+(x_i)} \\ &\cong \mathcal{F}|_{D_+(x_i)} \otimes \mathcal{O}_X|_{D_+(x_i)} \\ &\cong \mathcal{F}|_{D_+(x_i)} \end{aligned}$$

Since $\mathcal{F}(D_+(x_i))$ is generated as a $\mathcal{O}_X(D_+(x_i))$-module by the s_{ij}, it follows that $\mathcal{F}(n)(D_+(x_i))$ is generated by the $s_{ij} \dot{\otimes} \dot{x}_i^n|_{D_+(x_i)} = t_{ij}|_{D_+(x_i)}$, using the fact that $\mathcal{F}(n)(D_+(x_i)) \cong M_i'$, and the fact that since M_i' is f.g. so is $(M_i')_{\mathfrak{p}} \cong \mathcal{F}(n)_{\mathfrak{p}'}$ for $\mathfrak{p}' \in D_+(x_i)$, we see that the global sections $t_{ij} \in \mathcal{F}(n)(X)$ generate the sheaf $\mathcal{F}$. (by the images of the $t_{ij}|_{D_+(x_i)}$). In the above n was an arbitrary integer greater than all the integers used in the extensions to global sections, so the proof is complete. ☐

COROLLARY 5.18 Let X be projective over a noetherian ring A, and let $\mathcal{O}(1)$ be a very ample invertible sheaf on X relative to A. Then any coherent sheaf $\mathcal{F}$ on X can be written as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of sheaves $\mathcal{O}(n_i)$ for various integers n_i.

PROOF Let $n>0$ be such that $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$ is generated by a finite number of global sections. Then there is an epimorphism $\bigoplus_{i=1}^N \mathcal{O}_X \longrightarrow \mathcal{F}(n)$. For $n \in \mathbb{Z}$ let $\mathcal{O}(n)$ denote $\mathcal{O}(1)^{\otimes n}$ (defined for $n<0$ using $\mathcal{O}(1)^\vee$). Then it is shown in our "Exponential Tensor" notes that $\forall m,n \in \mathbb{Z}$ $\mathcal{O}(n+m) \cong \mathcal{O}(n) \otimes \mathcal{O}(m)$, and moreover $\mathcal{O}(n)$ is invertible.

Since $- \otimes \mathcal{O}(-n)$ is right exact we have a commutative diagram whose top row is epi:

$$\begin{array}{ccc}
\bigoplus_{i=1}^N \mathcal{O}_X \otimes \mathcal{O}(-n) & \longrightarrow & \mathcal{F}(n) \otimes \mathcal{O}(-n) \\
\wr\Vert & & \wr\Vert \\
\bigoplus_{i=1}^N (\mathcal{O}_X \otimes \mathcal{O}(-n)) & & \mathcal{F} \otimes (\mathcal{O}(n) \otimes \mathcal{O}(-n)) \\
\wr\Vert & & \wr\Vert \\
\bigoplus_{i=1}^N \mathcal{O}(-n) & \longrightarrow & \mathcal{F}
\end{array}$$

Giving the desired epimorphism $\bigoplus_{i=1}^N \mathcal{O}(-n) \longrightarrow \mathcal{F}$. $\square$

EXAMPLE Of course the main example is for a graded A-algebra with $S_0 \cong A$ and S f.g. by S_1 over S_0 and A noetherian (so $S \cong A[x_1,\dots,x_n]/\mathfrak{a}$ graded ideal $\mathfrak{a}$) and $\mathcal{O}(1) = \mathcal{O}_X(1)$ $(= \widetilde{S(1)})$. So any coherent sheaf on $\operatorname{Proj} S$ is a quotient of a module of the form $\bigoplus_{i=1}^N \mathcal{O}_X(m_i)$ $m_i \in \mathbb{Z}$. (in fact we can take all m_i equal).

THEOREM 5.19 Let k be a field, let A be a finitely generated k-algebra, and let X be a projective scheme over A with $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then $\Gamma(X,\mathcal{F})$ is a finitely-generated A-module. In particular, if $A=k$, $\Gamma(X,\mathcal{F})$ is a finite dimensional vector space.

PROOF By (5.16b) we can reduce to the case where $X = \operatorname{Proj} S$ for a graded A-algebra S with $S_0 \cong A$ and S f.g. by S_1 as an S_0-algebra. Let M be the graded S-module $\Gamma_*(\mathcal{F})$. Then by (5.15) $\widetilde{M} \cong \mathcal{F}$. On the other hand A is noetherian so by (5.17) for n sufficiently large, $\mathcal{F}(n)$ is generated by a finite number of global sections in $\Gamma(X,\mathcal{F}(n))$. Let M' be the submodule of M generated by (over S)
i.e. pick some n, consider the global generators, → these sections. Then M' is a finitely-generated S-module. The functor $\sim : S\text{GrMod} \longrightarrow \mathcal{O}_X\text{-}\underline{\text{Mod}}$ is exact (see our notes on Modules over Proj), so the inclusion $M' \longrightarrow M$ induces a monomorphism $\widetilde{M'} \longrightarrow \widetilde{M} \cong \mathcal{F}$. Since $\mathcal{O}_X(n)$ is invertible $- \otimes \mathcal{O}_X(n)$ is exact, so twisting by n we get a monomorphism $\widetilde{M'}(n) \longrightarrow \mathcal{F}(n)$ which is the bottom row in the following diagram (commutative by naturality of ρ)

$$\begin{array}{ccccc}
\widetilde{M'(n)} & \xrightarrow{i(n)} & \widetilde{M(n)} & & \\
\rho \Big\Vert & & \Big\Vert \rho & & \\
\widetilde{M'}(n) & \xrightarrow[\tau(n)]{} & \widetilde{M}(n) & \xRightarrow[\varepsilon(n)]{} & \mathcal{F}(n)
\end{array}$$

If $m_1,\dots,m_n$ are the generators of M' $(m_i \in \Gamma(X,\mathcal{F}(n)))$ and $\dot{m}_i$ denotes the global section of $\widetilde{M'(n)}$ corresponding to m_i, then we showed in our notes on the adjunction $\sim \dashv \Gamma_*$ that $\varepsilon(n)_X \rho_X(\dot{m}_i) = m_i$, it follows that $\widetilde{M'}(n) \longrightarrow \mathcal{F}(n)$ is surjective on stalks, and thus an isomorphism. Twisting by $-n$ we find that $\widetilde{M'} \cong \mathcal{F}$, and so we reduce to showing that if M is a finitely generated S-module, then $\Gamma(X,\widetilde{M})$ is a finitely-generated A-module.

This is trivial if $M=0$, so assume $M \neq 0$. Then by (I 7.4) there is a finite filtration

$$0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$$

of M by graded submodules, where for each i $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogenous prime ideal $\mathfrak{p}_i \subseteq S$ and $n_i \in \mathbb{Z}$ (iso as graded modules). For each i we have an exact sequence in $S\text{-}\underline{\text{GrMod}}$ $(i \geq 1)$

$$0 \longrightarrow M^{i-1} \longrightarrow M^i \longrightarrow (S/\mathfrak{p}_i)(n_i) \longrightarrow 0$$

and thus an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M^{i-1}} \longrightarrow \widetilde{M^i} \longrightarrow \widetilde{(S/\mathfrak{p}_i)(n_i)} \longrightarrow 0$$

giving rise to exact sequences of A-modules

$$0 \longrightarrow \Gamma(X, \widetilde{M^{i-1}}) \longrightarrow \Gamma(X, \widetilde{M^i}) \longrightarrow \Gamma(X, \widetilde{(S/\mathfrak{p}_i)(n_i)}) \qquad (2)$$

Suppose we could show that $\Gamma(X, \widetilde{(S/\mathfrak{p})(n)})$ was a finitely generated A-module for all homogenous primes $\mathfrak{p}$ and $n \in \mathbb{Z}$. Then we prove that $\Gamma(X, \widetilde{M^i})$ is a f.g. A-module by induction on i. The case $i = 0$ is trivial, and if $\Gamma(X, \widetilde{M^{i-1}})$ is f.g. for $i \geq 1$ then since A is noetherian and $\Gamma(X, \widetilde{(S/\mathfrak{p}_i)(n_i)})$ f.g. it follows that using (2) both $\Gamma(X, \widetilde{M^{i-1}})$ and $\Gamma(X, \widetilde{M^i})/\Gamma(X, \widetilde{M^{i-1}})$ are f.g. Hence $\Gamma(X, \widetilde{M^i})$ is f.g., as required. So finally with $i = r$ we see that $\Gamma(X, \widetilde{M})$ is a f.g. A-module.

So it only remains to show that $\Gamma(X, \widetilde{(S/\mathfrak{p})(n)})$ is a f.g. A-module for $n \in \mathbb{Z}$ and $\mathfrak{p}$ a homogenous prime. Let $\operatorname{Proj} S/\mathfrak{p} \xrightarrow{f} \operatorname{Proj} S$ be the morphism of schemes over A corresponding to $S \longrightarrow S/\mathfrak{p}$. Then by (5.12c) (or more precisely our Proj Tensor notes, which show $f_*(\widetilde{N}|_v) \cong ({}_S N)^\sim$) we have $f_*(\mathcal{O}_Y(n)) \cong \widetilde{(S/\mathfrak{p})(n)}$ where $Y = \operatorname{Proj} S/\mathfrak{p}$. Hence $\Gamma(Y, \mathcal{O}_Y(n)) \cong \Gamma(X, \widetilde{(S/\mathfrak{p})(n)})$ as A-modules. So we reduce finally to the special case where S is a graded integral domain, finitely generated by S_1 as a S_0-algebra and $S_0 \cong A$ where A is a finitely generated integral domain over k (i.e. $S/\mathfrak{p}_0$), and we have to show $\Gamma(X, \mathcal{O}_X(n))$ is a f.g. A-module for any $n \in \mathbb{Z}$.

Let $x_0, \ldots, x_r \in S_1$ generate the ring S as an A-algebra (equivalently, $x_0, \ldots, x_r$ generate S_1 as an A-module). Since S is an integral domain, multiplication by x_0 gives an injection $S(n) \longrightarrow S(n+1)$ for any n (of graded S-modules). This gives a monomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(n) \longrightarrow \mathcal{O}_X(n+1)$ and thus an injection of A-modules $\Gamma(X, \mathcal{O}_X(n)) \longrightarrow \Gamma_X(\mathcal{O}_X(n+1))$. Since A is noetherian it is sufficient to prove $\Gamma(X, \mathcal{O}_X(n))$ f.g. for sufficiently large n, say $n \gg 0$.

Since S is f.g. over A and A is f.g. over k, it follows that S is a f.g. k-algebra (if $S = A[x_1, \ldots, x_n]$ and $A = k[a_1, \ldots, a_n]$ then $S = k[a_1, \ldots, a_n, x_1, \ldots, x_n]$). By our notes on the ring $\Gamma_*(\mathcal{O}_X)$ (provided $X \neq \emptyset$ which we can safely assume) the subring $\Gamma_*(\mathcal{O}_X)' = \bigoplus_{n \geq 0} \Gamma(X, \hat{\mathcal{O}}_X(n))$ is integral over S, and is S-isomorphic to a subring of the localisation $S_{x_0 \ldots x_r}$ (where $S = A[x_0, \ldots, x_r]$). This in turn is S-isomorphic to a subring of the quotient field Q of S. By (I, 3.9), the theorem on finiteness of integral closure, the integral closure C of S in Q is a f.g. S-module. Since S is noetherian and $\Gamma_*(\mathcal{O}_X)' \subseteq C$ it follows that $\Gamma_*(\mathcal{O}_X)'$ is a f.g. S-module. Say $\Gamma_*(\mathcal{O}_X)'$ is generated over S by homogenous elements $m_1, \ldots, m_s$. For $n \gg 0$ take all the m_i of degree $\leq n$ and elements $t \cdot m_i$ where t is a monomial in $x_0, \ldots, x_r$ of degree $n - \deg(m_i)$. All these elements $t \cdot m_i \in \Gamma_*(\mathcal{O}_X)'_n$ generate $\Gamma_*(\mathcal{O}_X)'_n$ over S_0 – so $\Gamma_*(\mathcal{O}_X)'_n$ is a f.g. S_0-module. Via $A \Rightarrow S_0 \hookrightarrow S \longrightarrow \Gamma_*(\mathcal{O}_X)'$ it follows that for $n \gg 0$, $\Gamma_*(\mathcal{O}_X)'_n \cong \Gamma(X, \hat{\mathcal{O}}_X(n)) \cong \Gamma(X, \mathcal{O}_X(n))$ is a f.g. A-module. One checks this A-module structure coincides with the one coming from $A \cong \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \longrightarrow \mathcal{O}_X(X)$. Thus $\Gamma(X, \mathcal{O}_X(n))$ is a f.g. A-module, which completes the proof. □

COROLLARY 5.20 Let $f : X \longrightarrow Y$ be a projective morphism of schemes of finite type over a field k. Let $\mathcal{F}$ be a coherent sheaf on X. Then $f_* \mathcal{F}$ is a coherent sheaf on Y.

PROOF It suffices to show that $\forall y \in Y$ there is an open $y \in V \subseteq Y$ with $(f_*\mathcal{F})|_V$ coherent. But if $g : f^{-1}V \longrightarrow V$ is the restriction of f we have $(f_*\mathcal{F})|_V = g_*(\mathcal{F}|_V)$. Let V be an affine open neighborhood of y. Since X, Y are both of finite type over k, they are both noetherian, hence so are $f^{-1}V$, V. Moreover the inclusions $f^{-1}V \longrightarrow X$, $V \longrightarrow Y$ are quasi-compact open immersions, hence of finite type. Since g is projective, we have reduced to the case of $Y \cong \operatorname{Spec} A$. But isomorphisms preserve coherentness, so we reduce to the case $Y = \operatorname{Spec} A$, so A is a f.g. k-algebra. Then $f_*\mathcal{F}$ is quasi-coherent (5.8c) so $f_*\mathcal{F} \cong \Gamma(Y, f_*\mathcal{F})^\sim = \Gamma(X, \mathcal{F})^\sim$. But $\Gamma(X, \mathcal{F})$ is a f.g. A-module by the theorem, so $f_*\mathcal{F}$ is coherent. □

$\boxed{Q5.3}$ Let $X = \operatorname{Spec} A$ be an affine scheme. There are functors

$$\begin{tikzcd} \mathcal{O}_X\text{-}\underline{\mathrm{Mod}} \arrow[r, bend right, "\Gamma"'] & A\text{-}\underline{\mathrm{Mod}} \arrow[l, bend right, "\sim"'] \end{tikzcd}$$

We showed in (5.2) that $\sim$ is an exact, fully faithful, additive, colimit preserving functor. We claim that $\sim \dashv \Gamma$. Let an A-module M be given and define

$$\eta : M \longrightarrow \widetilde{M}(X)$$
$$\eta(m) = \dot{m}$$

We know from Prop. (5.1) that η is an isomorphism of A-modules. This morphism is clearly natural in M. Suppose we are given a morphism $\phi : M \longrightarrow \mathcal{F}(X)$ of A-modules, for an $\mathcal{O}_X$-module $\mathcal{F}$. For a prime $\mathfrak{p} \subseteq A$ define

$$\kappa_\mathfrak{p} : \mathcal{F}(X)_\mathfrak{p} \longrightarrow \mathcal{F}_\mathfrak{p}$$
$$\begin{aligned} \kappa_\mathfrak{p}(a/s) &= (X, \dot{s})^{-1} \cdot (X, a) \\ &= (D(s), \dot{1/s}) \cdot (X, a) = (D(s), 1/s \cdot a|_{D(s)}) \end{aligned}$$

To show $\kappa_\mathfrak{p}$ is well-defined, suppose $a/s = a'/s'$ in $\mathcal{F}(X)_\mathfrak{p}$. There is $t \notin \mathfrak{p}$ with $ts' \cdot a = ts \cdot a'$, that is, $\dot{t}\dot{s}' \cdot a = \dot{t}\dot{s} \cdot a'$. But $\dot{t}$ restricts to a unit in $D(t)$, $\dot{s}, \dot{s}'$ to units in $D(s), D(s')$ resp. Hence if $Q = D(s) \cap D(s') \cap D(t)$,

$$\begin{aligned} \dot{t}|_Q \cdot (\dot{s}'|_Q \cdot a|_Q) &= \dot{t}|_Q \cdot (\dot{s}|_Q \cdot a'|_Q) \\ \therefore \quad \dot{s}'|_Q \cdot a|_Q &= \dot{s}|_Q \cdot a'|_Q \\ \therefore \quad (\dot{\tfrac{1}{s}})|_Q \cdot a|_Q &= (\dot{\tfrac{1}{s'}})|_Q \cdot a'|_Q \\ \therefore \quad (\dot{\tfrac{1}{s}} \cdot a|_{D(s)})|_Q &= (\dot{\tfrac{1}{s'}} \cdot a'|_{D(s')})|_Q \end{aligned}$$

This shows that $\kappa_\mathfrak{p}$ is well-defined. If we make $\mathcal{F}_\mathfrak{p}$ into an $A_\mathfrak{p}$-module via $A_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}$ the map $\kappa_\mathfrak{p}$ is a morphism of $A_\mathfrak{p}$-modules. Composing with $\phi_\mathfrak{p}$ gives a morphism of $A_\mathfrak{p}$-modules

$$\psi_\mathfrak{p} : M_\mathfrak{p} \longrightarrow \mathcal{F}_\mathfrak{p}$$
$$\psi_\mathfrak{p}(m/s) = (D(s), \dot{1/s} \cdot \phi(m)|_{D(s)})$$

Now we define $\psi : \widetilde{M} \longrightarrow \mathcal{F}$ by

$$\operatorname{germ}_\mathfrak{p} \psi_U(s) = \psi_\mathfrak{p}(s(\mathfrak{p}))$$

This is well-defined since $\mathcal{F}$ is a sheaf. ψ is a morphism of $\mathcal{O}_X$-modules since for $r \in \mathcal{O}_X(U)$,

$$\begin{aligned} \operatorname{germ}_\mathfrak{p} \psi_U(r \cdot s) &= \psi_\mathfrak{p}(r(\mathfrak{p}) \cdot s(\mathfrak{p})) = r(\mathfrak{p}) \cdot \psi_\mathfrak{p}(s(\mathfrak{p})) \\ &= \operatorname{germ}_\mathfrak{p} r \cdot \operatorname{germ}_\mathfrak{p} \psi_U(s) \\ &= \operatorname{germ}_\mathfrak{p}(r \cdot \psi_U(s)) \end{aligned}$$

Next we show that

$$\begin{tikzcd} & M \arrow[dl, "\eta"'] \arrow[dr, "\phi"] & \\ \widetilde{M}(X) \arrow[rr, "\psi_X"'] & & \mathcal{F}(X) \end{tikzcd}$$

commutes. But this is immediate since $\operatorname{germ}_\mathfrak{p} \psi_X(\dot{m}) = \psi_\mathfrak{p}(m/1) = (X, \phi(m)) = \operatorname{germ}_\mathfrak{p} \phi(m)$. Proving uniqueness of ψ reduces to showing that if $\psi' : \widetilde{M} \longrightarrow \mathcal{F}$ has $\psi'_X = \psi_X$ then $\psi' = \psi$. Let $\psi''_\mathfrak{p} : M_\mathfrak{p} \Rightarrow \widetilde{M}_\mathfrak{p} \longrightarrow \mathcal{F}_\mathfrak{p}$ be the composite of $M_\mathfrak{p} \Rightarrow \widetilde{M}_\mathfrak{p}$ with $\psi'_\mathfrak{p}$. Then $\psi''_\mathfrak{p}$ is a morphism of $A_\mathfrak{p}$-modules, so

$$\begin{aligned} \psi''_\mathfrak{p}(m/s) &= 1/s \cdot \psi''_\mathfrak{p}(m/1) = 1/s \cdot \psi'_\mathfrak{p}(X, \dot{m}) = 1/s \cdot (X, \psi'_X(\dot{m})) \\ &= 1/s \cdot (X, \psi_X(\dot{m})) = 1/s \cdot (X, \phi(m)) = (D(s), \dot{1/s} \cdot \phi(m)|_{D(s)}) = \psi_\mathfrak{p}(m/s) \end{aligned}$$

Then if $s \in \widetilde{M}(U)$ and $p \in U$, we have $s(p) = m/s \in M_p$ for some $s \notin p$. Then

$$\begin{aligned} \operatorname{germ}_p \psi'_U(s) &= \psi'_p(\operatorname{germ}_p s) \\ &= \psi''_p(m/s) = \psi_p(m/s) \\ &= \operatorname{germ}_p \psi_U(s) \end{aligned}$$

Hence $\psi'_U = \psi_U$, as required. So $\psi = \psi'$ and ψ is unique. This completes the proof that $\sim \dashv \Gamma$.

$$\operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \overset{\alpha}{\cong} \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$$

$$\operatorname{germ}_p \alpha(\phi)_U(s) = \psi_p(s(p))$$

where $\psi_p : M_p \longrightarrow \mathcal{F}_p$ is $m/s \longmapsto (D(s), 1/s \cdot \phi(m)|_{D(s)})$

In particular the counit $\varepsilon : \widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F}$ is given by $\operatorname{germ}_p \varepsilon_U(s) = K_p(s(p))$ where K_p is defined as previously.

$\boxed{Q5.5}$ Let $f: X \longrightarrow Y$ be a morphism of schemes. Then

(b) A closed immersion is a finite morphism. Let f be a closed immersion, and let $\{V_\alpha\}$ be an affine open cover of Y. Since $f^{-1}V_\alpha$ is affine for all α (See Ex 4.3) and both "closed immersion" and "finite" are local on the base, we can reduce to the case where $f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a closed immersion, and thereby to the closed immersion $\operatorname{Spec} A/\mathfrak{a} \longrightarrow \operatorname{Spec} A$ for an ideal $\mathfrak{a} \subseteq A$, but this is clearly finite since $A/\mathfrak{a}$ is a f.g. A-module.

(c) Let $f: X \longrightarrow Y$ be a finite morphism of noetherian schemes and let $\mathcal{F}$ be coherent on X. Let $V \subseteq Y$ be an affine open set $h: V \Longrightarrow \operatorname{Spec} A$ an isomorphism. It suffices to show $(f_*\mathcal{F})|_V \cong \tilde{M}$ via h_* for a f.g. A-module M. Since f is finite $f^{-1}V \cong \operatorname{Spec} B$ for a f.g. A-module B. Let $g: f^{-1}V \longrightarrow V$ be induced from f. Since X is noetherian, if $k: f^{-1}V \Longrightarrow \operatorname{Spec} B$ is an isomorphism then there is a f.g. B-module N such that $k_*(\mathcal{F}|_{f^{-1}V}) \cong \tilde{N}$. But then

$$\begin{aligned} h_*(f_*\mathcal{F}|_V) &= h_*(g_*(\mathcal{F}|_{f^{-1}V})) \\ &= (hg)_*\mathcal{F}|_{f^{-1}V} \\ &= (\Upsilon k)_*\mathcal{F}|_{f^{-1}V} \\ &= \Upsilon_*(k_*\mathcal{F}|_{f^{-1}V}) \\ &\cong \Upsilon_*\tilde{N} \\ &\cong ({}_AN)^\sim \end{aligned}$$

$y: A \longrightarrow B$
$\Upsilon: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$
induced by g,

But if B is generated as an A-module by $b_1, \ldots, b_n$ and N as a B-module by $m_1, \ldots, m_d$ then the elements $b_i m_j$ generate N as an A-module, so ${}_AN$ is a f.g. A-module. <u>NOTE</u> We only use X noetherian.