2. SCHEMES

1. SHEAVES

We start by reviewing some sheaf theory, from our topos theory notes.

DEFINITION Let X be a topological space. A presheaf P of sets on X (abelian groups, rings, elements of a category $\mathscr{A}$) is a contravariant functor $P : \mathcal{O}(X) \longrightarrow \underline{Sets}$ ($\underline{Ab}$, $\underline{Rng}$, $\mathscr{A}$). Note that we do not require $P(\phi) = 0$, as Hartshorne does. A presheaf P is a sheaf if for any open set $U \subseteq X$ and open cover $\{V_i\}_{i \in I}$ of U, if $s_i \in P(V_i)$ is a matching family (i.e. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ all i,j) then there is a unique $s \in P(U)$ s.t. $s|_{V_i} = s_i \ \forall i \in I$.

NOTE Our convention is that the "empty cover" is an open cover of $\phi \subseteq X$, implying that if P is a sheaf of sets $P(\phi) = \{*\}$, similarly $P(\phi) = 0$ for abelian groups or rings. A presheaf on $X = \phi$ is just a set/group/ring. A sheaf on $X = \phi$ must be $\phi \mapsto \{*\}$ / 0. Generally we assume $X \neq \phi$ unless we say explicitly otherwise.

DEFINITION The category $Sh(X)$ of sheaves is a full subcategory of $\underline{Sets}^{\mathcal{O}(X)^{op}}$.

Since $\underline{Sets}$ is complete and cocomplete, so is $\underline{Sets}^{\mathcal{O}(X)^{op}}$ and limits and colimits are computed pointwise.

Let P be a presheaf of sets, then for $x \in X$ P_x is the usual stalk of germs. Let $\Lambda_P = \coprod_{x \in X} P_x$ (disjoint union) and $p : \Lambda_P \longrightarrow X$ map each germ to the point where it was taken. Then each $s \in P(U)$ determines a function

$$\dot{s} : U \longrightarrow \Lambda_P \qquad \dot{s}(x) = germ_x s \qquad x \in U$$

Topologise the set Λ_P by taking as a base of open sets all the image sets $\dot{s}(U) \subseteq \Lambda_P$ for $s \in P(U)$ and $U \subseteq X$. Any $\dot{s}$ is then continuous open and injective. Let $\Gamma\Lambda_P$ be the sheaf of sections of the bundle $\Lambda_P \longrightarrow X$, so $\ulcorner \Gamma\Lambda_P(\phi) = \{*\} \lrcorner$

$$(\Gamma\Lambda_P)(V) = \{ t : V \longrightarrow \Lambda_P \mid t \text{ is continuous and } pt = 1 \}$$

For a map $t : V \longrightarrow \Lambda_P$ the fact that $pt = 1$ simply means that for $x \in V$, $t(x) \in P_x$. It is then easily checked that

$$(\Gamma\Lambda_P)(V) = \{ t : V \longrightarrow \Lambda_P \mid t(x) \in P_x \text{ and every } x \in V \text{ has an open neighborhood } W \subseteq V \text{ s.t. } \forall y \in W \ t(y) = germ_y s \text{ where } s \text{ is some fixed } s \in P(W) \}$$

I.e. $\Gamma\Lambda_P$ formally amalgamates matching families in P. All of these uses the simple fact that if $t \in P(U)$, $s \in P(V)$ then $\{x \in U \cap V \mid germ_x t = germ_x s\}$ is open (by def^N). The above process gives a sheaf $\underline{a}P = \Gamma\Lambda_P$ and natural transformation

$$\eta : P \longrightarrow \underline{a}P \qquad \eta_V(s) = \dot{s}$$

Given a natural transformation $\phi : P \longrightarrow Q$ from a presheaf P to a sheaf Q, there is a unique $\hat{\phi} : \underline{a}P \longrightarrow Q$ s.t. the diagram

$$\begin{tikzcd} P \arrow[rr] \arrow[dr, "\phi"'] & & \underline{a}P \arrow[dl, "\hat{\phi}"] \\ & Q & \end{tikzcd}$$

commutes. Let $t \in \underline{a}P(V)$ and $V = \bigcup_i W_i$ be an open cover and $t_i \in P(W_i)$ s.t. $t(x) = germ_x t_i \ \forall x \in W_i$. Then $\phi(t_i) \in Q(W_i)$ form a matching family and hence have a unique amalgamation $t' \in Q(V)$. Then

$$\hat{\phi}(t) = t'$$

In particular any morphism $P \xrightarrow{\phi} P'$ of presheaves induces a morphism $\underline{a}P \xrightarrow{\underline{a}\phi} \underline{a}P'$ of sheaves, so there is a functor

$$\underline{a} : \underline{Sets}^{\mathcal{O}(X)^{op}} \longrightarrow Sh(X)$$

NOTE $(\underline{a}\phi)_U(\dot{s})(x) = \phi_x(s(x))$.

This is left adjoint to the inclusion $\underline{i} : Sh(X) \longrightarrow \underline{Sets}^{\mathcal{O}(X)^{op}}$. Hence $Sh(X)$ is a reflective subcategory of $\underline{Sets}^{\mathcal{O}(X)^{op}}$. The functor $\underline{a}$ is actually finite limit preserving, so $Sh(X)$ is a Giraud subcategory of $\underline{Sets}^{\mathcal{O}(X)^{op}}$. (Hence $\underline{a}$ preserves monomorphisms, using the pullback characterisation)

COROLLARY A morphism in $Sh(X)$ is a monomorphism in $Sh(X)$ iff. it is a monomorphism in $\underline{Sets}^{\mathcal{O}(X)^{op}}$, so iff. it is a pointwise injective map. A morphism in $Sh(X)$ is an isomorphism iff. it is an isomorphism of presheaves, so iff. it is pointwise bijective.

The same thing is not true of epimorphisms, although there is a condition which characterises these maps (see III topos notes).

LIMITS If D is a diagram of sheaves and transformations then the limit in $\underline{Sets}^{\mathcal{O}(X)^{op}}$ is a sheaf and is a limit in $Sh(X)$. So limits in $Sh(X)$ are computed pointwise.

COLIMITS If D is a diagram of sheaves and transformations, let L be the colimit in $\underline{Sets}^{\mathcal{O}(X)^{op}}$. Then $\underline{a}L$ together with the morphisms $D_i \longrightarrow L \longrightarrow \underline{a}L$ are a colimit for D in $Sh(X)$.

Let $f : X \longrightarrow Y$ be a continuous map of spaces. The direct image functor $f_* : Sh(X) \longrightarrow Sh(Y)$ is defined by $f_* F(V) = F(f^{-1}V)$. This functor has a left adjoint f^* called the inverse image functor, which preserves all finite limits. If X is a space and $x \in X$ then the continuous map $f : \{*\} \longrightarrow X$ $* \longmapsto x$ induces adjoint functors

$$\begin{tikzcd} \underline{Sets} \arrow[r, bend left, "f_*"] & Sh(X) \arrow[l, bend left, "f^*"] \end{tikzcd} \qquad f^* \dashv f_*$$

$$f_*(Z)(U) = Z$$

$$f^* Q = Q_x$$

$$f^*(\phi : Q \longrightarrow Q') = \phi_x$$

That is, f^* takes stalks at x and maps transformations to the induced maps on stalks.

SHEAVES OF GROUPS AND RINGS

There are two ways to treat this topic: via diagrams and bare hands. The former is done in our Thesis notes, and has the advantage of proving powerful theorems we will use here, where we take the latter approach. A presheaf of abelian groups (rings) is a contravariant functor $P : \mathcal{O}(X) \longrightarrow \underline{Ab}$ (resp. $\underline{Ring}$), and such a presheaf is a sheaf if it satisfies the necessary condition. A morphism of (pre)sheaves is a natural transformation. This defines the categories $\underline{Ab}(Sh(X))$ and $\underline{Ring}(Sh(X))$ as full subcategories of $\underline{Ab}^{\mathcal{O}(X)^{op}}$ and $\underline{Rng}^{\mathcal{O}(X)^{op}}$ resp.

Sheafification Let P be a presheaf of abelian groups; P^+ be the sheafification of P considered as a presheaf of sets. For $x \in X$ the stalk P_x is an abelian group in a canonical way, and by simply adding sections pointwise P^+ becomes a sheaf of abelian groups, and $\eta_P : P \longrightarrow P^+$ is a morphism of abelian presheaves. If $\phi : P \longrightarrow Q$ is a morphism from P to a sheaf Q of abelian groups, the induced map $\hat{\phi} : P^+ \longrightarrow Q$ is a morphism of sheaves of abelian groups, hence there is an adjoint pair

$$\begin{tikzcd} \underline{Ab}(Sh(X)) \arrow[r, bend right, "\underline{i}"'] & \underline{Ab}^{\mathcal{O}(X)^{op}} \arrow[l, bend right, "\underline{a}"'] \end{tikzcd} \qquad \underline{a} \dashv \underline{i}$$

Since $\underline{Ab}$ is complete and cocomplete, so is $\underline{Ab}^{\mathcal{O}(X)^{op}}$ and hence since $\underline{Ab}(Sh(X))$ is a reflective subcategory of $\underline{Ab}^{\mathcal{O}(X)^{op}}$ it follows that $\underline{Ab}(Sh(X))$ is complete and cocomplete:

- Limits in $\underline{Ab}(Sh(X))$ are computed in $\underline{Ab}^{\mathcal{O}(X)^{op}}$ hence are pointwise limits. (i.e. the pointwise limit is a sheaf)
- Colimits in $\underline{Ab}(Sh(X))$ are computed by reflecting the colimit in $\underline{Ab}^{\mathcal{O}(X)^{op}}$, i.e. sheafify the pointwise colimit C. The colimit morphisms are $\longrightarrow C \longrightarrow \underline{a}C$

Since $\underline{Rng}$ is also complete and cocomplete, all the above also applies to $\underline{Rng}(sh(X))$. That is:

$$\begin{tikzcd} \underline{Rng}(Sh(X)) \arrow[r, shift right, "i"'] & \underline{Rng}^{\mathcal{O}(X)^{op}} \arrow[l, shift right, "a"'] \end{tikzcd} \qquad \underline{a} \dashv \underline{i}$$

$\underline{Rng}(Sh(X))$ complete and cocomplete

- Limits pointwise
- Colimits pointwise and sheafify

⌈ The sheaf $U \mapsto 0$ is the terminal object and the sheafification of $U \mapsto \mathbb{Z}$ is the initial object ⌋

⌈ P_x ring in a canonical way ⌋

Moreover $\underline{Ab}(sh(X))$ is an additive category with zero (the presheaf $U \mapsto 0$). It is shown in our thesis notes that $\underline{a} : \underline{Ab}^{\mathcal{O}(X)^{op}} \longrightarrow \underline{Ab}(sh(X))$ preserves finite limits, so $\underline{Ab}(Sh(X))$ is actually a Giraud subcategory of $\underline{Ab}^{\mathcal{O}(X)^{op}}$, which is a Grothendieck abelian category. It follows that $\underline{Ab}(sh(X))$ is also Grothendieck abelian.

A morphism $\phi : F \longrightarrow G$ of presheaves (of sets, groups or rings) induces a morphism $\phi_x : F_x \longrightarrow G_x$ on the stalks (a morphism of sets, groups, rings resp.). The next result is true for any sheaves (sets, ab, ring)

<u>PROPOSITION 1.1</u> Let $\varphi : F \longrightarrow G$ be a morphism of sheares on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P : F_P \longrightarrow G_P$ is an isomorphism for all $P \in X$.

<u>PROOF</u> If φ is an isomorphism it is clear that each φ_P is an isomorphism. Conversely, assume that φ_P is an isomorphism for all $P \in X$. It suffices to show that $\varphi_U : F(U) \longrightarrow G(U)$ is bijective for all $U \subseteq X$. First we show that φ_U is injective. Let $s \in F(U)$ and $t \in F(U)$ and suppose $\varphi_U(s) = \varphi_U(t)$. It follows that s, t have the same germ at all $P \in U$ and hence are equal since F is a sheaf.

Next we show that φ_U is surjective. Suppose we have $t \in G(U)$. For each $P \in U$ let W_P be an open subset of U and $s \in F(W$ s.t. $\varphi_P(\text{germ}_P s) = t_P$. $(P \in W)$. Hence there is $P \in V_P \subseteq W_P$ s.t. if $a^P = s^P|_{V_P}$ then $\varphi_{V_P}(a^P) = t|_{V_P}$. The V_P form an open cover of U and $\varphi_{V_P \cap V_Q}(a^P|_{V_P \cap V_Q}) = \varphi_{V_P}(a^P)|_{V_P \cap V_Q} = t|_{V_P \cap V_Q} = \varphi_{V_P \cap V_Q}(a^Q|_{V_P \cap V_Q})$ so by injectivity of φ, $a^P|_{V_P \cap V_Q} = a^Q|_{V_P \cap V_Q}$. Since F is a sheaf there is $a \in F(U)$ s.t. $a|_{V_P} = a^P$ and hence $\varphi_U(a) = t$ since both sections agree on an open cover. □

Let $\phi : F \longrightarrow G$ be a morphism of presheaves of abelian groups. Since $\underline{Ab}^{\mathcal{O}(X)^{op}}$ is Grothendieck abelian, we have the following:

- The <u>kernel</u> of ϕ is the morphism $\kappa : K \longrightarrow F$ where

$$K(U) = Ker(\phi_U)$$
$$\kappa_U : Ker(\phi_U) \longrightarrow K(U)$$

- The <u>cokernel</u> of ϕ is the morphism $\theta : G \longrightarrow C$ where

$$C(U) = G(U)/Im\,\phi_U$$
$$\theta_U : G(U) \longrightarrow G(U)/Im\,\phi_U$$

- The <u>image</u> of ϕ is the morphism $i : I \longrightarrow G$ where

$$I(U) = Im\,\phi_U$$

$$\begin{tikzcd} & I \arrow[dr] & \\ F \arrow[ur] \arrow[rr] & & G \end{tikzcd} \qquad \begin{tikzcd} & Im\,\phi_U \arrow[dr] & \\ F(U) \arrow[ur] \arrow[rr, "\phi_U"'] & & G(U) \end{tikzcd}$$

These terms are used in the categorical sense, in the category $\underline{Ab}^{\mathcal{O}(X)^{op}}$.

⌈ <u>NOTE</u> Sheafification for groups, rings is just sheafification for sets. Hence $\underline{a} : \underline{Ab}^{\mathcal{O}(X)^{op}} \longrightarrow \underline{Ab}(sh(X))$ and $\underline{a} : \underline{Rng}^{\mathcal{O}(X)^{op}} \longrightarrow \underline{Rng}(sh(X))$ both preserve monomorphisms ⌋

Let $\phi : F \longrightarrow G$ be a morphism in $\underline{Ab}(Sh(X))$. Then

- The <u>kernel</u> of ϕ is the morphism $\kappa : K \longrightarrow F$ where

$$K(U) = Ker\phi_U$$
$$\kappa_U : Ker\phi_U \longrightarrow K(U)$$

i.e. the presheaf kernel is a sheaf.

- The <u>cokernel</u> of ϕ is the composite $G \longrightarrow C \longrightarrow \underline{a}C$ where C is the presheaf cokernel and $\underline{a}C$ its sheafification.

- The <u>image</u> of ϕ is the induced morphism $\underline{a}I \longrightarrow G$ where I is the presheaf image.

$$\begin{tikzcd} & \underline{a}I \arrow[dr] & \\ & I \arrow[u] & \\ F \arrow[ur] \arrow[uur] \arrow[rr] & & G \end{tikzcd}$$

⌜Since $\underline{a}I$ is isomorphic to a sheaf of G containing the subfunctor I, we may identify the categorical image with this subsheaf⌟

To see that $F \longrightarrow \underline{a}I$ is a sheaf epimorphism and $\underline{a}I \longrightarrow G$ is a monomorphism, we note that sheafification $\underline{a}$ preserves kernels, hence monics, and has a right adjoint so preserves epimorphisms. It follows that the composite $F \longrightarrow I \longrightarrow \underline{a}I$ is an epimorphism in $\underline{Ab}(Sh(X))$ and that the induced morphism $\underline{a}I \longrightarrow G$ is a monomorphism (hence pointwise injective) so that $\underline{a}I \longrightarrow G$ really is a categorical image, so $\underline{a}I$ can be identified with a subsheaf of G.

<u>NOTE</u> Hartshorne says ϕ is <u>surjective</u> if $Im\phi = G$ (sheaf image). We say that in such a case ϕ is an <u>epimorphism</u>, since by our topos notes ϕ is an epi of sheaves of sets iff. $\forall U \subseteq X$ and $t \in G(U)$ there is an open cover $U = \bigcup_i V_i$ s.t. each $t|_{V_i} \in Im\phi_{V_i}$ (i.e. ϕ is locally surjective). But by def^N of "image" in a category, ϕ is epi in $\underline{Ab}(Sh(X))$ iff. $Im\phi = G$. It is easy to check that ϕ is locally surjective iff. $\phi_x : F_x \longrightarrow G_x$ is surjective $\forall x \in X$. Hence the following are equivalent:

⌜ϕ epi means $Im\phi \to G$ is an iso ∴ bijective ∴ $Im\phi = G$ in the subsheaf sense⌟

⌜$t \in (Im\phi)(U)$ iff. there is an open cover $U = \bigcup_i V_i$ s.t. $t|_{V_i} \in Im\phi_{V_i}$ $\forall i$⌟

- $\phi : F \longrightarrow G$ is an epimorphism of sheaves of sets
- $\phi : F \longrightarrow G$ is an epimorphism of sheaves of abelian groups
- $Im\phi = G$
- For open $U \subseteq X$ and $t \in G(U)$ is $U = \bigcup_i V_i$ and $t|_{V_i} \in Im\phi_{V_i}$ $\forall i$
- ϕ_x surjective $\forall x \in X$.

⌜This is all for groups. For sets and rings see Ex1.2⌟

<u>DIRECT AND INVERSE IMAGE</u> Let $f : X \longrightarrow Y$ be a continuous map of spaces. There is an adjoint pair

$$\begin{tikzcd} Sh(X) \arrow[r, bend right, "f_*"'] & Sh(Y) \arrow[l, bend right, "f^{-1}"'] \end{tikzcd} \qquad f^{-1} \dashv f_*$$

where f_* is <u>direct image</u> and f^{-1} is <u>inverse image</u>. Define f_* by

$$(f_* F)(V) = F(f^{-1}V)$$
$$(f_*\phi)_V : (f_*F)(V) \longrightarrow (f_*G)(V)$$
$$F(f^{-1}V) \longrightarrow G(f^{-1}V) \quad \text{is } \phi_{f^{-1}V}$$

For $G \in Sh(Y)$ we define a presheaf G' on X as follows:

$$G'(U) = \varinjlim_{V \supseteq f(U)} G(V)$$

So elements of $G'(U)$ are pairs (V,t) where $f(U) \subseteq V$ and $t \in G(V)$, subject to the equivalence relation $(V,t) \sim (W,s)$ if $\exists Q \subseteq V \cap W$ s.t. $f(U) \subseteq Q$ and $t|_Q = s|_Q$. Note that $G'(\emptyset) = \{*\}$. If $U' \subseteq U$ in X then $f(U') \subseteq f(U)$ and the induced morphism $G'(U) \longrightarrow G'(U')$ of colimits maps $(V,t) \longmapsto (V,t)$. It is easy to verify that G' is a presheaf. Let $f^{-1}G$ be the associated sheaf. This defines the functor f^{-1} on objects. If $\phi : G \longrightarrow H$ is a morphism of sheaves the induced morphism of diagrams gives

$$G'(U) \longrightarrow H'(U) \qquad U \subseteq X$$
$$(V,t) \longmapsto (V, \phi_V(t))$$

This is clearly a morphism of presheaves, inducing a morphism $f^{-1}\phi : f^{-1}G \longrightarrow f^{-1}H$ of sheaves. With this definition f^{-1} is clearly a functor. To summarise

$\boxed{G}$ $\quad f^{-1}G$ is the associated sheaf of $U \longmapsto \varinjlim_{V \supseteq f(U)} G(V)$

$\boxed{\phi : G \longrightarrow H}$ $\quad f^{-1}\phi$ is the morphism of sheaves associated with the presheaf morphism defined for $U \subseteq X$ by

$$\begin{aligned} \varinjlim_{V\supseteq f(U)} G(V) &\longrightarrow \varinjlim_{V \supseteq f(U)} H(V) \\ (V,t) &\longmapsto (V, \phi_V(t)) \end{aligned}$$

If we consider sheaves of abelian groups or rings then the definitions of f_* and f^{-1} produce sheaves of abelian groups (resp. rings) and also morphisms of sheaves of abelian groups (rings).

Now we prove the adjunction $f^{-1} \dashv f_*$. Note that since for $G \in Sh(Y)$, $f^{-1}G = aG'$ there is a bijection between presheaf morphisms $G' \longrightarrow F$ and sheaf morphisms $f^{-1}G \longrightarrow F$ for a sheaf F. For $G \in Sh(Y)$ we define a morphism $\eta_G : G \longrightarrow f_*f^{-1}G$ as follows:

$$\begin{aligned} G &\longrightarrow f_*f^{-1}G \\ G(V) &\longrightarrow (f^{-1}G)(f^{-1}V) \\ t &\longmapsto \dot{(V,t)} \end{aligned}$$

where for a presheaf P, $\dot{}$ denotes $P \longrightarrow aP$. η_G is easily seen to be a morphism, and η_G is also trivially natural, in the sense that the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d] & H \arrow[d] \\ f_*f^{-1}G \arrow[r, "f_*f^{-1}\phi"'] & f_*f^{-1}H \end{tikzcd}$$

commutes for any morphism ϕ of sheaves on Y. So to show $f^{-1} \dashv f_*$ it only remains to show that $(f^{-1}G, \eta_G)$ is a reflection of G along f_*. Suppose we have a morphism $\phi : G \longrightarrow f_*F$ and define a morphism $\psi : G' \longrightarrow F$ of presheaves

$$\begin{aligned} \psi_U : G'(U) &\longrightarrow F(U) \\ \varinjlim_{f(U)\subseteq V} G(V) &\longrightarrow F(U) \\ (V,t) &\longmapsto \phi_V(t)|_U \end{aligned}$$

This induces $\psi' : f^{-1}G \longrightarrow F$. We claim the diagram

$$\begin{tikzcd} & G \arrow[dl, "\eta_G"'] \arrow[dr, "\phi"] & \\ f_*f^{-1}G \arrow[rr, "f_*\psi'"'] & & f_*F \end{tikzcd} \qquad (1)$$

commutes. But $(f_*\psi')_V(\eta_G)_V(t) = (f_*\psi')_V(\dot{(V,t)}) = \psi'_{f^{-1}V}(\dot{(V,t)}) = \psi_{f^{-1}V}(V,t) = \phi_V(t)|_{f^{-1}V} = \phi_V(t)$. Clearly ψ' is unique making (1) commute, so $(f^{-1}G, \eta_G)$ is indeed a reflection. (A morphism $G' \longrightarrow F$ is determined by its values on open sets of the form $f^{-1}V$).

If we consider sheaves of abelian groups or rings, η_G is a morphism of sheaves of groups (rings) and the assorted morphisms above are also. Hence $f^{-1} \dashv f_*$ in these cases also

$$\begin{tikzcd} \underline{Ab}(Sh(X)) \arrow[r, shift right, "f_*"'] & \underline{Ab}(Sh(X)) \arrow[l, shift right, "f^{-1}"'] \end{tikzcd} \quad f^{-1} \dashv f_* \qquad \begin{tikzcd} \underline{Rng}(Sh(X)) \arrow[r, bend right, "f_*"'] & \underline{Rng}(Sh(Y)) \arrow[l, bend right, "f^{-1}"'] \end{tikzcd} \quad f^{-1} \dashv f_*$$

Later we will need to know what the counit $\epsilon : f^{-1}f_* \longrightarrow 1$ of the adjunction is. Let a sheaf F on X be given. The counit $\epsilon_F : f^{-1}f_*F \longrightarrow F$ is the morphism induced by the identity $f_*F \longrightarrow f_*F$. So first put

$$G'(U) = \varinjlim_{V \supseteq f(U)} F(f^{-1}V)$$
$$\psi : G' \longrightarrow F$$
$$\psi_U((V,t)) = t|_U$$

Then ϵ is defined for $U \subseteq X$ and $y \in U$ by $\mathrm{germ}_y\, \epsilon_U(s) = \psi_y(s(y))$. If the function $s : U \longrightarrow \bigcup_{x \in U} G'_x$ is defined by $s(x) = (U_x, (V_x, t_x))$ with $t_x \in F(f^{-1}V_x)$ then

$$\mathrm{germ}_y\, \epsilon_U(s) = (U_y, t_y|_{U_y})$$

Let $f : X \longrightarrow Y$ be continuous, F a sheaf on X and G a sheaf on Y (of sets, groups or rings). If $\phi : G \longrightarrow f_*F$ is a morphism, the adjoint partner $\bar{\phi} : f^{-1}G \longrightarrow F$ is the composite

$$f^{-1}G \xrightarrow[f^{-1}\phi]{} f^{-1}f_*F \xrightarrow{\varepsilon_F} F$$

Using the above we see that for $U \subseteq X$ and $s \in (f^{-1}G)(U)$ if $s(x) = (U_x, (V_x, t_x))$, $V_x \supseteq f(U_x)$, $t_x \in G(V_x)$, then for $x \in U$,

$$\mathrm{germ}_x \bar{\phi}_U(s) = (U_x, \phi_{V_x}(t_x)|_{U_x}) \quad \text{in } F_x$$

Suppose that f is a homeomorphism, so that f_* is fully faithful and hence ε is a pointwise isomorphism. If ϕ is an isomorphism, then $\bar{\phi}$ is also an isomorphism. Let $t \in F(U)$ be given. Then let $s = \bar{\phi}_U^{-1}(t)$. By the above,

$$\mathrm{germ}_x t = (U_x, \phi_{V_x}(t_x)|_{U_x})$$

$(V_x, t_x) \in P(V_x)$

where $s(x) = (U_x, (V_x, t_x))$ with $x \in U_x$, $V_x \supseteq f(U_x)$ and $t_x \in G(V_x)$. That is,

$$\mathrm{germ}_x t = \mathrm{germ}_x \phi_{V_x}(t_x) \qquad (= (f^{-1}V_x, \phi_{V_x}(t_x)))$$
$$= \phi_{f(x)}\, \mathrm{germ}_{f(x)} t_x$$
$$\therefore\ \mathrm{germ}_{f(x)} t_x = \phi_{f(x)}^{-1}\, \mathrm{germ}_x t \qquad \phi_{f(x)} : G_{f(x)} \longrightarrow F_x$$

NOTE This is unnecessary: we already described the adjoint partner on the previous page.

NOTE Sheafification Commutes with Restriction.

Let X be a topological space, $U \subseteq X$ open. We claim the following diagram commutes up to isomorphism

$$\begin{tikzcd} P(X) \arrow[r, "a"] \arrow[d, "-|_U"'] & Sh(X) \arrow[d, "-|_U"] \\ P(U) \arrow[r, "a"'] & Sh(U) \end{tikzcd}$$

where P, Sh stand for presheaves and sheaves resp. (of sets, groups or rings). We prove the case of sets: the other cases follow immediately. Let F be a presheaf on P, and let

$$\phi_x : (F|_U)_x \longrightarrow F_x$$
$$(W,s) \longmapsto (W,s)$$

be the canonical isomorphism for $x \in U$. Then we define

$$\eta : a(F|_U) \longrightarrow (aF)|_U$$
$$\eta_W(s)(x) = \phi_x(s(x))$$

This is clearly a morphism of sheaves, which is an isomorphism due to the following commutative diagram:

$$\begin{tikzcd} a(F|_U)_x \arrow[r] \arrow[d, Rightarrow] & (aF)|_{U\,x} \arrow[d, Rightarrow] \\ (F|_U)_x \arrow[r, Rightarrow, "\phi_x"'] & F_x \end{tikzcd}$$

It is easily checked that the isomorphism η is natural in F, completing the proof.

EXAMPLES If $1 = \{*\}$ is the terminal space and $f: 1 \longrightarrow X$ a point $x \in X$ then $f^{-1} = \mathrm{Stalk}_x$ and $f_* = \mathrm{Sky}_x$ where

$$\mathrm{Sky}_x(A)(U) = \begin{cases} A & x \in U \\ \{*\} & \text{otherwise} \end{cases}$$

If $f: X \longrightarrow 1$ then f_* maps sheaves to sets (groups, rings) and $f_* F = F(X)$, i.e. f_* is the global sections functor Γ. The inverse image functor comes from the presheaf

$$A'(U) = \varinjlim_{V \supseteq f(U)} A'(V)$$

$$= \begin{cases} A & U \neq \emptyset \\ \{*\} & U = \emptyset \end{cases} \qquad \ulcorner\text{restrictions the identity}\lrcorner$$

Then f^{-1} is the associated sheaf of A'. For any $x \in X$, $A'_x = A$. So

$$f^{-1}A(U) = \{\text{locally constant functions } U \longrightarrow A\}$$

i.e. $f: U \longrightarrow A$ s.t. every $x \in U$ has an open neighbourhood V_x s.t. $f|V_x$ is constant.

If U is a connected open set, then for any $g \in f^{-1}A(U)$ we can partition U into open sets on which g takes distinct values. Hence g is constant on U, so $f^{-1}A(U) \cong A$.

NOTE If F is a pre sheaf (of sets, groups or rings) let $\eta: F \longrightarrow aF$ be canonical. For $P \in X$ the morphism $\eta_P: F_P \longrightarrow (aF)_P$ is a bijection, hence isomorphism. Surjectivity is obvious, and for injectivity if $(U,s),(V,t) \in F_P$ and $(U,s) = (V,t)$ then $\mathrm{germ}_Q s = \mathrm{germ}_Q t$ $\forall Q \in W$ where $W \subseteq U \cap V$ is open. Hence pick some Q — in F_Q $(U,s) = (V,t)$ so $s|_M = t|_M$ for some $Q \in M \subseteq U \cap V$. Since $P \in W$ we may assume $Q = P$ and $(U,s) = (V,t)$ in F_P as required. Note the inverse $(aF)_P \longrightarrow F_P$ is given by $(U,s) \longmapsto s(P)$.

NOTE Let F be a sheaf on a space X and let $Z \subseteq X$ be closed. We define a sheaf $F|_Z$ as follows: for an open subset $U \subseteq Z$ (open in the subspace topology)

$$F|_Z(U) = \{ s: U \longrightarrow \textstyle\bigcup_{P \in U} F_P \mid s(P) \in F_P \ \forall P \in U \text{ and for each } P \in U \text{ there is an open neighborboud } V \text{ of } P \text{ in } U \text{ and a set } Q \text{ open in } X, \text{ and } t \in F(Q) \text{ s.t. } V \subseteq Q \text{ and } s(Q) = \mathrm{germ}_Q t \ \forall Q \in V \}$$

If F is a sheaf of groups or rings, it is easily checked that $F|_Z(V)$ is resp. a group or ring under the pointwise operations. Restriction is defined in the obvious way, making $F|_Z$ into a presheaf (of groups or rings if F is). The sheaf condition is easily checked. See Ex1.19 for some more details.

NOTE Let $\phi: F \longrightarrow F'$ be a morphism of sheaves (of sets, groups or rings) on X. Then ϕ is locally surjective iff there is an open cover $X = \bigcup_i V_i$ s.t. $\forall i$, $\phi|_{V_i}: F|_{V_i} \longrightarrow F'|_{V_i}$ is a locally surjective morphism of sheaves on V_i.

NOTE Let $f: X \longrightarrow Y$ be a continuous map, F a sheaf on X and G a sheaf on Y (of sets, groups, rings) and suppose $f(X)$ closed. A morphism $G \longrightarrow f_* F$ is locally surjective if and only if there is a collection $\{V_i\}$ of open sets $V_i \subseteq Y$ such that $f(X) \subseteq \bigcup_i V_i$ and for each i, $\phi|_{V_i}: G|_{V_i} \longrightarrow (f_* F)|_{V_i}$ is locally surjective.

To prove this, let $U \subseteq Y$ and $s \in F(f^{-1}U)$ be given. Cover $U \cap V_i$ with W_{ij} s.t. $t_{ij} \in G(W_{ij})$ exist with $\phi_{W_{ij}}(t_{ij}) = s|_{f^{-1}(W_{ij})}$. Then if we put $Q = U - f(X)$, and select $t_Q = 0$, then $\{W_{ij}\}_{i,j} \cup \{Q\}$ is an open cover of U with the required property. This doesn't work if $f(X)$ is not closed.

NOTE A subsheaf of a sheaf F (of sets, groups, rings) is a sheaf P (of sets, groups, rings) s.t. $\forall U \subseteq X$, $P(U)$ is a subset (subgroup, subring) of $F(U)$ and $P \longrightarrow F$ defined by these inclusions is a morphism of sheaves. Equivalently $P(U) \subseteq F(U)$ and the restrictions agree. If we are given subsets $P(U) \subseteq F(U)$ (subgroups, subrings) for all $U \subseteq X$ then if $t|_V \in P(V)$ for all $V \subseteq U$, $t \in P(U)$ and if whenever $U \subseteq X$ is open and $\{V_i\}$ an open cover, if $s_i \in P(V_i)$ are a matching family then their amalgamation $s \in F(U)$ belongs to $P(U)$ (including empty cover, so $\{*\} = P(\emptyset)$) then P is a subsheaf (of sets, groups, rings).

Let $P \xrightarrow{\phi} F$ be a subsheaf (of sets, groups, rings). Then for $x \in X$ $P_x \xrightarrow{\phi_x} F_x$ is injective and we identify P_x with a subset (subgroup, subring) of F_x. Then for all $U \subseteq X$

$$P(U) = \{ s \in F(U) \mid germ_x s \in P_x \ \forall x \in U \}$$

NOTE Let $i : X \longrightarrow Y$ be a continuous map which gives a homeomorphism of X with a subspace of Y. We claim that for any sheaf F on X (sets, groups, rings) and $x \in X$ the map

$$\begin{aligned} (i_* F)_{i(x)} &\longrightarrow F_x \\ (U, s) &\longmapsto (f^{-1}U, s) \end{aligned} \qquad (1)$$

is an isomorphism (of sets, groups, rings). It is easy to see this is a well-defined map of sets, groups, or rings. Suppose (U,s), (V,t) both map to the same element, so there is $x \in W \subseteq i^{-1}U \cap i^{-1}V$ with W open and $s|_W = t|_W$. Then $i(W)$ is open in $i(X)$, say $W' \cap i(X) = i(W)$, $W' \subseteq Y$ open. Then $W' \subseteq U \cap V$ (one can assume) and in $i_* F$ we have $s|_{W'} = t|_{W'}$, so $(U,s) = (V,t)$ in $(i_* F)_{i(x)}$ and the map in (1) is injective. To see it is surjective let $W \subseteq X$ be open $s \in F(W)$. Find $W' \subseteq Y$ open s.t. $f^{-1}W' = W$ as above. Then $s \in (i_* F)(W')$ and $(W', s) \longmapsto (W, s)$, as required.

EXERCISES II §1

Q1.1 Done

Q1.2 (a) Let $\varphi : F \longrightarrow G$ be a morphism of sheaves on X, and let $P \in X$. Let $\varphi_P : F_P \longrightarrow G_P$ be the induced morphism on stalks. (we are dealing with sheaves of groups). Then

$$\operatorname{Ker}\varphi_P = \{(U,s) \in F_P \mid s|_W \in \operatorname{Ker}\varphi_W \text{ some } W \subseteq U\}$$

The sheaf $\operatorname{Ker}\varphi$ is a subsheaf of F, so $(\operatorname{Ker}\varphi)_P$ is a subgroup of F_P and a germ (U,s) belongs to this subgroup iff. there is (V,t) s.t. $\varphi_V(t) = 0$ and $(U,s) \sim (V,t)$, that is, s.t. $s|_W \in \operatorname{Ker}\varphi_W$ some $W \subseteq U$. Hence $(\operatorname{Ker}\varphi)_P = \operatorname{Ker}\varphi_P$ as subgroups of F_P.

If $t \in (\operatorname{Im}\varphi)(U)$ and $P \in U$ then there is open $P \in W \subseteq U$ s.t. $t|_W = \varphi_W(s)$ for some $s \in F(W)$. Hence the subsheaf $\operatorname{Im}\varphi$ of G yields the subgroup

$$\begin{aligned}(\operatorname{Im}\varphi)_P &= \{(W, \varphi_W(s)) \mid s \in F(W)\}\\ &= \operatorname{Im}\varphi_P\end{aligned}$$

(b) Since φ is monic ($=$ injective) iff. $\operatorname{Ker}\varphi = 0$ and φ is epi ($\neq$ surjective in the pt.wise sense) iff. $\operatorname{Im}\varphi = G$

$$\begin{aligned}\varphi \text{ monomorphism} &\iff \operatorname{Ker}\varphi_P = 0 \quad \forall P\\ &\iff \varphi_P \text{ injective} \quad \forall P\end{aligned}$$

$$\begin{aligned}\varphi \text{ epimorphism} &\iff \operatorname{Im}\varphi_P = G_P \quad \forall P\\ &\iff \varphi_P \text{ surjective}\end{aligned}$$

To do this we need only show that if K, L are subsheafs of a sheaf M s.t. $\forall U$ $K(U) \subseteq L(U)$ and if $K_P = L_P$ $\forall P$, then $K = P$. Say $t \in L(U)$ and for each $P \in U$ find $P \in V_P \subseteq U$ s.t. $t|_{V_P} \in K(V_P)$. (since $t_P \in K_P$). But the fact that K is a subsheaf means it contains amalgamations of its matching families, so $t \in K(U)$ as required.

<u>NOTE</u> The above only works for groups, but in our topos notes we show that for a morphism $\phi : F \longrightarrow G$ of sheaves of sets, ϕ is monic (resp. epi) in $Sh(X)$ iff. $\forall x \in X$ ϕ_x is injective (resp. surjective). Epimorphisms of rings are not the surjections, so there may not be a similar result for sheaves of rings.

(c) Let $\cdots F^{i-1} \xrightarrow{\varphi^{i-1}} F^i \xrightarrow{\varphi^i} F^{i+1} \xrightarrow{\varphi^{i+1}} \cdots$ be a sequence of sheaves of abelian groups and morphisms. ($\varphi^i\varphi^{i-1} = 0$) We claim that this sequence is exact in the abelian category $\underline{Ab}(Sh(X))$ iff. for each $P \in X$ the induced sequence on stalks is exact in $\underline{Ab}$. It suffices to prove the claim for a sequence $F \xrightarrow{\varphi} G \xrightarrow{\phi} H$, with $\phi\varphi = 0$. Then of course $\phi_P\varphi_P = 0$ for all P, and $\operatorname{Im}\varphi$ is a subsheaf of $\operatorname{Ker}\phi$ since the presheaf image of φ is contained in $\operatorname{Ker}\phi$, hence so is $\operatorname{Im}\phi$. By (b) we know that $F \to G \to H$ is exact iff. $\operatorname{Im}\varphi = \operatorname{Ker}\phi$ iff. $(\operatorname{Im}\varphi)_P = (\operatorname{Ker}\phi)_P$ $\forall P$ so iff. $F_P \longrightarrow G_P \longrightarrow H_P$ is exact for all P.

Q1.3 Done in notes

Q1.4 Done in notes

Q1.5 A morphism of sheaves of groups or rings is an isomorphism in $\underline{Ab}(Sh(X))$ or $\underline{Rng}(Sh(X))$ iff. it is an isomorphism in $\underline{Ab}^{\mathcal{O}(X)^{op}}$ or $\underline{Rng}^{\mathcal{O}(X)^{op}}$, since the former categories are full subcategories of the latter. But isomorphisms of presheaves are pointwise bijections, since isos in $\underline{Ab}$, $\underline{Rng}$ and $\underline{Sets}$ are the bijections. Hence

A morphism $\phi : F \longrightarrow G$ of sheaves of sets, groups or rings is an isomorphism iff. $\phi_U : F(U) \longrightarrow G(U)$ is an isomorphism $\forall U$, so iff. ϕ_U is bijective $\forall U$.

We know from Q1.2 that for sheaves of sets and groups ϕ iso $\iff$ ϕ_P bijective all $P \iff \phi_P$ inj + surj $\iff \phi$ monic & epic.

NOTE Morphism Cheat Sheet

Sets Presheaves A morphism $\phi: F \longrightarrow G$ is a

- monomorphism iff. ϕ_U is injective $\forall U \subseteq X$
- epimorphism iff. ϕ_U is surjective $\forall U \subseteq X$
- isomorphism iff. ϕ_U is bijective $\forall U \subseteq X$

- isomorphism iff. epimorphism and monomorphism

Sheaves A morphism $\phi: F \longrightarrow G$ is a

- monomorphism iff. ϕ_U is injective $\forall U \subseteq X$ ⊛
- epimorphism iff. ϕ is locally surjective : for all $t \in G(V)$ is an open cover $V = \bigcup_i W_i$ s.t. $\forall i$ $t|_{W_i} = \phi_{W_i}(s_i)$ some $s_i \in F(W_i)$.
- isomorphism iff. ϕ_U is bijective $\forall U \subseteq X$

⌈Note that in the definition of locall surjective it doesn't matter whether we allow empty covers⌋

Also

$$\text{monic} \iff \phi_P \text{ injective } \forall P \in X$$
$$\text{epi} \iff \phi_P \text{ surjective } \forall P \in X$$
$$\text{iso} \iff \phi_P \text{ bijective } \forall P \in X$$

Hence isomorphism = epi + monic.

Abelian Groups Presheaves as for Sets.

Sheaves as for Sets

Rings Presheaves A morphism $\phi: F \longrightarrow G$ is a

- monomorphism iff. ϕ_U is injective $\forall U \subseteq X$
- epimorphism iff. ϕ_U is a ring epimorphism $\forall U \subseteq X$
- isomorphism iff. ϕ_U is bijective $\forall U \subseteq X$

⊛ ⌈See Thesis notes on reflective subcategories⌋

Sheaves A morphism $\phi: F \longrightarrow G$ is a

- monomorphism iff. ϕ_U is injective $\forall U \subseteq X$ ⊛
- isomorphism iff. ϕ_U is bijective $\forall U \subseteq X$

Q1.6 (sheaves are of abelian groups) A monomorphism $\phi : F \to G$ is pointwise injective, so F is isomorphic to the subsheaf $U \mapsto \phi_U(F(U))$ of G. The cokernel of ϕ is the associated sheaf of the presheaf $U \mapsto G(U)/\mathrm{Im}\,\phi_U$. Any monic in an abelian category is the kernel of its cokernel. If F is a subsheaf of G and $G \to G/F$ is the cokernel there is trivially an exact sequence $0 \to F \to G \to G/F \to 0$.

(b) If $0 \to F' \to F \to F'' \to 0$ is exact then $F' \to F$ is monic, so F' is isomorphic to a subsheaf of F and $F \to F''$ is thus a cokernel for this subsheaf, so necessarily iso to the canonical F/sub.

Q1.7 (Sheaves of abelian groups) (a) $\varphi : F \to G$. The image of φ is the cokernel of φ's kernel, so trivially $\mathrm{Im}\,\varphi \cong F/\mathrm{Ker}\,\varphi$. (b) Similarly $\mathrm{Im}\,\varphi$ is the kernel of φ's cokernel, or put differently, the cokernel of φ is also the cokernel of $\mathrm{Im}\,\varphi \to G$, so $\mathrm{Coker}\,\varphi \cong G/\mathrm{Im}\,\varphi$.

Q1.8 Let $U \subseteq X$ be open. There is a functor $T'(U,-) : \underline{Ab}(Sh(X)) \to \underline{Ab}$ defined by $F \mapsto F(U)$. This functor is left exact: if $0 \to F' \xrightarrow{\phi} F \xrightarrow{\psi} F''$ is an exact sequence of sheaves we claim that

$$0 \longrightarrow T'(U,F') \xrightarrow{T\phi} T'(U,F) \xrightarrow{T\psi} T'(U,F'')$$

is an exact sequence of abelian groups. Clearly $T\psi \cdot T\phi = 0$ and $T\phi = \phi_U$ so $T\phi$ is a monomorphism.

The original sequence being exact means $\mathrm{Im}\,\phi = \mathrm{Ker}\,\psi$ as subsheaves of F, i.e. $(\mathrm{Im}\,\phi)(U) = (\mathrm{Ker}\,\psi)(U)$ $\forall U \subseteq X$. Hence $(\mathrm{Im}\,\phi)(U) = \mathrm{Ker}\,\psi_U$ for all $U \subseteq X$. Since ϕ is a monomorphism $(\mathrm{Im}\,\phi)(U) = \mathrm{Im}\,\phi_U$ – i.e. the image sheaf is just the presheaf image. Hence $\mathrm{Im}\,\phi_U = \mathrm{Ker}\,\psi_U$ as required.

Clearly $T'(U,-)$ is additive, $T'(U,-) : \underline{Ab}(Sh(X)) \to \underline{Ab}$ is a left-exact additive functor. (i.e. $T'(U,-)$ is kernel preserving).

Q1.9, **Q1.10**, **Q1.11**, **Q1.12** Done in notes.

Q1.13 See topos notes.

Q1.14 <u>Support</u> Let F be a sheaf on X, and let $s \in F(U)$ be any section. The <u>support</u> of s, denoted $\mathrm{Supp}\,s$, is

$$\mathrm{Supp}\ s = \{ P \in U \mid s_P \neq 0 \}$$

If $s_P = 0$ then $s|_{V_P} = 0$ for some open neighborhood V_P of P. Hence the complement of $\mathrm{Supp}\,S$ in U is an open set, so $\mathrm{Supp}\,s$ is closed in U. The <u>support</u> of F is $\mathrm{Supp}F = \{P \in X \mid F_P \neq 0\}$. It need not be closed.

Q1.15 <u>Sheaf Hom</u> Let F, G be sheaves of abelian groups on X. For any open set $U \subseteq X$, $\mathrm{Hom}(F|_U, G|_U)$ is naturally an abelian group and $U \mapsto \mathrm{Hom}(F|_U, G|_U)$ is a presheaf on X, where for $V \subseteq U$ and $\phi : F|_U \to G|_U$ the restriction $\phi|_V$ is defined in the obvious way. (Note $\phi \mapsto \mathrm{Hom}(F|_\emptyset, G|_\emptyset) = 0$). Clearly $\mathrm{Hom}(F|_X, G|_X) = \mathrm{Hom}(F,G)$. To show that this presheaf is a sheaf, let $V = \cup_i U_i$ and suppose we have $f_i : F|_{U_i} \to G|_{U_i}$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i,j$ – i.e. for $W \subseteq U_i \cap U_j$ $(f_i)_W = (f_j)_W$. Define a natural transformation $f : F|_V \to G|_V$ by

$$f_Q : F(Q) \longrightarrow G(Q)$$
$$f_Q(s) = \text{unique amalgamation of } (f_i)_{Q \cap U_i}(s|_{Q \cap U_i}), i \in I$$

The fact that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ shows that f_Q is well-defined, and it is easy to check that f_Q is a morphism of groups. If $Z \subseteq Q$ then $f_Q(s)|_Z$ is the unique amalg. of $(f_i)_{Q \cap U_i}(s|_{Q \cap U_i})|_{Z \cap U_i}$ in $G(Z)$, but

$$(f_i)_{Q \cap U_i}(s|_{Q \cap U_i})|_{Z \cap U_i} = (f_i|_{Z \cap U_i}(s|_{Z \cap U_i})$$

So clearly $f_Q(s)|_Z = f_Z(s|_Z)$ as required. It only remains to show that f is the unique amalgamation of the f_i. It is easily checked that $f|_{U_i} = f_i$ for all i. Moreover for $Q \subseteq X$ and $s \in F(Q)$, s is the unique amalgamation of the $s|_{Q \cap U_i}$ $i \in I$, so for <u>any</u> morphism $\phi : F \to G$ with $\phi|_{U_i} = f_i$, the element $\phi_Q(s)$ will be the unique amalgamation of $\phi_{Q \cap U_i}(s|_{Q \cap U_i}) = (f_i)_{Q \cap U_i}(s|_{Q \cap U_i})$, so $\phi_Q(s) = f_Q(s)$, as required.

Q1.16 Flasque Sheaves A sheaf F on a topological space X is flasque if for every inclusion $V \subseteq U$ of open sets the restriction map $F(U) \longrightarrow F(V)$ is surjective.

(a) Suppose X is irreducible and let F be the constant sheaf associated to a set (resp. group) A. If $f \in F(U)$ then either $U = \emptyset$ and $F(U) = \{*\}$ (resp. 0) or $U \neq \emptyset$ and thus U itself is irreducible. But U is partitioned into open sets on which f takes distinct values. Hence each of these sets is clopen in U – but the only clopen sets in an irred. space are $\emptyset$ and the whole space. So f is constant on U, and F is just given by $U \longmapsto A$ (or $\emptyset \longmapsto \{*\} / 0$). So the restriction maps are certainly surjective. (for sheaves of sets and groups)

(b) Let $0 \longrightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \longrightarrow 0$ be an exact sequence of sheaves of abelian groups, and suppose F' is flasque. For any $U \subseteq X$ we claim the sequence

$$0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U) \longrightarrow 0$$

is exact. We know from Ex1.8 that $0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U)$ is exact, so it only remains to show that $F(U) \longrightarrow F''(U)$ is surjective. It suffices to show that if $\psi : F \longrightarrow F''$ is a morphism of sheaves whose kernel is flasque, then $\mathrm{Im}\,\psi$ is just the presheaf image. Fix $U \subseteq X$, we show $(\mathrm{Im}\,\psi)(U) = \mathrm{Im}\,\psi_U$. Let $t \in (\mathrm{Im}\,\psi)(U)$ be given.

Let K denote the set of all pairs (V, q) where $V \subseteq U$, $q \in F(V)$ and $\psi_V(q) = t|_V$, and partially order K via $(V,q) \leq (W,s)$ iff. $V \subseteq W$ and $s|_V = q$.

Let L be the set of nonempty subsets of K with the property that the open sets in the pairs give a cover of U. Partially order L by setting $X \leq Y$ iff. $\forall (V,q) \in X$ there is $(W,s) \in Y$ s.t. $(V,q) \leq (W,s)$. Let L' be the set of equivalence classes of L under the relation $X \sim Y$ iff. $X \leq Y$ and $Y \leq X$. Then $\leq$ induces a partial order on L'. L is nonempty by definition of the subsheaf $\mathrm{Im}\,\psi$.

The poset L' is nonempty and by taking unions any chain has an upper bound. So by Zorn's Lemma there is a maximal element Z' of L'. Let Z be a representative of this maximal class. The set Z is itself a poset of pairs (V,q). Let $\mathcal{C}$ be a chain in Z, $\mathcal{C} = \{(V_i, q_i)\}_{i \in I}$. Suppose that $\mathcal{C}$ does not have an upper bound in Z. Since $\mathcal{C}$ is a chain, the sections q_i form a matching family on the open set $W = \bigcup_i V_i \subseteq U$, so since F is a sheaf there is $q \in F(W)$ s.t. $(V_i, q_i) \leq (W, q)$ $\forall i \in I$. If this belonged to Z it would be an upper bound for $\mathcal{C}$, so $(W,q) \notin Z$. But it is easily checked that $\psi_W(q) = t|_W$, so $Z \cup \{(W,q)\} > Z$, contradicting maximality of Z. Hence every chain in Z must have an upper bound, so by Zorn Z contains a maximal element (M, m). We claim that $M = U$, in which case $\psi_U(m) = t$, so $t \in \mathrm{Im}\,\psi_U$ and we are done.

Suppose otherwise that $M \subset U$ and let $(V,q) \in Z$ be s.t. $\emptyset \neq V \nsubseteq M$ (this is possible since the opens in Z cover U). If $M \cap V = \emptyset$ then the sheaf condition on F gives $m' \in F(M \cup V)$ s.t. $(M,m) < (M \cup V, m')$.

If $M \cap V \neq \emptyset$ then

$$\begin{aligned} \psi_{M\cap V}(m|_{M\cap V} - q|_{M\cap V}) &= \psi_M(m)|_{M\cap V} - \psi_V(q)|_{M\cap V} \\ &= t|_{M\cap V} - t|_{M\cap V} = 0 \end{aligned}$$

So $m|_{M\cap V} - q|_{M\cap V} \in \mathrm{Ker}\,\psi_{M\cap V} = (\mathrm{Ker}\,\psi)(M \cap V)$. By assumption the kernel of ψ is flasque so there is $\Delta \in \mathrm{Ker}\,\psi_M$ s.t. $\Delta|_{M\cap V} = m|_{M\cap V} - q|_{M\cap V}$. Then $m - \Delta, q$ is a matching family on $M \cup V$, so there is $m' \in F(M \cup V)$ s.t. $m'|_M = m - \Delta$ and $m'|_V = q$. Since $\psi_M(m'|_M) = \psi_M(m) = t|_M$ and $\psi_V(m'|_V) = \psi_V(q) = t|_V$ it follows that $\psi_{M\cup V}(m') = t|_{M\cup V}$, so $(M,m) < (M\cup V, m')$.

So in any case we have produced $(M', m') \in K$ s.t. $(M,m) < (M', m')$. Since (M,m) is maximal in Z, it follows that $Z \cup \{(M',m')\} > Z$, contradicting maximality of Z. So finally we conclude that $M \subset U$ is impossible, and the proof is complete.

(c) If $0 \longrightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \longrightarrow 0$ is exact and if F and F' are flasque, then F'' is flasque. Since ψ has flasque kernel, it follows that $(\mathrm{Im}\,\psi)(U) = \mathrm{Im}\,\psi_U$, so if $V \subseteq U$ and $t \in F''(V)$ then $t = \psi_V(s)$ for some $s \in F(V)$. But F is flasque so $s = m|_V$ some $m \in F(U)$. Consequently $t = \psi_V(m|_V) = \psi_U(m)|_V$ as required.

(d) If $f : X \longrightarrow Y$ is a continuous map, and if F is a flasque sheaf on X, then we claim $f_* F$ is flasque on Y. This is obvious since for $W \subseteq Q \subseteq Y$, $f_* F(Q) \longrightarrow f_* F(W)$ is just $F(f^{-1}Q) \longrightarrow F(f^{-1}W)$.

(e) Let F be any sheaf on X. We define a new sheaf G called the sheaf of <u>discontinuous sections</u> of F, as follows. For $U \subseteq X$, $G(U)$ is the set of maps $s : U \longrightarrow \bigcup_{P \in U} F_P$ s.t. for $P \in U$, $s(P) \in F_P$. This is clearly a sheaf (of abelian groups if F is) and G is flasque since if $U \subseteq V$ and $s : U \longrightarrow \bigcup_{P \in U} F_P$ is in $G(U)$, define $t : V \longrightarrow \bigcup_{P \in V} F_P$ by $t(Q) = s(Q)$ $\forall Q \in U$ and 0 otherwise. Then $t|_U = s$ and G is flasque. The map $F(U) \longrightarrow G(U)$ given by $s \longmapsto (P \longmapsto \mathrm{germ}_P s)$ is an injective morphism of sheaves.

Q1.17 <u>Skyscraper Sheaves</u> Let X be a topological space, $P \in X$, and let A be an abelian group. Define a sheaf $i_P(A)$ on X by $i_P(A)(U) = A$ if $P \in U$ and $i_P(A)(U) = 0$ otherwise. Let $Q \in \{P\}^-$ (closure of P). Then any open set containing Q contains P, and it follows that the stalk of i_P at Q is A. If $Q \notin \{P\}^-$ then every open set containing Q contains an open neighborhood of Q not containing P, so the stalk at Q is 0.

Let $i : \{P\}^- \longrightarrow X$ be the inclusion, and let F be the constant sheaf for A on $\{P\}^-$. The only open subsets of $\{P\}^-$ contain P, so if $V \subseteq \{P\}^-$ is open, a locally constant map $f : V \longrightarrow A$ is constant at the value $f(P)$. So clearly $i_* F \cong i_P(A)$.

Q1.18 Done in notes

Q1.19 <u>Extending a Sheaf by Zero</u> Let X be a topological space, let Z be a closed subset with inclusion $i : Z \longrightarrow X$, let $U = X - Z$ be the complementary open subset, and let $j : U \longrightarrow X$ be its inclusion.

(a) Let F be a sheaf on Z (of abelian groups). For a point $P \notin Z$ and any germ $(V, s) \in (i_* F)_P$ we have $(V, s) = (V \cap U, s|_{V \cap U}) = 0$ since $(i_* F)(V \cap U) = F(V \cap U \cap Z) = 0$. Hence $(i_* F)_P = 0$. For $P \in Z$ and a germ $(V, s) \in F_P$ (i.e. $V \subseteq Z$ is open in the subspace topology) let $Q \subseteq X$ be open s.t. $V = Q \cap Z$. Then $(i_* F)(Q) = F(Q \cap Z) = F(V)$ so $(Q, s) \in (i_* F)_P$. This defines a map $F_P \longrightarrow (i_* F)_P$ which is injective since if $(Q', s') = (Q, s)$ in $(i_* F)_P$ then there is $W \subseteq Q \cap Q'$ s.t. $s|_{W \cap Z} = s'|_{W \cap Z}$. But $W \cap Z \subseteq (Q \cap Z) \cap (Q' \cap Z)$ so $(Q \cap Z, s) = (Q' \cap Z, s')$ in F_P. (One also checks the map is well-defined). The map is trivially surjective and a morphism of groups. Hence $(i_* F)_P \cong F_P$ for $P \in Z$.

We call $i_* F$ the sheaf obtained by extending F by zero outside Z. By abuse of notation we will sometimes write F instead of $i_* F$ and say "consider F as a sheaf on X" when we mean "consider $i_* F$".

(b) Now let F be a sheaf on U. Let $j_!(F)$ be the sheaf on X associated to the presheaf $V \mapsto F(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. There is an isomorphism of abelian groups of $j_!(F)_P$ with the stalk at P of this presheaf: these stalks are clearly 0 if $P \notin U$ and F_P if $P \in U$. Hence $j_!(F)_P \cong F_P$ for $P \in U$ and 0 otherwise. Suppose G is another sheaf on X with $G|_U = F$ and $G_P = 0$ $\forall P \notin U$, with the canonical $F_P \longrightarrow G_P$ an isomorphism $\forall P \in U$.

There is a canonical morphism $\phi : \hat{F} \longrightarrow G$ where $\hat{F}$ is the presheaf above, hence $\phi' : j_!(F) \longrightarrow G$. For $V \subseteq U$ it is clear that ϕ'_V is an isomorphism. If $V \nsubseteq U$ and $s \in j_!(F)(V)$ then there is an open cover $V = \bigcup_i W_i$ and $s_i \in \hat{F}(W_i)$ s.t. $s|_{W_i} = s_i$. Dividing the W_i into those contained in U and those that aren't, we see that $\phi'_V(s)$ is the unique amalgamation in $G(V)$ of a bunch of sections over open sets in U, together with zero sections on a cover of $V - U$. Since $G|_U = F = \hat{F}|_U$ it is clear that if $\phi'_V(s) = 0$ then $s = 0$.
If $s \in G(V)$ then $s_P = 0$ for all $P \notin U$ so s is the unique amalgamation of $s|_{U \cap V}$ and a zero cover of $V - U$, so clearly ϕ' is surjective pointwise. Hence ϕ' is an isomorphism, so $j_! F$ is unique. We call $j_! F$ the sheaf obtained by <u>extending F by zero outside</u> U.

Note that if $Z = \emptyset$ then $i_* F = 0$ and if $U = \emptyset$ then $j_! F = 0$.

(c) Let F be a sheaf on X, let $Z \subseteq X$ be closed and $U = X - Z$. Then $F|_U$ is a sheaf on U and as above there is a morphism $\phi : j_!(F|_U) \longrightarrow F$. The sheaf $F|_Z$ was defined in notes and the morphism ψ is the canonical one

$$\psi : F \longrightarrow i_*(F|_Z)$$
$$\psi_V : F(V) \longrightarrow F|_Z(Z \cap V)$$
$$s \longmapsto \dot{s} \quad \text{where } \dot{s} : Z \cap V \longrightarrow \bigcup_{P \in Z \cap V} F_P$$
$$P \longmapsto \text{germ}_P\, s$$

This is a morphism of sheaves of abelian groups. By definition of $F|_Z$ every section is pieced together locally from images of sections under ψ, so ψ is an epimorphism of sheaves. We checked in (b) that ϕ was a monomorphism of sheaves. We claim that the sequence

$$0 \longrightarrow j_!(F|_U) \xrightarrow{\phi} F \xrightarrow{\psi} i_*(F|_Z) \longrightarrow 0 \qquad (1)$$

Trivially exact if $U = \emptyset$ or $Z = \emptyset$

is exact. To prove that (1) is exact it suffices by Ex1.2 to prove exactness on stalks. But for $P \in Z$, $j_!(F|_U)_P = 0$ and for $P \in U$, $i_*(F|_Z)_P = 0$. Moreover ϕ_P is injective and ψ_P surjective $\forall P$, so it suffices to show that for $P \in Z$ ψ_P is injective and for $P \in U$ that ϕ is surjective. For $P \in Z$ if $\psi_P(V, s) = 0$ then $\text{germ}_P s = \dot{s}(P) = 0$ so $(V, s) = 0$ and ψ_P is injective. For $P \in U$ and $(V, s) \in F_P$, $(V, s) = (V \cap U, s|_{U \cap V})$ so ϕ_P is surjective. Hence (1) is exact.

$\boxed{\text{Q1.20}}$ <u>Subsheaf with Supports</u> Let Z be a closed subset of X, and let F be a sheaf of abelian groups on X. We define

$$T_Z(X,F) = \{ s \in F(X) \mid \operatorname{Supp} s \subseteq Z \}$$

Clearly $T_Z(X,F)$ is a subgroup of $F(X)$.

note $\mathcal{H}^0_Z(F)(\phi) = 0$.

(a) Let $\mathcal{H}^0_Z(F)$ be the presheaf $V \longmapsto T_{Z\cap V}(V, F|_V) = \{ s \in F(V) \mid \operatorname{Supp} s \subseteq Z \cap V \}$. Then $\mathcal{H}^0_Z(F)$ is a subfunctor of F. To see that $\mathcal{H}^0_Z(F)$ is a sheaf, let $U = \bigcup_i V_i$ and say $s_i \in \mathcal{H}^0_Z(F)(V_i)$ are a matching family. Let $s \in F(U)$ be the amalgamation in F. Then $\operatorname{Supp} s = \bigcup \operatorname{Supp} s_i \subseteq Z$ so $s \in \mathcal{H}^0_Z(F)(U)$. Hence $\mathcal{H}^0_Z(F)$ is a sheaf, called the <u>subsheaf of F with support in Z</u>. Clearly $\mathcal{H}^0_\phi(F) = 0$.

(b) Let $U = X - Z$ and let $j : U \longrightarrow X$ be the inclusion. We assume $U \neq \phi$. We claim there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{H}^0_Z(F) \xrightarrow{\phi} F \xrightarrow{\psi} j_*(F|_U)$$

The morphism ϕ is trivially monic, so we just need to show that $\mathcal{H}^0_Z(F) = \operatorname{Ker}\psi$, where ψ is the morphism obtained by restriction. But

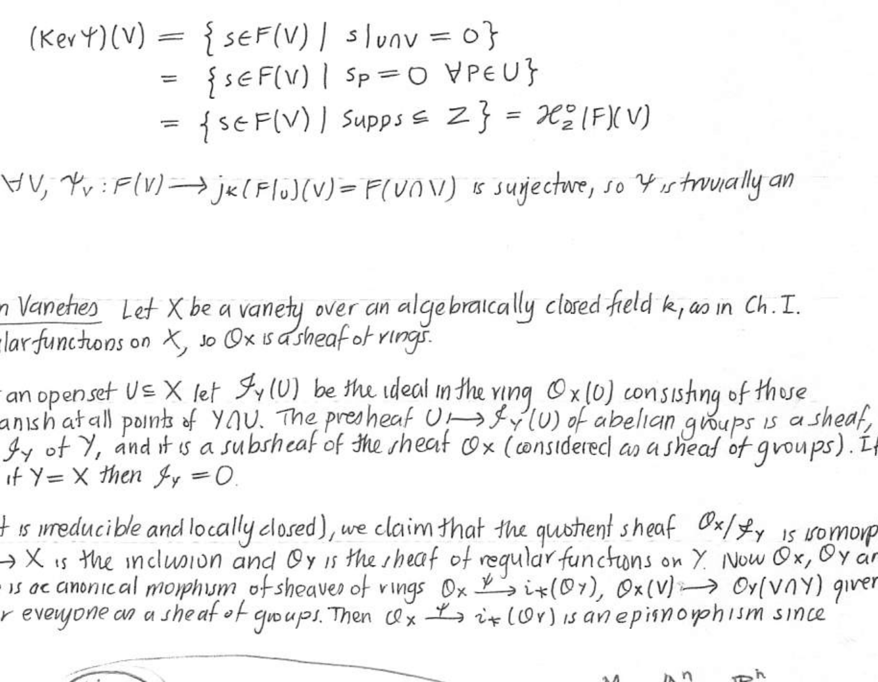
$$\begin{aligned}(\operatorname{Ker}\psi)(V) &= \{ s \in F(V) \mid s|_{U\cap V} = 0 \}\\ &= \{ s \in F(V) \mid s_P = 0 \ \forall P \in U \}\\ &= \{ s \in F(V) \mid \operatorname{Supp} s \subseteq Z \} = \mathcal{H}^0_Z(F)(V)\end{aligned}$$

If F is flasque then $\forall V$, $\psi_V : F(V) \longrightarrow j_*(F|_U)(V) = F(U \cap V)$ is surjective, so ψ is trivially an epimorphism.

$\boxed{\text{Q1.21}}$ <u>Some Examples of Sheaves on Varieties</u> Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X, so $\mathcal{O}_X$ is a sheaf of rings.

(a) Let $Y \subseteq X$ be closed. For an open set $U \subseteq X$ let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. The presheaf $U \longmapsto \mathcal{I}_Y(U)$ of abelian groups is a sheaf, called the <u>sheaf of ideals</u> $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf $\mathcal{O}_X$ (considered as a sheaf of groups). If $Y = \phi$ then $\mathcal{I}_Y = \mathcal{O}_X$ and if $Y = X$ then $\mathcal{I}_Y = 0$.

(b) If Y is a <u>subvariety</u> (i.e. it is irreducible and locally closed), we claim that the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \longrightarrow X$ is the inclusion and $\mathcal{O}_Y$ is the sheaf of regular functions on Y. Now $\mathcal{O}_X, \mathcal{O}_Y$ are sheaves of rings and there is ac canonical morphism of sheaves of rings $\mathcal{O}_X \xrightarrow{\psi} i_*(\mathcal{O}_Y)$, $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_Y(V \cap Y)$ given by $f \longmapsto f|_{V\cap Y}$. Now consider everyone as a sheaf of groups. Then $\mathcal{O}_X \xrightarrow{\psi} i_*(\mathcal{O}_Y)$ is an episnorphism since



$M = \mathbb{A}^n$ or $\mathbb{P}^n$

To define $\mathcal{O}_Y$ we use the original def^N of regular functions on subspaces of M, so $g : Q \longrightarrow k$ is regular iff. every point has an open neighborhood in Q, say V, and there are appropriate quotients of polynomials defined on V restricting to g on V. But $U \cap X$ is open in X, so it follows there is $Q' \subseteq X$ open and $f \in \mathcal{O}_X(Q')$ with $Q' \cap Y = V$ and $f|_V = g|_V$ – that is, $\psi_{Q'}(f) = g|_V$. We have shown that ψ is locally surjective, hence an epimorphism in the category $\underline{\mathrm{Ab}}(\mathrm{Sh}(X))$. Clearly $\operatorname{Ker}\psi = \mathcal{I}_Y$, so it follows immediately that $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}_Y$ as sheaves of abelian groups.

(c) Now let $X = \mathbb{P}^1$ and let Y be the union of two distinct points $P, Q \in X$. Then we claim there is an exact sequence of sheaves of abelian groups on X, where $F = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \xrightarrow{\psi} F \longrightarrow 0 \qquad (1)$$

Clearly $\mathcal{O}_P$ is the sheaf on $\{P\}$ given by $\emptyset \mapsto 0$, $\{P\} \mapsto k$. There is a canonical morphism of sheaves of abelian groups $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_P$ given by $f \longmapsto f(P)$ (similarly $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Q$). Since limits are taken pointwise

or 0 if $P \notin V$ so $i_*\mathcal{O}_P(V) = 0$

$F(V) = (i_*\mathcal{O}_P)(V) \oplus (i_*\mathcal{O}_Q)(V) = \mathcal{O}_P(V \cap \{P\}) \oplus \mathcal{O}_Q(V \cap \{Q\})$ and $\mathcal{O}_X \xrightarrow{\psi} F$ is given by $\mathcal{O}_X(V) \longrightarrow F(V)$, $f \longmapsto (f(P), f(Q))$. We prove that ψ is locally surjective. If $V \subseteq X$ is open but $\{P, Q\} \nsubseteq V$ then $\mathcal{O}_X(V) \longrightarrow F(V)$ is trivially surjective. If $P, Q \in V$ and $(a,b) \in F(V)$ $a, b \in k$ then let $U_1 = V - Q$ and $U_2 = V - P$. Clearly $(a,b)|_{U_1} = (a, 0)$ and $(a,b)|_{U_2} = (0, b)$. Hence $(a,b)|_{U_1} \in \operatorname{Im}\psi_{U_1}$ and $(a,b)|_{U_2} \in \operatorname{Im}\psi_{U_2}$ so ψ is locally surjective and hence an epimorphism. It is clear that $\mathcal{I}_Y = \operatorname{Ker}\psi$, so (1) is exact.

We claim however that $\mathcal{O}_X(X) \longrightarrow F(X) = k \oplus k$ is not surjective. That is, we cannot arbitrarily prescribe values at two distinct points of $\mathbb{P}^1$. But this is trivial since $\mathcal{O}(\mathbb{P}^1) = k$ – any regular function on $\mathbb{P}^1$ is constant. Hence the global sections functor $\Gamma(X, -)$ is not exact.

$\boxed{Q1.22}$ Glueing sheaves Let X be a topological space, $\{U_i\}_{i \in I}$ an open cover of X, and suppose that for each i we are given a sheaf $\mathcal{F}_i$ on U_i, and for each i,j an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j|_{U_i \cap U_j}$ s.t.

sheaves of sets, groups or rings

$$(\alpha) \qquad \forall i \ \ \varphi_{ii} = 1$$
$$(\beta) \qquad \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} \text{ on } U_i \cap U_j \cap U_k \quad \forall i,j,k$$

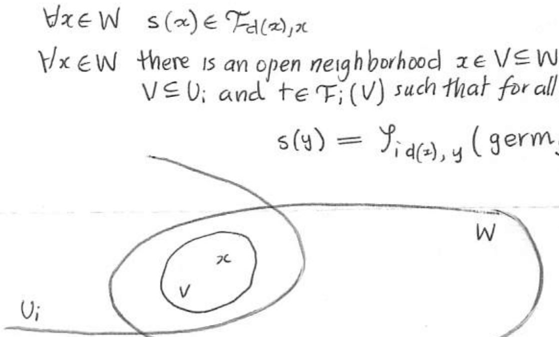
We claim that there exists a unique sheaf $\mathcal{F}$ for X with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{F}_i$ such that for each i,j $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We can clearly reduce to the case where no $U_i = \emptyset$. For each $x \in X$ let $d(x) \in I$ be some selected index with $x \in U_{d(x)}$. For $x \in U_i \cap U_j$ let $\varphi_{ij,x} : \mathcal{F}_{i,x} \xrightarrow{\sim} \mathcal{F}_{j,x}$ denote the isomorphism of (sets, groups, rings) induced by φ_{ij}. For an open set $W \subseteq X$ define $\mathcal{F}(W)$ to be the set of all functions $s : W \longrightarrow \bigcup_{x \in W} \mathcal{F}_{d(x),x}$ with the properties:

$$s : W \longrightarrow \bigcup_{x \in W} \mathcal{F}_{d(x),x}$$

$\forall x \in W \quad s(x) \in \mathcal{F}_{d(x),x}$

$\forall x \in W$ there is an open neighborhood $x \in V \subseteq W$ and $i \in I$ s.t. $V \subseteq U_i$ and $t \in \mathcal{F}_i(V)$ such that for all $y \in V$,

$$s(y) = \varphi_{i\,d(x),y}(\operatorname{germ}_y t)$$



By def^N if the $\mathcal{F}_i$ are sheaves of groups and φ_{ij} isomorphisms of groups, the $\mathcal{F}_{i,x}$ and $\varphi_{ij,x}$ are groups and morphisms of groups, so $\mathcal{F}$ becomes a presheaf of groups under the pointwise structure and restriction of functions. The same is true of rings. In a bit more detail: first notice that for i,j,k the following commutes due to (β) above (any $y \in U_i \cap U_j \cap U_k$)

$$\begin{tikzcd} \mathcal{F}_{i,y} \arrow[rr, "\sim"] \arrow[dr, "\wr"'] & & \mathcal{F}_{k,y} \\ & \mathcal{F}_{j,y} \arrow[ur] & \end{tikzcd}$$

We check the details for groups: the map $x \longmapsto 0$ belongs to $\mathcal{F}(W)$ for any $W \subseteq X$. If $s, t \in \mathcal{F}(W)$ and $x \in W$, let $V \subseteq U_i$ and $V' \subseteq U_k$ be given along with $t \in \mathcal{F}_i(V)$ and $t' \in \mathcal{F}_k(V')$ s.t. $\forall y \in V \cap V'$

$$\begin{aligned}(s+t)(y) &= \varphi_{i d(x),y}(\operatorname{germ}_y t) + \varphi_{k d(x),y}(\operatorname{germ}_y t') \\ &= \varphi_{i d(x),y}(\operatorname{germ}_y t) + \varphi_{i d(x),y}(\varphi_{ki,y}(\operatorname{germ}_y t')) \\ &= \varphi_{i d(x),y}(\operatorname{germ}_y t + \operatorname{germ}_y (\varphi_{ki})_{V' \cap U_i}(t'|_{V' \cap U_i})) \\ &= \varphi_{i d(x),y}(\operatorname{germ}_y \{ t + (\varphi_{ki})_{V' \cap U_i}(t'|_{V \cap U_i}) \})\end{aligned}$$

So $s+t \in \mathcal{F}(W)$, as required. If $s \in \mathcal{F}(W)$ it is clear that $-s \in \mathcal{F}(W)$, so $\mathcal{F}(W)$ is a group, and it is now easy to check that $\mathcal{F}$ is a presheaf of groups. The situation for rings is similar, and since the defining condition is local, it is easy to check that $\mathcal{F}$ is a sheaf (of sets, groups, rings).

$U \subseteq U_i$ open

Now to define the isomorphisms $\psi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{F}_i$. Let $s \in \mathcal{F}(U)$ be given. For each $x \in U$ let $j_x \in I$ and $V_x \subseteq U_{j_x}$ and $t_x \in \mathcal{F}_{j_x}(V_x)$ be s.t. $\forall y \in V_x$ $s(y) = \varphi_{j_x d(x), y}(\mathrm{germ}_y\, t_x)$. Let $W_x \in \mathcal{F}_i(V_x \cap U)$ be

$$W_x = (\varphi_{j_x i})_{V_x \cap U}\,(t_x|_{V_x \cap U})$$

i.e. $\psi_i(s)|_{V_x \cap U} = W_x$

Then $\forall z \in V_x \cap U$ we have $s(y) = \varphi_{i d(x), y}(\mathrm{germ}_y W_x)$. in $\mathcal{F}_{d(x), y}$. The $V_x \cap U$ cover U, and for $y \in (V_x \cap U) \cap (V_z \cap U)$ we have $\varphi_{i d(x), y}(\mathrm{germ}_y W_x) = s(y) = \varphi_{i d(x), y}(\mathrm{germ}_y W_z)$. Since $\varphi_{i d(x), y}$ is an isomorphism, $\mathrm{germ}_y W_x = \mathrm{germ}_y W_z$. Since $\mathcal{F}_i$ is a sheaf, there is unique $\psi_i(s) \in \mathcal{F}_i(U)$ with $\mathrm{germ}_x \psi_i(s) = \mathrm{germ}_x W_x$ for all $x \in U$. This defines the map ψ_i. That is, for $U \subseteq U_i$ open $(\psi_i)_U(s)$ is the unique element of $\mathcal{F}_i(U)$ with

$$\mathrm{germ}_x (\psi_i)_U(s) = \varphi_{d(x) i, x}(s(x)) \quad \forall x \in U$$

Consequently ψ_i is a morphism of sheaves (of groups, rings). The fact that the $\varphi_{d(x) i, x}$ are bijective means that $(\psi_i)_U$ is injective, and these maps are surjective by construction of $\mathcal{F}$, and the fact that $\varphi_{i d(x)} \varphi_{d(x) i} = \varphi_{ii} = 1$. To check that $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$, let $V \subseteq U_i \cap U_j$ and $s \in \mathcal{F}(V)$ be given. Then $\forall z \in V$,

$$\begin{aligned}
&\mathrm{germ}_z\big((\varphi_{ij})_V (\psi_i)_V(s) \big) \\
&= \varphi_{ij, z}(\mathrm{germ}_z (\psi_i)_V(s)) \\
&= \varphi_{ij, z}\, \varphi_{d(z) i, z}(s(z)) \\
&= \varphi_{d(z) j, z}(s(z)) = \mathrm{germ}_z (\psi_j)_V(s)
\end{aligned}$$

This completes the existence proof. For uniqueness, suppose S is another sheaf on X and $\theta_i : S|_{U_i} \longrightarrow \mathcal{F}_i$ s.t. $\theta_j = \varphi_{ij} \circ \theta_i$ on $U_i \cap U_j$. For an open subset $V \subseteq U_i$ define $\gamma_V : \mathcal{F}(V) \longrightarrow S(V)$ to be the composite

$$\mathcal{F}(V) \xrightarrow{(\psi_i)_V} \mathcal{F}_i(V) \xrightarrow{(\theta_i)_V^{-1}} S(V)$$

$$\gamma_V = (\theta_i)_V^{-1} (\psi_i)_V$$

Using the fact that $\theta_j = \varphi_{ij} \theta_i$ and $\psi_j = \varphi_{ij} \psi_i$ on V, we see that γ_V is independent of i (and is by assumption an isomorphism of groups, rings). It is natural with respect to $W \subseteq V \subseteq$ some U_i by construction. We then extend γ to any open set via defining $\gamma_V(s)$ to be unique s.t. $\gamma_V(s)|_{V \cap U_i} = \gamma_{V \cap U_i}(s|_{V \cap U_i})$ $\forall i \in I$. It is then easily checked that γ is an isomorphism of sheaves.