7. INTERSECTIONS IN PROJECTIVE SPACE

The purpose of this section is to study the intersection of varieties in a projective space. If Y, Z are varieties in $\mathbb{P}^n$, what can one say about $Y \cap Z$? We have already seen that $Y \cap Z$ need not be a variety (Ex 2.16). But it is an algebraic set, and we can ask first about its irreducible components. We take our cue from the theory of vector spaces: if U, V are subspaces of dimensions r, s of a vector space W of dimension n, then $U \cap V$ is a subspace of dimension $\geq r+s-n$, since

$$\frac{U}{U \cap V} \cong \frac{U+V}{V} \qquad \therefore \dim U - \dim U \cap V = \dim(U+V) - \dim V$$

Hence $r+s-\dim(U+V) = \dim(U \cap V)$, implying $\dim(U \cap V) \geq r+s-n$ and if U, V are in sufficiently general position (i.e. $U+V = W$) then the dimension of $U \cap V$ is $r+s-n$. Our first result in this section will be to prove that if Y, Z are projective varieties of dimensions r, s in $\mathbb{P}^n$ then every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$. Furthermore if $r+s-n \geq 0$ then $Y \cap Z$ is nonempty.

Knowing something about the dimension of $Y \cap Z$ we can ask for more precise information. Suppose for example that $r+s=n$, and that $Y \cap Z$ is a finite set of points. Then we can ask, how many points are there? Let us look at a special case. If Y is a curve of degree d in $\mathbb{P}^2$, and if Z is a line in $\mathbb{P}^2$, then $Y \cap Z$ consists of at most d points, and the number comes to d exactly if we count them with appropriate multiplicities. (Ex 5.4) This result generalises to the well-known theorem of Bézout, which says that if Y, Z are plane curves of degrees d, e with $Y \neq Z$, then $Y \cap Z$ consists of de points, counted with multiplicities. We will prove Bézout's Theorem later in this section.

The ideal generalisation of Bézout's Theorem to $\mathbb{P}^n$ would be this. First, define the degree of any projective variety. Let Y, Z be varieties of dimensions r, s and degrees d, e in $\mathbb{P}^n$. Assume that Y, Z are in sufficiently general position so that all irreducible components of $Y \cap Z$ have dimension $= r+s-n$, and assume that $r+s-n \geq 0$. For each irreducible component W of $Y \cap Z$, define the intersection multiplicity $i(Y, Z; W)$ of Y and Z along W. Then we should have

$$\sum i(Y, Z; W) \cdot \deg W = de$$

where the sum is taken over all irreducible components of $Y \cap Z$.

The hardest part of this generalisation is the correct definition of intersection multiplicity. We will define the intersection multiplicity only in the case where Z is a hypersurface. Our main task in this section will be the definition of the degree of a variety Y of dimension r in $\mathbb{P}^n$. Classically, the degree of Y is defined as the number of points of intersection of Y with a sufficiently general linear space L of dimension $n-r$. However, this definition is difficult to use. Cutting Y successively with r sufficiently general hyperplanes, one can find a linear space L of dimension $n-r$ which meets Y in a finite number of points (Ex 1.10). But the number of intersection points may depend on L, and it is hard to make precise the notion "sufficiently general".

Therefore we will give a purely algebraic definition of degree, using the Hilbert polynomial of a projective variety. This definition is less geometrically motivated, but it has the advantage of being precise. In an exercise (Ex 7.4) we show that it agrees with the classical definition in a special case.

<u>PROPOSITION 7.1</u> (Affine Dimension Theorem) Let Y, Z be varieties of dimensions r, s in $\mathbb{A}^n$. Then every irreducible component W of $Y \cap Z$ has dimension $\geq r+s-n$. (If $Y \cap Z \neq \phi$)

<u>PROOF</u> Assume $Y \cap Z$ nonempty throughout. We proceed in several steps. First suppose that Z is a hypersurface defined by an equation $f=0$. If $Y \subseteq Z$ there is nothing to prove. If $Y \nsubseteq Z$, we must show that each irreducible component W of $Y \cap Z$ has dimension $r-1$. Let $A(Y)$ be the affine coordinate ring of Y. Since $Y \nsubseteq Z$, $f \neq 0$ in $A(Y)$ and since $Y \cap Z \neq \phi$ f is not a unit in $A(Y)$ either. The irreducible components of $Y \cap Z$ correspond to the minimal prime ideals of the principal ideal (f) in $A(Y)$. By (1.11A) each such $\mathfrak{p}$ has height one in $A(Y)$, so by the dimension theorem (1.8A) $A(Y)/\mathfrak{p}$ has dimension $r-1$. By (1.7) this shows that each irreducible component W has dimension $r-1$.

Now for the general case. We consider the product $Y \times Z \subseteq \mathbb{A}^{2n}$, which is an affine variety of dimension $r+s$ (Ex 3.15). Let Δ be the diagonal $\{P \times P \mid P \in \mathbb{A}^n\} \subseteq \mathbb{A}^{2n}$. Then $\mathbb{A}^n$ is isomorphic to Δ by the map $P \mapsto P \times P$ (consider $k[x_1,\dots,x_n,y_1,\dots,y_n] \longrightarrow k[x_1,\dots,x_n]$ $x_i \mapsto x_i$, $y_i \mapsto x_i$ and $k[x_1,\dots,x_n] \longrightarrow k[x_1,\dots,x_n,y_1,\dots,y_n]$ canonical) and under this isomorphism, $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$. Since Δ has dimension n, and since $r+s-n = (r+s)+n-2n$, we reduce to proving the result for the two varieties $Y \times Z$ and Δ in $\mathbb{A}^n$. Now Δ is an intersection of n hypersurfaces $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$ so we can apply the special case n times (see our Note: Dimension and Q1.8 in §1 solutions). □

THEOREM 7.2 (Projective Dimension Theorem) Let Y, Z be varieties of dimensions r, s in $\mathbb{P}^n$. Then every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$. Furthermore, if $r+s-n \geq 0$ then $Y \cap Z$ is nonempty.

PROOF Firstly assume $Y \cap Z \neq \phi$ and let $Y \cap Z = W_1 \cup \dots \cup W_r$ be the decomposition into irreducible components. Let $U_i \subseteq \mathbb{P}^n$ be an affine open with $W_1 \cap U_i \neq \phi$. Then $(Y \cap Z) \cap U_i$ is a nonempty closed subset of $\mathbb{A}^n$, and

$$(Y \cap Z) \cap U_i = W_1 \cap U_i \cup \dots \cup W_r \cap U_i \qquad (1)$$

Omitting empty terms in union, the $W_j \cap U_i$ are nonempty, closed, irreducible sets in $\mathbb{A}^n$, no one contained in the other (closure $U_i \cap W_j$ in $\mathbb{P}^n$ is W_j, so $U_i \cap W_j \subseteq U_i \cap W_k \Rightarrow W_j \subseteq W_k$, a contradiction), hence (1) is the irreducible decomposition of the intersection of the two affine varieties $Y \cap U_i$, $Z \cap U_i$ which have dimensions r, s by the usual argument. By the previous Theorem

$$\dim W_1 = \dim W_1 \cap U_i \geq r+s-n$$

For the second result, let $C(Y)$ and $C(Z)$ be the cones over Y, Z in $\mathbb{A}^{n+1}$ (Ex 2.10). Then $C(Y), C(Z)$ have dimensions $r+1, s+1$ resp. Furthermore, $C(Y) \cap C(Z) \neq \phi$ since both contain the origin $P = (0,\dots,0)$. By the affine dimension theorem, $C(Y) \cap C(Z)$ has dimension $\geq (r+1)+(s+1)-(n+1) = r+s-n+1 > 0$. (See p17 of these notes). Hence $C(Y) \cap C(Z)$ contains some point $Q \neq P$, so $Y \cap Z \neq \phi$. □

COROLLARY Let Y, Z be curves in $\mathbb{P}^2$. Then $Y \cap Z$ is nonempty.

PROOF $r = s = 1$, $n = 2$ so $r+s-n = 0$. □

COROLLARY Let Y, Z be hypersurfaces in $\mathbb{P}^n$, $n \geq 2$. Then $Y \cap Z$ is nonempty.

Next, we come to the definition of the Hilbert polynomial of a projective variety. The idea is to associate to each projective variety $\mathbb{P}^n_k$ a polynomial $P_Y(z) \in \mathbb{Q}[z]$ from which we can obtain various numerical invariants. We will define P_Y starting from the homogenous coordinate ring $S(Y)$. In fact, more generally, we will define a Hilbert polynomial for any graded S-module, where $S = k[x_0,\dots,x_n]$.

DEFINITION A numerical polynomial is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all $n >> 0$, $n \in \mathbb{Z}$

PROPOSITION 7.3 (a) If $P(z) \in \mathbb{Q}[z]$ is a numerical polynomial, then there are integers $c_0, c_1, \dots, c_r$ such that

$$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \dots + c_r$$

where

$$\binom{z}{r} = \frac{1}{r!} z(z-1)\cdots(z-r+1)$$

is the binomial coefficient function. In particular $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) If $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ is any function, and if there exists a numerical polynomial $Q(z)$ such that the difference function $\Delta f = f(n+1) - f(n)$ is equal to $Q(n)$ for all $n >> 0$, then there exists a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for all $n >> 0$.

PROOF (a) By induction on the degree of P, the case of degree 0 being obvious. Since $\binom{z}{r} = \frac{1}{r!}z^r + \cdots$ using the division algorithm we can express any polynomial $P \in \mathbb{Q}[z]$ of degree r in the above form, with $c_0, \ldots, c_r \in \mathbb{Q}$. For any polynomial P we define the <u>difference polynomial</u> ΔP by

$$\Delta P(z) = P(z+1) - P(z)$$

Since $\Delta\binom{z}{r} = \binom{z}{r-1}$, we have

$$\Delta P = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \cdots + c_{r-1}$$

By induction $c_0, \ldots, c_{r-1} \in \mathbb{Z}$ (If P is numerical, clearly so is ΔP). But then $c_r \in \mathbb{Z}$ since $P(n) \in \mathbb{Z}$ for $n >> 0$.

(b) Write

$$Q = c_0\binom{z}{r} + \cdots + c_r$$

with $c_0, \ldots, c_r \in \mathbb{Z}$. Let

$$P = c_0\binom{z}{r+1} + \cdots + c_r\binom{z}{1}$$

Then $\Delta P = Q$, so $\Delta(f-P)(n) = 0$ for all $n >> 0$, so $(f-P)(n) =$ the constant c_{r+1} for all $n >> 0$, so

$$f(n) = P(n) + c_{r+1}$$

for all $n >> 0$, as required. The fact that $P(n) + c_{r+1}$ is numerical is immediate. $\square$

Next, we need some preparations about graded modules. Let S be a graded ring. A <u>graded S-module</u> is an S-module M, together with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ into abelian groups, s.t. $S_d M_e \subseteq M_{d+e}$. For any graded module M, and any $e \in \mathbb{Z}$, we define the <u>twisted</u> module $M(e)$ by shifting e places to the left, i.e. $M(e)_d = M_{d+e}$. If M is a graded S-module, we define the <u>annihilator</u> of M, $\mathrm{Ann}\, M = \{s \in S \mid s \cdot M = 0\}$. This is a homogenous ideal in S. The next result is the analogue for graded modules of a well-known result for f.g. modules over a Noetherian ring (our A & M notes p66). A submodule $N \subseteq M$ is a <u>graded submodule</u> if $N = \bigoplus_{d \in \mathbb{Z}}(N \cap M_d)$ — i.e., if $n \in N$ and $n = \sum n_i$ then each $n_i \in N$ (n_i hom.)

PROPOSITION 7.4 Let M be a nonzero finitely generated graded module over a noetherian graded ring S. Then there exists a filtration $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$ by graded submodules, such that for each i, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(\ell_i)$ as graded modules, where $\mathfrak{p}_i$ is a homogenous prime ideal of S, $\ell_i \in \mathbb{Z}$. The filtration is not unique, but for any such filtration we do have:

(a) If $\mathfrak{p}$ is a homogenous prime ideal of S, then $\mathfrak{p} \supseteq \mathrm{Ann}\, M \iff \mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. In particular, the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are just the minimal primes of M, i.e. the primes which are minimal containing $\mathrm{Ann}\, M$.

(b) For each minimal prime of M, the number of times which $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is equal to the length of $M_{\mathfrak{p}}$ over the local ring $S_{\mathfrak{p}}$. (and hence is independent of the filtration)

See printout overleaf for proof

PROOF Let N be any nonzero graded module over S, and let $\mathfrak{a} = \mathrm{Ann}(x)$ be maximal in S among all annihilators of nonzero homogenous elements of N. Clearly $\mathfrak{a}$ is proper, and we claim it is prime. For suppose a, b are homogenous and $ab \in \mathfrak{a}$ ($\mathfrak{a}$ is clearly homogenous), and $a \notin \mathfrak{a}$. Then $abx = 0$, $ax \neq 0$, so $\mathrm{Ann}(ax)$ is an annihilator of a nonzero homogenous element and $\mathfrak{a} \subseteq \mathrm{Ann}(ax)$, so $\mathfrak{a} = \mathrm{Ann}(ax)$ by maximality. Hence since $b \cdot (ax) = 0$, $b \in \mathfrak{a}$ as required. The submodule $L = (x) \subseteq N$ is a graded submodule and

$$(x) \cong (S/\mathfrak{a})(-e) \qquad x \in N_e$$

as graded S-modules.

Now consider the set of all nonzero graded submodules of M which admit a filtration of the desired type. The above shows that this set is nonempty. M is a noetherian module, so there is a maximal such submodule N. If $N = M$ we are done. Otherwise there is an opportunity to apply the above to the nonzero graded module M/N to find a homogenous $x \in M - N$ with $N + Sx/N \cong (S/\mathfrak{a})(-e)$ as graded modules, contradicting maximality of N. Hence $N = M$ and the desired filtration exists.

Proposition 1 (Hartshorne 7.4). *Let M be a nonzero finitely generated graded module over a noetherian graded ring S. Then there exists a filtration*

$$0 = M^0 \subset M^1 \subset \ldots \subset M^r = M$$

by graded submodules, such that for each i, M^i/M^{i-1} is isomorphic as a graded module to $(S/\mathfrak{p}_i)(\ell_i)$ where $\mathfrak{p}_i$ is a homogenous prime ideal of S and $\ell_i \in \mathbb{Z}$. The filtration is not unique, but for any such filtration we do have:

(a) If $\mathfrak{p}$ is any prime ideal of S, then $\mathfrak{p} \supseteq AnnM$ if and only if $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. In particular the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are just the minimal primes of M, i.e. the primes which are minimal containing $AnnM$. Hence the minimal primes of any finitely generated graded S-module are all homogenous.

(b) For each minimal prime $\mathfrak{p}$ of M, the number of times $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is equal to the length of $M_\mathfrak{p}$ over the local ring $S_\mathfrak{p}$ (and hence is independent of the filtration).

Proof. Let N be any nonzero graded module over S, and let $\mathfrak{a} = \mathrm{Ann}(x)$ be maximal in S amongst all annihilators of nonzero homogenous elements of N. Clearly $\mathfrak{a}$ is a proper homogenous ideal, and we claim it is prime. For suppose a, b are homogenous elements of S with $a \notin \mathfrak{a}$ and $ab \in \mathfrak{a}$. Then $ab \cdot x = 0$ and $a \cdot x \neq 0$, so $\mathrm{Ann}(ax)$ is an annihilator of a nonzero homogenous element and $\mathfrak{a} \subseteq \mathrm{Ann}(ax)$ so by maximality $\mathfrak{a} = \mathrm{Ann}(ax)$. Since $b \cdot (ax) = 0$ we have $b \in \mathrm{Ann}(ax)$ and thus $b \in \mathfrak{a}$, as required. The submodule $L = (x) \subseteq N$ is a graded submodule which is isomorphic as a graded module to $(S/\mathfrak{a})(-e)$ where e is the degree of x.

Now consider the set of all nonzero graded submodules of M which admit a filtration of the desired type. The above argument shows that this set is nonempty. Since M is noetherian, there is a maximal such submodule N. If $N \neq M$ there is an opportunity to apply the above to the nonzero graded module M/N to find a homogenous element $x \in M - N$ with $(N + Sx)/N \cong (S/\mathfrak{a})(-e)$ as graded modules, contradicting maximality of N. Hence $N = M$ and the desired filtration of M exists.

(a) Suppose we are given such a filtration of M. Then for each i,

$$\mathrm{Ann}(M^i/M^{i-1}) = \mathrm{Ann}(S/\mathfrak{p}_i)(\ell_i) = \mathfrak{p}_i$$

Consider the ideal $\mathfrak{b} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ of S. We claim that $\mathfrak{b}$ is the radical of the ideal $\mathrm{Ann}M$. Since $\mathfrak{b}$ is a radical ideal and $\mathfrak{b} \supseteq \mathrm{Ann}M$, one inclusion is clear. In the other direction, let $b \in \mathfrak{b}$. Then since $\mathfrak{p}_i = \mathrm{Ann}(M^i/M^{i-1})$ we have $b^r m = 0$ for any $m \in M$, so $\mathfrak{b}$ is contained in the radical of $\mathrm{Ann}M$, as required.

It follows that if $\mathfrak{p}$ is any prime ideal of S, then $\mathfrak{p}$ contains $\mathrm{Ann}M$ iff. it contains $\mathfrak{b}$ iff. it contains one of the $\mathfrak{p}_i$. The other claims now follow easily.

(b) We localise at the minimal prime $\mathfrak{p}$. Since M is finitely generated and S is noetherian, we can ignore the grading and apply (A&M 3.14) to see that in the filtration

$$0 = M^0_\mathfrak{p} \subseteq M^1_\mathfrak{p} \subseteq \ldots \subseteq M^r_\mathfrak{p} = M_\mathfrak{p} \tag{1}$$

we have

$$\begin{aligned}\mathrm{Ann}(M^i_\mathfrak{p}/M^{i-1}_\mathfrak{p}) &= \mathrm{Ann}\left((M^i/M^{i-1})_\mathfrak{p}\right) \\ &= \mathrm{Ann}(M^i/M^{i-1})_\mathfrak{p} = (\mathfrak{p}_i)_\mathfrak{p}\end{aligned}$$

But $(\mathfrak{p}_i)_\mathfrak{p}$ will be $S_\mathfrak{p}$, and hence $M^i_\mathfrak{p} = M^{i-1}_\mathfrak{p}$, unless $\mathfrak{p}_i \subseteq \mathfrak{p}$, which can only happen if $\mathfrak{p}_i = \mathfrak{p}$ since $\mathfrak{p}$ is minimal. In the case where $\mathfrak{p}_i = \mathfrak{p}$ we have $M^i_\mathfrak{p}/M^{i-1}_\mathfrak{p} \cong (S/\mathfrak{p})_\mathfrak{p} \cong k(\mathfrak{p})$ the quotient field of $S/\mathfrak{p}$. Throwing out the trivial links in (**??**) we end up with a filtration

$$0 = N^0 \subset N^1 \subset \ldots \subset N^s = M_\mathfrak{p}$$

where s is the number of times $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ (by (a) it occurs at least once). Since as an $S_\mathfrak{p}$ module $k(\mathfrak{p}) \cong S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$, $k(\mathfrak{p})$ is a simple $S_\mathfrak{p}$ module and hence has length 1. Using additivity of length and our modified filtration, we see that $M_\mathfrak{p}$ is an $S_\mathfrak{p}$ module of length s, as required. ☐

Corollary 2. *Let $\mathfrak{a}$ be a proper homogenous ideal in $S = k[x_0, \ldots, x_n]$ $(n \geq 0)$. Then the minimal primes of $\mathfrak{a}$ are all homogenous.*

Proof. Let M be the module $S/\mathfrak{a}$ which is finitely generated, graded, nonzero and has annihilator $\mathfrak{a}$. Now apply the previous Proposition. □

This means that we can apply the same techniques used in the affine case to find the irreducible decomposition of a nonempty algebraic set $Y = Z(\mathfrak{a})$ in $\mathbb{P}^n$. Take a primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$. Then $\sqrt{\mathfrak{a}}$ is the intersection of the minimal primes of $\mathfrak{a}$, which are all homogenous: $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m$. Hence

$$Y = Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{p}_1) \cup \ldots \cup Z(\mathfrak{p}_m) \tag{2}$$

is the irreducible decomposition of Y.

Definition 1. If $\mathfrak{p}$ is a minimal prime of a graded S-module M, we define the *multiplicity* of M at $\mathfrak{p}$, denoted $\mu_{\mathfrak{p}}(M)$ to be the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$. If M is nonzero and finitely generated then $0 < \mu_{\mathfrak{p}}(M) < \infty$.

If M is a finitely generated graded module over the polynomial ring $S = k[x_0, \ldots, x_n]$ with $M = \bigoplus_{d \in \mathbb{Z}} M_d$ then we can adapt the standard argument (see top of p.57 in our A&M notes) to see that each M_d is a finite dimensional vector space over k. So we can define the *Hilbert function* $\varphi_M : \mathbb{Z} \longrightarrow \mathbb{N}$ of M by

$$\varphi_M(\ell) = \dim_k M_\ell$$

The following Theorem is an analogue of the normal dimension theorem for finitely generated modules over noetherian local rings, which asserts an equality between the order of the Hilbert-Samuel polynomial and the Krull dimension of the module.

Theorem 3 (Hilbert-Serre). *Let M be a finitely generated graded module over $S = k[x_0, \ldots, x_n]$. Then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\varphi_M(\ell) = P_M(\ell)$ for all sufficiently large $\ell > 0$. Furthermore,*

$$deg P_M(z) = dim Z(Ann M)$$

where Z denotes the zero set in $\mathbb{P}^n$ of a homogenous ideal.

Proof. Throughout we use the convention that the degree of the zero polynomial is -1, and the dimension of the empty set is -1. So putting $P_M = 0$ the Theorem is true for $M = 0$.

Consider any exact sequence of nonzero finitely generated graded S-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Suppose the Theorem is true for M' and M''. Then $\varphi_M = \varphi_{M'} + \varphi_{M''}$ and we claim that $Z(\mathrm{Ann}M) = Z(\mathrm{Ann}M') \cup Z(\mathrm{Ann}M'')$. This will follow from the fact that for two homogenous ideals $\mathfrak{a}, \mathfrak{b}$ we have $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, and the equality

$$\sqrt{\mathrm{Ann}M} = \sqrt{\mathrm{Ann}M' \cdot \mathrm{Ann}M''}$$

One inclusion follows from $\mathrm{Ann}M' \cdot \mathrm{Ann}M'' \subseteq \mathrm{Ann}M$. If a annihilates M then aa belongs to the product $\mathrm{Ann}M' \cdot \mathrm{Ann}M''$ so clearly $\mathrm{Ann}M \subseteq \sqrt{\mathrm{Ann}M' \cdot \mathrm{Ann}M''}$.

Since the Theorem is true for M', M'' we have $\varphi_{M'}(\ell) = P_{M'}(\ell)$ and $\varphi_{M''}(\ell) = P_{M''}(\ell)$ for sufficiently large ℓ. Since these values are all positive, the leading coefficients of $P_{M'}, P_{M''}$ must be positive, so putting $P_M = P_{M'} + P_{M''}$ we have

$$\begin{aligned}
\mathrm{deg} P_M &= \deg(P_{M'} + P_{M''}) \\
&= \max\{\mathrm{deg} P_{M'}, \mathrm{deg} P_{M''}\} \\
&= \max\{\mathrm{dim} Z(\mathrm{Ann}M'), \mathrm{dim} Z(\mathrm{Ann}M'')\} \\
&= \mathrm{dim} Z(\mathrm{Ann}M)
\end{aligned}$$

where the last equality follows from a standard argument. Uniqueness of P_M follows immediately by considering the number of zeros possible for a polynomial over a field.

Let M be a nonzero finitely generated graded S-module. By (7.4) there is a filtration

$$\begin{gathered} 0 = M^0 \subset M^1 \subset \ldots \subset M^r = M \\ M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(\ell_i) \end{gathered} \tag{3}$$

where the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are the minimal primes over $\mathrm{Ann}M$. Suppose the Theorem were true for the modules $S/\mathfrak{p}_i$. Then for any $\ell \in \mathbb{Z}$ we could use the change of variables $z \mapsto z+\ell$ to show the Theorem holds for $(S/\mathfrak{p}_i)(\ell)$. Since $M^1 \cong (S/\mathfrak{p}_1)(\ell_1)$, the Theorem would be true for M^1. The above discussion could then be applied to the exact sequence

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow (S/\mathfrak{p}_2)(\ell_2) \longrightarrow 0$$

to show that the Theorem was also true for M^2. Proceeding in this way, we would see that the Theorem was true for M.

Now we have the necessary tools to prove the Theorem for all nonzero finitely generated graded S-modules by induction on the integer $\dim Z(\mathrm{Ann}M) \geq -1$. Firstly suppose that $\dim Z(\mathrm{Ann}M) = -1$, so the set $Z(\mathrm{Ann}M)$ is empty. Then since M is nonzero $\mathrm{Ann}M$ is proper and the only prime ideal containing $\mathrm{Ann}M$ is $(x_0, \ldots, x_n)$. Hence in the filtration of M all the $\mathfrak{p}_i$ are equal to $(x_0, \ldots, x_n)$, and to prove the Theorem for M it suffices to prove it for the module $N = S/(x_0, \ldots, x_n)$. But $\varphi_N(\ell) = 0$ for $\ell > 0$ so putting $P_N = 0$ shows that the Theorem is true for N.

For the induction step, suppose that the Theorem holds for all N with $\dim Z(\mathrm{Ann}N) < k$ and let M be a nonzero finitely generated graded S-module with $\dim Z(\mathrm{Ann}M) = k \geq 0$ and filtration given by (**??**). Since $\mathrm{Ann}M$ is a proper homogenous ideal, we can use (**??**) to write

$$Z(\mathrm{Ann}M) = Z(\sqrt{\mathrm{Ann}M}) = Z(\mathfrak{p}_1) \cup \ldots \cup Z(\mathfrak{p}_r)$$

So $\dim Z(\mathfrak{p}_i) \leq k$ for all i. To prove the result for M, it suffices to prove it for $N = S/\mathfrak{p}$ as $\mathfrak{p}$ ranges over $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. We can assume that $\mathfrak{p} \neq (x_0, \ldots, x_n)$ and hence that some $x_j \notin \mathfrak{p}$. Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{x_j} N \longrightarrow N'' \longrightarrow 0$$

where $N'' = N/x_jN$. Then $\varphi_{N''}(\ell) = \varphi_N(\ell) - \varphi_N(\ell-1) = (\Delta\varphi_N)(\ell-1)$. On the other hand $N'' = N/x_jN = S/(\mathfrak{p} + (x_j))$ so

$$Z(\mathrm{Ann}N'') = Z(\mathfrak{p} + (x_j)) = Z(\mathfrak{p}) \cap H$$

where H is the hypersurface $Z(x_j)$. By construction $Z(\mathfrak{p}) \nsubseteq H$ so (7.2) and Ex 1.10 imply that every irreducible component of $Z(\mathrm{Ann}N'')$ has dimension $\dim Z(\mathfrak{p}) - 1$. Hence

$$\dim Z(\mathrm{Ann}N'') = \dim Z(\mathfrak{p}) - 1 < k$$

Note that this holds even if the intersection $Z(\mathfrak{p}) \cap H$ is empty, since by (7.2) this can only occur if $\dim Z(\mathfrak{p}) = 0$. By the inductive hypothesis there is a polynomial $P_{N''} \in \mathbb{Q}[z]$ of degree $\dim Z(\mathfrak{p}) - 1$ with $\varphi_{N''}(\ell) = P_{N''}(\ell)$ for all sufficiently large $\ell > 0$. Since $\varphi_{N''}(\ell) = (\Delta\varphi_N)(\ell-1)$ we can use (7.3) to see that the Theorem is true for $N = S/\mathfrak{p}$, as required. ☐

Corollary 4. *If we have an exact sequence of finitely generated graded S-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

then $P_M = P_{M'} + P_{M''}$.

Proof. Clearly $\varphi_M(\ell) = \varphi_{M'}(\ell) + \varphi_{M''}(\ell)$ for all ℓ, so by considering sufficiently large ℓ we can show that the polynomial $P_M - P_{M'} - P_{M''}$ has infinitely many roots, hence is zero, as required. ☐

We have already seen that the Hilbert polynomial of the zero module is the zero polynomial. If M is a nonzero module, we can calculate the Hilbert polynomial as long as we can find a filtration:

Proposition 5. *Let M be a nonzero finitely generated graded module over $S = k[x_0, \ldots, x_n]$. Then M has a filtration by graded submodules:*

$$\begin{gathered} 0 = M^0 \subset M^1 \subset \ldots \subset M^r = M \\ M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(\ell_i) \end{gathered} \tag{4}$$

If P_i is the Hilbert polynomial of $(S/\mathfrak{p}_i)(\ell_i)$, then $P_M = P_1 + \ldots + P_r$.

Proof. Since $M^1 \cong (S/\mathfrak{p}_1)(\ell_1)$ it is clear that $P_{M_1} = P_1$. So beginning with $i = 1$ we can apply the Corollary to the exact sequence $0 \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow (S/\mathfrak{p}_{i+1})(\ell_{i+1}) \longrightarrow 0$. $\square$

DEFINITION The Polynomial P_M of the Theorem is the Hilbert polynomial of M.

DEFINITION If $Y \subseteq \mathbb{P}^n$ is an algebraic set of dimension r, we define the Hilbert polynomial of Y to be the Hilbert polynomial P_Y of its homogenous coordinate ring $S(Y)$. (By the Theorem, it is a polynomial of degree r) We define the degree of Y to be $r!$ times the leading coefficient of P_Y.

PROPOSITION 7.6 (a) If $Y \subseteq \mathbb{P}^n$, $Y \neq \emptyset$ then the degree of Y is a positive integer $(\deg \emptyset = 0)$

(b) Let $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 have the same dimension r and $\dim(Y_1 \cap Y_2) < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$

(c) $\deg \mathbb{P}^n = 1$

(d) If $H \subseteq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogenous irreducible polynomial of degree d, then $\deg H = d$ (In other words, this def^N of degree is consistent with the degree of a hypersurface as defined earlier (1.4.2])

PROOF Since $Y \neq \emptyset$, P_Y is a nonzero polynomial of degree $r = \dim Y$. By (7.3a), $\deg Y = c_0$, which is an integer. It is a positive integer because for $e \gg 0$ $P_Y(e) = \varphi_{S(Y)}(e) \geq 0$. Of course, $\deg \emptyset = 0$ (or is undefined)

(b) Let I_1, I_2 be the ideals of Y_1, Y_2. If either set is empty the result is trivial. Otherwise $I_1 \cap I_2$ is the ideal of $Y_1 \cup Y_2$. There is an exact sequence $(I = I_1 \cap I_2)$ of finitely generated graded modules

$$0 \longrightarrow S/I \longrightarrow S/I_1 \oplus S/I_2 \xrightarrow{\psi} S/(I_1 + I_2) \longrightarrow 0$$

Where $\psi(s + I_1, t + I_2) = s - t + I_1 + I_2$. Calculating dimensions we find that

$$\varphi_{S/I_1}(e) + \varphi_{S/I_2}(e) = \varphi_{S/I_1+I_2}(e) + \varphi_{S/I}(e)$$

For $e \gg 0$ we have $(P_{S/I_1} + P_{S/I_2})(e) = (P_{S/I_1+I_2} + P_{S/I})(e)$ so

$$P_{S/I_1} + P_{S/I_2} = P_{S/I_1+I_2} + P_{S/I}$$

Now $Z(I_1 + I_2) = Y_1 \cap Y_2$ which has dimension $< r$, so the leading coefficient of $P_{S/I}$ is the sum of the leading coefficients of P_{S/I_1} and P_{S/I_2}.

(c) We calculate the Hilbert polynomial of $\mathbb{P}^n$. It is the polynomial P_S, where $S = k[x_0, \ldots, x_n]$. For $e > 0$, $\varphi_S(e) = \binom{e+n}{n}$, so $P_S = \binom{z+n}{n}$. In particular its leading coefficient is $1/n!$ so $\deg \mathbb{P}^n = 1$.

(d) If $f \in S$ is homogenous irreducible poly of degree d then we have an exact sequence of graded S-modules

$$0 \longrightarrow S(-d) \xrightarrow{f} S \longrightarrow S/(f) \longrightarrow 0$$

Hence

$$\varphi_{S/(f)}(e) = \varphi_S(e) - \varphi_S(e-d)$$

Therefore we can find the Hilbert polynomial of H as

$$\begin{aligned} P_H(z) &= \binom{z+n}{n} - \binom{z-d+n}{n} \\ &= \tfrac{1}{n!}(z+n)(z+n-1)\cdots(z+1) - \tfrac{1}{n!}(z+n-d)\cdots(z-d+1) \\ &= \tfrac{1}{n!}(1+2+\cdots+n-1+n)z^{n-1} - \tfrac{1}{n!}(1-d+2-d+\cdots+n-1-d+n-d)z^{n-1} + \cdots \\ &= \tfrac{1}{n!}(d+\cdots+d)z^{n-1} + \cdots \\ &= nd/n!\, z^{n-1} + \cdots = \frac{d}{(n-1)!} z^{n-1} + \cdots \end{aligned}$$

Since $\dim H = n-1$ we have $\deg H = d$, as required. $\square$

NOTE Part (d) starts with $H \subseteq \mathbb{P}^n$ closed and says, if $I(H) = (f)$ where f is homogenous of degree d, and if $\dim H = n-1$ then $\deg H = d$. We assumed f irreducible just to make sure $\dim H = n-1$.

Note you can't start with f and say $\deg H = d$, because there is no gaurantee $I(Z(f)) = (f)$ (unless of course, f is irreducible)

Now we come to our main result about the intersection of a projective variety with a hypersurface, which is a partial generalisation of Bezout's Theorem to higher projective spaces. Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let H be a hypersurface not containing Y. Then by (7.2) $Y \cap H = Z_1 \cup \cdots \cup Z_s$ where Z_j are varieties of dimension $r-1$. If $r=0$ Y is a point so $Y \cap H = \emptyset$. Otherwise $Y \cap H$ is nonempty. Let $\mathfrak{p}_j$ be the homogenous prime ideal of Z_j. We define the intersection multiplicity of Y and H along Z_j to be

$$i(Y,H;Z_j) = \mu_{\mathfrak{p}_j}\left(S/(I_Y+I_H)\right)$$

⌈If $Y \cap H \neq \emptyset$ $I_Y + I_H \neq S$ so this is $0 < - < \infty$⌋

Here I_Y, I_H are the homogenous ideals of Y and H. The module $M = S/I_Y + I_H$ has annihilator $I_Y + I_H$, and $Z(I_Y+I_H) = Y \cap H$, so $\mathfrak{p}_j$ is a minimal prime of M, and μ is the multiplicity introduced above.

THEOREM 7.7 Let Y be a variety of dimension $r \geq 1$ in $\mathbb{P}^n$, and let H be a hypersurface not containing Y. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then

$$\sum_{j=1}^{s} i(Y,H;Z_j) \cdot \deg Z_j = (\deg Y)(\deg H)$$

PROOF Let H be defined by the homogenous irreducible polynomial f of degree d. Since $\dim Y \geq 1$ by (7.2) $Y \cap H$ is nonempty, so $I_Y + I_H$ is a proper ideal, and the intersection multiplicities $i(Y,H;Z_j)$ are all nonzero, positive integers. We consider the exact sequence of graded S-modules

$$0 \longrightarrow (S/I_Y)(-d) \xrightarrow{f} S/I_Y \longrightarrow M \longrightarrow 0$$

where $M = S/(I_Y + I_H)$. This is exact because $Y \not\subseteq H$. Taking Hilbert polynomials we find that

$$P_M(z) = P_Y(z) - P_Y(z-d)$$

Our result comes from comparing the leading coefficients of both sides of this equation. Let Y have degree e. Then $P_Y(z) = (e/r!)z^r + \cdots$ so on the right we have

$$(e/r!)z^r + \cdots - \left[(e/r!)(z-d)^r + \cdots \right] = \frac{de}{(r-1)!} z^{r-1} + \cdots \qquad (1)$$

Now consider the module M. By (7.4) M has a filtration $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^q = M$, whose quotients M^i/M^{i-1} are of the form $(S/q_i)(e_i)$. Hence $P_M = \sum_{i=1}^{q} P_i$ where P_i is the Hilbert polynomial of $(S/q_i)(e_i)$. If $Z(q_i)$ is a projective variety of dimension r_i and degree f_i, then

$$P_i = (f_i/r_i!)z^{r_i} + \cdots$$

Note that the shift e_i does not affect the leading coefficient of P_i. Since we are only interested in the leading coefficient of P_M, we can ignore those P_i of degree $< r-1$. (We know $\dim Y \cap H = r-1$ from the above discussion, so $\deg P_M = r-1$). Now $\sqrt{\operatorname{Ann} M} = q_1 \cap \cdots \cap q_q$, so $\sqrt{I_Y + I_H}$ is the intersection of the minimal primes in $\{q_1, \ldots, q_q\}$ (which are the minimal primes containing $I_Y + I_H$) and hence $Y \cap H$ is the union of the $Z(q_i)$ (q_i minimal). Any other closed irreducible subset of $Y \cap H$ is contained in one of these Z_i (the $Z_1, \ldots, Z_s$ must be the $Z(q_i)$, q_i minimal), hence is one of the Z_i or has dimension $< r-1$ (Ex 1.10). Hence in $\{q_1, \ldots, q_q\}$ the minimal elements are precisely those with $\dim Z(q_i) = r-1$. So we need only consider P_i with q_i a minimal prime of M, i.e., one of the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ corresponding to the Z_j. Each one of these occurs $\mu_{\mathfrak{p}_j}(M)$ times, so the leading coefficient of M is

$$\sum_{j=1}^{s} \frac{\deg Z_j \cdot i(Y,H;Z_j)}{(r-1)!}$$

Comparing with (1) we have our result. □

COROLLARY 7.8 (Bézout's Theorem) Let Y, Z be distinct curves in $\mathbb{P}^2$, having degrees d, e. Let $Y \cap Z = \{P_1, \dots, P_s\}$. Then

$$\sum i(Y, Z; P_j) = de$$

PROOF The intersection is gauranteed to be a finite nonempty set of points. We have only to observe that a point has Hilbert polynomial 1, hence degree 1. We have shown earlier that if $P = (a_0, \dots, a_n)$ $a_i \neq 0$ then $I(P) = (a_i x_0 - a_0 x_i, \dots, a_i x_n - a_n x_i)$ (see our notes on Linear Varieties in §2), so $I(P)$ is generated by n linearly independent linear polynomials, so there is an automorphism of $k[x_0, \dots, x_n]$ identifying $I(P)$ with $(x_0, \dots, x_{n-1})$. Moreover the automorphism is linear, hence preserves the degree of homogenous polynomials, so there is an isomorphism of

$$\left(S/I(P)\right)_e \cong \left(k[x_0, \dots, x_n]/(x_0, \dots, x_{n-1})\right)_e$$

$$= \frac{k[x_0, \dots, x_n]_e}{(x_0, \dots, x_{n-1})_e}$$

It is not hard to see that the latter vector space is free on x_n^e, hence has dimension 1, as required. □

REMARK 7.8.1 Our definition of intersection multiplicity in terms of the homogenous coordinate ring is different from the local definition given earlier (Ex 5.4). However, it is easy to show that they coincide in the case of intersections of plane curves.

REMARK 7.8.2 The proof of (7.8)

NOTE The multiplicities in (7.8) are integers ≥ 1. So if Y, Z are distinct curves of degree 1, then (in $\mathbb{P}^2$) $Y \cap Z$ is a single point. Of course, curves of degree 1 are hyperplanes (i.e. lines in $\mathbb{P}^2$) by (7.6). So two distinct lines in $\mathbb{P}^2$ meet at a distinct point.

NOTE If Y, Z are distinct