2. PROJECTIVE VARIETIES

To define projective varieties, we proceed in a manner analogous to the definition of affine varieties, except that we work in projective space. Let $k$ be our fixed algebraically closed field. We define projective $n$-space as follows:

**Definition**

We define projective $n$-space over $k$, denoted by $\mathbb{P}^n_k$ or just $\mathbb{P}^n$, to be the set of equivalence classes of $(n+1)$-tuples $(a_0, \ldots, a_n)$ of elements of $k$, not all zero, under the equivalence relation given by $(\lambda_0, \ldots, \lambda_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ if $\lambda \neq 0$.

Equivalently, $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1}$ \{ $(0, \ldots, 0)$ \} under the equivalence relation which identifies points lying on the same line through the origin. An element of $\mathbb{P}^n$ is called a point, and if $P$ is a point, then any $(n+1)$-tuple $(a_0, \ldots, a_n)$ in the equivalence class $P$ is called a set of homogeneous coordinates for $P$.

**Definition (Graded Ring)** A graded ring $S$, together with a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

of $S$ into a direct sum of abelian groups $S_d$, such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of $S_d$ is called a homogeneous element of degree $d$. Thus any element of $S$ can be written uniquely as a (finite) sum of homogeneous elements. An ideal $I \subseteq S$ is a homogeneous ideal if

$$I = \bigoplus_{d \geq 0} (I \cap S_d)$$

**Proposition** Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, then

(i) An ideal is homogeneous if and only if it can be generated by homogeneous elements (i.e., if $f \in S$ has homogeneous component at most $n$).

(ii) Let $a, b$ be homogeneous ideals, then

(i) $a + b$ is homogeneous
(ii) $a \cdot b$ is homogeneous
(iii) $a \cap b$ is homogeneous
(iv) $\sqrt{a} = a$ homogeneous

(iii) A homogeneous ideal $I$ is prime iff for any homogeneous elements $f, g \in S$, if $fg \in I$, either $f \in I$ or $g \in I$.

**Proof**

(i) Suppose $a$ is homogeneous. For any $f \in \mathbb{A}^n$, the homogeneous parts $f^{(d)} \in S_d, d \geq 0$, are $a$- and the collection of all such homogeneous parts generates $a$. Conversely suppose $a = \langle f_i \rangle$ for each $f_i$ homogeneous. Suppose $g \in a$, with say $g = g_1, g_2, \ldots, g_n$, each $g_i \in S$. Note that generally if $x = y + z$, $x^{(d)} = y^{(d)} + z^{(d)}$, and so

$$g^{(d)} = (g_1f_1)^{(d)} + \cdots + (g_nf_n)^{(d)}$$

But for $y$ homogeneous, $(xy)^{(d)} = x^{(d-m)}y$ where $y \in S_m$. Hence

$$g^{(d)} = g_1^{(d-m)}f_1 + \cdots + g_n^{(d-m)}f_n$$

But since $a$ is an ideal, this is in $a$. Hence $a$ is homogeneous.

(ii) Let $f, g \in a$, say $f = a + b$. Then $f^{(d)} = a^{(d)} + b^{(d)} \in a + b$.

(iii) Note that for $x, y \in S$

$$(xy)^{(d)} = \sum_{i+j=d} x^{(i)}y^{(j)}$$

And hence if $f = ab$, say $f = a_kb_k$ for each $a_kb_k$, each $a_kb_k$, $f^{(d)} \in a \cdot b$.

(iii) If $f \in a \cdot b$, say $f = \sum a_kb_k$, then $f^{(d)} \in a_kb_k$ each $d$.
(1) We induct on the number \( r \) of nonzero homogenous parts of an element \( f \in \sqrt{A} \). If \( r = 1 \), the only nonzero component

\[ \alpha f = f, \text{ so trivially } f^{(\alpha)} = \sqrt{A} \text{ all } \alpha. \]  

Now suppose the claim holds for all \( r < k \). Let

\[ f = f^{(1)} + \cdots + f^{(k)} \quad f^{(\alpha)} \in Sd, \]

be in \( \sqrt{A} \), so that \( \exists m \geq 0, f^m \in A \). We may suppose \( d_1 < d_2 < \cdots < d_m \) and hence that \( f^{(d_1 m)} = (f^{(d_1)})^m \). Hence \( f^{(d_1)} \in \sqrt{A} \).

But then \( f^{(d_2)} + \cdots + f^{(d_k)} = f - f^{(d_1)} \in \sqrt{A} \), and so by the inductive hypothesis we are done.

(3) Necessity is clear. Suppose \( \beta \) is s.t. whenever \( f, g \in S \) are homogeneous \( (f = f^{(a)}, g = g^{(b)}) \) and \( f, g \in \beta \), either \( f \) or \( g \) or \( \beta \) is not in \( \beta \).

Suppose \( \beta \) be \( \beta \), both \( a, b \neq 0 \), and \( \sigma = \sum a^{(\alpha)} \), \( \beta = \sum b^{(\alpha)} \). Then

\[ (a^{(\alpha)}) = \sum_{\delta} a^{(\delta)} b^{(\alpha - \delta)} \]

Since \( \alpha \) is homogeneous, each of these sums is in \( \beta \). Let \( d_n \) and \( e_n \) be the smallest degree of the components occurring in the expansion of \( a^{(\alpha)}, b^{(\alpha)} \) resp. Then \( (a^{(\alpha)})^{(d_n a)} = a^{(\delta)} b^{(e_n)} \). Hence either \( a^{(\alpha)} \) or \( b^{(\alpha)} \) is in \( \beta \). Suppose \( a^{(\alpha)} \) is in \( \beta \).

Working \( a = (a^{(\alpha)})^{(d_n a)} + (a^{(\alpha)})^{(d_n a + 1)} \cdots + (a^{(\alpha)})^{(d_n a + e_n - 1)} = \sum_{d_n a}^{d_n a + e_n - 1} a^{(\alpha)} b^{(\alpha - \delta)} \) we proceed by induction. Working \( b^{(\alpha)} \), \( (a^{(\alpha)})^{(d_n a + e_n)} = \sum_{d_n a + e_n} a^{(\alpha)} b^{(\alpha - \delta)} \). Hence \( a^{(\alpha)} \in \beta \). Proceeding in this fashion, each \( a^{(\alpha)} = b^{(\alpha)} \), \( a, b \in \beta \), and hence \( \beta \) is \( \beta \). But it is conversely true in the first step that \( b^{(\alpha)} \in \beta \) also.

Just keep showing \( a^{(\alpha)} \) is in \( \beta \), but replace \( b^{(\alpha)} \) by \( b^{(\alpha)} \) by minimal degree term out in \( \beta \). \( \Box \)

We make the polynomial ring \( S = k[x_0, \ldots, x_n] \) into a graded ring by taking \( Sd \) to be the set of all linear combinations of monomials of total weight \( d \) in \( x_0, \ldots, x_n \). If \( f \in S \) is a polynomial, we cannot use it to define a function on \( \mathbb{P}^n \), because of the nonuniqueness of the homogeneous coordinates. However, if \( f \) is a homogeneous polynomial of degree \( d \), then

\[ f(x_0, \ldots, x_n) = x^d f(x_0, \ldots, x_n) \]

so that the property of \( f \) being zero or not depends only on the equivalence class of \( (x_0, \ldots, x_n) \). Thus \( f \) gives a function from \( \mathbb{P}^n \) to \( \{0, 1\} \) by \( f(p) = 0 \) if \( f(x_0, \ldots, x_n) = 0 \) and \( f(p) = 1 \) if \( f(x_0, \ldots, x_n) \neq 0 \). Thus we can talk about the

\[ Z(f) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \} \]

If \( T \) is any set of homogenous elements of \( S \), we define the zero set of \( T \) to be

\[ Z(T) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \text{ for all } f \in T \} \]

If \( \alpha \) is a homogenous ideal of \( S \), we define \( Z(\alpha) = Z(T) \), where \( T \) is the set of all homogenous elements in \( \alpha \). Since \( S \) is a noetherian ring, any set of homogenous elements \( T \) has a finite subset \( f_1, \ldots, f_r \) such that \( Z(T) = Z(f_1, \ldots, f_r) \).

**Definition** A subset \( Y \) of \( \mathbb{P}^n \) is an algebraic set if there exists a set \( T \) of homogenous elements of \( S \) such that

\[ Y = Z(T). \]

**Proposition 2.1** The union of any two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

**Proof** If \( Y = Z(T) \) and \( Q = Z(S) \), \( Y \cup Q = Z(TS) \) and if \( Y_i = Z(T_i) \) all \( i \in I \), \( \cap Y_i = Z(\sum_{i \in I} T_i) \), where

the sum denotes all possible homogenous sums \( \sum_{i \in I} T_i, f_i \in T_i \). Since \( S \) is homogenous, \( S = Z(1) \), and \( P^n = Z(0) \). \( \Box \)

**Definition** We define the Zariski topology on \( \mathbb{P}^n \) by taking the open sets to be the complements of algebraic sets.

On a topological space, the notions of irreducible subset and the dimension of a subset, which we defined in §1, will apply.

**Occasionally used variant**

A projective algebraic variety (or simply projective variety) is an irreducible algebraic set in \( \mathbb{P}^n \), with the included topology. The dimension of a projective or quasi-projective variety is its dimension as a topological space.

If \( Y \) is any subset of \( \mathbb{P}^n \), we define the homogenous ideal of \( Y \) in \( S \), denoted \( I(Y) \), to be the ideal generated by \( \{ f \in S \mid f \text{ is homogenous and } f(p) = 0 \text{ for all } p \in Y \} \). If \( Y \) is an algebraic set we define the homogenous coordinate ring of \( Y \) to be \( S(Y) = \mathbb{P}(I(Y)) \). We refer to (Ex. 2.1—2.7) below for various properties of algebraic sets in projective space and their homogenous ideals.
Our next objective is to show that projective $n$-space has an open covering by affine $n$-space, and hence that every projective (respectively, quasi-projective) variety has an open covering by affine (resp. quasi-affine) varieties. First we introduce some notation. If $f \in S$ is a linear homogeneous polynomial, then the zero set of $f$ is called a hyperplane. In particular, we denote the zero set of $x_i$ by $H_i$, for $i = 0, \ldots, n$. Let $U_i$ be the open set $\mathbb{P}^n - H_i$. Then $\mathbb{P}^n$ is covered by the open sets $U_i$, because if $P = (a_0, \ldots, a_n)$ is a point, then at least one $a_i \neq 0$, hence $P \in U_i$. We define a mapping $\mathcal{P}: U_i \rightarrow \mathbb{A}^n$ as follows. If $P = (a_0, \ldots, a_n) \in U_i$, then $\mathcal{P}(P) = Q$, where $Q$ is the point with affine coordinates

$$(a_0 a_1 \ldots a_n)$$

with $a_i/\lambda$ omitted. Note that $\mathcal{P}$ is well-defined since the values $a_i/\lambda$ are independent of the choice of homogeneous coordinates.

**Proposition 2.2** The map $\mathcal{P}$ is a homeomorphism of $U_i$ with its induced topology to $\mathbb{A}^n$ with the Zariski topology.

**Proof** $\mathcal{P}$ is clearly bijective, so it will be sufficient to show that the closed sets of $U_i$ are identified with the closed sets of $\mathbb{A}^n$ by $\mathcal{P}$. We may assume $i = 0$, and we write simply $U$ for $U_0$ and $f: U \rightarrow \mathbb{A}^n$ for $\mathcal{P}$. Let $k$ be the aff. closed base field and set $A = k[x_0, \ldots, x_n]$. Define a map $\alpha$ from the set $S$ of homogeneous elements of $f$ to $A$, and a map from $A$ to $S$. Given $f \in S$, we set $\alpha(f) = (x_0, x_1, \ldots, x_n)$. On the other hand, given $g \in A$ of degree $e$, then $x_0^e g(x_0, x_1, \ldots, x_n)$ is a homogeneous polynomial of degree $e$ in the $x_i$, which we call $\beta(g)$. Now let $Y \subseteq U$ be a closed subset. Let $\bar{Y}$ be its closure in $\mathbb{P}^n$. This is an algebraic set, so $\overline{f} = z(T)$ for some subset $T \subseteq A$. Let $\bar{f} = \alpha(f) \in A$. Then $\alpha(f)$ is in the support of $\beta(g)$, so that $\beta(g)$ is also in $\bar{f}$. Hence $f \in T$, $\alpha(f)(x_0, x_1, \ldots, x_n) = 0$. Hence $f(1, x_1, \ldots, x_n) = 0$, so that $v = \overline{f}$ is a closed subset of $\mathbb{P}^n$. Then $\bar{f} = z(T)$, $\overline{f} = z(T)$. Conversely if $(x_0, x_1, \ldots, x_n) \in z(T)$, so that for all $f \in T$, $\alpha(f) \in \beta(g)$, then $\alpha(f) \in \beta(g)$, and hence $z(T) = \overline{f}$, so $f$ is closed.

Conversely, let $W$ be a closed subset of $\mathbb{A}^n$. Then $W = z(T)$ for some subset $T$ of $A$, and we claim $\overline{f}^{-1}(W) = z(\beta(T)) \cap U_i$. Let $(a_0, \ldots, a_n) \in T^{-1}(W)$, so that $(a_0 a_1 \ldots a_n) \in W$. Hence $a_i/\lambda \neq 0$ and for all $f \in T$, $a_0 \alpha(f)(a_0, a_1, \ldots, a_n) = 0$. Then $a_0 \alpha(f)(a_0, a_1, \ldots, a_n) = 0$. Then $a_0 \alpha(f)(a_0, a_1, \ldots, a_n) = 0$. Hence $f(a_0, a_1, \ldots, a_n) = 0$, and hence $f(a_0, a_1, \ldots, a_n) = 0$. Hence $f^{-1}(W)$ is closed in $U$. Thus $f$ and $f^{-1}$ are both closed maps, so $\mathcal{P}$ is a homeomorphism.

**Corollary 2.3** If $Y$ is a projective (resp. quasi-projective) variety, then $Y$ is covered by the open sets $Y \cap U_i$ which are homeomorphic to affine (resp. quasi-affine) varieties via the mapping $\mathcal{P}$ defined above.

**Proof** If $Y$ is closed, irreducible in $\mathbb{P}^n$, then $Y \cap U_i$ is a closed, irreducible (Ex. 1.6) subset. Hence under $\mathcal{P}: U_i \rightarrow \mathbb{A}^n$ identifies $Y \cap U_i$ with a closed, irreducible subset of $\mathbb{A}^n$ — an affine variety. If $Y$ is a projective variety and $z = Y \cap \mathbb{A}^n$ open in $\mathbb{P}^n$, then $Y \cap U_i$ is a closed, irreducible subset of $U_i$, and $z(\mathcal{P})$ is an open subset of $Y \cap U_i$, since $(a_0 a_1 \ldots a_n)$ is open in $U_i$ and $z(\mathcal{P}) = (Y \cap U_i) \cap (U_i \cap \mathbb{A}^n)$, hence $\mathcal{P}$ identifies $z$ with an affine variety. 

**Note** Cor 2.3 is true provided $Y \cap U_i$ is nonempty!

**Note** Let $T$ be the set of homogeneous elements in $S = k[x_0, x_1, \ldots, x_n]$, and denote the homogenous ideal generated by $T$ by $T$. Let $T'$ be the set of homogeneous elements in $l$ if $l = T$ so $z(T) = z(T')$. Hence $z(T) = (T \cap \mathbb{P}^n)$, $z(T')$ is an open subset of $Y \cap U_i$, since $(a_0 a_1 \ldots a_n)$ is open in $U_i$ and $z(\mathcal{P}) = (Y \cap U_i) \cap (U_i \cap \mathbb{A}^n)$, hence $\mathcal{P}$ identifies $z$ with an affine variety. 

Note Points are closed in $\mathbb{P}^n$: given $(x_0, \ldots, x_n) \in \mathbb{P}^n$ with $a_i \neq 0$ this point is the only solution to $T = \{x_0 = \frac{a_0}{a_1}, \ldots, x_n = \frac{a_n}{a_0} \}$ (exclude $x_i$).
Just as $\mathbb{R}[x]$ is defined as a set of functions $f: \mathbb{Z} \to \mathbb{R}$, we define:

**Definition.** The ring $\mathbb{R}[x,x^{-1}]$, called the ring of Laurent polynomials, is the set of all functions $f: \mathbb{Z} \to \mathbb{R}$ with finite support. Addition and multiplication are defined by

$$(f+g)(n) = f(n) + g(n)$$

$$(fg)(n) = \sum_{s+t=n} f(s)g(t)$$

$$(\delta(n)) = 0$$

$$(\delta'(n)) = \delta_{n0}$$

It is easy to see this makes $\mathbb{R}[x,x^{-1}]$ into a commutative ring with unit.

**Proposition.** There is an isomorphism of rings

$$\mathbb{R}[x,x^{-1}] \cong \mathbb{R}[x,y]/(xy-1) \cong \mathbb{R}[x]/(x-\lambda)$$

**Proof.** We already know that $\mathbb{R}[x]/(x-\lambda) \cong \mathbb{R}[x,y]/(xy-1)$. Since $x^{-1}$ is clearly a unit in $\mathbb{R}[x,y]$, we get $y: \mathbb{R}[x] \to \mathbb{R}[x,y]$ defined for $f \in \mathbb{R}[x]$ and $n \geq 0$ by

$$y(f/x^n) = f(x^{-1})^n$$

where $\phi: \mathbb{R}[x] \to \mathbb{R}[x,x^{-1}]$ is the obvious morphism $\phi(f)(n) = f(n)$, $n \geq 0$, $\phi(f)(n) = 0$, $n < 0$. $y$ is injective since $\phi(f)(x^{-1})^n = \phi(g)(x^{-1})^m$ implies that for $x \in \mathbb{Z}^+$,

$$f(x) = \phi(f)(x)$$

$$= (\phi(g)(x^{-1})^m)(x-n)$$

$$= \phi(g)(x+m-n)$$

$$= g(x+m-n)$$

Hence $x^n g = x^m f$, so $F/n = g/x^m$ in $\mathbb{R}[x]$. And $y$ is surjective since if $f \in \mathbb{R}[x,x^{-1}]$, let $a \in \mathbb{R}$ be least s.t. $f(a) \neq 0$. Then if $b$ is largest s.t. $f(b) \neq 0$, we suppose $a < 0.

$$f = \sum f(a) x^a + \cdots + f(b) x^b$$

$$= y\left(f(a)/x^a\right) + \cdots + y\left(f(b)/x^b\right) \in \text{Im} y$$

Since $(x^{-1})^n = x^{-n}$, hence $y$ is an isomorphism. □
\[ \text{Z-Graded Rings and their Localisations} \]

If we invert an element of a graded ring, even a homogenous element, we usually do not get a graded ring in the sense of Chapter I: Negative degrees will occur in the obvious grading. Thus we introduce the notion of a \( \text{Z-graded ring} \).

**Definition:** A \( \text{Z-graded ring} \) \( R \) is a ring \( R \) such that

\[
R = \cdots \oplus R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2 \oplus \cdots
\]

as abelian groups and \( R_i R_j \subseteq R_{i+j} \). The elements of \( R_i \) are called homogenous elements of degree \( i \).

A homogenous ideal in a \( \text{Z-graded ring} \) is simply an ideal generated by homogenous elements.

The case of ordinary graded rings is the case where \( R_i = 0 \) for \( i < 0 \).

**Exercise 2.16 (Characterisation of homogenous ideals)**

(a) An ideal \( I \) of a \( \text{Z-graded ring} \) is homogenous if, \( \forall f \in I \), all the homogenous components of \( f \) are in \( I \).

**Proof:** See Hartshorne notes. Same as graded case.

(b) \( \sqrt{I} \) is homogenous.

**Proof:** Again, same proof.

(c) If \( I, J \) are homogenous, so is \( \langle I : J \rangle = \{ f \in R \mid fJ \subseteq I \} \).

**Proof:** Suppose \( f = f^{(a)} + \cdots + f^{(d_k)} \in \langle I : J \rangle \), \( d_i \in \mathbb{Z} \) and \( f^{(a)} \in S_{d} \). Then since \( J \) is generated by homogenous elements, \( f^{(a)} \in (I : J) \) must lie in \( \langle I : J \rangle \). Hence \( \langle I : J \rangle \) is homogenous.

The condition is clearly necessary. The proof for the graded case works again.

Given a projective variety \( X \subseteq \mathbb{P}^r \), it is very useful to be able to write the localisations of the affine coordinate rings of the affine open pieces of \( X \) directly in terms of the homogenous coordinate ring of \( X \). The following exercise explains how to do this, in a form that works for arbitrary \( \text{Z-graded rings} \).

**Exercise 2.17 (Localisation of graded rings)** Suppose \( R \) is a \( \text{Z-graded ring} \) and \( 0 \neq f \in R \). Then \( R[f] \) is again a \( \text{Z-graded ring} \), where the grading of \( R[f] \) is inherited from that of \( R[x_i] \), which is canonically identified with \( R[f] \). That is, the grading of \( R/fm \) is \( d-n \) where \( r \) has grade \( d \). To see this is well-defined, suppose \( r/fm = s/fm \), so that \( f^m(r-fs) = 0 \), some \( q \geq 0 \). Then \( f^{m+q}r \) and \( f^{m+q}s \) have the same grade, so \( f \in R \) means the grade of \( s \) is \( d+m-q - (n+q) = d+m-n \). Hence \( s/fm \) has grade \( d+m-n = d-n \), as required.

(Above obviously \( f \) refers to monomials.) That is, for \( d \in \mathbb{Z} \), \( (R)[f] \) consists of all elements of the form

\[
\frac{r}{f^n} \text{ where } r \in R_{dm} \text{ (here } d \geq 0 \text{), for } n \geq 0. \]

The above shows these groups are disjoint. Suppose

\[
\frac{r}{f^n} \in (R[d] + \cdots + R[d_m]) \text{, } d, d_i \text{ all distinct}
\]

Then \( r/f^n \in (R[d] \cap \cdots \cap R[d_m]) \) implies \( r \in R_{d+n} \). Then if \( r_i \in R_{d_i+n} \) and

\[
\frac{r}{f^n} = \frac{r_1}{f^n} + \cdots + \frac{r_m}{f^n}
\]

Let \( k \geq \max \{ n_1, n_1, \ldots, n_m \} \). Then \( f^k \) is \( f^k \cdot n_1 - \cdots n_k - \cdots n_m \). Then \( f^k \cdot n_i \in R_{d_i+n_i} \) and \( f^k \cdot n_i \cdot f \in R_{d_i+n_i+k} \). Hence \( f^k \cdot n_i = 0 \), so that \( r_i/f^n = 0 \) in \( R[f] \). Hence we have defined a grading.
NOTE In a graded ring, principal ideals which are homogeneous are generated by a homogeneous elt.

Let \( S = \oplus_{d \geq 0} S_d \) be a graded domain, \( a \subseteq S \) a homogeneous ideal, \( f \in S \) with \( a = (f) \). Let
\[
f = f_0 + \cdots + f_m, \quad 0 \neq f_k \in S_n, \quad 0 \neq f_m \in S_m \text{ etc.}
\]

Since \( a \) is homogeneous \( f \in (f) \), so \( f = gf \) for some \( g = g_0 + \cdots + g_r \in S \). Then
\[
f = gf_0 + (g_1 g_0 + g_2 g_1 + \cdots + g_r g_m) + \cdots + g_r f_m
\]

Clearly we must have \( f + m = n \) since \( g \neq 0 \) and \( f_m \neq 0 \). But \( n \leq m \) so this implies \( n = m \) and \( f \) is homogeneous.

NOTE Projective Closure

Let \( Y \subseteq \mathbb{P}^n \) be a projective variety, and suppose \( \overline{Y} \cap U_0 \) is nonempty. Then \( \overline{Y} \cap U_0 \) is affine and its projective closure is \( \overline{Y} \).

NOTE If \( \mathfrak{a} \subseteq \mathbb{K}[x_0, \ldots, x_n] \) are homogeneous ideals, it is not hard to check that \( \mathcal{Z}(\overline{\mathfrak{a}}) = \overline{\mathcal{Z}(\mathfrak{a})} \).

NOTE Generating ideals in a Noetherian ring

Let \( R \) be a nonzero Noetherian ring \( \mathfrak{a} \) an ideal. Suppose \( \{ \mathfrak{a}_I \} \) is a set of ideals with \( \mathfrak{a} = (\{ \mathfrak{a}_I \}) \). We claim that \( \mathfrak{a} \) can be generated by a finite subset of the \( \mathfrak{a}_I \); if \( \mathfrak{a}_I \) is finite this is trivial. So assume \( \mathfrak{a}_I \) is infinite, and
\[
S = \{ J \subseteq I \mid J \neq \emptyset \text{ and } J \text{ is finite} \} \quad \text{(assume } I \neq \emptyset \text{)}
\]

For each \( J \in S \) let \( \alpha_J = (\{ J \}) \). Then \( \{ \alpha_J \} \subseteq \mathfrak{a}_I \) is a nonempty set of ideals in a Noetherian ring, hence has a maximal element \( \mathfrak{a}_\ell \). Clearly \( \mathfrak{a} = \mathfrak{a}_\ell \) and we are done.

NOTE Dimension and irreducible components

Let \( X \) be a nonempty Noetherian topological space, \( Y \subseteq X \) a nonempty closed subset and
\[
Y = Y_1 \cup \cdots \cup Y_k
\]

the decomposition of \( Y \) into its irreducible components. Then \( \dim Y = \sup \{ \dim Y_i \} \). Any chain of distinct irreducible closed sets in \( X \) becomes such a chain in \( Y \), so clearly \( \sup \dim Y_i = \dim Y \) (in particular, if this supremum is \( \infty \) then \( \dim Y = \infty \)). If \( Z_0 \subset Z_1 \subset \cdots \subset Z_k \) is a chain in \( Y \) then
\[
Z_0 = Z_0 \cap Y = Z_0 \cap Y_1 \cup \cdots \cup Z_0 \cap Y_k
\]

Since \( Z_0 \) is irreducible it is contained in some \( Y_i \), so the whole chain belongs to some \( Y_i \). This shows the reverse inequality, \( \dim Y = \sup \dim Y_i \).
NOTE Morphisms $\mathbb{P}^n \to \mathbb{P}^m$

Let $f_0(x_0, \ldots, x_n), f_1(x_0, \ldots, x_n), \ldots, f_m(x_0, \ldots, x_n)$ be homogeneous polynomials all of the same degree $d$, admitting no common solution in $\mathbb{P}^n$. (We assume all are nonzero, i.e., $d > 0$.) We define
\[ \mathcal{Y}: \mathbb{P}^n \to \mathbb{P}^m \]
\[ \mathcal{Y}(a_0, \ldots, a_n) = (f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n)) \]
\[ \phi: k[y_0, \ldots, y_m] \to k[x_0, \ldots, x_n] \]
Note $\phi$ preserves homogeneous polynomials of degree $d$, $\mathcal{Y}(a) = (f_0(a), \ldots, f_m(a))$.

Then $\mathcal{Y}$ is a morphism of $k$-algebras, and $\mathcal{Y}$ is a well-defined map of sets. We now show it is a morphism of varieties. Let $g(y_0, \ldots, y_m)$ be homogeneous. Then
\[ \mathcal{Y}(g) = \{ (a_0, \ldots, a_n) \mid g(a_0, \ldots, a_n) = 0 \} = \mathcal{Y}^{-1}(\{0\}) \]
Thus $\mathcal{Y}$ is continuous. To see it is a morphism, let $f: U \to k$ be regular, $U \subseteq \mathbb{P}^m$ open, and let $x \in f^{-1}(U)$. Say $f|_x \in k[y_0, \ldots, y_m]$ s.t. $\forall y \in V$, $f(y) = g(y)/h(y)$. Then for $y \in f^{-1}V, y = (a_0, \ldots, a_n)$
\[ f|_x(a_0, \ldots, a_n) = f(f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n)) \]
\[ = \mathcal{Y}(g)(a_0, \ldots, a_n) \]
Since $g, h$ are homogeneous of the same degree, so are $\mathcal{Y}(g), \mathcal{Y}(h)$. Hence $\mathcal{Y}$ is regular and $\mathcal{Y}$ is a continuous morphism of varieties.

NOTE Linear Morphisms $\mathbb{P}^{n-1} \to \mathbb{P}^n$ \((n \geq 2)\)

Consider the following special case. Let $(a_0, \ldots, a_n) \in \mathbb{P}^n$, so some element, say $a_i$, is nonzero. Put
\[ f_0(x_0, \ldots, x_{n-1}) = x_0 \]
\[ f_{i-1}(x_0, \ldots, x_{n-1}) = x_{i-1} \]
\[ f_i(x_0, \ldots, x_{n-1}) = \frac{1}{a_i} (a_0 x_0 + a_i x_i + \ldots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \ldots + a_n x_n) \]
\[ f_{n-1}(x_0, \ldots, x_{n-1}) = x_{n-1} \]
Then $\mathcal{Y}: \mathbb{P}^{n-1} \to \mathbb{P}^n$ is a morphism of varieties, $\mathcal{Y}(b_0, \ldots, b_{n-1}) = (b_0, \ldots, b_{i-1}, \frac{1}{a_i} (a_0 b_0 + a_i b_i + \ldots + a_{i-1} b_{i-1} + a_{i+1} b_{i+1} + \ldots + a_{n-1} b_{n-1}))$.

The hyperplane $a_0 x_0 + a_i x_i + \ldots + a_n x_n = 0$ is isomorphic as a variety to $\mathbb{P}^{n-1}$.
Automorphisms of $\mathbb{P}^n$

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Let $k$ be a field and $A \in GL_{n+1}(k)$ be $A = (a_{ij})$ and define

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

$$\varphi(x_1, \ldots, x_{n+1}) = (\ldots, \sum_j a_{ij} x_j, \ldots)$$

This is well-defined since if $\varphi(x_1, \ldots, x_{n+1}) = 0$ then since $A$ is invertible all the $x_i$ are zero. Clearly $\varphi(\lambda x_1, \ldots, \lambda x_{n+1}) = \varphi(x_1, \ldots, x_n)$. As we will show, $\varphi$ is an automorphism of varieties, and any morphism determined by an invertible matrix in this way is called a projective transformation.

First we check continuity. Define a morphism of $k$-algebras

$$\theta : k[x_1, \ldots, x_{n+1}] \longrightarrow k[x_1, \ldots, x_{n+1}]$$

$$\theta(x_i) = \sum_j a_{ij} x_j$$

This is clearly an automorphism of $k$-algebras. Note that for a homogenous polynomial $f \in k[x_1, \ldots, x_{n+1}]$ we have

$$\varphi^{-1}(Z(f)) = \{(a_1, \ldots, a_{n+1}) | f(\ldots, \sum_j a_{ij} x_j, \ldots) = 0\}$$

$$= \{(a_1, \ldots, a_{n+1}) | \theta(f)(a_1, \ldots, a_{n+1}) = 0\}$$

$$= Z(\theta(f))$$

Since $\theta$ maps a homogenous polynomial of degree $e$ to another homogenous polynomial of degree $e$, we see immediately that $\varphi$ is continuous.

Next we check that $\varphi$ is a morphism of varieties. Let $f : U \longrightarrow k$ be regular, where $U$ is an open subset of $\mathbb{P}^n$. Let $x$ be an element of the open set $\varphi^{-1}U$ and let $V$ be an open neighborhood of $\varphi(x)$ in $U$, $g, h$ homogenous polynomials of the same degree such that $f(v) = g(v)/h(v)$ for all $v \in V$. Then for $(a_1, \ldots, a_{n+1}) \in \varphi^{-1}V$

$$f(\varphi(a_1, \ldots, a_{n+1})) = \frac{g(\ldots, \sum_j a_{ij} x_j, \ldots)}{h(\ldots, \sum_j a_{ij} x_j, \ldots)}$$

$$= \frac{\theta(g)(a_1, \ldots, a_{n+1})}{\theta(h)(a_1, \ldots, a_{n+1})}$$

Hence $f \varphi$ is regular and so $\varphi$ is a morphism of varieties. Since the morphism induced by $A^{-1}$ is clearly inverse to $\varphi$, we have shown that $\varphi$ is an automorphism of the variety $\mathbb{P}^n$. 

1
It is clear that if \( \varphi, \phi \) are projective transformations determined by respective matrices \( A, B \) then the composition \( \varphi \phi \) is the projective transformation determined by the product \( AB \). So the composition of projective transformations is a projective transformation.

**Lemma 1.** Given two sets of three distinct points in \( \mathbb{P}^1 \)

\[
(P_1, P_2, P_3) \quad \text{and} \quad (Q_1, Q_2, Q_3)
\]

there is a unique projective transformation \( \varphi \) of \( \mathbb{P}^1 \) such that

\[
\varphi(P_i) = Q_i \quad i = 1, 2, 3
\]

**Proof.** First consider the case where \( P_1 = (1, 0), P_2 = (1, 1), P_3 = (0, 1) \). Let \( Q_i = (a_i, b_i), i = 1, 2, 3 \). Since the points \( Q_1 \) and \( Q_3 \) are distinct the matrix

\[
A = \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\]

has column rank 2 and is thus invertible. Let \( (\alpha, \beta) \) be such that

\[
\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}
\]

By scaling \( Q_1 \) by \( \alpha \) and \( Q_3 \) by \( \beta \) we may assume that \( \alpha = \beta = 1 \). It is then easily checked that the automorphism \( \varphi \) determined by \( A \) has the required property.

In the general case, let \( \psi \) be the transformation taking \( (1, 0), (1, 1), (0, 1) \) to \( P_1, P_2, P_3 \) and \( \varphi \) the transformation taking \( (1, 0), (1, 1), (0, 1) \) to \( Q_1, Q_2, Q_3 \). Then \( \varphi \psi^{-1} \) has the required property. If \( \psi \) is another transformation with the property that \( \psi(P_i) = Q_i, i = 1, 2, 3 \) then \( \psi^{-1} \psi \) maps the elements \( (1, 0), (1, 1), (0, 1) \) to themselves. This implies that the matrix determining the composite must be a scalar multiple of the identity, so \( \psi = \varphi \psi^{-1} \), proving uniqueness. \( \square \)

**Definition 1.** Points \( P_1, \ldots, P_n \) of \( \mathbb{P}^n \) are **collinear** if there is a linear polynomial \( a_0 x_0 + \ldots + a_n x_n \) which admits each \( P_i \) as a solution.

**Lemma 2.** Given two sets of four distinct points in \( \mathbb{P}^2 \)

\[
(P_1, P_2, P_3, P_4) \quad \text{and} \quad (Q_1, Q_2, Q_3, Q_4)
\]

which satisfy the condition that no three points in the set are collinear, there is a unique projective transformation \( \varphi \) of \( \mathbb{P}^2 \) such that

\[
\varphi(P_i) = Q_i \quad i = 1, 2, 3, 4
\]

**Proof.** First consider the case where

\[
P_1 = (1, 0, 0) \\
P_2 = (0, 1, 0) \\
P_3 = (0, 0, 1) \\
P_4 = (1, 1, 1)
\]

Let \( Q_i = (a_i, b_i, c_i) \). Since no three of the \( Q_i \) are collinear, the matrix

\[
A = \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\]

is invertible.
has column rank 3 and is thus invertible. As before, by scaling the $Q_i$ if necessary, we can assume that

$$
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
a_4 \\
b_4 \\
c_4
\end{pmatrix}
$$

If $\varphi$ is the projective transformation determined by $A$, then $\varphi$ has the required property. The general case and uniqueness follow in the same way as before. □
**UNDERSTANDING PROJECTIVE SPACE**

For \( n \geq 1 \) we identify \( \mathbb{A}^n \) with the tuples \((1, a_1, \ldots, a_n) \in \mathbb{P}^n\). This is an isomorphism of varieties. The points not belonging to \( \mathbb{A}^n \) are of the form \((0, a_1, \ldots, a_n)\), which we identify with the line

\[ a_1 x_1 + \cdots + a_n x_n = 0 \]

in \( \mathbb{A}^n \), and think of \( 0 \) as being an \( \infty \) in the direction of this line. So \( \mathbb{P}^n \) is \( \mathbb{A}^n \) together with an \( \infty \) for every line through the origin in \( \mathbb{A}^n \). Now consider Ex 2.4 and the note which shows that the projective closure of a hypersurface is a hypersurface. If we work now with \( n = 2 \), a line in \( \mathbb{A}^2 \) is the affine variety

\( a + bx + cy = 0 \)

coordinates: \( x, y \)

The projective closure of the line is the hypersurface \( aw + bx + cy = 0 \) (\( w \) the projective coordinate).

The solutions with \( w = 1 \) are precisely \( L \), so the only new solution is \( (0, 1, -b/c) \) which corresponds to the \( \infty \) in the direction of \( L \). (We order coordinates \( w, x, y \))

We have numerous other examples:

**Hyperbola** \( xy = 1 \). Projective closure is \( xy - w^2 = 0 \) which adds the infinities in the direction of the \( x \) and \( y \) axis:

\[ \infty = (0, 0, 1) \]

\[ \infty = (0, 1, 0) \]

**Parabola** \( y = x^2 \). Projective closure is \( wy - x^2 = 0 \) which adds the infinity \((0, 1, 0)\) in the direction of the \( y \)-axis:

\[ \infty = (0, 0, 1) \]

\[ \infty = (0, 1, 0) \]
**Ellipse** \( \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 1 \). Projective closure is \( \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = \omega^2 \), which adds no invariants.

From Exercise 1.1 we know that in \( \mathbb{A}^2 \) all conics are isomorphic either to \( y - x^2 \) or \( xy = 1 \). Since the discriminant of the ellipse is \( -b_2^2 a_2^2 = 0 \), any ellipse is isomorphic to the hyperbola, but we will see in Ex 3.1 that conics in \( \mathbb{P}^2 \) are all isomorphic (to \( \mathbb{P}^1 \) in fact). Visually, these make this apparent.

Any tuple \( (a_0, a_1, a_2) \in \mathbb{P}^2 \) determines a well-defined line \( \mathcal{L}_{a_0, a_1, a_2} \) and \( \mathcal{L}_{a_0, a_1, a_2} = \mathcal{L}_{b_0, b_1, b_2} \) if \( \mathcal{L}_{a_0, a_1, a_2} = \mathcal{L}_{b_0, b_1, b_2} \) in \( \mathbb{P}^2 \). Since \( \mathcal{L}_{a_0, a_1, a_2} = \mathcal{L}_{b_0, b_1, b_2} \), the tuple \( (a_0, a_1, a_2) = (b_0, b_1, b_2) \).

**Lemma** Two distinct lines in \( \mathbb{P}^2 \) meet at a point.

**Proof** Consider two lines:

\[
\begin{align*}
\alpha x_0 + \beta x_1 + \gamma x_2 &= 0 \\
\alpha' x_0 + \beta' x_1 + \gamma' x_2 &= 0
\end{align*}
\]

Since the lines are distinct the tuples \( (\alpha, \beta, \gamma) \) and \( (\alpha', \beta', \gamma') \) are linearly independent in \( \mathbb{A}^2 \). That is, the matrix

\[
\begin{pmatrix}
\alpha & \beta & \gamma \\
\alpha' & \beta' & \gamma'
\end{pmatrix}
\]

has rank 2. Note that two columns are linearly dependent if the determinant of the associated \( 2 \times 2 \) matrix is 0. Since rank = 2, the tuple

\[
\begin{pmatrix}
\beta & \gamma \\
\beta' & \gamma'
\end{pmatrix}
\]

has col-rank of 2. Note that two columns are linearly dependent if the determinant of the associated \( 2 \times 2 \) matrix is 0. Since col-rank = 2, the tuple

\[
\begin{pmatrix}
\alpha & \beta & \gamma \\
\alpha' & \beta' & \gamma'
\end{pmatrix}
\]

has col-rank = 2. Hence the kernel has dimension 1, implying that any other solution to (1) is equal to (2) in \( \mathbb{P}^2 \).

**Lemma 2.5** Given two distinct points in \( \mathbb{P}^2 \), there is a unique line going through them.

**Proof** Let \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) be distinct points. We want to find a line

\[\alpha x_0 + \beta x_1 + \gamma x_2\]

such that

\[
\begin{align*}
\alpha a_0 + \beta a_1 + \gamma a_2 &= 0 \\
\alpha b_0 + \beta b_1 + \gamma b_2 &= 0
\end{align*}
\]

That is, \((\alpha, \beta, \gamma)\) should be a nontrivial solution of \((a_0, a_1, a_2)\), which again has col-rank 2, hence a kernel of dimension 1, and so a unique solution in \( \mathbb{P}^2 \). The solution is

\[
(\alpha, \beta, \gamma) = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}
\]
Let us survey the homogenous ideals of $S = k[x_0, \ldots, x_n]$ we have the improper homogenous ideal $S$, and proper homogenous ideals $I$. For a proper homogenous ideal $I$, we cannot have $I \neq (x_0, \ldots, x_n)$, because this would imply that $f(0, \ldots, 0) \neq 0$, some $f \in I$, implying that $f$ has a nonzero component homogenous of degree $0$. Since $a$ is homogenous, this component would belong to $a$, contradicting the assumption that $I$ is proper. Further, if $a \subseteq (x_0 - a_0, \ldots, x_n - a_n)$ then $a$ belongs to $(x_0 - \lambda a_0, \ldots, x_n - \lambda a_n)$ for $a \in k$, since $f \in a$, $f = \Sigma i \in I f_i$ then $f_i \in a$, so

$$f(\lambda a_0, \ldots, \lambda a_n) = \Sigma i \in I f_i(\lambda a_0, \ldots, \lambda a_n) = \Sigma i \in I \lambda f_i(a_0, \ldots, a_n) = 0.$$ 

Hence for a proper and homogenous, the variety $V(I)$ contains 0 and for every point $(a_0, \ldots, a_n) \in V(I)$, also $(\lambda a_0, \ldots, \lambda a_n) \in V(I)$, hence $V(I) = 0$ implies the "ray" relation points $V(I) = 0$ into elements of $I^n$, and thus $I$ is the homogenous ideal of $I$ determined by $a$, as defined earlier. So $a \subseteq (x_0 - a_0, \ldots, x_n - a_n)$ if $V(F_a) = V(\text{homogenous } (a_0, \ldots, a_n)) = 0$.

Hence if $f \in S$, either $deg f = 0$ and the result is trivial, or $deg f > 0$. Assume $f(0) = 0$, $y \in E(I)$. (If homogenous, a homogenous, $Z(\overline{I})$ in $E^n$.) Then by the above $f$ belongs to all the maximal ideals containing $a$, (including $(a_0, \ldots, a_n)$ since $deg f > 0$). Hence $f \in \mathfrak{m}$ by the normal Nullstellensatz.

Let $a \subseteq S$ be homogenous. Suppose $Z(I) = \emptyset$. Then the only maximal ideal of $S$ containing $a$ is $(a_0, \ldots, a_n)$, if $a_i = 0$, and proper, and otherwise $a_i = S$. Hence $a_i$ is $S = (a_0, \ldots, a_n) = S + \mathfrak{m} = \mathfrak{m}$. Now, if $a_i$ is homogenous, then $S_i = \mathfrak{m}$. Then $S_i = S$ or $S_i = S + \mathfrak{m}$, then either $a_i = S$ or $a_i = S + \mathfrak{m}$. Then $S = S_i = S + \mathfrak{m}$, then $S = S_i = S + \mathfrak{m}$, then $a_i = S$. Hence, since $a_i$ is homogenous, both ideals contain the same homogenous elements, and are thus equal.

(a) Suppose $I \subseteq S$ is a subset of $S$, and that $(a_0, \ldots, a_n) \in Z(I)$. Then for $f \in I$, $f \in S$ implies $f(a_0, \ldots, a_n) = 0$, so $(a_0, \ldots, a_n) \in Z(I)$.

(b) Suppose $Y_i \subseteq Y_2$ are subset of $I^n$ and that $f \notin I(Y_2)$ is a homogenous element of $S$ s.t. $f(a_0, \ldots, a_n) = 0$. Then also this holds for $Y_i$, so $f \notin I(Y_i)$. Since $I(Y_i)$ is generated by such $f$, $I(Y_2) \subseteq I(Y_i)$.

(c) Let $Y_i \subseteq Y_2$. If $f \in I(Y_1) \cap I(Y_2)$ is homogenous, then clearly $f \in I(Y_1) \cap I(Y_2)$.

(4.2.4) Let $Y \subseteq I^n$. Clearly $Z(I)$ is a closed set containing $Y$. Suppose $Y \subseteq Y' \subseteq Z(I)$, $Y' = Z(I')$ a closed subset. Then $Z(I')$ implies that $f \notin I'$, $f(a_0, \ldots, a_n) = 0$ for $(a_0, \ldots, a_n) \in Y$. Hence since $f$ is homogenous, $f \notin I(Y)$.

(4.2.4) Let $Y \subseteq I^n$. Clearly $Z(I)$ is a closed set containing $Y$. Suppose $Y \subseteq Y' \subseteq Z(I)$, $Y' = Z(I')$ a closed subset. Then $Z(I')$ implies that $f \notin I'(Y)$, $f(a_0, \ldots, a_n) = 0$ for $(a_0, \ldots, a_n) \in Y$. Hence since $f$ is homogenous, $f \notin I(Y)$.

(b) Suppose $I(Y)$ is prime and that $Y = Y_1 \cup Y_2$, $Y_1$ a proper closed subset. Then $I(Y) = I(Y_1) \cap I(Y_2)$. Since the ideal $I(Y)$ is prime, why $I(Y_1) \subseteq I(Y)$. Hence $Z(I(Y)) \subseteq Z(I(Y_1))$. This shows that $Y$ is irreducible. Conversely, suppose $Y \subseteq I(Y)$ and that $f$ is homogenous with $f \notin I(Y)$. Then $Y \subseteq Z(f) = Z(f) \cup Z(f)$, or $Y \subseteq Z(f) \cup Z(f)$. Hence $Y \subseteq Z(f)$, or $Y \subseteq Z(f)$, so $Y \subseteq Z(f)$. Hence $f \in I(Y)$, and this shows that $I(Y)$ is prime.

(c) This is the old problem that says "there is no polynomial zero everywhere except $(0, \ldots, 0)$". Given this, it follows that $I(P^n) = \{ f \in S \mid f(0) = 0, \text{ all } \lambda \in k \}$, the intersection is over all maximal ideals other than $(x_0, \ldots, x_n)$, but by assumption, this is $S$. Since $S$ is Hilbert implies the Jacobson radical is $0$, and above claim $\ldots$ implies we cannot have $f$. If $f(0) = 0$, and $f \in I(P^n)$. Hence since $0$ is prime, $I(P^n) = 0$, $P^n$ is irreducible.
2.5 (a) $\mathbb{P}^n$ can be covered by the $U_i$, which are homeomorphic to $\mathbb{A}^n$ and are thus noetherian. Hence the result (and (b) using Prop. 1.5) follow from: If $X$ is a topological space with a finite open cover $X = U_1 \cup U_2 \cdots$ of closed sets in $X$. Then

$$Q_j \cup U_j = Q_k \cup U_j \cap \cdots$$

is a descending chain of closed sets in $U_i$. Let $N_i$ be s.t. $Q_j \cup U_j = Q_k \cup U_j \cap \cdots$ If $N = \max \{ N_i \}$ then $Q_j \cup U_j = Q_k \cup U_j \cap \cdots$. Thus $Q_j = U_j \cup Q_j \cup U_j = Q_k \cup U_j \cup \cdots$ $\forall j > N$. Thus

Let $Y \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y) = \mathbb{C}[1]$. We show in our Section 3 (Morphisms) (see proof at Theorem 4.3 and subsequent Notes) that if $f: \mathbb{P} \rightarrow \mathbb{P}$ are the usual isomorphisms and $\mathbb{Y}$ is the affine variety $((1,0,1), \{ 0 \} Y_1 \neq \emptyset)$ then $A(Y)$ is isomorphic as a $k$-algebra to the ring $S(Y)_{\text{aff}}$, which is the subring of $S(Y)$, consisting of pairs $g \in S(Y)$ with $g \neq 0 \in S(Y)$. Thus, the ring of polynomials $A(Y)_{\text{aff}}$ is isomorphic as a $k$-algebra to $S(Y)_{\text{aff}}$ with $S(Y)_{\text{aff}} = \mathbb{C}[z, z^{-1}]$ (homogeneous polynomials). As we have noted previously, $R[z, z^{-1}] \cong \mathbb{C}[z]_{\text{aff}}$, so

$$S(Y) = S(Y)_{\text{aff}} \cong A(Y)[z, z^{-1}]$$

as $k$-algebra. But if $R$ is a domain, $Q$ is quotient field of $k$ and $R'$ any subring $Q' \subseteq R'$ then the quotient field of $R'$ is isomorphic as a $k$-algebra to $Q$ (set $\varphi: \mathbb{C} \rightarrow Q'$ must be a quotient ring of $Q'$, field), and $R'$ is a field containing $R$ must be $Q$. Hence the quotient field of $Q$ of $S(Y)$ is isomorphic as a $k$-algebra to the quotient field of $A(Y)_{\text{aff}}$ as shown by (1.5a)

$$\dim S(Y) = \dim A(Y) + 1$$

as required.

2.7 (a) Consider the open cover $\mathbb{P}^n = \bigcup U_1 \cup U_2 \cup \cdots \cup U_n$, Each $U_i$ is homeomorphic to $\mathbb{A}^n$, so by Ex. 1.10

$$\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$$

2.7 (b) Let $Z \subseteq \mathbb{A}^n$ be a projective variety and $\bar{Y} \subseteq Z$ a nonempty open set. Let $T = \{ \emptyset \leq j \leq n \} \frac{Y}{(Y_j)_{k \neq j}}$. For each $j \in T$, $Y_j$ is homeomorphic to the open subset $Y_j$ of the affine variety $\{ (x_1, \ldots, x_n) \} \subseteq \mathbb{A}^n$. By Ex. 2.6 this implies $\dim Y_j = \dim Z \cap U_j = \dim Z$ (by 1.5b and Prop. 1.10). But then by Ex. 1.10

$$\dim Y = \sup \{ \dim Y_j : j \in T \}$$

as required. Note that $Y = Z$ using Ex. 1.6.

2.7 Let $Y \subseteq \mathbb{P}^n$ be a projective variety with ideal $\mathfrak{p} = Y(0) \subseteq \mathbb{C}[X_0, \ldots, X_n]$. Let $Z \subseteq \mathbb{A}^n$ be the affine variety defined by $\mathfrak{p}$. By Ex. 2.6

$$\dim Y = \dim \mathbb{A}[z] = 1$$

so $\dim Y = n - 1$.

2.7 (c) $\dim \mathbb{A}[x] = n = (n+1) - 1$ and by (Prop 1.13) this is iff $\mathfrak{p} = (f)$ where $f$ is a nonconstant irreducible polynomial. But since $Y$ is homogeneous, $f$ must be homogeneous (see an earlier note). This implies the proof. (Note the maximal prime $\{ x_0, \ldots, x_n \}$ can never be principal so we never have $Z(f) = \emptyset$ in $\mathbb{P}^n$ for an irreducible homogeneous $f$.)
Projective Closure of an Affine Variety

Let $Y \subseteq \mathbb{A}^n$ be an affine variety, and use $\mathbb{L} : \mathbb{A}^n \rightarrow U$, $(a_1, \ldots, a_n) \mapsto (1, a_1, \ldots, a_n)$ to identify $Y$ with a subset $W$ of $\mathbb{P}^n$. We call the closure $\overline{W}$ of $W$ in $\mathbb{P}^n$ the projective closure of $Y$. Since $\mathbb{L}$ is a homeomorphism $W$ is irreducible, so its closure is a projective variety.

(a) We claim that $I(\overline{W})$, which is the ideal generated by all the homogenous polynomials $f \in S = k[x_0, \ldots, x_n]$ which are zero on $\overline{W}$ (since $Z(f)$ is closed, this is if $f$ is zero on $W$), is generated by the collection

$$p(I(Y))$$

of homogenous polynomials (notation of the proof of (2.2)). Let $J$ denote this ideal i.e. $J = \langle p(I(Y)) \rangle$. Since for $f \in I(Y) S = k[x_0, \ldots, x_n]$, with order $e$, $p(f) = x_0^e f(x_0, \ldots, x_n)$, and hence for $(1, a_1, \ldots, a_n) \in W$, since $(1, a_1, \ldots, a_n) \in V$, $\beta(f)(1, a_1, \ldots, a_n) = f(a_1, \ldots, a_n) = 0$. Hence $Z(\beta(f)) \supseteq W$. Since $p(f)$ is homogenous $Z(\beta(f))$ is closed, hence $Z(\beta(f)) \supseteq W$, and so $f \in I(Y) \Rightarrow \beta(f) \in \overline{I(W)}$.

It follows that, as claimed, $\beta(I(Y)) \subseteq \overline{I(W)}$.

Now suppose $f \in I(\overline{W})$ that is, there are homogenous polynomials $p \in S$ and arbitrary polynomials $g_i \in S$ with $p_i$ zero on $W$, s.t. $f = \sum_{i=0}^n g_i p_i$ Suppose $p_i$ is homogeneous of degree $d$. Then $\alpha_0(p_i) = p_i(1, a_1, \ldots, a_n)$, and since $p_i$ is zero on $W$, $\alpha_0(p_i)$ is zero on $Y$. Hence $\alpha_0(p_i) \in I(Y)$. Now $\alpha_0(p_i) \in I(Y)$ is formed from $\alpha_0(p_i)$ by letting $f$ be the largest order of a monomial in $\alpha_0(p_i)$, and multiplying monomials by $x_0$ until we get a polynomial homogenous of order $f$. Since the monomial of $p_i$ which becomes $\alpha_0(p_i)$ must have the lowest power of $x_0$,

$$x_0^d \beta(\alpha_0(p_i)) = p_i$$

for some $k$. Hence $p_i \in J$, and so $f \in J$ i.e. $I(\overline{W}) = J$ as required.

(b) Finally, we claim that $I(Y) = (z - x^3, y - x^2)$. Clearly we have $\mathbb{Z}$. Suppose that $f \in I(Y)$, and

$$f = f_1(z - x^3) + f_2(y - x^2) + f_3(x)$$

then $f(b^3, b^3) = 0 \Rightarrow f_3(0, 0) = 0$ or $k$. Hence $f_3 = 0$, and $f \in (z - x^3, y - x^2)$. If $W = \mathbb{A}^n_0(Y)$, then by (a), $\overline{I(W)}$ is generated by $\beta(I(Y))$. We need

**Lemma** Let $f, g \in k[y_1, \ldots, y_n]$ where $f$ has order $e$ and $g$ has order $d, d > e$. Then

$$\beta(f + g) = x_0^{d - e} \beta(f) + \beta(g)$$

**Proof** Firstly, we can apply $\beta$ without collecting terms, since we are just multiplying each monomial by sufficient powers of $x_0$ to make $f + g$ be homogenous of degree $d$. Each monomial in $f$ gets multiplied by $x_0^{d - e}$, and its order is $d - e$. Each monomial in $g$ has its order increased to $d$ by $e$ - this can be achieved by getting them all to $e$ (this is $\beta(f)$) and then multiplying anything by $x_0^{d - e}$.

Note also that if $f \in k[y_1, \ldots, y_n]$ and $x_0^e$ is a monomial that $\beta(x_0^{e} f) = x_0^e \beta(f)$.

**Lemma** Let $f_1, \ldots, f_n \in k[y_1, \ldots, y_n]$ with the order of $f_i = e_i$ and $e_i \leq e_{i+1}$, $1 \leq i \leq n - 1$. Then

$$\beta(f_1 + \cdots + f_n) = x_0^{e_1 - e_2} \beta(f_1) + x_0^{e_2 - e_3} \beta(f_2) + \cdots + x_0^{e_{n-1} - e_n} \beta(f_{n-1}) + \beta(f_n)$$

$$= \sum_{i=1}^{n} x_0^{e_i - e_{i+1}} \beta(f_i)$$

**Proof** By induction on $n$. The above lemma handles $n = 2$. Suppose it holds for $n - 1$. Then

$$\beta(f_1 + \cdots + f_{n-1} + f_n) = \beta(\sum_{i=1}^{n-1} f_i + f_n)$$

$$= x_0^{e_1 - e_{n-1}} \beta(\sum_{i=1}^{n-1} f_i) + \beta(f_n)$$

$$= x_0^{e_1 - e_{n-1}} \sum_{i=1}^{n-1} x_0^{e_{i+1} - e_i} \beta(f_i) + \beta(f_n)$$

$$= \sum_{i=1}^{n} x_0^{e_i - e_{i+1}} \beta(f_i).$$
and finally,

**Lemma** If \( f, g \in k[y_1, \ldots, y_n] \) then if the monomials in \( g \) all have distinct degree,

\[
\beta(fg) = \beta(f) \beta(g)
\]

**Proof** Let \( g = \sum_{i=1}^{n} x_i x_i \) with the order of \( g \) being \( b \). Then assuming \( \lvert x_i \rvert < \lvert x_m \rvert \), \( i \leq n \), \( (s = b = \lvert x_n \rvert) \)

\[
\begin{align*}
\beta(fg) &= \beta\left( f \left( \sum_{i=1}^{n} x_i x_i \right) \right) \\
&= \beta\left( \sum_{i=1}^{n} x_i x_i \right) \\
&= \sum_{i=1}^{n} x_i x_i \beta(f) x_i \\
&= \sum_{i=1}^{n} x_i x_i \beta(f) x_i \\
&= \beta(f) \beta(g)
\end{align*}
\]

Now let \( f \in I(Y) \), \( f = f_1(x, y, z)(z-x^2) + f_2(x, y, z)(y-x^2) \). Then let \( e = \text{order } f_1 \), \( d = \text{order } f_2 \)

\[
\begin{align*}
\beta(f) &= \begin{cases} \\
\beta(f_1(z-x^2)) + x_0^{d-e-1} \beta(f_2(y-x^2)) & e + 3 > d + 2 \\
x_0^{d-e+1} \beta(f_1(z-x^2)) + \beta(f_2(y-x^2)) & e + 3 < d + 2 \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\beta(f) &= \begin{cases} \\
\beta(f_1(z-x^2)) + x_0^{d-e-1} \beta(f_2(y-x^2)) & e + 3 > d + 2 \\
\beta(f_1)(z-x^2) + x_0^{d-e-1} \beta(f_2)(y-x^2) & e + 3 < d + 2 \\
\end{cases}
\end{align*}
\]

Note we do not consider the case \( e + 3 = d + 2 \), because then we cannot apply \( \beta(fg) = x_0^{d-e}(\beta(f) + \beta(g)) \). So provided \( f \) can be written as above, with \( e + 3 \neq d + 2 \), \( \beta(f) \) is a member of \( (z x_0^2 - x^2, y x_0 - x^2) \). What happens when \( e + 3 = d + 2 ? \) For example, consider \( x z - y^2, y z \in I(Y) \)

\[
\begin{align*}
\alpha z - y^2 &= z(x - x^2) + \{-(x^2 + y)\}(y - x^2) \\
\end{align*}
\]

Then \( e = 0, d = 0 \), \( c = 0, b = 2, \) so \( c e + 1 \). Note that \( \beta(x^2 - y^2) = x z - y^2 \), but \( \beta(f_1(z-x^2)) + \beta(f_2(y-x^2)) \) is \( x_0^2(z-x^2) \). (Thus in particular we have answered the question, since \( x z - y^2 \) is \( (z x_0^2 - x^2, y x_0 - x^2) \) - to get \( z \) we would have to pick up \( x_0 \) as well, since everything else has a factor of \( x^1 \).) We finish with the claim that \( f_1 = f_2 \) - that is, we claim that \( I(Y) = (z x_0^2 - x^2, y x_0 - x^2, z - y^2) \). Consider \( f \in I(Y) \), \( f = f_1(z-x^2) + f_2(y-x^2) \). Then \( \beta(f) \) will be in \( (z x_0^2 - x^2, y x_0 - x^2) \) provided the highest order \( d \) of \( f_1(z-x^2) \) and \( f_2(y-x^2) \) don't cancel (although we required \( d > e \) in the first Lemma, the proof gave more). Let \( f_1 = f_1 + h \), where \( h \) is nonhomogeneous and contains all of the monomials of \( f \) of the highest order \( e = \text{order } f_1 \). Similarly, \( f_2 = f_2 + g \), then we are assuming \( h x = -g x^2 \), or \( h x = -g \). This means that

\[
\begin{align*}
f &= f_1(z-x^2) + f_2(y-x^2) = f_1(z-x^2) + f_2(y-x^2) + h(z-x^3) - h(x-y^2) \\
&= f_1(z-x^2) + f_2(y-x^2) + h(z-x^3) - h(x-y^2) \\
&= f_1(z-x^2) + f_2(y-x^2) + h(z-x^3) - h(x-y^2)
\end{align*}
\]

and we just repeat the process on \( f_1(z-x^2) + f_2(y-x^2) \), which have strictly smaller order.

**Note:** Above shows that the ideal of the twisted cubic in \( P^3 \) is \( (z x_0^2 - x^2, y x_0 - x^2, z - y^2) \).
NOTE The Projective Closure of a Hypersurface is a Hypersurface

Let \( Y \subseteq \mathbb{A}^n \) be the hypersurface \( Y = \{ f \} \) for a nonconstant irreducible polynomial \( f \in k[x_1, \ldots, x_n] \). We claim that the projective closure \( Z \subseteq \mathbb{P}^n \) of \( Y \) is the hypersurface \( Z(\beta(f)) \) (in particular, we will show \( \beta(f) \) is irreducible).

**Lemma** If \( f, g \in k[x_1, \ldots, x_n] \) and \( \beta : k[x_1, \ldots, x_n] \to k[x_1, x_2, \ldots, x_n] \) as before, then

\[
\beta(fg) = \beta(f)\beta(g)
\]

**Proof** Write \( f = f_1 \cdots f_d \), \( g = g_1 \cdots g_e \), assuming \( f_i \neq 0, g_j \neq 0 \) (mod if \( f = 0 \) or \( g = 0 \)). Then

\[
\beta(fg) = \beta(\sum_{i,j} f_i g_j) = \sum_{i,j} \beta(x_i^{d_i} x_j^{e_j} f_i g_j)
\]

\[
= \sum_{i,j} (x_i^{d_i} x_j^{e_j} f_i g_j) = \beta(f)\beta(g).
\]

It follows from Ex.2.9 that \( I(Z) = (\beta(f)) \), since \( I(Z) \) is prime, \( \beta(f) \) is irreducible (directly, there is at least one monomial in \( \beta(f) \) not involving \( x_0 \) if \( \beta(f) = \mathbb{H} \), \( f \in k[x_0] = \mathbb{H} \times \mathbb{C} \Rightarrow \) any \( \mathbb{H} \) involves only \( x_0 \) and therefore is a unit (otherwise contradict homog. of \( \beta(f) \)). This proves that the projective closure of a hypersurface is a hypersurface.

**Note** Projective Closure preserves dimension

Let \( Y \subseteq \mathbb{A}^n \) be affine of dimension \( r \). Let \( Z \) be the projective closure of \( Y \). Then \( Z \cap U_0 = Z(Y) \), so since \( Y \) is homeomorphic to \( Z(Y) \), we have by 02.7

\[
\dim Z = \dim Z \cap U_0 = \dim Y
\]

So the above note also follows by dimension calculations and Ex.2.8.

**Note** Alternative Generators of the Twisted Cubic in \( \mathbb{P}^3 \)

Let \( Y \subseteq \mathbb{A}^2 \) be the twisted cubic \( Y = \{(t, t^3, t^4) \mid t \in k\} \), \( Z \) the projective closure of \( Y \). We have shown in Ex.2.9 that \( I(Z) = (\beta(f)) \), the prime ideal \( (xy-xz, xz-y^2) \). (The projective coordinates \( x, y, z \).) We claim that

\[
I(Z) = (x^2-xy, xy-y^2, y^2-xz)
\]

Clearly \( xy-yz \) is zero on the triplets \((1, t^3, t^4)\) which are the image of \( Y \) in \( U_0 = \mathbb{P}^3 \). But \( x, y, z \) is closed and thus contains \( Z \), so a power of \( x, y, z \) belongs to \( I(Z) \). Hence \( Z \subseteq Z(x^2-xy, xy-yz, y^2-xz) \) and \( Z(x^2-xy, xy-yz, y^2-xz) \subseteq Z \), as required.

But then \( I(Z) = (x^2-xy, xy-yz, y^2-xz) \), so we have the equality in (1).

**Note** We have defined the projective closure of an affine variety \( Y \subseteq \mathbb{A}^n \) using the isomorphism \( \mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n \), but we may also use \( \mathbb{A}^n \cong U_i \) for \( 0 \leq i \leq n \). Let \( \beta : \mathbb{P}^n \to \mathbb{P}^{n+1} \) be the automorphism defined by the matrix

\[
\begin{pmatrix}
0 & \cdots & 0 & 0 \\
1 & \cdots & 0 & 0 \\
0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & \cdots & 1
\end{pmatrix}
\]

Then \( Y \) identifies \( Y \subseteq \mathbb{P}^n \) with \( Y \subseteq \mathbb{P}^n \), so everything we've proved about the projective closure is independent of \( i \). Including the projective closure of \( Z(f) \), being \( Z(\beta(f)) \), where \( \beta \) is defined using \( x_i \).
Linear Varieties

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A linear polynomial in \(k[x_0, \ldots, x_n]\) is a nonzero homogenous polynomial of degree 1:

\[ f = a_0 x_0 + \ldots + a_n x_n \]

Let \(f_0, \ldots, f_n\) be linear polynomials with associated tuples \((a_{i0}, \ldots, a_{in})\) for \(0 \leq i \leq n\). Assume that these tuples are linearly independent in \(k^{n+1}\) so that the matrix \(A = (a_{ij})\) has row rank \(n + 1\), hence column rank \(n + 1\) and so the associated endomorphism of \(k^{n+1}\) is an isomorphism, implying that \(A\) is invertible. Let \(B = (b_{ij})\) be the inverse of \(A\). We define morphisms of \(k\)-algebras:

\[
\varphi : k[x_0, \ldots, x_n] \rightarrow k[x_0, \ldots, x_n]
\]

\[
\varphi(x_i) = f_i = \sum_j a_{ij} x_j
\]

and

\[
\phi : k[x_0, \ldots, x_n] \rightarrow k[x_0, \ldots, x_n]
\]

\[
\phi(x_i) = \sum_k b_{ik} x_k
\]

It is easy enough to check that \(\varphi \phi = \phi \varphi = 1\). This isomorphism identifies the ideal \((f_0, \ldots, f_n)\) with \((x_0, \ldots, x_n)\), which is a prime ideal. We can now prove

**Lemma 1.** Any ideal in \(k[x_0, \ldots, x_n]\) generated by linear polynomials is prime.

**Proof.** Let \(\mathfrak{a}\) be an ideal generated by linear polynomials. Clearly \(\mathfrak{a}\) is a proper homogenous ideal. Even if \(\mathfrak{a}\) is generated by an infinite number of linear polynomials, we can find a finite subset which generate \(\mathfrak{a}\) (see an earlier Note). If linear polynomials \(g_1, \ldots, g_s\) generate \(\mathfrak{a}\) but are linearly dependent, then we can omit one of the \(g_i\) and still have a generating set. In this way we produce a set of generators \(f_0, \ldots, f_r\) for \(\mathfrak{a}\) which are linearly independent (since the coefficients form tuples in \(k^{n+1}\) it is clear that \(r \leq n\). Say

\[
f_i = a_{i0} x_0 + \ldots + a_{in} x_n
\]

for \(0 \leq i \leq r\). Then the set of linearly independent vectors \(a_i = (a_{i0}, \ldots, a_{in})\) \((0 \leq i \leq r)\) can be extended to a basis \(a_0, \ldots, a_r, a_{r+1}, \ldots, a_n\) for \(k^{n+1}\) and we define linear polynomials \(f_{r+1}, \ldots, f_n\) using these new tuples. Then \(f_0, \ldots, f_n\) induces the isomorphism \(\varphi\) of the above discussion, under which the ideal \(\mathfrak{a} = (f_0, \ldots, f_r)\) corresponds to the prime ideal \((x_0, \ldots, x_r)\). Hence \(\mathfrak{a}\) is prime. \(\square\)
The above proof also shows the following:

**Corollary 2.** If \( \mathfrak{a} \) is an ideal in \( k[x_0, \ldots, x_n] \) generated by \( k \) linearly independent linear polynomials, then the height of \( \mathfrak{a} \) is \( k \). In particular, the number of elements in any set of linearly independent linear generators is the same.

**Definition 1.** A linear variety in \( \mathbb{P}^n \) is a projective variety of the form \( Y = Z(\mathfrak{a}) \) where \( \mathfrak{a} \) is an ideal generated by linear polynomials.

Since any ideal generated by linear polynomials is prime homogenous, if \( Y = Z(\mathfrak{a}) \) is a linear variety then \( I(Y) = \mathfrak{a} \). So a projective variety \( Y \) is linear if and only if \( I(Y) \) can be generated by linear polynomials.

**Definition 2.** A hyperplane in \( \mathbb{P}^n \) is a projective variety of the form \( Y = Z(f) \) where \( f \) is a linear polynomial. By Corollary 2 or Exercise 2.8 of Hartshorne, any hyperplane has dimension \( n - 1 \).

Note that if \( f \) is any linear polynomial, \( Z(f) \) is empty iff. \( (f) \) is the irrelevant maximal ideal \( (x_0, \ldots, x_n) \) which is impossible since \( f \) cannot be a unit multiple of \( x_0, \ldots, x_n \). So \( Z(f) \) is a hyperplane.

The following gives our solution to Exercise 2.11(a) of Hartshorne.

**Lemma 3.** Any linear variety in \( \mathbb{P}^n \) is an intersection of hyperplanes. Conversely, any nonempty intersection of hyperplanes is a linear variety.

**Proof.** Let \( Y \) be a linear variety, and suppose \( \mathfrak{a} = (f_0, \ldots, f_r) \) where the \( f_i \) are linear polynomials. Then

\[
Y = Z(\mathfrak{a}) = Z(\sum_i (f_i)) = \bigcap_i Z(f_i)
\]

So \( Y \) is an intersection of hyperplanes. Conversely if \( f_0, \ldots, f_r \) are linear polynomials and \( Q = Z(f_0) \cap \ldots \cap Z(f_r) \) is nonempty, then \( Q = Z(f_0, \ldots, f_r) \) and since the ideal \( (f_0, \ldots, f_r) \) is prime we have \( I(Q) = (f_0, \ldots, f_r) \). Hence \( Q \) is a linear variety.

For general homogenous irreducible polynomials \( p_0, \ldots, p_r \), it is not true that

\[
I(Z(p_0) \cap \ldots \cap Z(p_r)) = (p_0, \ldots, p_r)
\]

See Exercise 2.16 for a counterexample. But if the \( p_i \) are linear and the intersection is nonempty, then this equality holds, since \( (p_0, \ldots, p_r) \) is prime:

\[
I(Z(p_0) \cap \ldots \cap Z(p_r)) = I(Z(p_0, \ldots, p_r)) = \sqrt{(p_0, \ldots, p_r)} = (p_0, \ldots, p_r)
\]

**Lemma 4.** A linear variety \( Y \) in \( \mathbb{P}^n \) has dimension \( r \) if and only if \( I(Y) \) is minimally generated by \( n - r \) linear polynomials (equivalently, is generated by \( n - r \) linearly independent linear polynomials).

**Proof.** By Exercise 2.6 we have

\[
r = \dim Y = \dim S(Y) - 1 = n - \text{height} I(Y)
\]

But by Corollary 2 the height of \( I(Y) \) is the unique integer \( k \) for which there exists a set of linearly independent linear generators. Any minimal generating set consisting of linear polynomials must be linearly independent, so this completes the proof.

\( \square \)
Suppose \(a\) can be generated by \(n+1\) linearly independent linear polynomials \(f_0, \ldots, f_n\). Then the associated tuples \(a_0, \ldots, a_n\) must span \(k^{n+1}\), implying that
\[
a = (f_0, \ldots, f_n) = (x_0, \ldots, x_n)
\]
And hence \(Z(a) = \emptyset\) in \(\mathbb{P}^n\). This proves

**Lemma 5.** If \(f_0, \ldots, f_r\) are linear polynomials and \(Z(f_0) \cap \ldots \cap Z(f_r)\) is empty then \(r \geq n\).

**Proof.** By assumption \(a = (f_0, \ldots, f_r) = (x_0, \ldots, x_n)\). The set \(f_0, \ldots, f_r\) can be refined to a linearly independent set of linear generators for \(a\), which must have \(n+1\) elements. Hence \(r \geq n\).

Finally we answer part (d) of the Exercise.

**Lemma 6.** If \(Y, Z\) are linear varieties in \(\mathbb{P}^n\) of respective dimensions \(r, s\) and \(r+s \geq n\), then \(Y \cap Z\) is nonempty and is a linear variety of dimension \(\geq r+s-n\).

**Proof.** We can write \(Y = Z(f_1) \cap \ldots \cap Z(f_{n-r})\) and \(Z = Z(g_1) \cap \ldots \cap Z(g_{n-s})\) where the \(f_i\) and \(g_j\) are linearly independent generators for the ideals \(I(Y), I(Z)\) respectively. Thus \(Y \cap Z\) is the intersection of \(2n-r-s\) hyperplanes. Provided \(r+s \geq n\) it follows from the previous Lemma that this intersection is nonempty. Then we can refine the list \(f_1, \ldots, f_{n-r}, g_1, \ldots, g_{n-s}\) to find a set of linearly independent generators of \(I(Y \cap Z) = I(Y) + I(Z)\) with \(q \leq 2n-r-s\) elements. Then
\[
\dim(Y \cap Z) = n - q \geq r + s - n
\]

This completes Exercise 2.11. We keep our old proofs because they use different techniques which may be useful at some point. Note only that the proof of (a) part (ii) implies (i) is incorrect in the written notes. Other than that, the solutions are valid.

The ideal of a point in affine space is a maximal ideal in \(k[x_1, \ldots, x_n]\), which must have the form \((x_1 - a_1, \ldots, x_n - a_n)\) for some \(a_i\) since \(k\) is algebraically closed. These maximal ideals are certainly not homogenous! So no maximal ideal of \(k[x_0, \ldots, x_n]\) can occur as the ideal of any algebraic set in \(\mathbb{P}^n\). Instead, the ideals of projective points are prime ideals of coheight 1:

**Lemma 7.** If \(P = (a_0, \ldots, a_n)\) is a point of \(\mathbb{P}^n\) with \(a_i \neq 0\), then
\[
I(P) = (a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)
\]
Moreover \(I(P)\) is a prime ideal of coheight 1 in \(k[x_0, \ldots, x_n]\).

**Proof.** The ideal \(a = (a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)\) is homogenous and \(Z(a) = \{P\}\). Since \(a\) is generated by \(n\) linearly independent linear polynomials it is a prime ideal of height \(n\). Hence \(a = I(P)\) and \(I(P)\) has coheight 1.

The maximal ideals containing \(I(P)\) are those ideals corresponding to tuples \((b_0, \ldots, b_n)\) which are equal to \(P\) in \(\mathbb{P}^n\), together with the irrelevant maximal ideal which corresponds to \((0, \ldots, 0)\).
The following is proved earlier in our notes:

**Lemma 8.** Any hyperplane in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^{n-1}$.

**Proof.** Let the hyperplane be $Z(f)$ where $f = a_0x_0 + \ldots + a_nx_n$ is a linear polynomial. We select some $i$ with $a_i \neq 0$ and define the isomorphism $\varphi : \mathbb{P}^{n-1} \rightarrow Z(f)$ by mapping $(c_0, \ldots, c_{n-1})$ to

$$(c_0, \ldots, c_{i-1}, -\frac{1}{a_i}(c_0a_0 + \cdots + c_{i-1}a_{i-1} + c_ia_{i+1} + \cdots + a_nc_{n-1}), c_i, \ldots, c_{n-1})$$

\[ \Box \]

**Lemma 9.** Let two hyperplanes $H, K$ in $\mathbb{P}^n$ have nonempty intersection. Identifying $H$ with $\mathbb{P}^{n-1}$ induces an isomorphism of $H \cap K$ with a hyperplane of $\mathbb{P}^{n-1}$.

**Proof.** Let $H = Z(f)$ and $K = Z(g)$ where

$$f = a_0x_0 + \cdots + a_nx_n$$
$$g = b_0x_0 + \cdots + b_nx_n$$

Select some $i$ with $a_i \neq 0$ and let $\varphi : \mathbb{P}^{n-1} \rightarrow H$ be the isomorphism of Lemma 8. It is not difficult to check that $\varphi$ identifies $H \cap K$ with the hyperplane

$$(b_0 - \frac{b_i}{a_i}a_0)x_0 + \cdots + (b_{i-1} - \frac{b_i}{a_i}a_{i-1})x_{i-1} +$$
$$(b_{i+1} - \frac{b_i}{a_i}a_{i+1})x_i + \cdots + (b_n - \frac{b_i}{a_i}a_n)x_{n-1} = 0$$

\[ \Box \]

**Corollary 10.** A linear variety of dimension $r \geq 1$ in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^r$.

**Proof.** By induction on $n$. If $n = 1$ then this is trivial, since there can be no such linear variety. If $n = 2$ then we need only consider linear varieties of dimension $r = 1$. But these are hyperplanes in $\mathbb{P}^2$, so we use Lemma 8. So assume the result is true for $n - 1$ where $n > 2$ and let $Y \subseteq \mathbb{P}^n$ be a linear variety of dimension $r \geq 1$. Note that $r \leq n - 1$ by Ex 1.10. If $r = n - 1$ then we are in the situation of Lemma 8. So assume $I(Y)$ is generated by $n - r \geq 2$ linearly independent linear polynomials

$$I(Y) = (f_1, \ldots, f_{n-r})$$

Then $Z(f_1)$ is a hyperplane in $\mathbb{P}^n$ and is thus isomorphic to $\mathbb{P}^{n-1}$. Now

$$Y = \bigcap_{i=1}^{n-r} Z(f_i) = \bigcap_{i=2}^{n-r} Z(f_1) \cap Z(f_i)$$

Considered as closed subsets of $\mathbb{P}^{n-1}$ the $n - r - 1$ sets $Z(f_1) \cap Z(f_i)$ are hyperplanes by Lemma 9, and it follows that $Y$ is a linear variety in $\mathbb{P}^{n-1}$. Hence by the inductive hypothesis $Y$ is isomorphic to $\mathbb{P}^r$.

\[ \Box \]

**Definition 3.** A line in $\mathbb{P}^n$ is a linear variety of dimension 1.
So lines arise as the zero sets of ideals \( \mathfrak{a} \) generated by \( n - 1 \) linearly independent linear polynomials, and any line is isomorphic to \( \mathbb{P}^1 \). There are no lines in \( \mathbb{P}^1 \).

**Proposition 11.** Two distinct lines in \( \mathbb{P}^n \) meet at most one point \((n \geq 2)\).

**Proof.** Let \( H, K \) be two distinct lines in \( \mathbb{P}^n \) (we must have \( n \geq 2 \)). Then \( I(H) = (f_1, \ldots, f_{n-1}) \) and \( I(K) = (g_1, \ldots, g_{n-1}) \). Write

\[
\begin{align*}
  f_i &= a_{i0}x_0 + \cdots + a_{in}x_n \\
  g_i &= b_{i0}x_0 + \cdots + b_{in}x_n
\end{align*}
\]

The fact that the lines \( H, K \) are distinct means that the matrix

\[
A = \begin{pmatrix}
  a_{10} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{(n-1)0} & \cdots & a_{(n-1)n} \\
  b_{10} & \cdots & b_{1n} \\
  \vdots & \ddots & \vdots \\
  b_{(n-1)0} & \cdots & b_{(n-1)n}
\end{pmatrix}
\]

has row-rank \( \geq n \). Hence the kernel of \( A \) has dimension \( \leq 1 \) and so the two lines meet at most one point of \( \mathbb{P}^n \). \( \square \)

**Proposition 12.** Given two distinct points in \( \mathbb{P}^n \) \((n \geq 2)\) there is a unique line going through them.

**Proof.** Let \((a_0, \ldots, a_n)\) and \((b_0, \ldots, b_n)\) be distinct points. We want to find a linear variety of dimension 1 in \( \mathbb{P}^n \) containing both points. That is, we need \( n - 1 \) linearly independent linear polynomials

\[
f_i = c_{i0}x_0 + \cdots + c_{in}x_n \quad 1 \leq i \leq n - 1
\]

with

\[
\begin{align*}
  c_{i0}a_0 + \cdots + c_{in}a_n &= 0 \\
  c_{i0}b_0 + \cdots + c_{in}b_n &= 0
\end{align*}
\]

for \( 1 \leq i \leq n - 1 \). That is, the tuples \((c_{i0}, \ldots, c_{in})\) should be nonzero solutions of the matrix

\[
A = \begin{pmatrix}
  a_0 & \cdots & a_n \\
  b_0 & \cdots & b_n
\end{pmatrix}
\]

Since the two points are distinct this matrix has col-rank 2 and hence a kernel of dimension \( n - 1 \). Take a basis for the kernel and use these vectors to produce the required polynomials \( f_i \). This gives a line in \( \mathbb{P}^n \) going through both points, and this line is unique by the previous Proposition. \( \square \)

Let \( H \) be a hyperplane in \( \mathbb{P}^n \) and \( L \) a line. Then \( H, L \) are linear varieties of respective dimensions \( n - 1, 1 \) so that the intersection \( H \cap L \) is nonempty by Lemma 6. If \( I(H) = (f) \) and \( I(L) = (g_1, \ldots, g_{n-1}) \) then

\[
\begin{align*}
  H \cap L &= Z(f) \cap Z(g_1) \cap \cdots \cap Z(g_{n-1}) \\
  I(H \cap L) &= (f, g_1, \ldots, g_{n-1})
\end{align*}
\]
Provided $L$ is not contained in $H$, the polynomials $f,g_1,\ldots,g_{n-1}$ are linearly independent, so $H \cap L$ has dimension 0. Since $H \cap L$ is a projective variety, it is irreducible and hence $H \cap L$ is a single point. So provided the line is not contained in the hyperplane, the two meet at precisely one point.
Linear Varieties in $\mathbb{P}^n$

A linear polynomial in $k[x_0, \ldots, x_n]$ is a homogenous polynomial (nonzero)

$$a_0 x_0 + \cdots + a_n x_n$$
of degree 1. A hypersurface defined by a linear polynomial is called a hyperplane.

Hence any linear polynomial is irreducible if its nonzero. Hence any hyperplanes has dimension $n - 1$ by Ex 2.8.

(a) Let $Y \subseteq \mathbb{P}^n$ be a nonempty closed set. We claim the following are equivalent:

(i) $I(Y)$ can be generated by linear polynomials

(ii) $Y$ can be written as an intersection of hyperplanes

PROOF (i) $\Rightarrow$ (ii)

Say $I(Y) = (f_1, \ldots, f_m)$ where $f_i$ are nonzero linear polynomials (we may assume the list finite, since if $I(Y)$ is generated by an infinite list it is generated by a finite subset - see the earlier note). Then $Y = Z(I(Y)) = Z(f_1) \cap \cdots \cap Z(f_m).

(ii) $\Rightarrow$ (i)

If $Y = \emptyset$, then $I(Y) = \{0\}$ so we may assume $Y = Z(f_1) \cap \cdots \cap Z(f_m), n - 1.

In either case, we call $Y$ a linear algebraic set, or linear variety. $Y$ is irreducible (in what follows we say "linear variety" but people often use the term "linear algebraic set")

**WARNING** Our definition of a linear algebraic set includes the "nonempty" condition. For example $Z = \{x_0 = x_n = 0\}$ is clearly generated by linear polynomials but $Z(\mathbb{P}^n) = \emptyset$.

Our definition requires that $I(Y)$ be a finite list of linear polynomials, but $Z(\mathbb{P}^n) = \emptyset$.

Every $(a_0, \ldots, a_n) \in \mathbb{P}^n$ determines a hyperplane (well-defined) and this assignment is injective.

(b) First we recall a Note: Morphisms $\mathbb{P}^m \to \mathbb{P}^n$. First we prove

**LEMMA** Let $f = a_0 x_0 + \cdots + a_n x_n$ for $(a_0, \ldots, a_n) \in \mathbb{P}^n$ and let $\varphi : \mathbb{P}^n \to Z(f) \subseteq \mathbb{P}^m$ be the isomorphism discussed earlier. If $(b_0, \ldots, b_n) \in \mathbb{P}^n$ then $\varphi$ identifies the closed set $Z(f) \cap Z(g)$ with the following hyperplane in $\mathbb{P}^{m+1}$:

$$\begin{align*}
(b_0 - b_i a_0) x_0 + \cdots + (b_m - b_i a_m) x_m &= 0 \\
+ \cdots + (b_n - b_i a_n) x_n &= 0
\end{align*}
$$

(Assuming $Z(f) \cap Z(g) \neq \emptyset$ (and $(a_0, \ldots, a_n) \neq (b_0, \ldots, b_n)$ in $\mathbb{P}^n$)

PROOF Assuming $Z(f) \cap Z(g) \neq \emptyset$, $Z(f) \cap Z(g)$ is a nonempty closed subset of $Z(f)$. Since $Z(f) \subseteq \mathbb{P}^m$ under $Y$, we can write (taking $q_i = 0$)

$$\begin{align*}
Y' \cap (Z(f) \cap Z(g)) &= \{ (c_0, \ldots, c_n) \mid f(c_0, \ldots, c_n) = 0 \} \\
&= \{ (c_0, \ldots, c_n) \mid (c_0, \ldots, c_1, a_0 c_0 + \cdots + a_n c_n) \in Z(g) \} \\
&= Z(h) \text{ where } h \text{ is the polynomial (i)}
\end{align*}
$$

Notice that all the coefficients in (i) are zero off $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$.

Let $Y \subseteq \mathbb{P}^n$ be a linear variety. Then we can write $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ for a finite collection of linear polynomials. Let $m = \min \{ \deg(f_i) \mid i = 1, \ldots, r \}$. Then $Y$ is a linear variety of dimension $m$.

LEMMA If $Y \subseteq \mathbb{P}^n$ is a linear variety with minimal decomposition $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ then $r \leq n$ and dim $Y = n - r$.

PROOF By induction on $n$. In $\mathbb{P}^1$ a linear variety is a point (and points are linear varieties), so the intersection of two hyperplanes (a line) is empty. So $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ is minimal $r = 1$ and clearly dim $Y = 0 = 1$.

Now assume the result holds for $n - 1$ with $n > 1$. Let $Y = Z(f_1) \cap \cdots \cap Z(f_r)$. Then $Z(f_i)$, $i = 1, \ldots, r$, is a linear hyperspace of dimension $n - 1$. Let $Y_i = Z(f_1) \cap \cdots \cap Z(f_i)$ be a minimal decomposition of $Y_i$. Identifying $Y_i$ with a closed subset of $\mathbb{P}^{n - 1}$, then the above list forms a minimal decomposition of $Y$ in $\mathbb{P}^n$ by the inductive hypothesis $r - 1 \leq n - 1$, or $r = n$, and

$$\text{dim } Y = (n - 1) - (r - 1) = n - r. \square$$
Now to the actual Exercise! Let \( Y \subseteq \mathbb{P}^n \) have dimension \( n-r \) (\( 0 \leq r < n \)). If \( Y \) is a linear variety we can write \( Y = Z(f_1) \cap \cdots \cap Z(f_t) \) for some linear polynomials \( f_1, \ldots, f_t \). Assume the \( f_1, \ldots, f_t \) are a minimal generating set of \( I(Y) \). Then their decomposition of \( Y \) must also be minimal, for otherwise we could omit terms and generate \( Y \) with fewer of the \( f_i \) (we may assume \( r > 0 \) and \( I(Y) \neq 0 \) and hyperplanes are proper). Then by the Lemma \( t = n \) and \( n-r = \dim Y = n-t \Rightarrow t = r \). So \( I(Y) \) is minimally generated by \( r \) elements. (Any generating set contains a minimal one.)

For (b) we need the following Lemma:

**NOTE**. The above shows that a linear variety of dimension \( r \) in \( \mathbb{P}^n \) (\( 0 \leq r < n \)) has a minimal decomposition as the intersection of \( n-r \) hyperplanes.

**LEMMA**. Let \( f = a_0 x_0 + \cdots + a_n x_n \) be a linear polynomial (nonzero). Then \( Z(f) \) is nonempty, in \( \mathbb{P}^n \).

**PROOF**. The element \( f \) is reducible, so \( f = p \cdot q \). If \( Z(f) \neq \emptyset \), \( p = S \cdot \mathbb{Z} \cdot x \). But this impossible, since \( f \) cannot be a unit multiple of \( x_0, \ldots, x_n \) (\( n \geq 1 \)). Hence \( p = S \) and \( Z(f) \) is nonempty. \( \square \)

**LEMMA**. If \( f = a_0 x_0 + \cdots + a_n x_n \) and \( g = b_0 x_0 + \cdots + b_n x_n \) are linear polynomials, then \( Z(f) \cap Z(g) \) if nonempty (the hyperplanes meet) in \( \mathbb{P}^n \).

**PROOF**. If \( (a_0, \ldots, a_n) \) and \( (b_0, \ldots, b_n) \) determine the same point in \( \mathbb{P}^n \) then \( Z(f) = Z(g) \) so the result follows from the previous Lemma. Otherwise there are two cases:

(a) There is no \( 0 \leq j < n \) with both \( a_j \neq 0 \) and \( b_j \neq 0 \). If \( a_i = b_i = 0 \) for some \( i \) then \( (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in Z(f) \cap Z(g) \) and we are done. Otherwise not all the \( b_j \) are zero, so the nonzero coefficients define a linear polynomial in \( n-1 \) variables, which admits a nonzero solution by the above. Pad this solution with zeros (in all the spots \( a_j \neq 0 \) and you have an element of \( Z(f) \cap Z(g) \).

(b) \( a_i \neq 0 \) and \( b_i \neq 0 \) for some \( 0 \leq j < n \). Since \( (a_0, \ldots, a_n) \neq (b_0, \ldots, b_n) \) in \( \mathbb{P}^n \) the linear polynomial in \( n-1 \) variables

\[
\begin{align*}
(b_0 - b_i a_i) y_0 + (b_i - b_i a_i) y_1 + \cdots + (b_n - b_i a_i) y_n \\
\end{align*}
\]

is nonzero, hence has a nonzero solution \( (y_0, y_1, \ldots, y_{n-1}) \). Put

\[
c = -b_i a_i + a_i c_0 + \cdots + a_i c_{n-1} + b_i c_{n-1} + \cdots + b_i c_0
\]

and we have an element in \( Z(f) \cap Z(g) \). \( \square \)

**LEMMA**. If \( f_1, \ldots, f_r \) are linear polynomials and \( Z(f_1) \cap \cdots \cap Z(f_r) \) is nonempty in \( \mathbb{P}^n \) then \( r \geq n+1 \).

**PROOF**. By induction on \( n \geq 1 \). If \( n = 1 \) then for any linear polynomial \( f \) \( Z(f) \) is a point, so to get \( \emptyset \) you must intersect \( \geq 2 \) points them. Assume it is true for \( n-1 \). If \( Z(f_1) \cap \cdots \cap Z(f_r) = \emptyset \) we may assume this is a minimal intersection. Otherwise there is some \( s \) where \( Z(f_s) \neq \emptyset \) and \( Z(f_1) \cap \cdots \cap Z(f_{s-1}) \cap Z(f_{s+1}) \cap \cdots \cap Z(f_r) \) is nonempty, \( s \neq r \). By the previous Lemma \( (s-1) \) distinct hyperplanes in \( \mathbb{P}^n \), \( Z(f_s) \cap \cdots \cap Z(f_r) \) is nonempty (by previous Lemma), closed, distinct hyperplanes in \( \mathbb{P}^n \), \( Z(f_s) \cap \cdots \cap Z(f_r) \) with this intersection to give \( \emptyset \) in \( \mathbb{P}^n \). By the inductive hypothesis \( r-1 \geq n \) is not required. \( \square \) (Ex. \( n = 3 \) \( \emptyset = \mathbb{Z}(x) \cap \mathbb{Z}(x) \cap \mathbb{Z}(x) \)).

(b) Let \( Y, Z \) be linear varieties in \( \mathbb{P}^n \), of dimensions \( s, t \) respectively. Then \( Y \) is the intersection of \( n-s \) hyperplanes, so \( \dim Y \geq n-s \). \( \dim \mathbb{P}^n = n \) means \( r = n \). Thus \( n-r-s = n-t \leq n-t \leq n-s \leq n-s \). Hence if \( r < s \geq n \) then \( Y \cap Z = \emptyset \). If \( Y \cap Z \) is nonempty we can write the intersection to get a minimal decomposition of \( Y \cap Z \) with \( q \leq 2n-r-s \) hyperplanes. Then

\[
\dim Y \cap Z = n-q \\
\]

\[
\geq r+s-n \square \]
The \(d\)-Uple Embedding

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For given \(n, d > 0\) let \(M_0, M_1, \ldots, M_N\) be the monomials of degree \(d\) in the \(n+1\) variables \(x_0, \ldots, x_n\) where \(N = \binom{n+d}{n} - 1\) (this is justified at the end of our Section 1.1 solutions). For a given monomial \(f\) of degree \(d\) let \([f]\) denote the index \(0 \leq |f| \leq N\) with \(M_{|f|} = f\). The assignment of indices to the monomials may be completely arbitrary.

Given \(n, d > 0\) and the ordering \([-\) on the monomials, we define a map

\[
\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N
\]

\[
\rho_d(a_0, \ldots, a_n) = (M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n))
\]

This is called the \(d\)-Uple embedding of \(\mathbb{P}^n\) in \(\mathbb{P}^N\). It is easily seen that this map is well-defined. The map is also injective, since if \(\rho_d(a_0, \ldots, a_n) = \rho_d(b_0, \ldots, b_n)\) then there is \(0 \neq \lambda \in k\) such that

\[
(M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n)) = (\lambda M_0(b_0, \ldots, b_n), \ldots, \lambda M_N(b_0, \ldots, b_n))
\]

Let \(0 \leq i \leq n\) be such that \(a_i \neq 0\). Then \(a_i^d = M_{[x_i^d]}(a_0, \ldots, a_n) = \lambda b_i^d\), so \(a_i = \mu b_i\) where \(\mu = \lambda a_i^{1-d}b_i^{d-1}\). Then for any \(0 \leq j \leq n\) we have

\[
\begin{align*}
\mu b_j &= \lambda a_i^{1-d}b_i^{d-1}b_j \\
&= a_i^{1-d} \lambda M_{[x_i^{d-1}x_j]}(b_0, \ldots, b_n) \\
&= a_i^{1-d} M_{[x_i^{d-1}x_j]}(a_0, \ldots, a_n) \\
&= a_i^{1-d} a_i^{-1} a_j \\
&= a_j
\end{align*}
\]

so \((a_0, \ldots, a_n) = (b_0, \ldots, b_n)\) and \(\rho_d\) is injective. We claim that the image of \(\rho_d\) in \(\mathbb{P}^N\) is a projective variety. To prove this, first consider the morphism of \(k\)-algebras

\[
\theta : k[y_0, \ldots, y_N] \longrightarrow k[x_0, \ldots, x_n]
\]

\[
y_i \mapsto M_i
\]

Let \(\mathfrak{a}\) be the prime ideal \(\text{Ker}\theta\). This ideal is homogenous since if \(f\) is homogenous of degree \(e\) then \(\theta(f)\) is homogenous of degree \(de\). It is easy to see that \(\text{Im}(\rho_d) \subseteq Z(\mathfrak{a})\) since if \(g \in k[y_0, \ldots, y_n]\),

\[
g(\rho_d(a_0, \ldots, a_n)) = g(M_0(a_0, \ldots, a_n), \ldots, M_N(a_0, \ldots, a_n))
\]

\[
= \theta(g)(a_0, \ldots, a_n)
\]
In particular $Z(a)$ is nonempty, so $Z(a)$ is a projective variety in $\mathbb{P}^N$. The hard part is to show that $Z(a) \subseteq \text{Im}(\rho_d)$. We proceed as follows: assume $(b_0, \ldots, b_N) \in Z(a)$, and let $b_i \neq 0$. Let $h$ be the monomial with $[h] = i$. Then the polynomial $h^d$ can be written as a product $h^d = x^d_{i_1} \cdots x^d_{i_s}$ (where we may have $x_{ik} = x_{ij}$ for $j \neq k$). Thus

$$y^d_{i_1} - y^d_{x^d_{i_1}} \cdots y^d_{x^d_{i_s}} \in a$$

and since $(b_0, \ldots, b_N) \in Z(a)$ we see that $b_i^d = b_{x^d_{i_1}} \cdots b_{x^d_{i_s}}$. The fact that $b_i \neq 0$ implies that $b_{x^d_{i_j}} \neq 0$ for all $0 \leq j \leq s$. We have just shown that for any $(b_0, \ldots, b_N) \in Z(a)$ there is $0 \leq K \leq n$ with $b_{x^d_K} \neq 0$.

Given such $K$, suppose we could find $(a_0, \ldots, a_n) \in \mathbb{P}^n$ with $b_i = \lambda M_i(a_0, \ldots, a_n)$ for all $0 \leq i \leq N$. In particular this would imply

$$b_{x^d_K} = a^d_K, \quad b_{x^d_{i_1}} = a^{d-1}_{x^d_{i_1}} a_i \quad i \neq K$$

So that for $i \neq K$

$$a_i = \frac{a_K b_{x^d_{i_1}}}{a^d_K} = \frac{a_K b_{x^d_{i_1}}}{b_{x^d_K}}$$

So the obvious plan of attack is to try putting $a_K = 1$ and $a_i = b_{x^d_{i_1}}/b_{x^d_K}$ and try to show that

$$\rho_d(a_0, \ldots, a_n) = (b_0, \ldots, b_N)$$

And this is precisely what we are going to do. For any $0 \leq j \leq N$ there are nonnegative integers $m_0, \ldots, m_n$ with $m_0 + \ldots + m_n = d$ and

$$M_j = x^{m_0}_0 \cdots x^{m_n}_n$$

Then

$$x^d_K M_j = (x^d_K x_0)^{m_0} \cdots (x^d_K x_n)^{m_n}$$

This implies that

$$y^d_{x^d_K} y_j = \prod_{i=0}^{n} y^m_{x^d_{i_1}} \in a$$

Since $(b_0, \ldots, b_N) \in Z(a)$ we can replace “$y$’s by “$b$’s in the above polynomial and divide through by $b_{x^d_K}$ to obtain (using the fact that $m_0 + \ldots + m_n = d$)

$$b_j = \frac{b_{x^d_K}}{b_{x^d_K}} = \prod_{i=0}^{n} \frac{b_{x^d_{i_1}}}{b_{x^d_K}}$$

Since $j$ was arbitrary, we have for any $0 \leq j \leq n$

$$b_j = b_{x^d_K} M_j \left( \frac{b_{x^d_{i_1}}}{b_{x^d_K}}, \ldots, \frac{b_{x^d_{i_s}}}{b_{x^d_K}} \right)$$

$$= b_{x^d_K} M_j (a_0, \ldots, a_n)$$

Since $b_{x^d_K} \neq 0$ it follows that $\rho_d(a_0, \ldots, a_n) = (b_0, \ldots, b_N)$ in $\mathbb{P}^N$, as required. Hence $\text{Im}(\rho_d) = Z(a)$ and the map $\rho_d$ gives a bijection of $\mathbb{P}^n$ and $Z(a)$. 

2
In fact, the above argument defines a map \( \psi : Z(a) \to \mathbb{P}^n \) in which 
\[ \psi(b_0, \ldots, b_N) = (b_{[x^{d-1}]}, \ldots, b_{[x^{d-1}]}) \]
where \( 0 \leq K \leq n \) is such that \( b_{[x^K]} \neq 0 \). Notice that the definition is actually
independent of \( K \), since \( \rho_d \) is injective and \( K \) was chosen arbitrarily in the
proof that \( Im(\rho_d) = Z(a) \). By construction \( \rho_d \psi = 1 \) and it is easily seen that
\( \psi \rho_d = 1 \). We claim that \( \rho_d \) defines a morphism of varieties \( \mathbb{P}^n \to Z(a) \)
and that \( \psi : Z(a) \to \mathbb{P}^n \) is also a morphism.

Continuity of \( \rho_d \) is immediate, for if \( g(y_0, \ldots, y_N) \) is a homogenous polynomial
\[ \rho_d^{-1}(Z(g)) = Z(\theta(g)) \]
where \( \theta(g) \) is also homogenous. To show continuity of \( \psi \), consider the homomorphisms of \( k \)-algebras defined for \( 0 \leq K \leq n \) by
\[ \theta'_K : k[x_0, \ldots, x_n] \to k[y_0, \ldots, y_N] \]
\[ x_i \mapsto y_{[x^{d-1}]} \]
Given a homogenous polynomial \( f(x_0, \ldots, x_n) \) to show that \( \psi^{-1}Z(f) \) is closed
it suffices to show that \( U_{[x^K]} \cap \psi^{-1}Z(f) \) is closed in \( U_{[x^K]} \) for all \( 0 \leq K \leq n \),
so we have already shown that every \( (b_0, \ldots, b_N) \in Z(a) \) has some \( b_{[x^K]} \neq 0 \).
But \( U_{[x^K]} \cap \psi^{-1}Z(f) \) is the set
\[ \{(b_0, \ldots, b_N) \in Z(a) | b_{[x^K]} \neq 0 \text{ and } f(b_{[x^{d-1}]}, \ldots, b_{[x^{d-1}]}) = 0 \} \]
which is the intersection of the closed set \( Z(\theta'_K(f)) \) with \( U_{[x^K]} \). Hence \( \psi \) is also
continuous.

A standard argument using the morphism \( \theta \) shows that \( \rho_d \) is a morphism
of varieties, and by using the morphisms \( \theta'_K \) and considering the restrictions
\( \psi|_{U_{[x^K]}} \) it is also straightforward to check that \( \psi \) is a morphism of varieties.

Hence \( \rho_d \) gives rise to an isomorphism of varieties \( \mathbb{P}^n \cong Z(a) \). In particular,
the \( d \)-uple embedding of \( \mathbb{P}^n \) in \( \mathbb{P}^N \) is a projective variety of dimension \( n \).

**Example 1.** With \( n = 1 \) and \( d = 2 \) the relevant monomials are \( x_0^2, x_0x_1, x_1^2 \).
Depending on the way we order the monomials, we obtain 6 embeddings of \( \mathbb{P}^1 \)
in \( \mathbb{P}^2 \). For example, the following 2-uple embedding
\[ \rho_d(a, b) = (a^2, ab, b^2) \]
gives an isomorphism of \( \mathbb{P}^1 \) with the conic \( xz - y^2 \) in \( \mathbb{P}^2 \) (see our typed notes
on conics for a proof).

**Example 2.** Recall the twisted cubic curve in \( \mathbb{A}^3 \) is the set of all tuples \( (t, t^2, t^3) \)
with \( t \in k \), which is equal to the affine variety \( Z(y^2 - x, z^3 - x) \). If we identify
\( \mathbb{A}^3 \) with the open set \( U_0 \subseteq \mathbb{P}^1 \) and take the closure \( W \) of these points, we obtain
the *twisted cubic curve in \( \mathbb{P}^3 \).* In Exercise 2.9 we showed that
\[ I(W) = (wy - x^2, zw^2 - x^3, xz - y^2) \]
where the coordinates of $\mathbb{P}^3$ are $w, x, y, z$. The claim that $W$ is the image of the 3-Uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, given by

$$\rho_d(a, b) = (a^3, a^2b, ab^2, b^3)$$

By setting $a = 1$ and noting that $\rho_d(1, b) = (1, b, b^2, b^3)$ see that $Im(\rho_d)$ contains the twisted cubic curve of $\mathbb{A}^3$ and hence contains the closure $W$ of these points. To prove the reverse inclusion $Im(\rho_d) \subseteq W$ we note that $Im(\rho_d) = Z(a)$ and $W = Z(I(W))$, so it would suffice to show that $\theta$ maps the polynomials of $I(W)$ to zero. Here $\theta : k[w, x, y, z] \rightarrow k[t, u]$ is the map $w \mapsto t^3, x \mapsto t^2u, y \mapsto tu^2, z \mapsto u^3$ and by considering the generators of $I(W)$ it is clear that $I(W) \subseteq a$.

So the twisted cubic curve in $\mathbb{P}^3$ is of dimension 1 and is isomorphic to $\mathbb{P}^1$. 

4
The cone over a Projective Variety. Let \( Y \subseteq \mathbb{P}^n \) be a nonempty algebraic set, and let \( G : \mathbb{A}^{n+1} \to \mathbb{P}^n \) be the map which sends the point with affine coordinates \((a_0, \ldots, a_n)\) to the point with homogeneous coordinates \((a_0, \ldots, a_n)\). \( G \) is continuous, since for \( g \in \mathbb{R}[X_0, \ldots, X_n] \) homogeneous, \( G^{-1}(z(\mathbf{g})) = z(\mathbf{g}) \cap \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\} \).

The Zariski topology on \( \mathbb{P}^n \) is the quotient topology, which follows from:

**Proposition** Let \( k \) be an algebraically closed field, then an ideal \( \mathfrak{a} \subseteq k[x_0, \ldots, x_n] \) is homogeneous iff

\[ \mathfrak{a} = (x_0 - a_0, \ldots, x_n - a_n) \implies \mathfrak{a} = (x_0 - \lambda a_0, \ldots, x_n - \lambda a_n), \quad \lambda \neq 0 \]

for all \( a_0, \ldots, a_n \in k \).

**Proof.** We have already noted that this condition is necessary. We use induction on \( n \) to show that if a sum of

\[ m_1 + m_2 \in \mathfrak{a}, \quad \mathfrak{a} = (x_0 - a_0, \ldots, x_n - a_n) \]

(\( a_0 \) is improper, it is homogeneous trivially), so that \( m_1(a_0, \ldots, a_n) = -m_2(a_0, \ldots, a_n). \) Then for \( 0 \neq \lambda \in k \), we also have by hypothesis

\[ \lambda^d m_1(a_0, \ldots, a_n) = -\lambda^d m_2(a_0, \ldots, a_n) \]

where \( m_1 \in S_d \) and \( m_2 \in S_d \). Either \( m_1(a_0, \ldots, a_n) = 0 \), \( \lambda = 2 \) or one of them is nonzero - wlog suppose \( m_1(a_0, \ldots, a_n) \neq 0 \). Then

\[ \lambda^d - d = -m_2(a_0, \ldots, a_n) / m_1(a_0, \ldots, a_n) \]

Thus this holds for any \( \lambda \), we must have \( d_1 = d_2 \), which is a contradiction.

We define the affine cone over \( Y \) to be

\[ C(Y) = \mathbb{A}^n Y \cup \{(0, \ldots, 0)\} \]

(a) Let \( I(Y) \) be the homogeneous ideal of \( Y \). Then \( I(Y) \subseteq (x_0 - a_0, \ldots, x_n - a_n) \) and for \( (a_0, \ldots, a_n) \in Y \), \( I(Y) \subseteq (x_0 - a_0, \ldots, x_n - a_n) \).

(b) By \( \text{Q2.4} \) \( Y \) is irreducible if and only if \( I(Y) \) is prime, so iff \( Z(I(Y)) \) is irreducible.

(c) By \( \text{Q2.4} \) \( S(Y) = k[x_0, \ldots, x_n] / I(Y) \), so by \( \text{Q2.6} \) \( \dim C(Y) = \dim k[x_0, \ldots, x_n] / I(Y) = \dim S(Y) = \dim Y + 1 \).

**Q2.12** The d-Uple Embedding. For given \( n, d \geq 0 \), let \( M_0, M_1, \ldots, M_n \) be all the monomials of degree \( d \) in the \( n + 1 \) variables \( x_0, \ldots, x_n \), where \( N = \binom{n + d}{n} - 1 \). We define a mapping (see the note at the end of this sequence) which is used to define the monomial \( M_j \),

\[ \rho_d : \mathbb{P}^n \to \mathbb{P}^N \]

by sending the point \( P = (a_0, \ldots, a_n) \) to the point \( \rho_d(P) = (M_0(a), \ldots, M_N(a)) \) obtained by substituting the \( a_i \) in the monomials \( M_j \). This is called the d-uple embedding of \( \mathbb{P}^n \) in \( \mathbb{P}^N \). For example, if \( n = 1, d = 2 \) then \( N = 2 \) (the three monomials are \( x_0^2, x_1, x_0 x_1 \), and the image \( Y \) of the 2-uple embedding of \( \mathbb{P}^1 \) in \( \mathbb{P}^3 \) is a conic

\[ (a_0, a_1) \mapsto (a_0^2, a_1, a_0 a_1) \]

since the \( M_j \) are monomials, \( \rho_d \) is always well defined, since \( \rho_d((\lambda a_0, \ldots, \lambda a_n)) = (M_0(\lambda a_0, \ldots, \lambda a_n), \ldots, M_N(\lambda a_0, \ldots, \lambda a_n)) \) is a monomial of degree \( d \), \( \rho_d \) is always bijective if \( \rho_d(a_0, \ldots, a_n) = (b_0, \ldots, b_n) \) then there is \( \lambda \in k \) s.t.

\[ M_j(a_0, \ldots, a_n) = \lambda M_j(b_0, \ldots, b_n), \quad 0 \leq j \leq N \]
in particular, let $0 \leq i \leq n$ be s.t. $a_i \neq 0$ and hence $a_i \neq 0$, implies $a_i = \{a_i \}^{(d-i)} b_i \neq 0$. Relabel $X_i = a_i \{a_i \}^{(d-i)} b_i$ (since $a_i \neq 0$, hence $X_i \neq 0$). Then for any $0 \leq j \leq n$,

$$X_j b_i = a_i \{a_i \}^{(d-i)} b_i \{a_i \}^{(d-i)} b_j = a_i \{a_i \}^{(d-i)} a_j$$

$$= a_j$$

$$\{ X_i \}^{(d-i)} x_j \text{ is a monomial degree}$$

hence $(a_0, \dots, a_n) = (b_0, \dots, b_n)$.

(a) Let $\mathcal{O} : k[y_0, \dots, y_n] \rightarrow k[x_0, \dots, x_n]$ be $y_0 \mapsto M_0$, and $a = \text{Ker} \mathcal{O}$. Then $a$ is prime, and if

$$f = \sum m_e m_e \in S$$

is the unique expansion of $f$ as a sum of homogeneous polynomials, then $f \in \mathcal{O}$ implies

$$0 = \mathcal{O}(f) = \sum e \mathcal{O}(m_e)$$

for any monomial of order $e \in k[y_0, \dots, y_n]$. Hence the $\mathcal{O}(m_e)$ are all of degree $d$, and so $\mathcal{O}(m_e) = 0$, each $e$. Hence $M_e \in R$, and $a$ is homogeneous. Hence $Z(a)$ is a projective variety in $\mathbb{P}^n$.

(b) Let $g(y_0, \dots, y_n) \in \mathcal{O}$. If $(a_0, \dots, a_n) \in \mathbb{P}^n$, then $p(a_0, \dots, a_n) = (M_0(a_0, \dots, a_n), \dots, M_n(a_0, \dots, a_n))$, and

$$g(p(a_0, \dots, a_n)) = g(M_0(a), \dots, M_n(a))$$

$$= \mathcal{O}(g)(a_0, \dots, a_n) = 0 \text{ since } a = \text{Ker} \mathcal{O}$$

hence $\text{Im} \mathcal{O} \subseteq Z(a)$. To prove that $Z(a) = \text{Im} \mathcal{O}$, note that since $a = \text{Ker} \mathcal{O}$, and hence $k[y_0, \dots, y_n]/a$ is isomorphic to a subring of $k[x_0, \dots, x_n]$, it is a radical ideal. Hence $\text{Im} \mathcal{O} \subseteq Z(a)$ implies $a \subseteq I(\text{Im} \mathcal{O})$. We prove the converse inclusion (i.e. this that $Z(a) = \text{Im} \mathcal{O}$) by showing that if $f \in I(\text{Im} \mathcal{O})$, then $\mathcal{O}(f) = 0$. But this is obvious, since

$$\mathcal{O}(f)(a_0, \dots, a_n) = f(M_0(a_0, \dots, a_n), \dots, M_n(a_0, \dots, a_n))$$

$$= f(p(a_0, \dots, a_n)) = 0$$

and hence $\mathcal{O}(f) = 0$.

NOTE: Previous paragraph is irrelevant - don't read it. Suppose $(b_0, \dots, b_n) \in Z(a)$. For a monomial $M$ of degree $d$, let $g(M)$ denote the index it is assigned to in $M = g(M)$, then some $b_i$ is nonzero. The key is to notice that if we raise the corresponding monomial to the $d$th power, it can be written as the product of $x_j b_i$ for $0 \leq j \leq n$. Say

$$M_i^d = \prod_{j=0}^n x_j b_i$$

then this means that $y_j^d = \prod_{i=0}^n y_i b_i \in \mathcal{O}$, and so $b_i^d = \prod_{j=0}^n b_j(x_i)$. Hence since $b_i \neq 0$, some $b_j(x_i) \neq 0$. Relabel if necessary to arrange $j = 0$, so $b_j(x_i) \neq 0$. If indeed we find $(a_0, \dots, a_n) \in \mathbb{P}^n$, s.t. $(b_0, \dots, b_n) = p(a_0, \dots, a_n)$, then we would have $b_j(x_i) = a_0^d$, and $b_j(x_i) = a_j$, for $1 \leq i \leq n$, hence we would have

$$a_i = a_0 b_j(x_i) = a_0 \frac{b_j(x_i^d - x_i)}{a_0^d} b_j(x_i^d)$$

since we're working in projective coordinates, we can arrange for $a_0 = 1$. Hence, we claim, setting $a_0 = 1$, $a_i = \frac{b_j(x_i^d - x_i)}{b_j(x_i^d)}$.

Then we need to find $\lambda \in k$ s.t.

$$b_i = \lambda M_i(l, a_0, \dots, a_n) \text{ s.t. } i \leq N$$
To this end, let $D \subseteq \mathbb{N}$ and suppose

$$M_i = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n}$$

then $x_0 \cdots x_n = c$. Hence, the polynomial

$$y \mapsto \prod_{j=1}^{m_j} \frac{y_j}{g(x_0^{d-1} x_j)} \in \mathbb{A}$$

since by assumption $(b_0, \ldots, b_n) \in Z(\mathbb{N})$, we have

$$b_j(x_0^{d-1}) = b_j(x_0^{d-1} x_j) \cdots b_j(x_0^{d-1} x_n)$$

Equating both sides by $b_j(x_0^{d-1})$, gives

$$\frac{b_i}{b_0(x_0^{d-1})} = \left\{ \frac{b_j(x_0^{d-1} x_j)}{b_j(x_0^{d-1})} \right\} \cdots \left\{ \frac{b_j(x_0^{d-1} x_n)}{b_j(x_0^{d-1})} \right\}$$

since $m_0 + \cdots + m_n = m$. Hence, you = $b_j(x_0^{d-1})$, and we're done!

(c) We already know that $\rho_1$ is a bijection. We first show that it is also continuous. Since $Z(\mathbb{N})$ is covered by the open subsets $Z(\mathbb{N}) \cup U_i$, $i = 0, \ldots, n$, it suffices to see that

$$\rho_1^{-1}(Z(\mathbb{N}) \cup U_i) = \{ (a_0, \ldots, a_n) \mid (a_0, a_1, \ldots, a_n) \in Z(\mathbb{N}) \}$$

$$= \{ (a_0, \ldots, a_n) \mid a_i = 0 \}$$

which is open. Hence $\rho_1$ is continuous. We define the inverse $\hat{\rho} : \mathbb{N} \to \mathbb{N}$ by $\rho_1$ in that $(b_0, \ldots, b_n)$ in $Z(\mathbb{N})$ if selected some $j$, $0 \leq j \leq n$ s.t. $b_j(x_j) \neq 0$. Recall $g(x_j)$ is an integer $0 \leq j \leq n$ telling us which index in the $N$-tuple is assigned to the monomial $x_j^i$. Then $\rho_1$ is mapped $(b_0, \ldots, b_n)$ to

$$\left( \frac{b_j(x_j^{d-1} x_0)}{b_j(x_0^{d-1})}, \ldots, \frac{b_j(x_j^{d-1} x_n)}{b_j(x_0^{d-1})} \right)$$

To see that this is continuous, we take for $0 \leq i \leq n$,

$$\hat{\rho}^{-1}(U_i) = \{ (b_0, \ldots, b_n) \mid \frac{b_j(x_j^{d-1} x_0)}{b_j(x_0^{d-1})} = 0 \}$$

$$= Z(\mathbb{N}) \cup U_j(x_j^{d-1} x_0)$$

so $\hat{\rho}$ is also continuous, and thus $\rho_1$ is a homeomorphism.

(d) Here we want $\rho_1 : \mathbb{N} \to \mathbb{N}$ so $n = 2$ and $N = 3$. The variables $x_0, x_1$, and all monomials of degree $d = 2$ with the following order:

$$0 \quad 1 \quad 2 \quad 3$$

$$x_0 \quad x_1 \quad x_0 x_1 \quad x_1^2 \quad x_1^3$$

Hence, $\text{Imp}_d = Z(\mathbb{N})$. $\rho_1 : \mathbb{N} \to \mathbb{N}$ given by $\rho_1(a_0, a_1) = (a_3, a_2 a_1, a_0 a_1^2, a_1^3)$ and $\pi = \ker \hat{\rho}$,

$\hat{\rho} : \mathbb{N}[y_0, y_1, y_2, y_3] \to \mathbb{N}[x_0, x_1]$ given by

$$y_0 \mapsto x_0^3$$
$$y_1 \mapsto x_0^2 x_1$$
$$y_2 \mapsto a_0 x_0 x_1^2$$
$$y_3 \mapsto x_1^3$$
By putting \( a_0 = 1 \) we see that \( \mathcal{A}(1, k) = (1, b_0, b_0, b_0) \), so since the twisted cubic curve in \( \mathbb{P}^3 \) (henceforth denoted \( C \)) is the closure of all these points \( C \subseteq \mathbb{P}(a) \). To show \( \mathbb{P}(a) \subseteq C \) we simply show that \( \mathcal{I}(C) \subseteq \mathcal{I}_a \) that is, that the generators

\[
y_1 y_2 y_3 - y_2^3, \quad y_2 y_0 - y_1^2, \quad y_1 y_3 - y_2^2
\]

for \( \mathcal{I}(C) \) (obtained in Q2.8) are contained to \( 0 \) by \( \mathcal{I}_a \). But this obvious by inspection—hence \( C = \mathbb{P}(a) \), as required. \( \Box \)

**Note** All the above remains true no matter how we order the monomials.

**Q2.13** Let \( Y \) be the image of the 2-sphere embedding \( \rho_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5 \) given by

\[
\rho_2((a_0, a_1, a_2)) = ([a_0^2, a_0 a_1, a_0 a_2, a_1^2, a_1 a_2, a_2])
\]

Then \( k[y_0, y_1, y_2, y_3, y_4, y_5] \rightarrow k[x_0, x_1, x_2] 

\[
y_0 \mapsto x_0^2, \quad y_1 \mapsto x_2, \quad y_2 \mapsto x_2, \quad y_3 \mapsto x_1 x_2, \quad y_4 \mapsto x_1 x_2, \quad y_5 \mapsto x_1 x_2
\]

This is the **Veronese surface**. If \( Z \subseteq Y \) is a closed curve (note \( \rho_2 \) is a homeomorphism, we can unambiguously talk about \( \rho_2 \) as a closed curve), where a curve is a variety of dimension 1, then since \( \dim \mathbb{P}^5 = 5 \),

by Q2.8 \( Z = Z(f) \) for an irreducible homogeneous \( f \in k[x_0, x_1, x_2] \). Since \( Z(f) = Z(f^2) \), and \( f^2 \) will be a polynomial in the monomials \( x_0^2, x_1^2, x_2^2, x_0 x_1, x_0 x_2, x_1 x_2 \) (all the polynomial \( h \in k[y_0, y_1, y_2] \)), we have \( Z \subseteq Z(h) \) in \( \mathbb{P}^5 \), since

\[
h(\rho_2((a_0, a_1, a_2))) = h(a_0^2, a_0 a_1, a_2, a_0 a_2, a_1^2, a_1 a_2) = f^2(a_0, a_1, a_2) = 0 \quad (1)
\]

If we write \( Z(h) = \cup V_i Y_i \) as a finite union of irreducible algebraic sets, then since \( Z \) is closed irreducible, it must be contained in some \( Y_i \) (else we can adjoin to the list and contradict uniqueness in Q2.5). Since we can assume the decompostion of \( Z(h) \) is given by the irreducible factors of \( h \), this yields that there is a hypersurface \( V \subseteq \mathbb{P}^5 \) with \( Z \subseteq V \). Since \( Z(h) \not\subseteq V \) since \( Z(h) \not\subseteq Y \) by Eq (1), we have shown that \( Z = Y \cap V \), as required. \( \Box \)

**Q2.14** The Segre embedding Let \( Y : \mathbb{P}^r \times \mathbb{P}^s \rightarrow \mathbb{P}^n \) be the map

\[
(a_0, \ldots, a_r) \times (b_0, \ldots, b_s) \mapsto (a_0 b_0, a_0 b_1, \ldots, a_0 b_s, a_1 b_0, \ldots, a_1 b_s, \ldots, a_r b_0, \ldots, a_r b_s)
\]

in lexicographic order. \( n = rs + rs + rs \) \( Y \) is well-defined since if \( (\lambda a_0, \ldots, \lambda a_r) = (a_0, \ldots, a_r) \) and \( (\mu b_0, \ldots, \mu b_s) = (b_0, \ldots, b_s) \) then the RHS of (1) becomes \( \lambda \mu a_0 b_0, \lambda \mu a_1 b_0, \ldots, \lambda \mu a_r b_0, \ldots, \lambda \mu a_0 b_s, \ldots, \lambda \mu a_r b_s \), which is \( \mu(\lambda a_0, \ldots, \lambda a_r) \times (\lambda b_0, \ldots, \lambda b_s) \). If it is injective since

\[
(a_0 b_0, a_0 b_1, \ldots, a_0 b_s, \ldots, a_r b_0, \ldots, a_r b_s) = (a_0 b_0, \ldots, a_0 b_s, \ldots, a_r b_0, \ldots, a_r b_s)
\]

Then \( a_0 b_0 = \lambda \mu a_0 b_0 \), \( a_i b_j = \lambda \mu a_i b_j \) so either \( a_0 = 0 \) or \( b_j = \lambda \mu a_0 b_0 \), \( b_0 = \lambda \mu a_0 b_0 \), \( b_s = \lambda \mu a_0 b_s \), \( b_0 = (b_0, b_s) = (b_0, b_s) \). If \( a_0 = 0 \) keep going till some \( a_i \neq 0 \) and do the same thing. Some deal shows \( (a_0, \ldots, a_r) = (a_0, \ldots, a_r) \).

Let the homogenous coordinates of \( \mathbb{P}^n \) be \( z_j \), \( 0 \leq i < r \), \( 0 \leq j < s \), and define the map \( \Phi \) \( \mathbb{P}^n \rightarrow \mathbb{P}^n \)

\[
(0 : k[z_0, \ldots, z_r, y_0, \ldots, y_s]) \mapsto (z_0, \ldots, z_r) \mapsto (z_0, \ldots, z_r)
\]

and \( \mathcal{R} = \ker \Phi \), which is then prime homogenous. We claim that \( \mathcal{I}(Y) = Z(a) \). If \( f \in \mathcal{R} \) then

\[
f(Y((a_0, \ldots, a_r) \times (b_0, \ldots, b_s))) = f(a_0, a_0, b_0, \ldots, b_r) = 0
\]

\[
= \Phi(f)(a_0, \ldots, a_r, b_0, \ldots, b_r) = 0
\]
Here $\text{Im} \psi \subseteq Z(\sigma)$. For the converse inclusion, note that for $0 \leq y < r$ and $0 \leq i, d \leq 5$ we have $Z(y2d - 2i2y) \in \sigma$. Suppose that $a_{cd} = 0$. Then since

$$a_{ij} = \frac{1}{a_{cd}} a_{ij}$$

we put $a_i = a_{id}$ and $b_j = a_{jd}$ so that in $P^5$ we have $\psi((a_{i0} x \ldots a_{ij} y) x (b_{0j} \ldots b_{jd})) = (a_{i0} \ldots a_{ij})$ as required. Hence $\text{Im} \psi = Z(\sigma)$, so $\text{Im} \psi$ is a subvariety of $P^5$. (Note that $\text{Im} \psi$ is a projective variety.)

**Q 2.15** The Quadric Surface in $P^3$ consider the surface $Q$ (a surface is a variety of dimension 2) in $P^3$ defined by the equation $x^2 + z^2 = 0$, where we order the variables $w, x, y, z$.

(a) Consider the Segre embedding \( \psi : P^1 \times P^1 \longrightarrow P^3 \) which is defined by

$$\psi((a_{00} x, a_{01} x) \times (b_{00} y, b_{01} y)) = (a_{00} b_{00}, a_{00} b_{01}, a_{01} b_{00}, a_{01} b_{01})$$

By 2.24 the image of $\psi$ is a projective variety in $P^3$. Obviously $\text{Im} \psi \subseteq Z(z^2)$, so the product inclusion, suppose $(a_{00}, a_{01}, a_{10}, a_{11}) \in Z(z^2)$, then $a_{00} a_{10} = a_{01} a_{11}$ and by cases:

- If $a_{10} = 0$, then $a_{00} = 0$ put $(a_{00}, a_{10}) \times (1, 0)$.
- If $a_{11} = 0$, put $(a_{11}, a_{01}) \times (0, 1)$.

In any case the constructed pair map onto $(a_{00}, a_{01}, a_{10}, a_{11})$, so $\text{Im} \psi = Z(z^2)$. Since $Q = Z(x^2 + z^2)$ is the locus of a nonconstant, irreducible homogeneous polynomial, by Ex. 2.8 $\dim Q = 2$.

(b) For $t \in P^1$ let

$$L_t = \psi(t \times P^1)$$

$$H_t = \psi(P^1 \times t)$$

If $t = (a_{00}, a_{01})$ then we claim $L_t = Z(a_{00} x - a_{01} y) \cap Z(a_{10} x - a_{11} y)$. It is clear that $L_t \subseteq \text{Im} \psi$. For the reverse inclusion, if $(a_{00,01}, c_{00}, c_{01}) \in \text{Im} \psi$ belongs to the RHS, then either $a_{00} \neq 0$ or $a_{11} \neq 0$. Wlog $a_{00} \neq 0$ (case $a_{11} \neq 0$ is similar), then

$$a_{10} = a_{11} \cdot \frac{a_{10}}{a_{11}} = a_{11} \cdot \frac{a_{10}}{a_{11}} \cdot \frac{a_{00}}{a_{01}} = a_{11} \cdot \frac{a_{00}}{a_{01}}$$

and $(a_{00}, c_{00}, c_{01}, c_{01}) \in Q$.

In any case, $(a_{00}, c_{00}, c_{01}, c_{01}) \in Q$, say \( \psi((d_0, d_1) \times (b_{00} y, b_{01} y)) = (c_{00}, c_{01}, c_{01}) \). This implies $(d_0, d_1) = (c_{00}, c_{01}, c_{01}) = (c_{00}, c_{01}, c_{01})$. Similarly $M_t$ is the linear variety $Z(b_{00} y - a_{00} x) \cap Z(b_{01} y - a_{11} x)$. So both $M_t, L_t$ are linear schemes for $t \in P^1$. Both intersections are "minimal" in the sense used in Ex. 2.11, so our notes there show that

$$\dim L_t + \dim M_t = 1$$

If $t, u$ are distinct then the fact that $\psi$ is injective implies that $L_t \cap L_u = \emptyset$ and $M_t \cap M_u = \emptyset$. Clearly $L_t \cap M_t = \{ (x, x) \}$.

(c) The curve $Q \cap Z(x - y)$ is not one of these lines, and the topology on $Q$ is not the one induced by $\tau$; and the product topology on $P^1 \times P^1$, because the set $Q \cap Z(x - y) \subseteq Q$ corresponds to the diagonal in $P^1 \times P^1$ which is closed in the product topology $\tau$. (If it is Hausdorff.

But by Ex. 7.9 it is impossible.)
Q.16 (a) The intersection of two conics is not a variety. For example consider the following quadrics in $\mathbb{P}^3$

\[ Q_1 = Z(x^2 - yw) \]
\[ Q_2 = Z(xy - zw) \]

Both polynomials are irreducible, so $Q_1 Q_2$ are quadric surfaces in $\mathbb{P}^3$. Let $T$ be the twisted cubic curve in $\mathbb{P}^3$ – $T$ is a projective variety of dimension 1,

\[ I(T) = \{ zw^3 - x^3, yw - x^2, xz - y^3 \} \]

(see Q.9), and let $L$ be the line $Z(w) \cap Z(x)$ (under $w, x, y, z$) this is a minimal decomposition by 2.1! $L$ has dimension 1, so really it is a line. Note that $L$ is a projective variety since $\langle x, w \rangle = I(L)$ is prime. It is clear that $L \subseteq Q_1 \cap Q_2$, and $T \subseteq \mathbb{P}^3$ since $x^2 - yw \in I(T)$, suppose $(a_0, a_1, a_2, a_3) \in T$. If $a_0 = a_1 = 0$ then the point is in $Q_2$, since $L \subseteq Q_2$. Otherwise we assume $a_0 \neq 0$ (curve $a_0 \neq 0$ similar). Then we may assume $a_0 = 1$, so

\[ a_1 a_2 = a_0 a_1^2 = (x^3 = yw) \]
\[ = a_1^3 \]
\[ = a_3 a_2 = (x^3 = z y w) \]
\[ = a_4 a_0 = (a_0 = 1) \]

Hence $(a_0, a_1, a_2, a_3) \in Q_2$. Hence $L \cap T \subseteq Q_1 \cap Q_2$, we claim this is an equality. Suppose $(a_0, a_1, a_2, a_3) \in Q_1 \cap Q_2$ and say $a_0 \neq 0$ (similar). It is easy to check that $a_1 a_2 = a_1^3, a_3 a_2 = a_4 a_0 = a_2$. So

\[ Q_1 \cap Q_2 = L \cap T \]

This is a decomposition of $Q_1 \cap Q_2$ into its irreducible components. So $Q_1 \cap Q_2$ is not reducible, hence not a variety.

(b) Let $C$ be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$ and $L$ the line $y = 0$. Then clearly

\[ C \cap L = \{ (0, 0, 0) \} = \{ p \} \]

we know that $P = Z(x^2 - yz) \cap Z(y) = Z(y, x^2 - yz) = Z(0, 0, 0)$. But $I(P) \neq I((y, x^2 - yz))$ since $(x^2, y)$ is not radical ($x^2 \notin (x y)$ but $x \notin (x, y)$). So even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals.

Q.17 (Complete Intersections) A projective variety $Y$ of dimension $n$ in $\mathbb{P}^n$ is a (first) complete intersection if $I(Y)$ can be generated by $n$ elements. $Y$ is a set-theoretic complete intersection if $Y$ can be written as the intersection of $n$ hypersurfaces. (Hypersurfaces in the loose sense, $Z(f)$, where $f$ need not be prime.)

(a) Let $Y \subseteq \mathbb{P}^n$ be a projective variety, $Y = Z(x)$. If it can be generated by $n$ elements then by Knaf's PI theorem (Prop A.4.1) $\dim Y = n - 1$.

(b) If $Y$ is a first complete intersection with $Z(Y) = (f_1, \ldots, f_r)$ then $Y = Z(f_1) \cap \cdots \cap Z(f_r)$ so $Y$ is a set-theoretic complete intersection.

(c) Let $T$ be the twisted cubic curve in $\mathbb{P}^3$. By a note following Ex. 2.4

\[ I(T) = (x^2 - wy, xy - wz, y^2 - wz) \]

we claim that $Z$ is a set-theoretic complete intersection but not a strict complete intersection. Since $\dim Z = 1$ (Ex. 1.2) it is shown that the coordinate ring of the normal twisted cubic is 1, and the projective closure preserves dimension 1. It suffices to show that $Z$ is the intersection of $2$ hypersurfaces but $I(Y)$ cannot be generated by 2 elements.

We claim $Z = Z(x^2 - wy) \cap Z(y^2 + w^2 - 2xyz)$. This will follow from $I(T) = I(Z)$ where $I(T) = (x^2 - wy, y^2 + w^2 - 2xyz)$, which will follow from $I(Z)$ and $I(T)$ as $Z$ is the intersection of $2$ hypersurfaces but $I(Y)$ cannot be generated by 2 elements. But these inclusions follow from

\[ y^3 + w^2 z^2 - 2xyz = y(y^2 - xz) + w z^2 - xz y \]
\[ (xy - wz)^2 = w(y^3) + w^2 w^2 (y^2z^2 - 2xyz) + \frac{y}{y^3} \]
\[ y^2 - xz = y^3 + w^2 + 2y^2 z^2 + (y^2 w z) \]

Since $I(Z)$ contains no homogeneous elements of degree 0 or 1, if two elements generate $I(Z)$ they would be of order 2. Suppose $V(Z)$ is the 2-dimensional vector space of all homogeneous order 2 polynomials in $x, y, z, x^2 - wy, xy - wz, y^2 - xz$ are L2, and this space cannot be spanned by two elements.

\[ \text{I.e. I-module gen by } x^2 - wy, xy - wz, y^2 - xz \]
NOTE (Counting Monomials)

We claim that

**LEMMA** The number of monomials of weight $d$ in $x_1, \ldots, x_n$ is

$$\binom{n+d-1}{n-1} = \binom{n+d-1}{d} \quad (n \geq 1, d \geq 1)$$

**PROOF** Given a monomial $x_1^2 x_2 x_3^3$, for example $(n=3, d=6)$ we can write

$$x_1 x_1 x_2 x_2 x_3 x_3$$

which means $d$ stars will go. This is $\binom{n+d-1}{d}$. $\square$

**LEMMA** The number of monomials of weight $\leq d$ in $x_1, \ldots, x_n$ is

$$\binom{n+d}{d}$$

**PROOF** So the weight is $0 \leq d$. We define a bijection between monomials of weight $\leq d$ in $x_0, x_1, \ldots, x_n$ and monomials of weight $d$ in $x_0, x_1, \ldots, x_n$ by

$$x_1^{d-1} \lambda x_n \rightarrow x_0^{d-1} x_1 \lambda x_n$$

Hence the Lemma follows immediately. $\square$
NOTE Testing for Irreducibility

**Theorem** (Eisenstein's Criterion) Let $R$ be a UFD with quotient field $F$. Let $f = a_0 + a_1 x + \ldots + a_n x^n$ \((a_n \neq 0)\) be in $R[x]$ and suppose that $p \in R$ is a prime such that

\[
\begin{align*}
p & \text{ does not divide } a_n \\
p & \text{ divides } a_i \text{ for } 0 \leq i \leq n-1 \\
p^2 & \text{ does not divide } a_0
\end{align*}
\]

Then $f(x)$ is irreducible in $F[x]$, hence in $R[x]$.

**Proof.** See Atkin. (p 47.)

If $k$ is a field, $k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_{n-1}][x_n]$ is a UFD, so put $R = k[x_1, \ldots, x_{n-1}]$ in the above.

**Example.**

1. \(y^2 + x^2 + x \) is irreducible in $k[x,y]$. Consider $(x^2 + x) + y^2 \in k[x][y]$ and $p = x \in k[x]$

   This works for any \(y^m + a_1 x + \ldots + a_n x^n\) with $a_1 \neq 0$ and $n \geq 1$.

2. In fact it works for \(y^m + f_1(x)y^{m-1} + \ldots + f_n(x)y + f_n(x)\) provided $x \mid f_1(x)$ and $f_2(x)$ has a nonzero $x^2$ term. So $y$ for example

\[
y^{120} - x^3 y^5 + x y^5 + 3x^4 y - 17x^2 + x
\]

is irreducible in $k[x,y]$.\(\)

4. So provided \(f(x,y)\) has

\[
\begin{align*}
& \text{(i) no constant term} \\
& \text{(ii) let } g(x) \text{ be the collection of } x \text{ only terms. Then } x \mid g(x) \text{ but } x^2 \mid g(x) \\
& \text{(or the same with } x \rightarrow y)\)
\]

Then $f$ is irreducible.

Of course, this doesn't work for \(y^2 - x^2 (x+1)\). So here's the tough way. Suppose \(f(x,y) \in k[x,y]\)

is nonzero, nonconstant and involves no monomials of order 4 or more -- so $f = f_3 + f_4 + f_5 + f_6$.

To be explicit, say

\[
\begin{align*}
f_3 & = f_x x + f_y y \\
f_4 & = f_{xx} x^2 + f_{xy} x y \\
f_5 & = f_{xx} x^3 + f_{xy} x^2 y + f_{yy} x y^2 + f_{xx} y^2 x^2
\end{align*}
\]

Now suppose $f = GH$. Then $G = G_0 + G_1$ and $H = H_0 + H_1 + H_2$ \((0 = f_G = G_0 H_0 \text{ and } G_2 = 0)\)

\[
\begin{align*}
f_3 & = G_0 H_0 \\
f_4 & = f_3 x + f_4 y = G_0 H_1 + G_1 H_0 \\
& = G_0 (H_2 x + H_2 y) + H_1 G_1 (G_0 H_2 x + G_0 H_2 y) \\
& = \{G_0 H_2 x + G_0 H_2 y\} x + \{G_0 H_2 y + G_0 H_2 y\} y \\
f_5 & = f_5 x^2 + f_5 y^2 + f_{xy} x y \\
& = G_0 H_2 + G_1 H_1 \\
& = G_0 (H_2 x^2 + H_2 y^2 + H_2 x y + H_2 y^2) \\
& + (G_2 x + G_2 y) (H_1 x + H_1 y)
\end{align*}
\]
\[ f_x x^2 + f_y y^2 + f_{xy} xy + f_{xx} y^2 + f_{yy} x^2 + f_{xx} y^2 + f_{xy} x y + f_{yy} y^2 = \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]

This leads to:

1. \( f_0 = G_0 H_0 \)
2. \( f_x = G_0 H_x + H_0 G_x \)
3. \( f_y = G_0 H_y + H_0 G_y \)
4. \( f_{x2} = G_0 H_{x2} + G_x H_{x2} \)
5. \( f_{xy} = G_0 H_{xy} + H_0 G_{xy} + H_y G_x + H_x G_y \)
6. \( f_{y2} = G_x H_{y2} + G_y H_{y2} \)
7. \( f_{x2} = G_x H_{x2} + G_y H_{x2} \)
8. \( f_{xy} = G_y H_{xy} \)
9. \( f_{x2} = G_x H_{xy} + G_y H_{y2} \)
10. \( f_{y2} = G_x H_{xy} + G_y H_{y2} \)

**EXAMPLES**

1. \( y^2 - x^2(x+1) = y^2 - x^2 - x^2 \). The equations become

   \[ 0 = G_0 H_0 \quad \text{(we put } G_0 = 0) \]
   \[ 0 = H_0 G_x \quad \text{(since } f \neq 0, G \neq 0 \text{ so } G_1 \neq 0 \text{ if } H_0 = 0 \text{ is impossible}) \]
   \[ 0 = H_0 G_y \]
   \[ -1 = G_x H_n \]
   \[ 0 = H_y G_n + H_n G_y \]
   \[ 1 = G_y H_y \]
   \[ -1 = G_x H_{y2} \]
   \[ 0 = G_y H_{y2} \]
   \[ 0 = G_x H_{xy} + G_y H_{xy} \]
   \[ 0 = G_x H_{y2} + G_y H_{y2} \]

**Case 2** \( G_y = 0 \). Then \( G = G_1 = G_x x \) with \( G_x \neq 0 \). But \( x \neq y \), so this is impossible.

2. **Case 2** \( G_y \neq 0 \) and \( H_{xy} = 0 \). Then \( G_y H_y = 0 \Rightarrow H_y = 0 \), and \( G_x H_{x2} = 0 \Rightarrow H_{x2} = 0 \) so \( H_z = 0 \) which is impossible since \( x \neq z \) and \( f \) cannot come from somewhere (i.e., \( f = G_x H_{x2} = 0 \)).