Sheaves of Sets: part two

Recall that in the last talk, we explored presheaves on a category $\mathcal{C}$. We introduced the Yoneda Lemma, and investigated the Yoneda embedding. In the course of this study, we used sieves, which correspond to sub-objects of representable presheaves, to define sheaves on a topological space purely in the language of presheaves and covering sieves. This lead us to the conclusion that we may be able to define sheaves on an arbitrary category, not just on the category of open subsets of a topological space. In this talk we will define the appropriate notion of covering sieves for objects of an arbitrary category, finally realising the idea of defining sheaves on a category.

To this end, we introduce the concept of a Grothendieck topology on a category, generalising the definition of a topology on a set. We will end the talk by discovering how Grothendieck topologies arise naturally in algebraic geometry. This is discovery will lead us define the Zariski site.

1 Grothendieck topologies

In this talk, all categories will be small. We also assume that all rings are commutative with unit.

Definition 1.1. Let $\mathcal{C}$ be a category and $C \in \text{ob}(\mathcal{C})$. Given a sieve S on C and a morphism $h : D \to C$, the **pullback** of S along h is the sieve

$$h^*(S) := \{g \mid \text{cod}(g) = D,\ h \circ g \in S\}$$

on D.

Definition 1.2. A **Grothendieck topology**, on a small category $\mathcal{C}$ is a function τ which assigns to each object $C \in \text{ob}(\mathcal{C})$ a set $\tau(C)$ of sieves on C satisfying the following axioms:

1. The *maximal sieve* $t_C = \{f \mid \text{cod}(f) = C\} \in \tau(C)$.
2. (stability) If $S \in \tau(C)$, and if $h : D \to C$ is a morphism into C, then $h^*(S) \in \tau(D)$.
3. (transitivity) If R is a sieve on C such that, for every morphism $h : D \to C$ into C, $h^*(R) \in \tau(D)$, then $R \in \tau(C)$.

A **site** is a pair $(\mathcal{C}, \tau)$ consisting of a small category $\mathcal{C}$ and a Grothendieck topology τ on $\mathcal{C}$. If $C \in \text{ob}(\mathcal{C})$ then we call a sieve $S \in \tau(C)$ a **covering sieve** of C.

Definition 1.3. Let τ be a Grothendieck topology on a small category $\mathcal{C}$ with pullbacks. A **basis** for τ is a function $\mathcal{B}$ which assigns to each $C \in \text{ob}(\mathcal{C})$ a set $\mathcal{B}(C)$ of sets of morphisms into C satisfying the following properties:

1. If $f : D \to C$ is an isomorphism, then $\{f : D \to C\} \in \mathcal{B}(C)$.
2. If $\{f_i : C_i \to C\}_{i\in I} \in \mathcal{B}(C)$, and $h : D \to C$ is a morphism into C, then $\{\pi_2 : C_i \times_C D \to D\}_{i\in I} \in \mathcal{B}(D)$. Here π_2 is the projection onto the second factor.
3. If $\{f_i : C_i \to C\}_{i\in I} \in \mathcal{B}(C)$ and, for each $i\imath I$, $\{g_{ij} : D_{ij} \to C_i\}_{j\in J_i} \in \mathcal{B}(C_i)$, then $\{f_i \circ g_{ij} : D_{ij} \to C\}_{i,j} \in \mathcal{B}(C)$.

The following example illustrates that Grothendieck topologies on categories are a generalisation of topologies on sets.

Example 1.1. Let X be a topological space, and let $O(X)$ be the category of open subsets of X. Then the topology on X induces a Grothendieck topology τ on $O(X)$. For each open subset $U \subseteq X$, define

$$\tau(U) = \{\text{sieves } S \text{ on } U \mid \bigcup_{f\in S} \text{dom}(f) = U\}.$$

Elements of $\tau(U)$ are covering sieves, in the notation of the first talk, which motivates the terminology for Grothendieck topologies.

We will show that τ is indeed a Grothendieck topology. First, observe that the maximal sieve is obviously in $\tau(U)$ for each U. Next, recall that for any open subsets $U, V \subseteq X$, a morphism $h : V \to U$ in $O(X)$ is the inclusion $V \subseteq U$. By identifying a sieve S on U with a a cover of U which is closed under taking open subsets, we can identify $h^*(S)$ with $\{V \cap U_i\}$. This certainly covers V, to the stability axiom holds. The transitivity axiom follows similarly.

Remark 1.1. Given a category $\mathcal{C}$ with pullbacks and Grothendieck topology τ on $\mathcal{C}$, there is a maximal basis $\mathcal{B}$ which generates τ, given on objects $C \in \text{ob}(\mathcal{C})$ by

$$R \in \mathcal{B}(C) \Leftrightarrow (R) \in \tau(C)$$

where

$$(R) := \{f \circ g \mid f \in R \text{ and } \text{dom}(f) = \text{cod}(g)\}.$$

is the sieve generated by R.

Grothendieck topologies allow us to give a definition of sheaves on an arbitrary category $\mathcal{C}$.

Definition 1.4. Let $(\mathcal{C}, \tau)$ be a site, and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if, for every $C \in \text{ob}(\mathcal{C})$ and every covering sieve $S \in \tau(C)$, the monomorphism $i_S : S \to h_U$ induces an isomorphism

$$(i_S)^* : \text{Hom}(h_U, \mathcal{F}) \to \text{Hom}(S, \mathcal{F}).$$

We write $\text{Sh}_\tau(\mathcal{C})$ for the full subcategory of sheaves in $\mathbf{Set}^{\mathcal{C}^{op}}$.

2 Algebraic geometry

Algebraic geometry is the study of spaces carved out by system of polynomial equations. More precisely, let $\Bbbk$ be a field, and let $F = (f_1, \ldots, f_m) \in \Bbbk[x_1, \ldots, x_n]$ be a system of polynomial equations with coefficients in $\Bbbk$. We want to study the set

$$Z(F) = \{\mathbf{p} = (p_1, \ldots, p_n) \in \Bbbk^n \mid f_1(\mathbf{p}) = \cdots = f_m(\mathbf{p}) = 0\} \subseteq \Bbbk^n.$$

called the **zero locus** of F.

To do geometry, we need to put a topology on the zero locus.

Definition 2.1. The **Zariski topology** on $\Bbbk^n$ is defined by declaring that the zero loci of systems of polynomial equations are closed. Equivalently, the Zariski topology has a basis of open sets of the form

$$D(g) := \{\mathbf{p} \in \Bbbk \mid g(\mathbf{p}) \neq 0\}, \qquad g \in \Bbbk[x_1, \ldots, x_n].$$

We give the zero loci the subspace topology. Open covers are given by sets $\{D(g_1), \ldots, D(g_k)\}$ where $(g_1, \ldots, g_k) = A$.

Lemma 2.1. *Let $F = (f_1, \ldots, f_m)$ be a system of polynomial equations. There is a bijection*

$$Z(F) \longleftrightarrow \mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n]/F, \Bbbk).$$

Proof. First, recall that $\mathbb{Z}$ is the initial object in the category of commutative rings, so there is a unique morphism $\mathbb{Z} \to \Bbbk$. It follows that a morphism $h : \mathbb{Z}[x_1, \ldots, x_n]/F \to \Bbbk$ is completely determined by the images of the generators x_i. Therefore, h is determined by a choice of points $p_1, \ldots, p_m \in \Bbbk$ satisfying the same relations as the generators x_i. This is precisely the data of a point $\mathbf{p} \in Z(F)$.

Conversely a point $\mathbf{p} \in Z(F)$ determines a morphism $h : \mathbb{Z}[x_1, \ldots, x_n]/F \to \Bbbk$ by $h(x_i) = p_i$. □

This lemma raises a question: how do we realise the Zariski topology on $\mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n]/F, \Bbbk)$? In light of the lemma, a point in $Z(F)$ can be described as a morphism $\mathbb{Z}[x_1, \ldots, x_n] \to \Bbbk$ which factors as

$$\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]/F \to \Bbbk.$$

Now observe that the open sets $D(g)$ can be realised as the image of the projection

$$Z(ug - 1) \subseteq \Bbbk^{n+1} \to \Bbbk^n$$

onto the first n coordinates. Using the lemma we can therefore understand the inclusion $D(g) \subseteq \Bbbk^n$ via the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n, u]/(ug - 1), \Bbbk) & \longrightarrow & \mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n], \Bbbk) \\
\updownarrow & & \updownarrow \\
D(g) & \longrightarrow & \Bbbk^n
\end{array}$$

Here, the top arrow in induces from $\mathbb{Z}[x_1, \ldots, x_n] \hookrightarrow \mathbb{Z}[x_1, \ldots, x_n, u] \to \mathbb{Z}[x_1, \ldots, x_n, u]/(ug - 1)$. Now to understand the Zariski topology on $\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]/F$, we need to interpret the intersection $D(g) \cap Z(F)$ as above. Since intersections are given by fiber products in the ambient space, we need to understand the pullback of the following diagram:

$$\begin{array}{ccc}
 & & \mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n, u]/(ug - 1), \Bbbk) \\
 & & \downarrow \\
\mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n]/F, \Bbbk) & \longrightarrow & \mathrm{Hom}_{\mathrm{Rng}}(\mathbb{Z}[x_1, \ldots, x_n, \Bbbk)
\end{array}$$

This corresponds precisely to the following pushout in *Rng*:

$$\begin{array}{ccc}
\mathbb{Z}[x_1, \dots, x_n] & \longrightarrow & \mathbb{Z}[x_1, \dots, x_n, u]/(ug-1) \\
\big\downarrow & & \\
\mathbb{Z}[x_1, \dots, x_n]/F & &
\end{array}$$

But pushouts in Rng are precisely given by tensor products. This means that a point in the intersection $D(g) \cap Z(F)$ corresponds to a morphism $\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{k}$ which factors as

$$\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]/F \otimes_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z}[x_1, \dots, x_n, u]/(ug-1) \to \mathbb{k}$$

Writing $A = \mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]/F$, the above morphism takers the form

$$A \to A[u]/(ug-1) \cong A_g$$

where A_g is the localisation of A at g. For non-experts, A_g can be though of as the ring A with inverses to all powers of g attached.

Upshot: The inclusion $D(g) \cap Z(F)$ corresponds to the morphism

$$A \otimes_{\mathbb{Z}[x_1,\dots,x_n]} A_g.$$

Definition 2.2. A finitely presented $\mathbb{Z}$-algebra is a ring of the form $A = \mathbb{Z}[x_1, \dots, x_n]/F$ for some system of polynomials F. We denote the category of finitely presented $\mathbb{Z}$-algebras by $(\mathbb{Z}\text{-}alg)_{fp}$.

Form our investigation so far, it follows that the study of zero loci of systems of polynomials takes place in $(\mathbb{Z} - alg)_{fp}$. The basis of the Zariski topology on $Z(F)$ is captured by the family of morphisms

$$\{\{A \to A_{g_i}\}_{g_1,\dots,g_k \in A^*} \mid (g_1, \dots, g_k) = A\}.$$

which we call *cocovers*. In other words, suitable families of morphisms in $(\mathbb{Z}\text{-}alg)_{fp}$ encode open covers in the Zariski topology.

Theorem 2.1. *The assignment* $\mathcal{B}(A) = \{\{A \to A[u]/(ug_i - 1)\}_{g_1,\dots,g_k \in A^*} \mid (g_1, \dots, g_k) = A\}$ *is a basis for a Grothendieck topology on* $(\mathbb{Z}\text{-}alg)^{op}_{fp}$.

Proof. We will leave the verification of the first two properties as an easy exercise. To show that the third property is satisfied, we need to show that if we have a cocover

$$\{A \to A_{g_i}\}_{i=1,\dots,n}, \qquad (g_1, \dots, g_n) = A$$

and, for each $i \in I$, cocovers

$$\{A_{g_i} \to (A_{g_i})_{f_{i,j}}\}_{j=1,\dots,m_i}, \qquad (f_{i,1}, \dots, f_{i,m_i}) = A_{g_i}$$

then the composites

$$\{A \to A_{g_i} \to (A_{g_i})_{f_{i,j}}\}$$

are cocovers of A for each i, j.

First we observe that each $f_{i,j} \in A_{g_i}$ can be written as

$$f_{i,j} = \frac{\widetilde{f_{i,j}}}{g_i^{k_{i,j}}}$$

where $\widetilde{f_{i,j}} \in A$ and $k_{i,j} \in \mathbb{Z}_{\geq 0}$. It follows that there is an isomorphism

$$(A_{g_i})_{f_{i,j}} \cong A_{g_i \widetilde{f_{i,j}}}.$$

Thus, we are reduced to showing that for each i, j,

$$\{A \to A_{g_i} \to A_{g_i \widetilde{f_{i,j}}}\}$$

is a cocover. We therefore need to show that $(\{g_i \widetilde{f_{i,j}}\}_{i,j}) = A$, or equivalently, that this ideal contains the unit $1 \in A$.

Now by assumption there exist $\gamma_{i,j} \in A_{g_i}$ such that

$$1 = \sum_{j=1}^{m_i} \gamma_{i,j} f_{i,j} = \sum_{j=1}^{m_i} \frac{\widetilde{\gamma_{i,j}}}{g_i^{l_{i,j}}} \frac{\widetilde{f_{i,j}}}{g_i^{k_{i,j}}}$$

where $\widetilde{\gamma_{i,j}} \in A$ and $l_{i,j} \in \mathbb{Z}_{\geq 0}$. Choose $k_i > \max_j \{k_{i,j} + l_{i,j}\}$, so that multiplication with $g_i^{k_i}$ cancels out the denominators on the right hand side. This implies that

$$g_i^{k_i} \in (g_i \widetilde{f_{i,1}}, \ldots, g_i \widetilde{f_{i,m_i}}) \subseteq A$$

Since $(g_1, \ldots, g_n) = A$, there exist $\alpha_i \in A$ such that

$$1 = \sum_{i=1}^{n} \alpha_i g_i,$$

so by choosing $K > n \cdot \max_i k_i$ we have

$$1 = \left(\sum_{i=1}^{n} \alpha_i g_i \right)^K \in (g_1^{k_1}, \ldots, g_n^{k_n}) \subseteq (\{g_i \widetilde{f_{i,j}}\}_{i,j})$$

in A. This finishes the proof.

□

Definition 2.3. The site $((\mathbb{Z}\text{-}alg)_{fp}, \tau)$, where τ is the Grothendieck topology generated by $\mathcal{B}$, is called the **Zariski site**, and the associated category of sheaves is called the **Zariski topos**.