Sheaves of Sets: part one

1 Presheaves

We begin by recalling the classical definition of a presheaf on a topological space:

Definition 1.1. Let X be a topological space, and let $\mathcal{O}(X)$ be the poset of open subsets of X. A presheaf $\mathcal{F}$ on $\mathcal{O}$ then consists of the following data:

1. For each open set $U \subseteq X$, a set $\mathcal{F}(U)$,
2. For each inclusion of open subsets $U \subseteq V$, a function

$$\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U).$$

This is subject to the requirements

1. if $U \subseteq X$ then $\mathrm{res}_{U,U} = \mathrm{id}$,
2. if $U \subseteq V \subseteq W$ then $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$.

Example 1.1. Let X be a smooth manifold. For each open subset of X, the correspondence

$$U \mapsto \{f : U \to \mathbb{R} \mid f \text{ smooth}\}$$

defines a presheaf on X. In this case the restriction maps are given by classical restriction of functions.

Remark 1.1. Motivated by the above example, we will refer to element of $s \in \mathcal{F}(V)$ as sections of $\mathcal{F}$ over V, and we will write $\mathrm{res}_{V,U}(s) = s|_U$.

We can restate the definition of presheaves in a much cleaner and compact way, namely as functors:

Definition 1.2. Let $\mathcal{C}$ be a category. A presheaf on $\mathcal{C}$ is a set valued functor $\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Set}$.

Remark 1.2. We can recover the classical definition of a presheaf on a topological space X by taking $\mathcal{C} = O(X)$, the poset of open subsets of X. Using this abstract definition, we define a morphism of preseaves $\mathcal{F}$ and $\mathcal{G}$ as a natural transformation $\mathcal{F} \to \mathcal{G}$. The resulting category of presheaves on $\mathcal{C}$ is denoted $\mathbf{Set}^{\mathcal{C}^{op}}$

Theorem 1.1. *On any category $\mathcal{C}$, the category of presheaves $\boldsymbol{Set}^{\mathcal{C}^{op}}$ is a topos.*

2 The Yoneda Lemma

Let $\mathcal{C}$ be a category and let $C \in \text{ob}(\mathcal{C})$. We can define a contravariant functor

$$h_C : \mathcal{C} \to \mathbf{Set},$$

which acts on objects by $h_C(D) = \text{Hom}_{\mathcal{C}}(D, C)$ and on morphisms $f : D_1 \to D_2$ by $h_C(f)(g) = g \circ f$. Now suppose we have a morphism $f : C \to C'$ in $\mathcal{C}$. We can then define a natural transformation

$$h_f : h_{C_1} \to h_{C_2},$$

given by

$$(h_f)_D : h_{C_1}(D) \to h_{C_2}(D), \qquad g \mapsto f \circ g.$$

Upshot: There is a functor $h_{(-)} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$, called the Yoneda embedding.

Theorem 2.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, and $F : \mathcal{C}^{op} \to$ **Set** a presheaf. Then for each object C of $\mathcal{C}$, the map*

$$\Psi_{C,\mathcal{F}} : \text{Nat}(h_C, \mathcal{F}) \to \mathcal{F}(C), \qquad \eta \mapsto \eta_C(\text{id}_C)$$

is a bijection, and is natural in both variables.

Corollary 2.1. *The functor $h_{(-)} : \mathcal{C} \to$ **Set**$^{\mathcal{C}^{op}}$ is fully faithful.*

Proof. Let $C, D \in \text{ob}(\mathcal{C})$, and let $\mathcal{F} = h_D$. Then the Yoneda Lemma gives a bijection

$$\Psi_{C,D} : \text{Nat}(h_C, h_D) \to h_D(C) = \text{Hom}_{\mathcal{C}}(C, D).$$

□

3 Sieves

Now that we have a natural way to embed $\mathcal{C}$ into the larger category $\mathbf{Set}^{\mathcal{C}^{op}}$, is is natural to ask what new information we can obtain. A natural first question in this direction is the following: given an object C of $\mathcal{C}$, what are the subobjects of h_C in $\mathbf{Set}^{\mathcal{C}^{op}}$? To answer this question, we need to understand the monomorphisms in $\mathbf{Set}^{\mathcal{C}^{op}}$.

Lemma 3.1. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ in* **Set**$^{\mathcal{C}^{op}}$ *is a monomorphism if and only if $\phi_C : \mathcal{F}(C) \to \mathcal{G}(C)$ is a monomorphism in* **Set** *for every object C of $\mathcal{C}$.*

Proof. The morphism ϕ is a monomorphism precisely if

$$\phi \circ \psi_1 = \phi \circ \psi_2 \Rightarrow \psi_1 = \psi_2.$$

for any $\psi_1, \psi_2 : \mathcal{H} \to \mathcal{F}$, where $\mathcal{H}$ is any object of $\mathbf{Set}^{\mathcal{C}^{op}}$. Unpacking the definitions, we see that if ϕ is a monomorphism then for every $C \in \text{ob}(\mathcal{C})$,

$$\phi(C) \circ \psi_1(C) = \phi(C) \circ \psi_2(C) \Rightarrow \psi_1(C) = \psi_2(C).$$

This is precisely means that $\phi(C)$ is a monomorphism in **Set** for every $C \in \text{ob}(\mathcal{C})$.

Conversely, suppose that $\phi(C) : \mathcal{F}(C) \to \mathcal{G}(C)$ is a monomorphism for every object C of $\mathcal{C}$. Then given any set S and any morphisms of$f, g : S \to \mathcal{F}(C)$,

$$\phi(C) \circ f = \phi(C) \circ g \Rightarrow f = g.$$

In particular, given a sheaf $\mathcal{H}$ and a pair of morphisms $\psi_1, \psi_2 : \mathcal{H} \to \mathcal{F}$,

$$\phi(C) \circ \psi_1(C) = \phi(C) \circ \psi_2(C) \Rightarrow \psi_1(C) = \psi_2(C)$$

for every object C of $\mathcal{C}$. Thus

$$\phi \circ \psi_1 = \phi \circ \psi_2 \Rightarrow \psi_1 = \psi_2,$$

so $\phi : \mathcal{F} \to \mathcal{G}$ is a monomorphism in $\mathbf{Set}^{\mathcal{C}^{op}}$.

□

Corollary 3.1. *A subobject of h_C in $\mathbf{Set}^{\mathcal{C}^{op}}$ can always be represented by a subfunctor of h_C.*

To understand the subfunctors of h_C, it is helpful to introduce the auxiliary notion of sieves:

Definition 3.1. Let C be an object of a category $\mathcal{C}$. A sieve on C is a set S of morphisms of $\mathcal{C}$ into C which is closed under pre-composition, i.e. if $f \in S$ and $f \circ h$ is defined, then $f \circ h \in S$.

Proposition 3.1. *Let $C \in \mathrm{ob}(\mathcal{C})$. There is a bijection*

$$\{\textit{sieves on } C\} \leftrightarrow \{\textit{subfunctors of } h_C\}.$$

Proof. Let S be a sieve on C, and let $D \in \mathrm{ob}(\mathcal{C})$. Define

$$\mathcal{F}(D) = S \cap \mathrm{Hom}_{\mathcal{C}}(D, C).$$

This is clearly a subfunctor of h_C. Conversely, given a subfunctor $\mathcal{F} \subseteq h_C$, consider the set

$$S = \{f \mid \exists D \in \mathrm{ob}(\mathcal{C}),\ f \in \mathcal{F}(D)\}.$$

Here we identify $\mathcal{F}(D)$ with its image in $h_C(D)$. If $f : D \to C \in S$ and $h : D' \to D$ is a morphism such that $f \circ h$ is defined, then $f \circ h \mathrm{res}_{D,D'}(f) = \mathcal{F}(D')$, so $f \circ g \in S$, making S a sieve on C.

□

Example 3.1. Let X be a topological space, and let $\mathcal{C} = \mathcal{O}(X)$. For any open subset $U \subseteq X$, a sieve on U is collection of inclusions of open subsets $S = \{V \subseteq U\}$ such that if $V \in S$ and $W \subseteq V$ is open, then the inclusion $W \subseteq U \in S$.

Definition 3.2. A sieve on an open subset $U \subseteq X$ is a covering sieve if

$$\bigcup_{f \in S} \mathrm{domain}(f) = U.$$

Remark 3.1. From the examples above, it is clear that sieves give a categorical generalisation of the idea of an open cover of a topological space. The slogan to keep in mind is the following: sieves encode the way in which objects of a category hang together.

4 Sheaves

By definition, presheaves of sets on a category $\mathcal{C}$ encode collections of sets parametrised by $\mathcal{C}$. Informally, this means that a presheaf can be thought of as a generalised object of $\mathcal{C}$. This is not sufficient if one wishes to have a notion of a generalised element which is locally modelled on $\mathcal{C}$, since very little information about the sections of a presheaf over C can be obtained from know about sections over other objects. General presheaves simply do not have enough structure.

This is a problem in geometry, for instance, because we would like to be able to "glue" from local data: we would like our presheaves to "know" about the topology. As we now know, sieves give a generalisation of the notion of open covers, and hence can be though of as keeping track of local information on categories.

Idea: We need to specialise general presheaves to those which are sensitive to sieves. We will call such presheaves sheaves.

Definition 4.1. Let X be a topological space. A presheaf of set $\mathcal{F}$ on $O(X)$ is called a sheaf if, given any open cover $\{U_i\}_{i\in I}$, it satisfies the following two properties:

1. If $r, s \in \mathcal{F}(X)$ are a pair of sections such that $s|_{U_i} = r|_{U_i}$ for all $i \in I$, then $s = r$.
2. Given a family of sections s_i, one for each $\mathcal{F}(U_i)$, such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}, \qquad \forall i, j \in I,$$

there exists a section $s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$.

Remark 4.1. It is worth unpacking this definition. The second condition says that sections on X can be glued from sections on the open cover, provided they are compatible with the cover, and the first condition ensures that this gluing is unique.

Example 4.1. Let X be a topological space and let $\mathcal{F}$ be the presheaf from example one:

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{smooth}\}.$$

Then $\mathcal{F}$ is a prototypical example of a sheaf. Certainly if two functions agree locally, then they agree globally. Moreover, if one has locally defined functions f_i which agree on overlaps of the cover, then one can define a global function f by

$$f(x) = f_i(x), \qquad x \in U_i.$$

Remark 4.2. We can actually rewrite the sheaf conditions purely in categorical terms. A presheaf $\mathcal{F}$ is a sheaf if and only if for each open cover $\{U_i\}_{i\in I}$ of an open subset $U \subseteq X$, the following diagram is an equaliser in **Set**:

$$\mathcal{F}(U) \xrightarrow{\quad d \quad} \prod_{i\in I} \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

Here $d(s) = (s|_{U_i})_{i\in I}$, $p((s_i)) = (s_i|_{U_i\cap U_j})_{i,j\in I}$, and $p((s_j)) = (s_j|_{U_i\cap U_j})_{i,j\in I}$. To see this, first note that, by definition, any family $(s_i)_{i\in I}$ such that $p((s_i)) = q((s_i))$, must agree on overlaps, so the sheaf condition ensures there is a section $s \in \mathcal{F}(U)$ such that $d(s) = (s_i)_{i\in I}$. This same property also implies universality, with uniqueness a consequence of the first axiom.

The characterisation of the sheaf condition in terms of an equaliser diagrams provided a convenient categorification of the sheaf axioms, but in order to generalise sheaves to arbitrary categories we need a way to rephrase the sheaf condition in terms of sieves on objects. This is provided by the next theorem.

Theorem 4.1. *Let X be a topological space, and $\mathcal{F}$ a presheaf on $O(C)$. Then $\mathcal{F}$ is a sheaf if for every open subset $U \subseteq X$, and every covering sieve S of U, the inclusion $i_S : S \to h_U$ induces an isomorphism*

$$(i_S)^* : \mathrm{Nat}(h_U, \mathcal{F}) \to \mathrm{Nat}(S, \mathcal{F}).$$

Proof. Identify S with the covering $\{U_i\}_{i \in I}$, and consider the equaliser diagram

$$E \xrightarrow{\quad e \quad} \prod_{i \in I} \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

where $E = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i)\} \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\ \forall i, j \in I\}$. For each $i \in I$, replace U_i with all open sets $V \subseteq U_i$, and write $x_V = x_i|_V$. Because the x_i in E agree on overlaps, x_V is independent of the choice of index with $V \subseteq U_i$. The equaliser E can then be described as

$$E = \{(x_V)_{V \in S} \mid x_V|_{V'} = x_{V'}\ V' \subseteq V\}.$$

Using Proposition 3.1, regard S as a functor $O(X)^{op} \to \mathbf{Set}$,

$$S(V) = \begin{cases} 1, & V \in S, \\ 0, & \text{else.} \end{cases}$$

Each section $x_V \in \mathcal{F}(V)$ can then be identified with a map $S(V) \to \mathcal{F}(V)$, to E can be reinterpreted as $\mathrm{Nat}(S, \mathcal{F})$. We can now augment the equaliser diagram to

$$\begin{array}{ccccc} \mathrm{Nat}(S, \mathcal{F}) & \xrightarrow{\quad e' \quad} & \prod_{i \in I} \mathcal{F}(U_i) & \underset{q}{\overset{p}{\rightrightarrows}} & \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \\ {\scriptstyle (i_S)^*} \uparrow & & \uparrow {\scriptstyle d} & & \\ \mathrm{Nat}(h_U, \mathcal{F}) & \xrightarrow[\Phi_{U,\mathcal{F}}]{\qquad} & \mathcal{F}(U) & & \end{array}$$

Here $e(\eta) = (\eta_{U_i}(1))_{i \in I}$. We claim that the square commutes. Indeed, proceeding from the lower left corner, going clockwise we have

$$\eta \mapsto (i_S)^*(\eta) = \eta \circ i_S \mapsto ((\eta_{U_i} \circ (i_S)_{U_i})(1))_{i \in I} = (\eta_{U_i}(1))_{i \in I}$$

and in the other direction we have

$$\eta \mapsto \eta_U(1) \mapsto (\eta_{U_i}(1))_{i \in I}.$$

This shows that d always factors through the equaliser, and when $(i_S)^*$ is an isomorphism this implies that $\mathcal{F}$ is a sheaf.

□