Recall the following definitions.

Definition 1.1. A **category** $\mathscr{C}$ consists of a collection $\operatorname{ob}\mathscr{C}$ of **objects**, and for each pair of objects A, B a collection $\operatorname{Hom}_{\mathscr{C}}(A, B)$ of **morphisms** $f : A \to B$, such that

- for each object $A \in \operatorname{ob}\mathscr{C}$, there exists a morphism $\mathrm{id}_A : A \to A$ called the **identity morphism** for A,
- for each pair of morphisms $f : A \to B$ and $g : B \to C$, there exists a morphism $g \circ f : A \to C$ called the **composition** of f and g,
- for each morphism $f : A \to B$, we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$, and
- for each triple of morphisms $f : A \to B, g : B \to C, h : C \to D$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 1.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $\operatorname{ob}\mathscr{C} \to \operatorname{ob}\mathscr{D}$, written $A \mapsto FA$, and for each pair of objects $A, B \in \operatorname{ob}\mathscr{C}$ a function $\operatorname{Hom}_{\mathscr{C}}(A, B) \to \operatorname{Hom}_{\mathscr{D}}(FA, FB)$, written $f \mapsto Ff$ such that:

- for each $A \in \operatorname{ob}\mathscr{C}$ we have $F\mathrm{id}_A = \mathrm{id}_{FA}$, and
- for each $f : A \to B$ and each $g : B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = Fg \circ Ff$.

A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. In other words, F *reverses morphisms.*

Definition 1.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** $\alpha : F \Rightarrow G$ consists of, for each object $A \in \operatorname{ob}\mathscr{C}$, a morphism $\alpha_A : FA \to GA$, such that for any $f : A \to B$ in $\mathscr{C}$, the following diagram commutes.

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB \end{array}$$

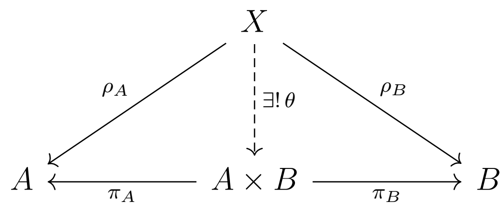
1.1 Limits and colimits

Recall that the cartesian product of two sets A, B is defined as the set of ordered pairs of an element from A and an element from B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

However, the definition of a category does not assume any kind of internal structure of its objects, and so one cannot define products in category theory by referring to the 'elements' of objects. Instead, one must frame the definition only in terms of objects and morphisms. This can be achieved as follows. Let $\mathscr{C}$ be a category.

Definition 1.4. Let $A, B \in \mathrm{ob}\,\mathscr{C}$. The **product** of A and B, if it exists, is the data of an object $A \times B$ together with two morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ satisfying the following *universal property*: If we are given $X \in \mathrm{ob}\,\mathscr{C}$ and $\rho_A : X \to A, \rho_B : X \to B$, then there exists a unique map θ such that the following diagram commutes.

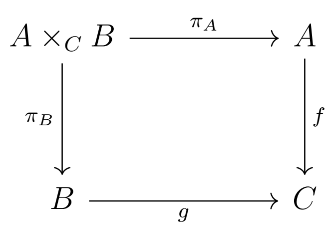


If products exist, they will be unique up to unique isomorphism. In **Set**, products exist between any pair of objects; the morphisms π_A and π_B are the projection maps.

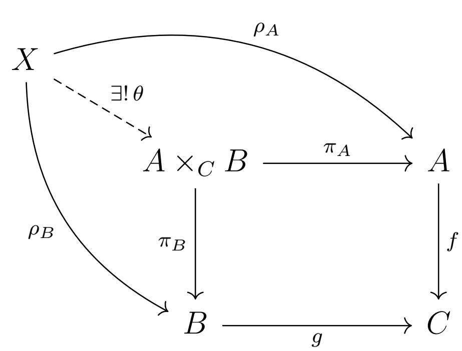
Through the language of universal properties, many other constructions from mathematics can be realised in category theory:

Definition 1.5. Let $f : A \to C, g : B \to C$ be morphisms. The **pullback** of f and g, if it exists, is the data of an object $A \times_C B$ together with two morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$ such that

(1) The following diagram commutes:



(2) We have the following *universal property*: If we are given $X \in \mathrm{ob}\,\mathscr{C}$ and $\rho_A : X \to A, \rho_B : X \to B$ such that $g \circ \rho_B = f \circ \rho_A$, then there exists a unique map θ such that the following diagram commutes.



This is also sometimes called the **fiber product**. In **Set**, the pullback of f and g always exists; it is the set

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

Of particular interest is the special case where $A \subseteq C$, with f the inclusion. We then have

$$A \times_C B \cong g^{-1}(A).$$

Definition 1.6. The **terminal object** of $\mathscr{C}$, if it exists, is an object $\mathbb{1}$ such that if $A \in \operatorname{ob}\mathscr{C}$ then there exists a unique map $A \to \mathbb{1}$.

In **Set**, this is the singleton set $\{*\}$.

The common feature of each of these constructions is that:

1. We begin with some small diagram.
2. We ask for the existence of some object C, together with maps from C into each of the each of the nodes in the diagram.
3. We assert that C should satisfy a universal property: any other object C' with maps into the diagram should factor uniquely through C.

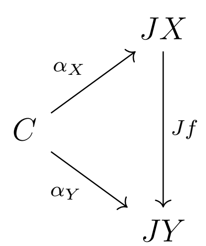
We would like to generalise this. To do this, we need to formalise the notion of a diagram in a category.

Definition 1.7. Let $\mathscr{C}$ be a category, and $\mathscr{J}$ a small 'index category'. A **diagram** in $\mathscr{C}$ of shape $\mathscr{J}$ is a functor $J : \mathscr{J} \to \mathscr{C}$.

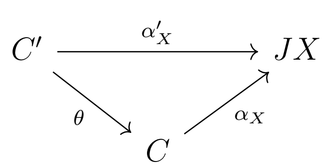
The condition that J is a functor ensures that any commuting triangles in the category $\mathscr{J}$ are sent to commuting triangles in $\mathscr{C}$.

Definition 1.8. Let J be a diagram in $\mathscr{C}$ of shape $\mathscr{J}$. A **cone** over J consists of

- an object $C \in \mathscr{C}$,
- for each object $X \in \mathscr{J}$ a morphism $\alpha_X : C \to JX$ in $\mathscr{C}$, such that for each morphism $f : X \to Y$ in $\mathscr{J}$, the following triangle commutes.



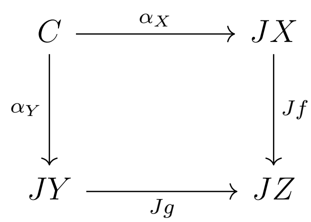
Definition 1.9. We say that a cone $(C, \{\alpha_X\}_{X \in \operatorname{ob}\mathscr{J}})$ over J is the **limit** of J if any other cone $(C', \{\alpha'_X\}_{X \in \operatorname{ob}\mathscr{J}})$ over J factors uniquely through C. Spelt out concretely, this means that there exists a unique morphism $\theta : C' \to C$ such that the following diagram commutes for each $X \in \operatorname{ob}\mathscr{J}$.



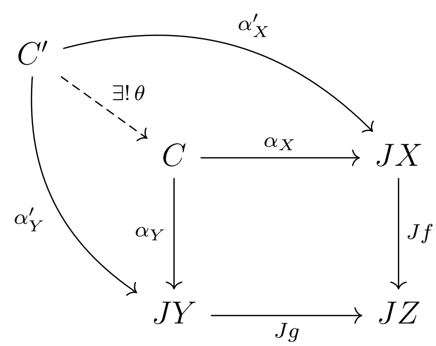
If J has a limit, it is unique up to unique isomorphism.

Colimits also deserve mention. The notion of a **cocone** is the same as that of a cone, except that the morphisms α_X are reversed; that is, $\alpha_X : JX \to C$. A cocone C is called a **colimit** if any other cocone over J factors uniquely through C.

Example 1.10. Let $\mathscr{J}$ be the category with three objects X, Y, Z, and two non-identity morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$, let $\mathscr{C}$ be a category, and let $J : \mathscr{J} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ of shape $\mathscr{J}$. A cone over J is the data of an object $C \in \mathscr{C}$ and two[1] morphisms $\alpha_X : C \to JX, \alpha_Y : C \to JY$ such that the following diagram commutes:



The cone C is a limit if for any other cone C', there exists a unique morphism $\theta : C' \to C$ such that the following diagram commutes.



In other words, the limit of diagram J is precisely the pullback of Jf and Jg.

In much the same way:

- The binary product is the limit of the diagram with two objects X, Y, and no morphisms except identities.
- The terminal object is the limit of the empty diagram.
- The equaliser is the limit of the diagram with two objects X, Y and two morphisms $X \to Y$.

The definition of a topos requires that the category in question has all finite limits; that is, limits of finite diagrams. The following lemma gives us sufficient conditions for this.

[1]Note that specifying the morphism $\alpha_Z : C \to JZ$ is superfluous; it is fixed by the choices of α_X and α_Y.

Lemma 1.11. For a category $\mathscr{C}$, the following are equivalent:

(1) $\mathscr{C}$ has all finite limits.
(2) $\mathscr{C}$ has all equalisers, binary products and a terminal object.
(3) $\mathscr{C}$ has pullbacks and a terminal object.

1.2 Exponentials

Let $\mathscr{C}$ be a category which has all binary products, and let $X \in \operatorname{ob}\mathscr{C}$.

Definition 1.12. An **exponential for** X is a family of objects $\{Z^X\}_{Z \in \operatorname{ob}\mathscr{C}}$ such that there exists a family of bijections (for $Y, Z \in \operatorname{ob}\mathscr{C}$)

$$\operatorname{Hom}_{\mathscr{C}}(Y \times X, Z) \cong \operatorname{Hom}_{\mathscr{C}}(Y, Z^X), \tag{$\dagger$}$$

which is natural in Y and Z.

Remark 1.13. In category theory, a family of bijections as above is called an **adjunction** between the functors $- \times X : \mathscr{C} \to \mathscr{C}$ and $-^X : \mathscr{C} \to \mathscr{C}$.

The motivating example is that seen in **Set**, where Z^X is defined as the set of all functions $f : X \to Z$. The bijection ($\dagger$) then sends a function $f : Y \times X \to Z$ in two variables to the function $\overline{f} : Y \to Z^X$ given by $\overline{f}(y) = f(y, -)$. Note that in fact Z^X *unique* set (up to bijection) for which there exists a family of bijections ($\dagger$), for we have:

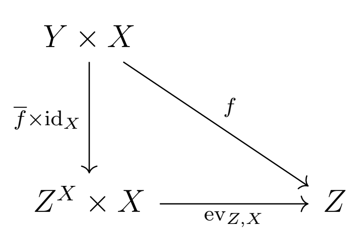
$$Z^X \cong \operatorname{Hom}_{\mathbf{Set}}(\mathbb{1}, Z^X) \cong \operatorname{Hom}_{\mathbf{Set}}(\mathbb{1} \times X, Z) \cong \operatorname{Hom}_{\mathbf{Set}}(X, Z)$$

One can equivalently define exponentials via the *counit* of the above adjunction. This is a family of morphisms

$$\operatorname{ev}_{Z,X} : Z^X \times X \to Z$$

called the **evaluation map**, obtained by applying the inverse of the bijection ($\dagger$) with $Y = Z^X$ to the identity map $\operatorname{id}_{Z^X} : Z^X \to Z^X$.

Lemma 1.14. Let $X \in \operatorname{ob}\mathscr{C}$. A family of objects $\{Z^X\}_{Z \in \operatorname{ob}\mathscr{C}}$ is an exponential for X if and only if for each function $f : Y \times X \to Z$, there exists a unique map $\overline{f} : Y \to Z^X$ such that the following diagram commutes



Proof. (Sketch): Let $\Theta_{X,Y,Z}$ denote the bijection in ($\dagger$). Given $f : Y \times X \to Z$, define $\overline{f} = \Theta_{X,Y,Z}(f)$. Then $\operatorname{ev}_{Z,X} \circ (\overline{f} \times \operatorname{id}_X) = \Theta^{-1}_{X,Z^X,Z}(\operatorname{id}_{Z^X}) \circ (\Theta_{X,Y,Z}(f) \times \operatorname{id}_X) = \Theta^{-1}_{X,Y,Z}(\operatorname{id}_{Z^X} \circ \Theta_{X,Y,Z}(f)) = f$ by naturality of ($\dagger$). The converse is similar; define $\Theta_{X,Y,Z}(f) = \overline{f}$. This is natural in Y and Z by uniqueness of $\overline{f}$. $\square$

Definition 1.15. We say that $\mathscr{C}$ is **cartesian closed** if it has all binary products, a terminal object, and all exponentials.

Unsurprisingly, the category **Set** is cartesian closed. For a simple non-example, consider the category **Ab** of abelian groups. Given two abelian groups G and H, it is easily verified that $\mathrm{Hom}_{\mathbf{Ab}}(G, H)$ can be given the structure of an abelian group, by defining $(\varphi + \psi)(g) = \varphi(g) + \psi(g)$. However, **Ab** does *not* possess an exponential for any nontrivial group G. To see this, recall that the terminal object in **Ab** is the zero group. If H^G existed, we would therefore have

$$\mathrm{Hom}_{\mathbf{Ab}}(G, H) \cong \mathrm{Hom}_{\mathbf{Ab}}(0 \times G, H) \cong \mathrm{Hom}_{\mathbf{Ab}}(0, H^G) = \{0\}$$

for any group H, a contradiction.

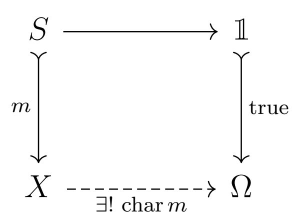
1.3 Subobject classifiers

Let $\mathscr{C}$ be a category with all finite limits.

Definition 1.16. A morphism $f : X \to Y$ in $\mathscr{C}$ is a **monomorphism** if for all morphisms $g, h : W \to X$ such that $fg = fh$, we have $g = h$.

Definition 1.17. A **subobject** of $X \in \mathrm{ob}\,\mathscr{C}$ is an equivalence class of monomorphisms $m : S \rightarrowtail X$, where $m : S \rightarrowtail X$ and $m' : S' \rightarrowtail X$ are equivalent if there exists an isomorphism $f : S \to S'$ such that $m' \circ f = m$. We write $\mathrm{Sub}_{\mathscr{C}}(X)$ to mean the set of all subobjects of X.

Definition 1.18. A **subobject classifier** for $\mathscr{C}$ consists of an object Ω and a monomorphism $\mathrm{true} : \mathbb{1} \rightarrowtail \Omega$, such that for each monomorphism $m : S \rightarrowtail X$, there exists a unique morphism $\mathrm{char}\, m : X \to \Omega$ such that the following square is a pullback square.



If a subobject classifier exists, it is unique up to unique isomorphism.

A subobject classifier can also be characterised in terms of subobjects of an object $X \in \mathrm{ob}\,\mathscr{C}$.

Proposition 1.19. Let $\mathscr{C}$ have all finite limits and small Hom-sets. Then $\mathscr{C}$ has a subobject classifier if and only if there exists an object $\Omega \in \mathrm{ob}\,\mathscr{C}$ and a family of isomorphisms

$$\mathrm{Sub}_{\mathscr{C}}(X) \cong \mathrm{Hom}_{\mathscr{C}}(X, \Omega),$$

which is natural in X.

We now work through some examples of subobject classifiers.

Example 1.20. Let $\mathscr{C} = \mathbf{Set}$, in which the terminal object is the one element set $\mathbb{1} = \{*\}$. The subobject classifier is

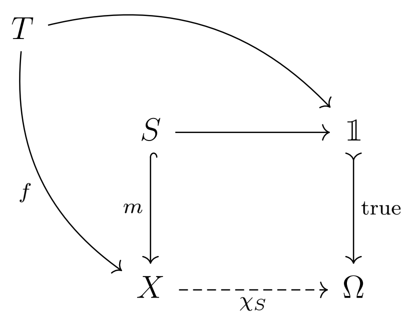
$$\Omega = \{0, 1\} \qquad \text{true}(*) = 1.$$

Suppose that we are given a monomorphism $m : S \rightarrowtail X$. For simplicity, assume without loss of generality that $S \subseteq X$ and m is the inclusion. Define $\operatorname{char} m$ as the characteristic function χ_S of S in X:

$$\chi_S(x) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}.$$

Claim 1: The diagram of Definition 1.18 is a pullback square.

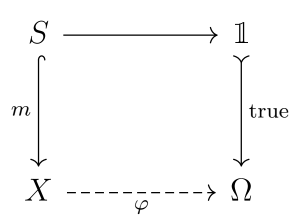
Proof. It is easy to see that the diagram commutes; we need to show that it is universal. Suppose we are given a set T and a function $f : T \to X$ such that the following diagram commutes:



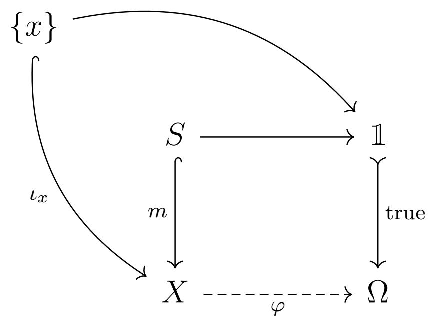
For any $t \in T$, we therefore have $\chi_S \circ f(t) = \text{true}(*) = 1$, and hence the image of f is a subset of S. It follows that T factors (uniquely) through S, via the map $\theta : T \to S$ given by $\theta(t) = f(t)$. □

Claim 2: The map $\operatorname{char} m = \chi_S$ is the *unique* map which makes the diagram of Definition 1.18 a pullback square.

Proof. Suppose we are given a map $\varphi : X \to \Omega$ such that the following diagram is a pullback square:



The fact that the square commutes tells us that for any $s \in S \subseteq X$, we must have $\varphi(s) = \text{true}(*) = 1$. Now suppose that $x \in X$. If one has $\varphi(x) = 1$, then it follows that the diagram

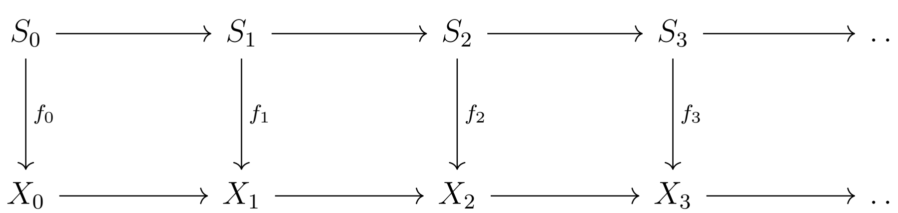


commutes, where $\iota_x : \{x\} \to X$ is the inclusion. By the universal property of the pullback it follows that ι_x factors through m, but clearly this is only possible if $x \in S$, since m is the inclusion. It therefore follows that $\varphi(x) = 0$ for each $x \in X \setminus S$, and hence $\varphi = \chi_S$. □

Example 1.21. Let $\mathbf{Set}^{\mathbb{N}}$ denote the category whose objects are infinite sequences of functions[2]

$$X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} X_3 \xrightarrow{\sigma_3} \dots$$

A morphism $f : S \to X$ in $\mathbf{Set}^{\mathbb{N}}$ is a sequence of morphisms $f_i : S_i \to X_i$ in $\mathbf{Set}$ such that the following diagram commutes.



In particular, such a morphism $f : S \to X$ gives a subobject of Y if each f_i in the above diagram is a monomorphism. In particular, if these monomorphisms are inclusions then commutativity of the diagram says that $\sigma S_i \subseteq S_{i+1}$.

It is easily seen that the terminal object $\mathbb{1}$ in $\mathbf{Set}^{\mathbb{N}}$ is the sequence of singletons $\{*\} \to \{*\} \to \cdots$. In order to describe the subobject classifier, let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and define Ω as the sequence

$$\overline{\mathbb{N}} \xrightarrow{\omega} \overline{\mathbb{N}} \xrightarrow{\omega} \overline{\mathbb{N}} \xrightarrow{\omega} \dots, \qquad \omega(n) = \begin{cases} 0 & n = 0 \\ \infty & n = \infty \\ n-1 & \text{otherwise.} \end{cases}$$

The morphism $\mathrm{true} : \mathbb{1} \to \Omega$ is the map which sends $*$ to $0 \in \overline{\mathbb{N}}$ in each factor.

[2] We consider 0 to be a natural number.

Suppose we are given a subobject $m : S \rightarrowtail X$, and assume for simplicity that each $m_i : S_i \to X_i$ is the inclusion of a subset $S_i \subseteq X_i$. We define the characteristic map $\chi : X \to \Omega$ as the map with components

$$\chi_i : X_i \to \overline{\mathbb{N}}, \qquad \chi_i(x_i) = \begin{cases} \min \{n \in \mathbb{N} \mid \sigma^n(x_i) \in S_{i+n}\} & \text{if such } n \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

The maps χ_i can be understood as measuring the 'time' before the element x_i becomes included in S under the action of the horizontal maps σ. Note that χ is indeed a morphism in $\mathbf{Set}^{\mathbb{N}}$, since for any $s_i \in S_i$, we have

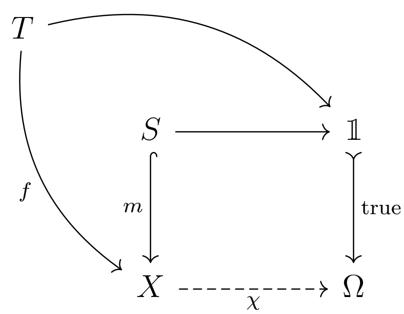
$$\chi_{i+1} \circ \sigma(s_i) = 0 = \omega \circ \chi_i(s_i)$$

since $\sigma S_i \subseteq S_{i+1}$, and for any $x_i \in X_i \setminus S_i$ we have

$$\chi_{i+1} \circ \sigma(x_i) = \min \left\{n \mid \sigma^{n+1}(x_i) \in S_{i+n+1}\right\} = \min \{n \mid \sigma^n(x_i) \in S_{i+n}\} - 1 = \omega \circ \chi_i(x_i).$$

Claim 1: The diagram of Definition 1.18 is a pullback square.

Proof. The square clearly commutes, since for any $i \in \mathbb{N}$ and any $s_i \in S_i$, we have $\chi_i(s_i) = 0$ by definition. To see it is universal, suppose that we are given a commuting diagram



Then for each $i \in \mathbb{N}$ and each $t_i \in T_i$, we have $\chi_i \circ f_i(t_i) = 0$, and thus $f_i(t_i) \in S_i$. Thus $\operatorname{im} f_i \subseteq S_i$ for each i, and hence one can define the factorisation $\theta : T \to S$ as having components $\theta_i = f_i$. □

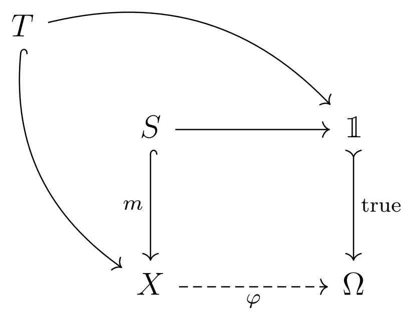
Claim 2: The map χ is the *unique* map which makes the diagram of Definition 1.18 a pullback square.

Proof. Suppose $\varphi : X \to \Omega$ also gave a pullback square. Note first that any $s_i \in S_i$ must necessarily have $\varphi_i(s_i) = 0$ by virtue of the fact that the pullback square commutes. Let $x_i \in X_i$, and let n be minimal such that $\sigma^n x_i \in S_{i+n}$. Since φ is a morphism in $\mathbf{Set}^{\mathbb{N}}$, we therefore have $\omega^n \circ \varphi_i(x_i) = \varphi_{i+n}(\sigma^n x_i) = 0$, so $\varphi_i(x_i) \leq n$ by definition of ω.

Now suppose $\varphi_i(x_i) = k < \infty$, so that $\varphi_{i+k}(\sigma^k x_i) = \omega^k \circ \varphi_i(x_i) = \omega^k(k) = 0$. In order to show that $k \geq n$, consider the sequence T given by

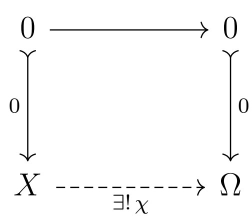
$$\emptyset \longrightarrow \emptyset \longrightarrow \dots \longrightarrow \emptyset \longrightarrow \{\sigma^k x_i\} \longrightarrow \{\sigma^{k+1} x_i\} \longrightarrow \dots$$

where there are $i+k$ copies of $\emptyset$. There is an obvious inclusion $T \hookrightarrow X$, and furthermore the following diagram commutes precisely because $\varphi_{i+k}(\sigma^k x_i) = 0$:



So by the universal property of the pullback we have a unique factorisation $\theta : T \to S$. In particular, we have a map $\theta_{i+k} : \{\sigma^k x_i\} \to S_{i+k}$ satisfying $\theta_i(\sigma^k x_i) = \sigma^k x_i$, which is of course only possible if $\sigma^k x_i \in S_{i+k}$, and thus $k \geq n$ by minimality. It follows that $\varphi = \chi$. $\square$

Example 1.22. The category **Ab** of abelian groups does not admit a subobject classifier. Suppose for a contradiction that (Ω, true) were a subobject classifier for **Ab**. Since the terminal object in **Ab** is the zero group, the map $\mathrm{true} : 0 \to \Omega$ must be the zero homomorphism. Let G be an abelian group, and let χ be the characteristic map of the zero map $0 \rightarrowtail G$.



We therefore have $\ker \chi \cong 0$, and hence χ is injective. But this is impossible, as it implies that every group G embeds into Ω.

1.4 Topoi

Definition 1.23. An (elementary) **topos** is a cartesian closed category which has all finite limits and a subobject classifier.

One can equivalently define an elementary topos as a category with finite limits and a *power object*, which is a family of objects $\{PX\}_{X \in \mathrm{ob}\,\mathscr{C}}$ such that there exist bijections

$$\mathrm{Hom}_{\mathscr{C}}(Y, PX) \cong \mathrm{Sub}_{\mathscr{C}}(X \times Y)$$

natural in Y.