Classifying topos for rings

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Throughout, all rings will be taken to be commutative and unital.

Ring objects

Definition 1. *Let $\mathscr{C}$ be a category which admits all finite products (including a terminal object 1), then a* ***group object*** (A, a, u, i) *consists of*

- *an object $A \in \mathscr{C}$, and*
- *three morphisms,*
  - $a : A \times A \to A$,
  - $i : A \to A$
  - $u : 1 \to A$

such that the following diagrams commute,

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{a\times id} & A\times A \\ {\scriptstyle id\times a}\downarrow & & \downarrow{\scriptstyle a} \\ A\times A & \xrightarrow[a]{} & A \end{array} \qquad \begin{array}{ccccc} 1\times A & \xrightarrow{u\times id} & A\times A & \xleftarrow{id\times u} & A\times 1 \\ & \searrow^{\cong} & \downarrow{\scriptstyle a} & \swarrow^{\cong} & \\ & & A & & \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A\times A & \xrightarrow{i\times id} & A\times A \\ \downarrow & & & & \downarrow{\scriptstyle a} \\ 1 & & \xrightarrow[u]{} & & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\Delta} & A\times A & \xrightarrow{id\times i} & A\times A \\ \downarrow & & & & \downarrow{\scriptstyle a} \\ 1 & & \xrightarrow[u]{} & & A \end{array}$$

These diagrams respectively express, associativity, left and right identity, and left and right inverse. An **abelian group object** (A, a, u, i) is a group object which also satisfies commutativity of the following diagram,

$$\begin{array}{ccc} A\times A & & \\ {\scriptstyle \sigma}\downarrow & \searrow^{a} & \\ A\times A & \xrightarrow[a]{} & A \end{array}$$

where $\sigma = \langle \pi_2, \pi_1 \rangle$ is the "swap map".

There is a notion of a morphism between abelian group objects,

Definition 2. *Let $\mathscr{C}$ be a category and let (A, a, u, i) and (A', a', u', i') be abelian group objects in $\mathscr{C}$. Then a* ***group morphism*** $f : A \to A'$ *is a morphism in $\mathscr{C}(A, A')$ such that the following diagrams commute,*

$$\begin{array}{ccc} A\times A & \xrightarrow{f\times f} & A'\times A' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a'} \\ A & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{u} & A \\ & \searrow_{u'} & \downarrow{\scriptstyle f} \\ & & A' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

Definition 3. *Let $\mathscr{C}$ be a category which admits all finite products, then a* ***ring object*** (R, a, m, u_a, u_m, i) *consists of*

- *an object $R \in \mathscr{C}$, and*
- *five morphisms*
  - a,$m : R \times R \to R$,
  - $u_a, u_m : 1 \to R$
  - $i : R \to R$

such that

- *(R,a,u_a,i) forms an abelian group object,*
- *m is associative and commutative (in the above sense),*
- *u_m is the multiplicative identity (in the above sense),*
- *the following diagrams expressing distributivity commutes,*

$$\begin{array}{ccccc} R^3 & \xrightarrow{\langle \pi_1, \pi_2, \pi_1, \pi_3 \rangle} & R^4 & \xrightarrow{m \times m} & R^2 \\ {\scriptstyle id \times a}\downarrow & & & & \downarrow{\scriptstyle a} \\ R^2 & & \xrightarrow[m]{\qquad\qquad} & & R \end{array} \qquad \begin{array}{ccccc} R^3 & \xrightarrow{\langle \pi_1, \pi_3, \pi_2, \pi_3 \rangle} & R^4 & \xrightarrow{m \times m} & R^2 \\ {\scriptstyle a \times id}\downarrow & & & & \downarrow{\scriptstyle a} \\ R^2 & & \xrightarrow[m]{\qquad\qquad} & & R \end{array}$$

There is also a notion of a morphism of ring objects,

Definition 4. *Let $\mathscr{C}$ be a category and let (R, a, m, u_a, u_m, i) and $(R', a', m', u'_a, u'_m, i')$ be two ring obejects in $\mathscr{C}$. Then a* ***morphism of rings*** *is a morphism $f \in \mathscr{C}(R, R')$ which is a morphism of abelian groups $f : (R, a, u_a, i) \to (R', a', u'_a, i')$, and is such that the following diagrams commute,*

$$\begin{array}{ccc} R \times R & \xrightarrow{f \times f} & R' \times R' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ R & \xrightarrow[f]{} & R' \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{u_m} & R \\ & {\scriptstyle u'_m}\searrow & \downarrow{\scriptstyle f} \\ & & R' \end{array}$$

This gives rise to the following category,

Definition 5. *Let $\mathscr{C}$ be a category with all finite products. Then the category Ring($\mathscr{C}$) has as objects all ring objects of $\mathscr{C}$, and all morphisms of rings in $\mathscr{C}$ are morphisms in Ring($\mathscr{C}$).*

Classifying topos for the theory of rings

A simple example of a classifying topos is the classifying topos of rings. In Dan's lecture "Classifying Topoi I", the notion of a geometric morphism was defined in terms of "geometric theories" and "models" of such theories. This definition, when considered in the context of rings specifically, can be phrased as thus,

Definition 6. *A* ***classifying topos for the theory of rings*** *is a topos $\mathscr{R}$, such that for any cocomplete topos $\mathscr{E}$, there is a bijection*

$$Geo(\mathscr{E}, \mathscr{R}) \cong Ring(\mathscr{E})$$

which is natural in $\mathscr{E}$.

To describe the classifying topos for rings, first recall that a ring is finitely presented over $\mathbb{Z}$ if it is isomorphic to one of the form

$$\frac{\mathbb{Z}[x_1, ..., x_n]}{(f_1, ..., f_m)}$$

The collection of rings which are finitely presented over $\mathbb{Z}$, and ring homomorphisms between them forms a category, fp-rings. For convenience, denote fp-rings by $\mathscr{A}$.

In Moerdijk and MacLane, it is proved that the classifying topos for rings is the category $\underline{\text{Sets}}^{\mathscr{A}^{\text{op}}}$.

Proof sketch

The proof comes down to proving two categorical equivalences,

$$\mathrm{Geo}(\mathscr{E}, \underline{\mathrm{Sets}}^{\mathscr{A}^{\mathrm{op}}}) \cong \mathrm{lex}(\mathscr{A}^{\mathrm{op}}, \mathscr{E}) \cong \mathrm{Ring}(\mathscr{E}) \qquad (\dagger)$$

where $\mathrm{Geo}(\mathscr{E}, \underline{\mathrm{Sets}}^{\mathscr{A}^{\mathrm{op}}})$ is the category of geometric functors from $\mathscr{E} \to \underline{\mathrm{Sets}}^{\mathscr{A}^{\mathrm{op}}}$ and $\mathrm{lex}(\mathscr{A}^{\mathrm{op}}, \mathscr{E})$ is the category of limit preserving functors from $\mathscr{A}^{\mathrm{op}} \to \mathscr{E}$.

The goal of this talk is to prove the second equivalence. The first equivalence is a corollary to a very technical proof outlined in chapter VII, section 9 in Moerdijk and MacLane, and will not be outlined here.

The bulk of the work in proving the second equivalence will be in giving $\mathbb{Z}[x] \in \mathscr{A}^{\mathrm{op}}$ a ring structure (which is equivalent to giving $\mathbb{Z}[x]$ a co-ring structure in $\mathscr{A}$), and then proving the following,

Theorem 1. *Let $\mathscr{C}$ be a category which admits all finite limits, and $\mathcal{P}$ be the smallest subcategory of $\mathscr{A}$ containing the ring object $\mathbb{Z}[x] \in \mathscr{A}^{op}$. Then any limit preserving functor $F : \mathcal{P} \to \mathscr{C}$ can be extended uniquely to a limit preserving functor $\overline{F} : \mathscr{A}^{op} \to \mathscr{C}$.*

There are two immediate observations to make, 1) for this statement to even make any sense, $\mathscr{A}^{\mathrm{op}}$ better be a category which admits all finite limits. This is indeed the case and will be proved first. 2) This result is slightly more general than what is needed for the current purpose, as at the end of the day, $\mathscr{C}$ will be replaced with an arbitrary topos, but there is no great effort saved in only considering the case when $\mathscr{C}$ is a topos, so this statement is presented here.

$\mathscr{A}^{\mathrm{op}}$ admits all finite limits

As mentioned, it ought first be shown that $\mathscr{A}^{\mathrm{op}}$ admits all finite limits. This will be done by showing that the category $\mathscr{A}$ admits an initial object, binary coproducts, and coequalisers.

For any finitely presented ring $\mathbb{Z}[\underline{x}]/I$, the map

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[\underline{x}] \twoheadrightarrow \mathbb{Z}[\underline{x}]/I$$

exists and is unique as ring morphisms are determined by where 1 is mapped to.

Since the tensor product is the coproduct in the category of rings, it requires only to show that the tensor product of two finitely presented rings is again finitely presented. To this end, let $\mathbb{Z}[\underline{x}]/I$ and $\mathbb{Z}[\underline{y}]/J$ be two finitely presented rings, then consider the map

$$\Psi : \mathbb{Z}[\underline{x}]/I \otimes \mathbb{Z}[\underline{y}]/J \to \mathbb{Z}[\underline{x}, \underline{y}]/(I, J)$$
$$f \otimes g \mapsto f \cdot g$$

By the universal property of the kernel, the map

$$\Gamma : \mathbb{Z}[\underline{x}, \underline{y}] \to \mathbb{Z}[\underline{x}]/I \otimes \mathbb{Z}[\underline{y}]/J$$
$$x_i \mapsto x_i \otimes 1$$
$$y_i \mapsto 1 \otimes y_i$$

descends to a map $\Psi^{-1} : \mathbb{Z}[\underline{x}, \underline{y}]/(I, J) \to \mathbb{Z}[\underline{x}]/I \otimes \mathbb{Z}[\underline{y}]/J$ which is clearly an inverse for Ψ.

The coequaliser of a diagram $A \overset{\phi}{\underset{\psi}{\rightrightarrows}} B$ in the category of rings is the ring B/L, where L is the smallest ideal containing all elements of the form $\phi(a) - \psi(a)$, $a \in A$. In the finitely presented case, $\mathbb{Z}[\underline{x}]/I \overset{\phi}{\underset{\psi}{\rightrightarrows}} \mathbb{Z}[\underline{y}]/J$, these elements can be always be written as a linear combination of elements of the form $\phi(x_i) - \psi(x_i)$, where $x_i \in \underline{x}$. So the ring $\frac{\mathbb{Z}[\underline{y}]/J}{L}$, where L is the smallest ideal containing all elements of the form $\phi(f) - \phi(f)$, $f \in \mathbb{Z}[\underline{x}]/I$, is isomorphic to the ring

$$\frac{\mathbb{Z}[\underline{y}]}{(J, \phi(x_1) - \psi(x_1), ..., \phi(x_n) - \psi(x_n))}$$

□

Proof of theorem 1

Definition 7. *Define the following morphisms in $\mathscr{A}$,*

- $m : \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes \mathbb{Z}[x]$, $x \mapsto x \otimes x$.
- $a : \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes \mathbb{Z}[x]$, $x \mapsto x \otimes 1 + 1 \otimes x$.
- $u_a : \mathbb{Z}[x] \to \mathbb{Z}$, $x \mapsto 0$.
- $u_m : \mathbb{Z}[x] \to \mathbb{Z}$, $x \mapsto 1$.
- $i : \mathbb{Z}[x] \to \mathbb{Z}[x]$, $x \mapsto -x$.

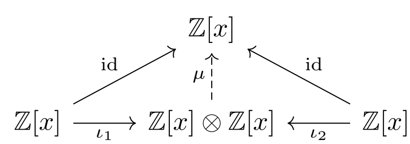
Then $(\mathbb{Z}[x], m, a, u_m, u_a, i)$ is a ring object in the cateogry $\mathscr{A}^{\mathrm{op}}$, this could equivalently be thought of as a "co-ring" object in $\mathscr{A}$.

Idea: Imagine there is a tool kit, and inside this tool kit there is everything inside the category $\mathcal{P}$, the initial object of $\mathscr{A}$, the ability to "take coproducts", and the ability to "take coequalisers". Then the proof of the theorem amounts to showing that every object and morphism in $\mathscr{A}$ can be built using the tools inside this kit. Why? Because, to see where an arbitrary object or morphism in $\mathscr{A}^{\mathrm{op}}$ is sent, look at (one of) its constructions using this tool kit, and then use limit preservation, and functoriality to see where this arbitrary object or morphism must be sent.

Morphisms $\mathbb{Z}[x] \to \mathbb{Z}[x]$

Any morphism $\varphi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ is determined by the polynomial $\varphi(x)$, and any polynomial $\varphi(x)$ is a finite sum of monomials. So what needs to be described, is how to obtain the morphism which sends x to any monomial, as well as sums of such maps.

The very first thing to notice is that the map $\psi : \mathbb{Z}[x] \otimes \mathbb{Z}[x] \to \mathbb{Z}[x]$, $f \otimes g \mapsto f \cdot g$ can be obtained from the following diagram,



Constant maps

The constant map $1 : \mathbb{Z}[x] \to \mathbb{Z}[x]$ is $\mathbb{Z}[x] \to \mathbb{Z} \hookrightarrow \mathbb{Z}[x]$.

Then, constant maps $x \mapsto k$, $k \in \mathbb{Z}_{k>1}$ are obtained by composing,

$$\mathbb{Z}[x] \xrightarrow{a} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \overset{(k-1)\otimes 1}{\longrightarrow} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

Maps $x \mapsto x^n$

Obviously, the map $(\_)^1 = \mathrm{id}_{\mathbb{Z}[x]}$.

Then for $n > 1$, $(\_)^n$ is the composite

$$\mathbb{Z}[x] \xrightarrow{m} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \overset{(\_)^{n-1}\otimes(\_)^1}{\longrightarrow} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

Monomials

Any monomial map $f : x \mapsto kx^n$, $n \in \mathbb{Z}_{n>2}$ is obtained from the following composite

$$\mathbb{Z}[x] \xrightarrow{m} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \overset{k\otimes(\_)^n}{\longrightarrow} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

Polynomials

Finally, given a morphims $f_1 : x \mapsto k_1 x^{n_1}$ and $f_2 : x \mapsto k_2 x^{n_2}$, the function $f_1 + f_2$ can be obtained from the following composition,

$$\mathbb{Z}[x] \xrightarrow{a} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{f_1 \otimes f_2} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

from which, arbitrary morphisms $\mathbb{Z}[x] \to \mathbb{Z}[x]$ can be obtained.

Objects $\mathbb{Z}[\underline{x}]$

Since all binary products are allowed, all finite products are allowed. Then, the object $\mathbb{Z}[\underline{x}]$, where $\underline{x} = (x_1, ..., x_n)$ is obtained by the coproduct of n copies of the identity map $\mathbb{Z}[x] \to \mathbb{Z}[x]$. Strictly, this gives the object $\mathbb{Z}[x]^{\otimes n}$, but, $\mathbb{Z}[x] \otimes \mathbb{Z}[x] = \mathbb{Z}[x \otimes 1, 1 \otimes x]$.

Morphisms $\mathbb{Z}[x] \to \mathbb{Z}[\underline{y}]$

Say $f : x \mapsto k y_1^n y_2^m$, then if $f_1 : x \mapsto k y_1^n$ and $f_2 : x \mapsto y_2^m$, (both of which are known how to obtain), then f is the following composite

$$\mathbb{Z}[x] \xrightarrow{m} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{f_1 \otimes f_2} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

The sum of two polynomials $f_1 : x \mapsto k_{ij} y_i^n y_j^n$ and $f_2 : x \mapsto k_{kl} y_k^p y_l^q$ is the composite

$$\mathbb{Z}[x] \xrightarrow{a} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{f_1 \otimes f_2} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \xrightarrow{\mu} \mathbb{Z}[x]$$

From which, arbitrary polynomials can be obtained.

Morphisms $\mathbb{Z}[\underline{x}] \to \mathbb{Z}[\underline{y}]$

This is easy, as the universal property of the coproduct implies that it suffices to give n maps $\mathbb{Z}[x] \to \mathbb{Z}[\underline{y}]$, where n is the length of the vector $\underline{x}$.

Objects $\frac{\mathbb{Z}[\underline{x}]}{I}$

Again this is easy, just take the coequaliser of a presentation of $\frac{\mathbb{Z}[\underline{x}]}{I}$,

$$\mathbb{Z}[\underline{z_1}] \underset{0}{\overset{\psi_1}{\rightrightarrows}} \mathbb{Z}[\underline{x}] \overset{\pi_I}{\dashrightarrow} \frac{\mathbb{Z}[\underline{x}]}{I}$$

Morphisms $f : \frac{\mathbb{Z}[\underline{x}]}{I} \to \frac{\mathbb{Z}[\underline{y}]}{J}$

Take a presentation of each ring,

$$\mathbb{Z}[\underline{z_1}] \underset{0}{\overset{\psi_1}{\rightrightarrows}} \mathbb{Z}[\underline{x}] \overset{\pi_I}{\dashrightarrow} \frac{\mathbb{Z}[\underline{x}]}{I}$$

$$\mathbb{Z}[\underline{z_2}] \underset{0}{\overset{\psi_2}{\rightrightarrows}} \mathbb{Z}[\underline{y}] \overset{\pi_J}{\dashrightarrow} \frac{\mathbb{Z}[\underline{y}]}{I}$$

Then for every $[x_i]$, let f_i be any representative of $f([x_i])$, and define the map $\hat{f} : \mathbb{Z}[\underline{x}] \to \mathbb{Z}[\underline{y}]$ which maps $x_i \to f_i$. Then the following diagram commutes

$$\begin{array}{ccccc} \mathbb{Z}[\underline{z_1}] & \underset{0}{\overset{\psi_1}{\rightrightarrows}} & \mathbb{Z}[\underline{x}] & \overset{\pi_I}{\dashrightarrow} & \frac{\mathbb{Z}[\underline{x}]}{I} \\ & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle f} \\ \mathbb{Z}[\underline{z_2}] & \underset{0}{\overset{\psi_2}{\rightrightarrows}} & \mathbb{Z}[\underline{y}] & \overset{\pi_J}{\dashrightarrow} & \frac{\mathbb{Z}[\underline{y}]}{I} \end{array}$$

So by the universal property of the top coequaliser row, f is the unique such arrow which makes that above diagram commute, so in fact it is determined by $\hat{f}$, which has already been shown how to be obtained.

Upshot

This implies the existence of two functors,

$$\text{ev} : \text{lex}(\mathscr{A}^{\text{op}}, \mathscr{C}) \to \text{Ring}(\mathscr{C})$$

$$\Psi : \text{Ring}(\mathscr{C}) \to \text{lex}(\mathscr{A}^{\text{op}}, \mathscr{C})$$

where $ev(F) = F(\mathbb{Z}[x])$, and $\Psi(R) = \psi_R$, where $\psi_R(\mathbb{Z}[x]) = R$. There are some technical concerns, but it can be shown that this is in fact an equivalence of categories, which proves the second equivalence in (†).