1 Quantifiers as adjoints

Let $S(x, y)$ be a predicate, where x, y are elements of sets X and Y respectively. One can interpret S as a subset of $X \times Y$, namely the set of pairs for which $S(x, y)$ is true.

For a set X, we write $\mathcal{P}X$ for the Boolean algebra of all subsets of X. This forms a category whose arrows are inclusions. Let $p : X \times Y \to Y$ denote the projection.

Definition 1.1. For a relation $S \subseteq X \times Y$, let

$$\forall_p S = \{y \in Y \mid (x, y) \in S \text{ for all } x \in X\}.$$

For an inclusion $S \subseteq S'$, note that $\forall_p S \subseteq \forall_p S'$, and hence the above defines a functor $\forall_p : \mathcal{P}(X \times Y) \to \mathcal{P}Y$. Similarly, we define

$$\exists_p S = \{y \in Y \mid (x, y) \in S \text{ for some } x \in X\}.$$

which gives a functor $\exists_p : \mathcal{P}(X \times Y) \to \mathcal{P}Y$.

Theorem 1.2. With p the projection, let $p^{-1} : \mathcal{P}Y \to \mathcal{P}(X \times Y)$ be the inverse image functor. Then the functors $\exists_p$ and $\forall_p$ are respectively the left and right adjoints of p^{-1}.

Proof. Recall that adjunctions $\exists_p \dashv p^{-1} \dashv \forall_p$ consist of bijections

$$\mathrm{Hom}(\exists_p S, T) \cong \mathrm{Hom}(S, p^{-1}T) \qquad \text{and} \qquad \mathrm{Hom}(p^{-1}T, S) \cong \mathrm{Hom}(T, \forall_p S)$$

natural in $S \subseteq X \times Y$ and $T \subseteq Y$. Since the Hom sets in question are either singletons or empty, this amounts to showing the following equivalences:

$$\exists_p S \subseteq T \Leftrightarrow S \subseteq p^{-1}T \qquad \text{and} \qquad p^{-1}T \subseteq S \Leftrightarrow T \subseteq \forall_p S.$$

We have:

$$\begin{aligned}
p^{-1}T \subseteq S \quad &\Leftrightarrow \quad \text{if } p(x, y) \in T \text{ then } (x, y) \in S \\
&\Leftrightarrow \quad \text{if } y \in T \text{ then } (x, y) \in S \text{ for all } x \in X \\
&\Leftrightarrow \quad T \subseteq \forall_p S.
\end{aligned}$$

$$\begin{aligned}
S \subseteq p^{-1}T \quad &\Leftrightarrow \quad \text{if } (x, y) \in S \text{ then } p(x, y) \in T \\
&\Leftrightarrow \quad \text{if } (x, y) \in S \text{ for some } x \in X \text{ then } y \in T \\
&\Leftrightarrow \quad \exists_p S \subseteq T.
\end{aligned}$$

□

By replacing the projection p with an arbitrary morphism $f : Z \to Y$, we obtain the following generalisation. For a subset $S \subseteq Z$, let

$$\begin{aligned}
\forall_f S &= \{y \in Y \mid \text{ for all } z \in Z \text{ if } f(z) = y \text{ then } z \in S\}, \\
\exists_f S &= \{y \in Y \mid \text{ there exists } z \in S \text{ such that } f(z) = y\}.
\end{aligned}$$

Theorem 1.3. Let $f : Z \to Y$ be a morphism, and let $f^{-1} : \mathcal{P}Y \to \mathcal{P}Z$ be the inverse image functor. Then the functors $\exists_f, \forall_f : \mathcal{P}Z \to \mathcal{P}Y$ are respectively the left and right adjoints of f^{-1}.

Proof. Essentially the same as Theorem 1.2. □

The same idea applies to a topos $\mathcal{E}$, with the poset $\mathrm{Sub}_{\mathcal{E}}(X)$ taking the role of $\mathcal{P}X$. Recalling the natural isomorphism $\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Hom}_{\mathcal{E}}(X, \Omega)$, and noting that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a poset for any X we likewise obtain a poset structure on $\mathrm{Hom}_{\mathcal{E}}(X, \Omega)$.

Definition 1.4. Let Y, Z be objects in $\mathcal{E}$, and let $\varphi : \Omega^Y \to \Omega^Z$ and $\psi : \Omega^Y \to \Omega^Z$ be morphisms. We say that φ is **internally left adjoint** to ψ if, for each object $A \in \mathcal{E}$, the maps φ_* and ψ_* induced on Hom-sets form an adjoint pair, with $\varphi_* \dashv \psi_*$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(A, \Omega^Y) \arrow[r, shift left, "\varphi_* = \varphi \circ -"] & \mathrm{Hom}_{\mathcal{E}}(A, \Omega^Z). \arrow[l, shift left, "\psi_* = \psi \circ -"] \end{tikzcd}$$

Theorem 1.5. Let $f : Z \to Y$ be a morphism in $\mathcal{E}$. Then $\Omega^f : \Omega^Y \to \Omega^Z$ has internal left and right adjoints $\exists_f, \forall_f : \Omega^Z \to \Omega^Y$ respectively.

Proof. Let A be an object of $\mathcal{E}$, and consider the inverse image functor

$$(f \times \mathrm{id})^{-1} : \mathrm{Sub}_{\mathcal{E}}(Y \times A) \to \mathrm{Sub}_{\mathcal{E}}(Z \times A).$$

This is natural in A, since it is constructed by pullback. In addition, $(f \times \mathrm{id})^{-1}$ has left and right adjoints $\exists_{f\times\mathrm{id}}, \forall_{f\times\mathrm{id}}$ by (a generalisation of) Theorem 1.3. By composing with the natural isomorphism $\mathrm{Sub}_{\mathcal{E}}(- \times A) \cong \mathrm{Hom}_{\mathcal{E}}(- \times A, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega^{-})$, we therefore obtain natural transformations $(\exists_f)_*, (\forall_f)_* : \mathrm{Hom}_{\mathcal{E}}(-, \Omega^Z) \to \mathrm{Hom}_{\mathcal{E}}(-, \Omega^Y)$, as in the following diagram:

$$\begin{tikzcd} \mathrm{Sub}_{\mathcal{E}}(Y \times A) \arrow[r, shift left, "(f\times\mathrm{id})^{-1}"] \arrow[d, "\cong"] & \mathrm{Sub}_{\mathcal{E}}(Z \times A) \arrow[l, shift left, "\exists_{f\times\mathrm{id}}"] \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathcal{E}}(A, \Omega^Y) \arrow[r, shift left, "(\Omega^f_*)_A"] & \mathrm{Hom}_{\mathcal{E}}(A, \Omega^Z) \arrow[l, shift left, "((\exists_f)_*)_A"] \end{tikzcd} \qquad \text{(and similarly for } (\forall f)_*\text{)}$$

Note that since we have adjoint pairs $\exists_{f\times\mathrm{id}} \dashv (f \times \mathrm{id})^{-1} \dashv \forall_{f\times\mathrm{id}}$, we also have adjoint pairs $((\exists_f)_*)_A \dashv (\Omega^f)_A \dashv ((\forall_f)_*)_A$ for all A.

Now, by the Yoneda lemma natural transformations $\mathrm{Hom}_{\mathcal{E}}(-, \Omega^Z) \to \mathrm{Hom}_{\mathcal{E}}(-, \Omega^Y)$ are in bijection with $\mathrm{Hom}_{\mathcal{E}}(\Omega^Z, \Omega^Y)$, and hence from $(\exists_f)_*, (\forall_f)_*$ we obtain uniquely determined maps

$$\begin{tikzcd} \Omega^Z \arrow[r, shift left=2, "\exists_f"] \arrow[r, shift right=2, "\forall_f"'] & \Omega^Y. \arrow[l, "\Omega^f"'] \end{tikzcd}$$

The fact that these maps are internal left and right adjoints to Ω^f is by design. □

2 The Mitchell-Bènabou language

Throughout, let $\mathcal{E}$ be a topos. Recall (Higher-order logic & topoi II) that a **type theory** consists of

- a class of types including special types $\mathbb{1}, \Omega$,
- a class of terms of each type, including countably many variables of each type,
- for each finite set X of variables, a binary relation $\vdash_X$ of entailment.

We will describe in this section a canonical type theory which arises from a topos. With the ability to encode logical formulas in a topos, this will allow us to specify subobjects of a topos through the use of set-builder notation.

If σ is a term, we write $\mathrm{FV}\,\sigma$ for its set of free variables, and if $S = \{x_1, ..., x_n\}$ is a finite set of variables , we write $\overline{S}$ for the product $X_1 \times ... \times X_n$.

Definition 2.1. The **Mitchell-Bènabou language** $\mathcal{L}(\mathcal{E})$ associated to $\mathcal{E}$ is defined as follows. The types of $\mathcal{L}(\mathcal{E})$ are the objects of $\mathcal{E}$. The terms of $\mathcal{L}(\mathcal{E})$ are defined recursively below. Associated to each term σ of type X is a morphism in $\mathcal{E}$

$$\overline{\sigma} : \overline{\mathrm{FV}\,\sigma} \to X,$$

called its **interpretation**.

The term construction rules and their interpretations are as follows.

- For each type X there are variables $x_1, x_2, ...$ of type X, each of which are interpreted by the identity $\overline{x_i} = \mathrm{id}_X : X \to X$.
- Given terms σ of type X and τ of type Y, there is a term $\langle \sigma, \tau \rangle$ of type $X \times Y$. It is interpreted by the morphism

  $$\overline{\langle \sigma, \tau \rangle} : \overline{\mathrm{FV}\,\sigma \cup \mathrm{FV}\,\tau} \xrightarrow{\langle \overline{\sigma} p, \overline{\tau} q \rangle} X \times Y,$$

  where $p : \overline{\mathrm{FV}\,\sigma \cup \mathrm{FV}\,\tau} \to \overline{\mathrm{FV}\,\sigma}$ and $q : \overline{\mathrm{FV}\,\sigma \cup \mathrm{FV}\,\tau} \to \overline{\mathrm{FV}\,\tau}$ are the projections.
- Given terms σ and τ of type X, there is a term $\sigma = \tau$ of type Ω, interpreted by the composite

  $$\overline{\sigma = \tau} : \overline{\mathrm{FV}\,\sigma \cup \mathrm{FV}\,\tau} \xrightarrow{\langle \overline{\sigma} p, \overline{\tau} q \rangle} X \times X \xrightarrow{\delta_X} \Omega,$$

  where p, q are as above, and δ_X is the characteristic map of the diagonal $X \to X \times X$.

- Given terms σ of type Y^X and τ of type X, there is a term $\sigma(\tau)$ of type Y whose interpretation is

$$\overline{\sigma(\tau)} : \overline{\mathrm{FV}\,\sigma \cup \mathrm{FV}\,\tau} \xrightarrow{\langle \overline{\sigma}p, \overline{\tau}q \rangle} Y^X \times X \xrightarrow{\mathrm{ev}_{X,Y}} Y.$$

  where $\mathrm{ev}_{X,Y}$ is the evaluation map. In the particular case where $Y = \Omega$, we write this term as $\tau \in \sigma$ instead.

- Given a term σ of type X and a morphism $f : X \to Y$ in $\mathcal{E}$, there is a term $f \circ \sigma$ of type Y, with the interpretation

$$\overline{f \circ \sigma} : \overline{\mathrm{FV}\,\sigma} \xrightarrow{\overline{\sigma}} X \xrightarrow{f} Y.$$

- Given a term σ of type Z containing a free variable of type X, and given a variable x of type X, there is a term $\lambda x.\sigma$ of type Z^X, which is interpreted as the transpose of the map σ:

$$\overline{\lambda x.\sigma} : \overline{\mathrm{FV}\,\sigma \setminus \{x\}} \to Z^X.$$

  Note that x no longer occurs free in the term $\lambda x.\sigma$.

A term of type Ω is called a **formula**. A formula $\sigma : U \to \Omega$ is **true** if it factors through $\mathrm{true} : \mathbb{1} \to \Omega$.

Part of the appeal of defining the internal language of a topos in this way is that the logical connectives are immediately dealt with via the internal Heyting algebra structure of Ω. For example, conjunction: given $B \in \mathcal{E}$, define $\wedge_B : \mathrm{Hom}_{\mathcal{E}}(B, \Omega \times \Omega) \to \mathrm{Hom}_{\mathcal{E}}(B, \Omega)$ as the map making the following commute, where $\cap_B$ is the (external) meet defined on subobjects.

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(B) \times \mathrm{Sub}_{\mathcal{E}}(B) \arrow[r, "\cap_B"] & \mathrm{Sub}_{\mathcal{E}}(B) \arrow[dd, "\cong"] \\
\mathrm{Hom}_{\mathcal{E}}(B, \Omega) \times \mathrm{Hom}_{\mathcal{E}}(B, \Omega) \arrow[u, "\cong"] & \\
\mathrm{Hom}_{\mathcal{E}}(B, \Omega \times \Omega) \arrow[u, "\cong"] \arrow[r, dashed, "\wedge_B"] & \mathrm{Hom}_{\mathcal{E}}(B, \Omega)
\end{tikzcd}$$

Since $\wedge_B$ is composed of maps which are natural in B, we obtain a natural transformation $\wedge : \mathrm{Hom}(-, \Omega \times \Omega) \to \mathrm{Hom}(-, \Omega)$, and hence (by Yoneda) a morphism

$$\wedge : \Omega \times \Omega \to \Omega$$

explicitly given by $\wedge = \wedge_{\Omega \times \Omega}(\mathrm{id})$. Given two formulas $\sigma : U \to \Omega$, $\tau : V \to \Omega$, one can then define their conjunction as the obvious composite

$$\sigma \wedge \tau : W \xrightarrow{\langle \overline{\sigma}p, \overline{\tau}q \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

In particular, if σ, τ are the characteristic maps for subobjects $S, T \in \mathcal{E}$, then $\sigma \wedge \tau$ is the characteristic map of their intersection. The other propositional connectives are defined in much the same way.

We now move to the task of defining quantifiers. Suppose that $\sigma : X \times U \to \Omega$ is a formula containing a free variable x of type X, together with possibly other free variables. The formula $\forall x\, \sigma$ should therefore be interpreted by an arrow $U \to \Omega$. Let $p : X \to \mathbb{1}$ be the unique map, and consider the induced map $\Omega^p : \Omega \to \Omega^X$ and its internal adjoints from Theorem 1.5:

$$\begin{tikzcd} \Omega^X \arrow[r, shift left=8, "\exists_p"] \arrow[r, shift right=8, "\forall_p"'] & \Omega \arrow[l, "\Omega^p"'] \end{tikzcd}$$

The interpretation of $\forall x\, \sigma$ is given by the composite

$$\overline{\forall x\, \sigma} : U \xrightarrow{\overline{\lambda x.\sigma}} \Omega^X \xrightarrow{\forall_p} \Omega,$$

and $\exists x\, \sigma$ is the same except with $\exists_p$ replacing $\forall_p$.

Definition 2.2. If σ is a formula with a free variable x of type X, we write

$$\{x \in X \mid \sigma(x)\}$$

for the subobject classified by its interpretation. Explicitly, this means that we have a pullback square

$$\begin{tikzcd} \{x \in X \mid \sigma(x)\} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "\mathrm{true}"] \\ X \arrow[r, "\sigma"'] & \Omega. \end{tikzcd}$$

Upshot: this allows us to specify subobjects of a given object $X \in \mathcal{E}$ just 'as if' they have elements x!

Example 2.3. One can define the 'object of epimorphisms' $\mathrm{Epi}(X, Y) \rightarrowtail Y^X$ as the following subobject

$$\mathrm{Epi}(X, Y) = \left\{ f \in Y^X \mid \forall y \in Y\, \exists x \in X\, f(x) = y \right\}.$$