The category of sheaves is a topos part 2

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Recall from the last talk that for a small category $\mathcal{C}$, the category $\mathrm{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ is an elementary topos. Explicitly, $\mathrm{PSh}(\mathcal{C})$ has the following structure:

- Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{C}$, the exponential $\mathcal{G}^{\mathcal{F}}$ is the presheaf defined on objects $C \in \mathrm{ob}\mathcal{C}$ by
$$\mathcal{G}^{\mathcal{F}}(C) = \mathrm{Hom}(h_C \times \mathcal{F}, \mathcal{G}),$$
where $h_C = \mathrm{Hom}(-, C)$ is the representable functor associated to C, and the product $\times$ is defined object-wise.

- Writing 1 for the constant presheaf of the one object set, the subobject classifier true : $1 \to \Omega$ in $\mathrm{PSh}(\mathcal{C})$ is defined on objects by
$$\Omega(C) := \{S \mid S \text{ is a sieve on } C \text{ in } \mathcal{C}\},$$
and $\mathrm{true}_C : * \to \Omega(C)$ sends $*$ to the maximal sieve $t(C)$.

The goal of this talk is to refine this structure to show that the category $\mathrm{Sh}_\tau(\mathcal{C})$ of sheaves on a site $(\mathcal{C}, \tau)$ is also an elementary topos. To do this we must make use of the sheafification functor defined at the end of the first talk:

Theorem 0.1. *The inclusion functor* $i : \mathrm{Sh}_\tau(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ *has a left adjoint*
$$\boldsymbol{a} : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}_\tau(\mathcal{C}),$$
called sheafification, or the associated sheaf functor. Moreover, this functor commutes with finite limits.

Explicitly, $\mathbf{a}(\mathcal{F}) = (\mathcal{F}^+)^+$, where
$$\mathcal{F}^+(C) := \mathrm{colim}_{S \in \tau(C)} \mathrm{Match}(S, \mathcal{F}),$$
where $\mathrm{Match}(S, \mathcal{F})$ is the set of matching families for the cover S of C, and the colimit is taken over all covering sieves of C, ordered by reverse inclusion.

1 The basic structure of $\mathrm{Sh}_\tau(\mathcal{C})$

Recall that the category $\mathrm{PSh}(\mathcal{C})$ of presheaves of sets on a small category $\mathcal{C}$ has all small limits, and that they are computed object-wise:
$$(\lim \mathcal{F}_i)(C) = \lim \mathcal{F}_i(C),$$
where the limit on the right hand side is taken in **Set**. This can be immediately extended to the category of sheaves on a site. Suppose $I \to \mathrm{Sh}_\tau(\mathcal{C})$, $i \mapsto \mathcal{F}_i$, is a diagram of sheaves, and $\mathcal{F} = \lim \mathcal{F}_i$ in the category of presheaves. If S is a covering sieve of an object C of $\mathcal{C}$, then the sheaf condition means that there is an equaliser

$$\mathcal{F}_i(C) \longrightarrow \prod_{f\in S} \mathcal{F}_i(\mathrm{dom}(f)) \rightrightarrows \prod_{f\circ g\in S} \mathcal{F}_i(\mathrm{dom}(g))$$

for each $i \in I$. But from basic category theory we know that limits commute with limits, so taking the limit of all such equalisers gives an equaliser

$$\mathcal{F}(C) \longrightarrow \prod_{f\in S} \mathcal{F}(\mathrm{dom}(f)) \rightrightarrows \prod_{f\circ g\in S} \mathcal{F}(\mathrm{dom}(g))$$

which implies that $\mathcal{F}$ is a sheaf. The upshot is that limits computed in the category of sheaves are identical to limits computed in the category of presheaves.

Next, the adjunction of Theorem 0.1 gives a recipe to define all small colimits in $\mathrm{Sh}_\tau(\mathcal{C})$: first compute the colimit in the category of presheaves,using the inclusion ι, then take the sheafifcation. This is indeed the limit because a left adjoint preserves colimits, so we may write

$$\mathrm{colim}\mathcal{F}_i := \mathbf{a}\left(\mathrm{colim}\, \iota(\mathcal{F}_i)\right).$$

Next note that morphisms in $\mathrm{Sh}_\tau(\mathcal{C})$ are simply morphisms of presheaves. In fact, another consequence of the adjunction is that a morphism of sheaves if a monomorphism if and only if it is a monomorphism as a morphism of presheaves. This means that $\phi : \mathcal{F} \to \mathcal{G}$ is a monomorphism iff $\phi_C : \mathcal{F}(C) \to \mathcal{G}(C)$ is a monomorphism for every object C of $\mathcal{C}$.

WARNING: The story is not the same for epimorphisms: a morphism of sheaves may be an epimorphism when considered as a presheaf but may fail to be an epimorphism of sheaves.

2 Exponential objects in $\mathrm{Sh}_\tau(\mathcal{C})$

To show that $\mathrm{Sh}(\mathcal{C})$ is a topos we must show that it has exponential objects and a subobject classifier. It will turn out that the exponential in sheaves is the same as the presheaf exponential, but the subobject classifier is a more complicated story.

First we observe that if exponentials exist in $\mathrm{Sh}_\tau(\mathcal{C})$ then they must be of the same form as the exponentials in $\mathrm{PSh}(\mathcal{C})$.

Lemma 2.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $\mathcal{C}$, and write $\overline{\mathcal{G}^{\mathcal{F}}}$ for the conjectural exponential object in $\mathrm{Sh}_\tau(\mathcal{C})$. Then*

$$\iota\left(\overline{\mathcal{G}^{\mathcal{F}}}\right) = \iota(\mathcal{G})^{\iota(\mathcal{F})},$$

where ι is the inclusion $\mathrm{Sh}_\tau(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$.

Proof. Let $\mathcal{H}$ be an arbitrary presheaf on $\mathcal{C}$. Using Theorem 0.1, and definition of the exponential, we have the following sequence of natural bijections:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\left(\mathcal{H}, \iota\left(\overline{\mathcal{G}^{\mathcal{F}}}\right)\right) &\cong \mathrm{Hom}_{\mathrm{Sh}_\tau(\mathcal{C})}\left(\mathbf{a}(\mathcal{H}), \overline{\mathcal{G}^{\mathcal{F}}}\right) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_\tau(\mathcal{C})}(\mathbf{a}(\mathcal{H}) \times \mathcal{F}, \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_\tau(\mathcal{C})}(\mathbf{a}(\mathcal{H} \times \iota(\mathcal{F})), \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{H} \times \iota(\mathcal{F}), \iota(\mathcal{G})) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\left(\mathcal{H}, \iota(\mathcal{G})^{\iota(\mathcal{F})}\right).
\end{aligned}$$

On the third line we have used the fact that $\mathbf{a}$ preserves products and that $\mathbf{a}\circ\iota \cong \mathrm{id}$. The lemma then follows from the Yoneda lemma.

□

As a result of the is lemma, we know that if exponential objects exist in $\mathrm{Sh}_\tau(\mathcal{C})$, they must be given by

$$\mathcal{G}^{\mathcal{F}}(C) = \mathrm{Hom}_{\mathrm{Sh}_\tau(\mathcal{C})}(h_C \times \mathcal{F}, \mathcal{G}).$$

Next we must show that these prototypical exponentials are actually sheaves.

Theorem 2.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $\mathcal{C}$. If $\mathcal{G}$ is a sheaf, then so is $\mathcal{G}^{\mathcal{F}}$.*

Proof. We will proceed in two steps: first we will show that if $\mathcal{G}$ is a separated presheaf then so is $\mathcal{G}^{\mathcal{F}}$. Using that, it will suffice to show the existence of amalgamations for matching families of elements of $\mathcal{G}^{\mathcal{F}}$, since the uniqueness of any amalgamation follows from $\mathcal{G}^{\mathcal{F}}$ being separated.

Recall that an element $\theta \in \mathcal{G}^{\mathcal{F}}(C)$ is a natural transformation $\theta : h_C \times \mathcal{F} \to \mathcal{G}$ which assigns to any $g : D \to C$ in $h_C(D)$ and any $x \in \mathcal{F}(D)$ an element $\theta(g, x) \in \mathcal{F}(D)$. The naturatity condition means that, for any $h : E \to D$, we have the identity

$$\theta(g, x)|_h = \theta(gh, x|_h).$$

Moreover, given any morphisms $f : C' \to C$ and $g' : D' \to C'$, and any $x \in \mathcal{F}(D')$, the restriction map gives

$$\theta|_f(g', x) = \theta(fg', x).$$

In other words, $\theta|_f = \theta \circ (h_f \times 1)$.

Fix a covering sieve $S \in \tau(C)$ and suppose that $\theta, \sigma \in \mathcal{G}^{\mathcal{F}}(C)$ satisfy

$$\theta|_f = \sigma|_f, \qquad \forall f \in S.$$

This means that $\theta(fg', x) = \sigma(fg', x)$ for all g' and x as above, so in the case when $g' = 1$ we have

$$\theta(f, x) = \sigma(f, x), \qquad \forall f \in S,\ x \in \mathcal{F}(\mathrm{dom}(f)).$$

Next take a morphism $k : C' \to C$ and $x \in \mathcal{F}(C')$. Then for any $g' \in k^*(S)$ we have

$$\begin{aligned} \theta(k, x)|_{g'} &= \theta(kg', x|_{g'}) \\ &= \sigma(kg', x|_{g'}) \\ &= \sigma(k, x)|_{g'}. \end{aligned}$$

But $k^*(S)$ is a cover of C', so if we assume that $\mathcal{G}$ is separated then the matching family $\{\sigma(k, x)|_{g'}\}_{g' \in k^*(S)}$ must have a unique amalgamation. Both $\theta(k, x)$ and $\sigma(k, x)$ amalgamate this matching family, so this implies that $\theta(k, x) = \sigma(k, x)$. Since we chose k and x arbitrarily, this means that $\theta = \sigma$. Hence $\mathcal{G}^{\mathcal{F}}$ is separated whenever $\mathcal{G}$ is.

It remains now to show that amalgamations of matching families of elements of $\mathcal{G}^{\mathcal{F}}$ exist. Fix a covering sieve $S \in \tau(C)$ and, matching family $\{\theta_f \in \mathcal{G}^{\mathcal{F}}(\mathrm{dom}(f))\}_{f \in S}$. The matching property means that

$$\theta_{fg}(h, x) = \theta_f|_g(h, x) = \theta_f(gh, x), \qquad x \in \mathcal{F}(\mathrm{dom(g)})$$

whenever the composition makes sense.

To find an amalgamation of this matching family, we will construct from S a natural transformation $\theta' : h_C \times \mathcal{F} \to \mathcal{G}^+$ such that, for all $f \in S$, the following diagram commutes:

$$\begin{array}{ccc} h_D \times \mathcal{F} & \xrightarrow{\theta_f} & \mathcal{G} \\ {\scriptstyle h_f \times 1}\downarrow & & \downarrow{\scriptstyle \eta_{cG}} \\ h_C \times \mathcal{F} & \xrightarrow[\theta']{} & \mathcal{G}^+ \end{array}$$

Indeed, the hypothesis that $\mathcal{G}$ is a sheaf means that $\eta_{\mathcal{G}}$ is an isomorphism, so θ' will provide an amalgamation $\theta = (\eta_{\mathcal{G}})^{-1} \circ \theta'$ of $\{\theta_f\}_{f \in S}$.

For $k : B \to C$ and $x \in \mathcal{F}(B)$, define

$$\theta'(k, x) := \{\theta_{kh}(1, x_h)\}_{h \in k^*(S)}.$$

It is immediate the that right hand side is a matching family. All that remains is to check that this makes the above diagram commute. Pick $f \in S$, so that $f^*(S)$ is a maximal sieve. Then for any (k, x) as above, going anticlockwise around the square gives

$$\begin{aligned}\theta'|_f(k, x) &= (\theta' \circ (h_f \times 1))(k, x)\\ &= \theta'(fk, x)\\ &= \{\theta_{fkh}(1, x_h)\}_{h \in (fk)^*(S)}\\ &= \{\theta_{fkh}(1, x_h)\}_{h \in t_B}.\end{aligned}$$

On the other hand, going clockwise gives

$$(\eta_{\mathcal{G}} \circ \theta_f)(k, x) = \eta_{\mathcal{G}}(\theta_{fk}(1, x)) = \{\theta_{fkh}(1, x_h)\}_{h \in t_B},$$

so the diagram commutes.

□

This confirms that $\mathcal{G}^{\mathcal{F}}$ is a sheaf whenever $\mathcal{G}$ and $\mathcal{F}$ are sheaves, so it gives an exponential object in $\mathrm{Sh}_\tau(\mathcal{C})$.

3 A subobject classifier in $\mathrm{Sh}_\tau(\mathcal{C})$

See the success in adapting the exponential objects from presheaves to sheaves, we may be tempted to believe that the same holds for the subobject classifier. Unfortunately, sieves do not glue locally, so the assignment of all sieves does not form a sheaf in general.

Recall that on any site $(\mathcal{C}, \tau)$, and for any sieve S on C and any morphism $f : D \to C$, $f \in S$ iff $f^*(S) = t_D$. We say that S covers f if $f^*(S) \in \tau(D)$. Obviously if $f \in S$ then S covers f, but the converse is not true in general.

It turns out that this fact is the key obstruction to the presheaf of sieves forming a sheaf. It is therefore natural to restrict our set of sieves.

Definition 3.1. A sieve S on C is *closed* with respect to τ if for every morphism f in $\mathcal{C}$, S covers f iff $f \in S$.

Remark 3.1. The nomenclature here is an unfortunate artefact of history, and a sieve being closed has no connection to the topological notion of a closed set.

Observe that if S is a closed sieve on C, and $h : B \to C$ is a morphism, then $h^*(S)$ is also closed on B. This means that the assignment of closed sieves to an object is functorial, so we can define a presheaf

$$\Omega^{cl}(C) := \{S \mid S \text{ a closed sieve on } C\}.$$

Definition 3.2. If S is a sieve on C, its *closure* is

$$\widehat{S} := \{h \mid \mathrm{cod}(h) = C,\ S \text{ covers } h\}.$$

We leave it as an exercise to confirm that the closure of a sieve is indeed a closed sieve, and that it is actually the smallest closed sieve containing S. As a consequence of this universal property, we see that taking closure and pullbacks of sieves commute, so

$$\widehat{g^*(S)} = g^* \left(\widehat{S} \right)$$

for any morphism g into C.

Lemma 3.1. *The presheaf Ω^{cl} is a sheaf.*

Proof. We will prove first that Ω^{cl} is a separated presheaf. Fix a covering sieve on C. Take two closed sieves $M, N \in \Omega^{cl}(C)$ such that

$$g^*(M) = g^*(N), \qquad \forall g \in S.$$

Then in particular, $M \cap S = N \cap S$. If $f \in M$ then M covers f, and S covers f because S covers C, and hence $M \cap S$ covers f. But $M \cap S = N \cap S \subseteq N$, so N covers f and, since N is closed, $f \in N$. Therefore $M \subseteq N$. Running this argument again with N in place of M shows that $N \subseteq N$, so $M = N$ and Ω is separated.

It remains to show that every matching family has an amalgamation. Let $S \in \tau(C)$, and pick a matching family $\{M_f \in \Omega(\mathrm{dom(f)})\}_{f \in S}$. The matching property means that

$$g^* M_f = M_{fg}$$

whenever the composition is defined. Now consider the sieve

$$M := \{f \circ g \mid g \in M_f,\ f \in S\}.$$

This is not generally a closed sieve, but we will show that its closure $\widehat{M}$ is an amalgamation of $\{M_f\}_{f \in S}$.

It is immediate that $M_f \subseteq f^*(M)$. Conversely, if $g \in f^*(M)$, so $fg \in M$, then there exists some $f' \in S$ and $g' \in M_{f'}$ such that $fg = f'g'$, so $M_{fg} = M_{f'g'}$. It follows that $g^* M_f = g'^* M_{f'}$. But $g' \in M_{f'}$, so $g'^* M_{f'}$ is a maximal sieve, hence so is $g^* M_f$. Therefore $g \in M_f$, and $M_f = f^*(M)$. Finally, since M_f is closed,

$$f^*(\widehat{M}) = \widehat{f^*(M)} = \widehat{M}_f = M_f,$$

so $\widehat{M}$ is indeed an amalgamation of $\{M_f\}_{f \in S}$.

□

Now maximal sieves are obviously closed, so as in the presheaf case, define a natural transformation true : $1 \to \Omega^{cl}$ by $\mathrm{true}_C : * \to \Omega^{cl}(C)$, $* \mapsto t_C$.

Lemma 3.2. *The monomorphism* true : $1 \to \Omega^{cl}$ *is a subobject classifier in* $\mathrm{Sh}_\tau(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be a sheaf and $\mathcal{G} \subseteq \mathcal{F}$ a sub-sheaf. We define a characteristic function $\phi : \mathcal{F} \to \Omega$ in the same was as the presheaf case:

$$\phi_C(x) := \{f \in t_C \mid x|_f \in \mathcal{G}(\mathrm{dom(f)})\}.$$

The fact that true : $1 \to \Omega^{cl}$ is a subobject classifier will follows identically to the presheaf case once we show that $\phi_C(c)$ is a closed sieve for all C in $\mathcal{C}$ and $x \in \mathcal{F}(C)$.

Fix $f : D \to C$ and suppose that $\phi_C(x)$ covers f. Then $f^*(\phi_C(x)) \in \tau(D)$. Now

$$\begin{aligned} f^*(\phi_C(x)) &= \{h \in t_D \mid f \circ h \in \phi_C(D)\} \\ &= \{h \in t_D \mid (x|_f)|_h \in \mathcal{G}(\mathrm{dom(h)})\}. \end{aligned}$$

It is easy to see that $\{(x|_f)|_h\}_{h \in f^*(\phi_C(x))}$ now forms a matching family with amalgamation x_f. Moreover, since $\mathcal{G}$ is a sheaf it follows that $x_f \in \mathcal{G}(\mathrm{dom}(f))$ is the unique amalgamation, and so $x_f \in \phi_C(x)$. Hence $\phi_C(x)$ is closed.

□

Putting all of these results together, we have proven the following:

Theorem 3.1. *For any site $(\mathcal{C}, \tau)$ with $\mathcal{C}$ a small category, the category $\mathrm{Sh}_\tau(\mathcal{C})$ of sheaves of sets on $\mathcal{C}$ is a topos.*