Vertex algebras, Hopf algebras and lattices

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- Goal 1: Understand this construction.
- Goal 2: Construct lattice vertex algebras via the universal measuring algebra associated to the formal additive group and group algebra of the lattice. Then by a universal construction, lift the pairing on the lattice to a bicharacter inducing a singular multiplication map.

Introduction

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- correlation functions of 2-dimensional conformal field theories,
- an infinite dimensional $\mathbb{Z}$-graded representation of the Monster group with modular functions j as the generating function of the dimensions of its homogeneous subspaces,
- singular commutative algebras in a certain category.

Introduction

"... in vertex operator algebra theory, there are essentially no examples, [...] that are easy to construct and for which the axioms can be easily proved." - Lepowsky and Huang

Definition

The space of A-valued formal distributions is defined by:

$$A[[z^{\pm 1}]] = \{\sum_{n \in \mathbb{Z}} a_n z^{-n-1} \mid a_n \in A\}$$

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The commutation of two formal distributions is defined by:

$$[a(z), b(w)] = \sum_{m,n \in \mathbb{Z}} [a_m, b_n] z^{-m-1} w^{-n-1}$$

Definition: Mutually local

Two formal distributions $\alpha(z), \beta(z) \in End(V)[[z^{\pm 1}]]$ are said to be mutually local if there exists some integer $n \in \mathbb{Z}$ such that:

$$(z-w)^n [a(z), b(w)] = 0(z)$$

This is denoted by $a(z) \sim b(z)$.

Definition

The data $(V, \mathbb{1}, T, Y)$ is called a **vertex algebra**, where V is a vector space, $\mathbb{1} \in V$ is a distinguished vacuum vector, $T \in End(V)$ is a linear endomorphism, and Y is a linear map $Y(\_, z) : V \to End(V)[[z^{\pm 1}]]$, satisfying the following axioms:

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3. translation covariance: $[T, Y(v, z)] = \partial Y(v, z)$, $T(\mathbb{1}) = 0$

Alternative definition

The data $(V, \mathbb{1}, \mathcal{F})$ is called a **vertex algebra**, where V is a vector space, $\mathbb{1} \in V$ is a distinguished vacuum vector, and $\mathcal{F}$ is a collection of fields, satisfying the following axioms:

1. creativity: for all $v \in V$, there exists precisely one field $C_v(z) \in \mathcal{F}$ such that $C_v(z)(\mathbb{1}) = v + O(z)$,
2. locality: $C_v(z) \sim C_w(z)$ for all $v, w \in V$

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Theorem: Uniqueness theorem

If $(V, \mathbb{1}, \mathcal{F})$ is a vertex algebra, then there exists at most one translation operator $T \in End(V)$ compatible with these fields.

Example 1: Heisenberg vertex algebra

Definition

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with Lie bracket defined by:

$$[h_m, h_n] = m\delta_{m,-n}\mathbb{1}, \quad [\mathbb{1}, h_n] = 0$$

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$$M_0 := U(H)/U(H_+) = span\{h_{-n_1}...h_{-n_k}(\mathbb{1}) \mid n_1, ..., n_k \geq 1\}$$

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with vacuum vector $h_{-1}(\mathbb{1})$, translation operator $T = \sum_{n\in\mathbb{Z}} h_{-n-1}h_n$, and the field for $h_{-1}(\mathbb{1})$ defined by:

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By the reconstruction theorem, the other fields are determined by:

$$Y(h_{-n_1}...h_{-n_k}(\mathbb{1})) = (h *_{n_1} (h *_{n_2} ...(h *_{n_k} I)))(z)$$

Example 2: Lattice vertex algebras

Let N be even and $L = \sqrt{N}\mathbb{Z}$ the one-dimensional lattice. The lattice vertex algebra V_L is vector space:

$$V_L = \bigoplus_{\lambda \in L} M_\lambda$$

where $M_\lambda = M_0$ for all $\lambda \in L$.

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$$V_L = \bigoplus_{\lambda \in L} M_\lambda$$

where $M_\lambda = M_0$ for all $\lambda \in L$. The field for $|\lambda\rangle \in M_\lambda$ is given by:

$$V_\lambda(z) = S_\lambda z^{\lambda h_0} exp(-\lambda \sum_{n<0} \frac{h_n}{n} z^n) exp(-\lambda \sum_{n>0} \frac{h_n}{n} z^n)$$

called the "bosonic vertex operator".

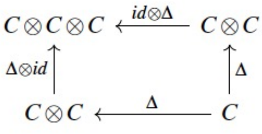
Lattice vertex algebras

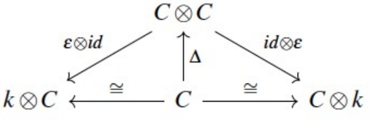
Lemma (Borcherds, Quantum vertex algebras)

The lattice vertex algebra V_L has the universal measuring algebra $\beta(H, \mathbb{C}[L])$ as the underlying vector space, where $H = \mathbb{C}[x]$ is the Hopf algebra associated to the algebraic group $(\mathbb{C}, +)$ and $\mathbb{C}[L]$ is the group ring of L.

Definition

A **coalgebra** is the triple (C, Δ, ϵ) with C a vector space, a map $\Delta : C \to C \otimes C$ called *comultiplication*, and a map $\epsilon : C \to k$ called the *counit*, making the following diagrams commute:





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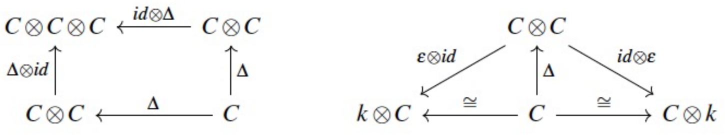

Notation

For $x \in C$, we denote the image of the comultiplication by:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$$

Definition

The group-like elements of a coalgebra are the elements $x \in C$ satisfying:

$$\Delta(x) = x \otimes x, \quad \epsilon(x) = 1$$

Denote $G(C)$ the set of group-like elements of C.

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Example

Let S be a set and k a field. Denote kS the vector space with S as a basis. The comultiplication and counit are defined by:

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There is a bijection:

$$(kS)^* = Hom_k(kS, k) \cong Maps(S, k)$$

that identifies $(kS)^*$ as an algebra.

Definition

The Sweedler dual space is defined for an algebra A by:

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Theorem

The Sweedler dual $(\ )^\circ$ is a functor from the category of algebras to the category of coalgebras that is adjoint to the dual $(\ )^*$. In other words, we have the adjuction:

$$Alg_k(A, C^*) \cong CoAlg_k(C, A^\circ)$$

Definition

A **bialgebra** is the data $(H, \nabla, \eta, \Delta, \epsilon)$ where (H, ∇, η) is an algebra and (H, Δ, ϵ) is a coalgebra such that Δ, ϵ are algebra maps.

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Definition

A **Hopf algebra** is a bialgebra with an antipode $S : H \to H$.

Example

The polynomial algebra $\mathbb{C}[x]$ forms a Hopf algebra with:

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \Delta(1) = 1 \otimes 1$$
$$\epsilon(x) = 0, \quad \epsilon(1) = 1$$
$$S(1) = 1, \quad S(x) = -x$$

Lemma

There is a bijection:

$$G(H^\circ) \cong CoAlg_k(k, H^\circ) \cong Alg_k(H, k)$$

and the set $Alg_k(H, k)$ forms an algebraic group with the convolution product defined by:

$$(f * g)(x) = \sum f(x_{(1)})g(x_{(2)})$$

Proposition

There is a functor $G(\_^\circ)$ from the category of Hopf algebras to the category of algebraic groups:

$$G(\_^\circ) : \mathbf{Hopf}_k \to \mathbf{AlgGrp}.$$

Example

The associated Hopf algebra to the algebraic additive group $(\mathbb{C}, +)$ is given by $(\mathbb{C}[x], \Delta, \epsilon)$ where $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\epsilon(x) = 0$.

Universal measuring algebra

Suppose A is an algebra, B is a commutative algebra and C is a coalgebra:

$$\Phi : Hom_k(A \otimes C, B) \xrightarrow{\cong} Hom_k(A, Hom_k(C, B))$$

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Question

What maps in $Hom_k(A \otimes C, B)$ correspond to algebra maps in $Hom_k(A, Hom_k(C, B))$?

Proposition

Suppose $\psi \in Hom_k(A \otimes C, B)$. Then $\rho := \Phi(\psi) \in Hom_k(A, Hom_k(C, B))$ is an algebra morphism if and only if ψ makes the following diagrams commute:

$$\begin{array}{ccccc}
A \otimes A \otimes C & \xrightarrow{id_{A \otimes A} \otimes \Delta} & A \otimes A \otimes C \otimes C & \xrightarrow{\cong} & (A \otimes C) \otimes (A \otimes C) \\
\downarrow{\scriptstyle \nabla \otimes id} & & & & \downarrow{\scriptstyle \psi \otimes \psi} \\
A \otimes C & \xrightarrow{\psi} & B & \xleftarrow{\nabla} & B \otimes B
\end{array}$$

$$\begin{array}{ccc}
C \cong C \otimes k & \xrightarrow{id_C \otimes \eta} & C \otimes A \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \psi} \\
k & \xrightarrow{\eta} & B
\end{array}$$

Definition

The pair (C, ψ), where C is a coalgebra and $\psi : A \otimes C \to B$ is a map satisfying the previous diagrams, is said to **measure** A to B.

Theorem

Suppose A is a commutative algebra and C is a cocommutative coalgebra. Then there exists a commutative algebra $\beta(C, A)$ with a measuring:

$$\theta : A \otimes C \to \beta(C, A)$$

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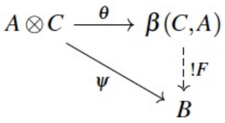
which is universal, in the sense that for any $\psi \in Meas(A, C; B)$ there is a unique algebra morphism $F : \beta(C, A) \to B$ that makes the following diagram commute:



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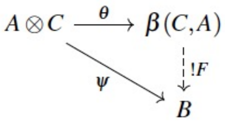

This algebra with the morphism θ is defined to be the **universal measuring algebra**.

Proof

Construct $\beta(C, A)$ as a quotient of $Sym(C \otimes A)$ making the following diagrams commute:

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$$\begin{array}{ccccc}
A \otimes A \otimes C & \xrightarrow{id_{A \otimes A} \otimes \Delta} & A \otimes A \otimes C \otimes C & \xrightarrow{\cong} & (A \otimes C)^{\otimes 2} \\
\downarrow{\scriptstyle \nabla \otimes id} & & & & \downarrow{\scriptstyle i \otimes i} \\
A \otimes C & \xhookrightarrow{i} & Sym(A \otimes C) & \xleftarrow{m} & Sym(A \otimes C)^{\otimes 2}
\end{array}$$

$$\begin{array}{ccc}
C \cong C \otimes k & \xrightarrow{id_C \otimes \eta} & C \otimes A \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle i} \\
k & \xrightarrow{\eta} & Sym(A \otimes C)
\end{array}$$

Universal property

$$Alg_k(\beta(C, A), k) \cong Alg_k(A, C^*)$$

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Lemma

This induces the bijection:

$$Hom_{Sch_k}(Spec(k), Spec(\beta(C, A)) \cong Hom_{Sch_k}(Spec(C^*), Spec(A))$$

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Lemma

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Example

Consider the scheme $X = Spec(A)$ and $C = (k[t]/t^2)^*$. Then the scheme $TX = Spec(\beta(C, A))$ has the k-points given by:

$$(TX)_k = Hom_{Sch_k}(Spec(k[t]/t^2), X) = Alg_k(A, C^*)$$

Jets

Definition

The **n-jet algebra** is defined $J_nA = \beta(C, A)$ for $C = (k[t]/(t^{n+1}))^*$, and observe that the scheme $J_nX = Spec(J_nA)$ has the k-points:

$$(J_nX)_k = \mathbf{Alg}_k(A, (k[t]/(t^{n+1}))$$

Example

- Consider an one-dimensional lattice L, its group ring $A = \mathbb{C}[L]$ and the scheme $X = Spec(A)$.

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- Consider an one-dimensional lattice L, its group ring $A = \mathbb{C}[L]$ and the scheme $X = Spec(A)$.
- Recall the Hopf algebra $H = \mathbb{C}[x]$ viewed as a coalgebra.
- The universal measuring algebra $\beta(H, A)$ can be considered the coordinate ring of the infinite jet space of the scheme X.

- Identify $H_n = \mathbb{C}[x]_{\leq n} \cong (\mathbb{C}[t]/t^{n+1})^*$ then $H \cong \lim_{n \to \infty} H_n$.

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- Consider the scheme $J_nX = Spec(J_nA) = Spec(\beta(H_n, A))$
- The inverse limit $J_\infty X = \varprojlim J_nX$ identifies the scheme $Spec(\beta(H, C))$ as the infinite jet scheme.

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- Take the group ring $A = \mathbb{C}[L]$ and construct the universal measuring algebra $\beta(H, A)$.

Summary

- Take an even integral one-dimensional lattice L.
- Take the algebraic additive group $(\mathbb{C}, +)$ and identify the associated Hopf algebra $H = \mathbb{C}[x]$.
- Take the group ring $A = \mathbb{C}[L]$ and construct the universal measuring algebra $\beta(H, A)$.
- The algebra $\beta(H, A)$ then forms the underlying vector space of the lattice vertex algebra which can be described as the coordinate ring of the infinite jet space of the scheme $Spec(A)$.

References

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