# Minimal models for MFs 9 (checked)

ainfmf9

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(Note that throughout all $W \in m^3$).

We continue ainfmf7 with a more careful study of the symmetry factors that were left out of the Feynman diagrams. Our starting point is (2.1) there, namely

$$(Hd_{End})^m \beta(w \otimes f\Psi) = \sum_{\underline{j},\underline{z}} \sum_{\underline{\gamma}} (-1)^m C_{p+b}(|\gamma_1|+1, \ldots, |\gamma_m|+1) \prod_{i=1}^{m} \left( W^{j_i}(\gamma_i + e_{z_i}) [(\gamma_i)_{z_i}+1] x^{\gamma_i} \right) \prod_{i=1}^{m} \left( [\psi_{j_i}, -]\theta_{z_i} \right) \beta(w \otimes f\Psi) \tag{1.1}$$

where $\beta$ is either $H$ or $\sigma$ and

$$H_\infty = \sum_{m \geq 0} (-1)^m (Hd_{End})^m H$$
$$\sigma_\infty = \sum_{m \geq 0} (-1)^m (Hd_{End})^m \sigma \tag{1.2}$$

Other notation is as on p. 1 of ainfmf7. The "symmetry factors" are the coefficients

$$C_{p+b}(l_1, \ldots, l_m) = \frac{1}{p+b+l_m} \frac{1}{p+b+l_m+l_{m-1}} \cdots \frac{1}{p+b+l_1+\cdots+l_m} \tag{1.3}$$

present in (1.1).

<u>Example</u> Set $W = y^d - x^d$ so $W_1 = -x^{d-1}$, $W_2 = y^{d-1}$ and consider $\delta_{\infty}$. The first two terms in the expansion (i.e. $m=1$ and $2$) are (omitting the global sign) as follows.

For $m=1$ we have

$$(H d_{End})\,\delta(\Psi) = \sum_j \sum_z \sum_{\gamma} (-1)\, C_0(|\gamma|+1)\, W^j(\gamma + e_z)\,[(\gamma)_z + 1]\, x^{\gamma}\, [\psi_j, -]\, \theta_z(\Psi) \tag{2.1}$$

First consider the case where $d=3$, so $W_1 = -x^2$ and $W_2 = y^2$. Now only $j=1, z=1, \gamma=(1,0)$ and $j=2, z=2, \gamma=(0,1)$ contribute, and in both cases $C_0(|\gamma|+1) = C_0(2) = \frac{1}{2}$. This cancels the $[(\gamma)_z+1] = 2$ factor, so

$$(H d_{End})\,\delta(\Psi) = -\Big[ W^1(2,0)\, x\, [\psi_1, -]\, \theta_1 + W^2(0,2)\, y\, [\psi_2, -]\, \theta_2 \Big](\Psi) \tag{2.2}$$

Now consider the $d=4$ case. The formula (2.1) becomes nonzero only when $j=1, z=1, \gamma=(2,0)$ and $j=2, z=2, \gamma=(0,2)$. In both cases $C_0(|\gamma|+1) = C_0(d-1) = \frac{1}{d-1}$ which cancels $[(\gamma)_z+1]$.

So for any value of $d$ we get

$$(H d_{End})\,\delta(\Psi) = -\Big[ W^1(d-1,0)\, x^{d-2}\, [\psi_1, -]\, \theta_1 + W^2(0,d-1)\, y^{d-2}\, [\psi_2, -]\, \theta_2 \Big](\Psi) \tag{2.3}$$

$$= -\Big[ -x^{d-2} [\psi_1, -]\, \theta_1 + y^{d-2} [\psi_2, -]\, \theta_2 \Big](\Psi).$$

For $m=2$ we have

(3.1)

$$(\mathrm{HdEnd})^2 \delta(\Psi) = \sum_{j_1,j_2} \sum_{z_1,z_2} \sum_{\gamma_1,\gamma_2} C_0(|\gamma_1|+1, |\gamma_2|+1)$$

$$W^{j_1}(\gamma_1+e_{z_1})\, W^{j_2}(\gamma_2+e_{z_2}) \left[(\gamma_1)_{z_1}+1\right]\left[(\gamma_2)_{z_2}+1\right]$$

$$x^{\gamma_1} x^{\gamma_2}\, [\Psi_{j_1}, -]\, O_{z_1}\, [\Psi_{j_2}, -]\, O_{z_2}\, (\Psi)$$

Again the only $\gamma$'s that can be nonzero here have $|\gamma| = d-2$ so

$$C_0(|\gamma_1|+1, |\gamma_2|+1) = C_0(d-1, d-1)$$

(3.2)

$$= \frac{1}{d-1} \cdot \frac{1}{2(d-1)} = \frac{1}{2} \cdot \frac{1}{(d-1)^2}$$

while $[(\gamma_1)_{z_1}+1][(\gamma_2)_{z_2}+1]$ will always be $(d-1)^2$ so overall the factor is $1/2$. We get nonzero contributions from

(3.3)

$$j_1 = 1,\ z_1 = 1,\ \gamma_1 = (d-2, 0) \ \& \ j_2 = 2,\ z_2 = 2,\ \gamma_2 = (0, d-2)$$

$$j_1 = 2,\ z_1 = 2,\ \gamma_1 = (0, d-2) \ \& \ j_2 = 1,\ z_2 = 1,\ \gamma_2 = (d-2, 0)$$

each of which contribute the same term

$$\frac{1}{2} W^1(d-1, 0)\, W^2(0, d-1)\, [\Psi_1, -]\, O_1\, [\Psi_2, -]\, O_2\, (\Psi) \cdot x^{d-2} y^{d-2}$$

So overall this doubling cancels with the $1/2$, so

(3.4)

$$(\mathrm{HdEnd})^2 \delta(\Psi) = -\left[\, x^{d-2} y^{d-2}\, [\Psi_1, -]\, O_1\, [\Psi_2, -]\, O_2 \right]$$

We see how the general case works. For general $m$, the sum restricts to $\underline{j} = (j_1, \dots, j_m)$ and $\underline{z} = (z_1, \dots, z_m)$ **equal**, both to $(\sigma(1), \dots, \sigma(m))$ for some $\sigma \in S_m$. Then also

$$\gamma_i = (d-2) \cdot e_{\sigma(i)}$$

and hence

$$C_0(|\gamma_1|+1, \dots, |\gamma_m|+1) = C_0(\overbrace{d-1, \dots, d-1}^{m})$$

$$= \frac{1}{d-1} \cdot \frac{1}{2(d-1)} \cdots \frac{1}{m(d-1)}$$

$$= \frac{1}{(d-1)^m} \frac{1}{m!}$$

Then the $(d-1)^m$ cancels with $[(\gamma_1)_{z_{\sigma(1)}}+1] \cdots [(\gamma_m)_{z_{\sigma(m)}}+1] = (d-1)^m$ and the $\frac{1}{m!}$ cancels with the repetitions. Of course here everything is zero for $m > 2$, but the above would apply e.g. to $W = x_1^d + \dots + x_n^d$. More precisely

__Lemma__ For $W = \sum_{i=1}^n \lambda_i x_i^d$ where $\lambda_i \in k$ (possibly zero) we have for $m \leq n$

$$(H d \mathrm{End})^m \delta(\Psi) = (-1)^m \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| = m}} \prod_{i \in I} (\lambda_i x_i^{d-2}) \cdot \prod_{i \in I} ([\psi_i, -]\partial_i)(\Psi)$$

__Proof__ We take $W_i = \lambda_i x_i^{d-1}$. Then the proof follows from the above. $\square$

**lemma** Given integers $a_1, \dots, a_r > 0$

$$\sum_{\sigma \in S_r} \frac{1}{a_{\sigma(1)}} \frac{1}{a_{\sigma(1)} + a_{\sigma(2)}} \cdots \frac{1}{a_{\sigma(1)} + \cdots + a_{\sigma(r)}} = \frac{1}{a_1 \cdots a_r} \qquad (5.1)$$

**Proof** By induction on $r$. For $r = 1$ this is clear. For $r > 1$ let the LHS of (5.1) be denoted $A(a_1, \dots, a_r)$. Then

$$A(a_1, \dots, a_r) = \frac{1}{a_1 + \cdots + a_r} \sum_{\sigma \in S_r} \prod_{i=1}^{r-1} \frac{1}{a_{\sigma(1)} + \cdots + a_{\sigma(i)}}$$

$$= \frac{1}{a_1 + \cdots + a_r} \sum_{\ell=1}^{r} \sum_{\substack{\sigma \in S_r \\ \sigma(r) = \ell}} \prod_{i=1}^{r-1} \frac{1}{a_{\sigma(1)} + \cdots + a_{\sigma(i)}}$$

$$= \frac{1}{a_1 + \cdots + a_r} \sum_{\ell=1}^{r} \sum_{\sigma \in S_{r-1}} \prod_{i=1}^{r-1} \frac{1}{b_{\sigma(1)} + \cdots + b_{\sigma(r-1)}}$$

with $(b_1, \dots, b_{r-1}) = (a_1, \dots, a_r) \setminus a_\ell$

$$\overset{\text{ind.hyp}}{=} \frac{1}{a_1 + \cdots + a_r} \sum_{\ell=1}^{r} \frac{1}{b_1 \cdots b_{r-1}}$$

$$= \frac{1}{a_1 + \cdots + a_r} \sum_{\ell=1}^{r} \frac{1}{a_1 \cdots \hat{a}_\ell \cdots a_r}$$

$$= \frac{\sum_{\ell=1}^{r} a_\ell}{(a_1 + \cdots + a_r) a_1 \cdots a_r} = \frac{1}{a_1 \cdots a_r}. \qquad \square$$

Lemma For $W=\sum_{i=1}^n \lambda_i x_i^{d_i}$ where $\lambda_i \in k$ and $d_i \geq 3$ we have for $m \leq n$

$$(H d_{End})^m \sigma(\Psi) = (-1)^m \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=m}} \prod_{i\in I} (\lambda_i x_i^{d_i-2}) \prod_{i \in I} ([\psi_i, -]\theta_i)(\Psi) \tag{6.1}$$

Proof The only question is the coefficient, which is

$$\sum_{\sigma \in S_m} C_0(d_{\sigma(1)}-1, \dots, d_{\sigma(m)}-1)[d_{\sigma(1)}-1]\cdots[d_{\sigma(m)}-1]$$

$$= 1$$

by the lemma on p. 5. □

where $j_1 < \dots < j_m$
$z_1 < \dots < z_m$

Generally, the sum in (1.1) breaks into sums of permutations of possibly different subsets. Thus with $\underline{j}$, $\underline{z}$ being permutations

so $\underline{j} = (j_{\sigma(1)}, \dots, j_{\sigma(m)})$
$\underline{z} = (z_{\tau(1)}, \dots, z_{\tau(m)})$

$$(H d_{End})^m \beta(\omega \otimes f\Psi) = (-1)^m \sum_{\substack{J \subseteq \{1,\dots,n\} \\ |J|=m}} \sum_{\substack{Z \subseteq \{1,\dots,n\} \\ |Z|=m}} \sum_{\sigma,\tau \in S_m} \tag{6.2}$$

$$\sum_{\gamma} C_{p+b}(|\gamma_1|+1, \dots, |\gamma_m|+1) \prod_{i=1}^m \left( W^{j_{\sigma(i)}}(\gamma_i + e_{z_{\tau(i)}}) [(\gamma_i)_{z_{\tau(i)}} + 1] x^{\gamma_i} \right)$$

$$\prod_{i=1}^m ([\psi_{j_{\sigma(i)}}, -]\theta_{z_{\tau(i)}}) \beta(\omega \otimes f\Psi)$$

$$= (-1)^m \sum_{J,Z} \sum_{\sigma, \tau \in S_m} \sum_{\underline{\gamma}} C_{p+b}(|\gamma_1|+1, \dots, |\gamma_m|+1)$$

$$\prod_{i=1}^{m} \left( W^{j_{\sigma(i)}}(\gamma_i + e_{z_{\tau(i)}}) \left[ (\gamma_i)_{z_{\tau(i)}} + 1 \right] x^{\gamma_i} \right)$$

$$(-1)^{\binom{m}{2}} \prod_{i=1}^{m} [\psi_{j_{\sigma(i)}}, -] \prod_{i=1}^{m} \theta_{z_{\tau(i)}} \, \beta(w \otimes f \Psi) \qquad (7.1)$$

$$= (-1)^m \sum_{J, Z, \sigma, \tau} \sum_{\underline{\gamma}} C_{p+b}(\cdots) \prod_{i=1}^{m} \left( W^{j_{\sigma(i)}}(\gamma_i + e_{z_{\tau(i)}}) \left[ (\gamma_i)_{z_{\tau(i)}} + 1 \right] x^{\gamma_i} \right)$$

$$(-1)^{\binom{m}{2} + |\sigma| + |\tau|} \prod_{i=1}^{m} [\psi_{j_i}, -] \prod_{i=1}^{m} \theta_{z_i} \cdot \beta(w \otimes f \Psi)$$

$$= \overset{(-1)^m}{\sum_{J,Z}} \left\{ \sum_{\sigma, \tau, \underline{\gamma}} (-1)^{|\sigma|+|\tau|} C_{p+b}(|\gamma_1|+1, \dots) \prod_{i=1}^{m} W^{j_{\sigma(i)}}(\gamma_i + e_{z_{\tau(i)}}) \left[ (\gamma_i)_{z_{\tau(i)}} + 1 \right] x^{\gamma_i} \right\} \prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_i} \beta(w \otimes f \Psi)$$

$$= \overset{(-1)^m}{\sum_{J,Z}} \left\{ \sum_{\sigma, \tau, \underline{\gamma}} (-1)^{|\sigma|+|\tau|} C_{p+b}(|\gamma_1|+1, \dots) \prod_{i=1}^{m} W^{j_i}(\gamma_{\sigma^{-1}(i)} + e_{z_{\tau\sigma^{-1}(i)}}) \left[ (\gamma_{\sigma^{-1}(i)})_{z_{\tau\sigma^{-1}(i)}} + 1 \right] x^{\gamma_{\sigma^{-1}(i)}} \right\} \prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_i} \beta$$

$$= \overset{(-1)^m}{\sum_{J,Z}} \left\{ \sum_{\sigma, \tau, \underline{\gamma}} (-1)^{|\sigma|+|\tau|} C_{p+b}(|\gamma_{\sigma(1)}|+1, \dots, |\gamma_{\sigma(m)}|+1) \prod_{i=1}^{m} W^{j_i}(\gamma_i + e_{z_{\tau\sigma^{-1}(i)}}) \left[ (\gamma_i)_{z_{\tau\sigma^{-1}(i)}} + 1 \right] x^{\gamma_i} \right\}$$

$$\cdot \prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_i} \beta(w \otimes f \Psi)$$

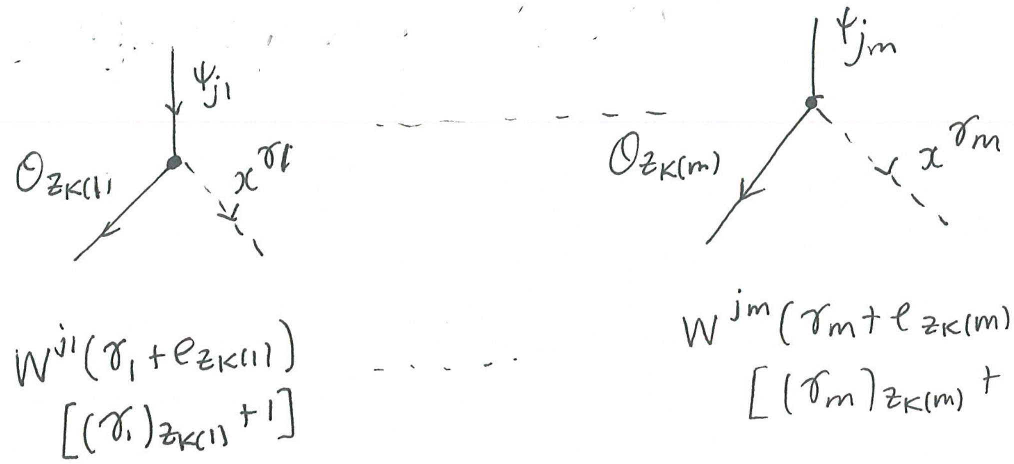
But now we can reindex to sum over $\sigma, \kappa \in S_m$ but using $\kappa\sigma$ in the formula, to obtain

$$(-1)^m \qquad\qquad (8.1)$$

$$= \sum_{J,Z} \Big\{ \sum_{\sigma,\kappa \in S_m} \sum_{\gamma} (-1)^{|\sigma|+|\kappa|+|\sigma|} C_{p+b}(|\gamma_{\sigma(1)}|+1, \dots) \prod_{i=1}^m \{ W^{ji}(\gamma_i + e_{z_{\kappa(i)}}) [(\gamma_i)_{z_{\kappa(i)}} + 1] x^{\gamma_i} \} \prod_{i=1}^m [\psi_{ji}, -] \theta_{z_i} \beta(w \otimes f \Upsilon)$$

$$(-1)^m$$

$$= \sum_{J,Z} \Big\{ \sum_{\kappa \in S_m} \sum_{\gamma} \Big( \sum_{\sigma \in S_m} C_{p+b}(|\gamma_{\sigma(1)}|+1, \dots, |\gamma_{\sigma(m)}|+1) \Big) \prod_{i=1}^m \{ W^{ji}(\gamma_i + e_{z_{\kappa(i)}}) [(\gamma_i)_{z_{\kappa(i)}} + 1] x^{\gamma_i} \} \prod_{i=1}^m [\psi_{ji}, -] \theta_{z_{\kappa(i)}} \beta(w \otimes f \Upsilon) \Big\}$$



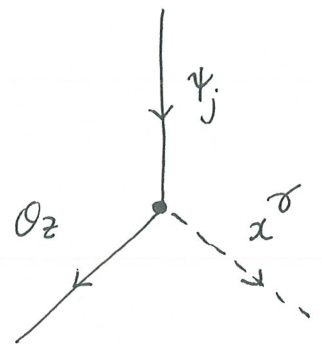

(in the last line we swallow the $(-1)^{|\kappa|}$ and rearrange $\theta$'s.)

<u>Lemma</u> In general, for $m \le n$

$$(H d_{\mathrm{End}})^m \delta(\Psi) = (-1)^m \sum_{\substack{J \subseteq \{1,\dots,n\} \\ |J| = m}} \ \sum_{\substack{Z \subseteq \{1,\dots,n\} \\ |Z| = m}} \ \sum_{\kappa \in S_m} \ \sum_{\gamma_1,\dots,\gamma_m \in \mathbb{Z}^n_{\geq 0}} \prod_{i=1}^{m} \Big\{ W^{j_i}(\gamma_i + e_{z_{\kappa(i)}})\, x^{\gamma_i} [\psi_{j_i}, -]\, \theta_{z_{\kappa(i)}} \Big\} \cdot \prod_{i=1}^{m} \frac{(\gamma_i)_{z_{\kappa(i)}} + 1}{|\gamma_i| + 1}\, \delta(\Psi) \tag{9.1}$$

<u>Proof</u> This just applies (5.1) to (8.1), since $p+b=0$. $\square$

Note that the factor $\dfrac{(\gamma_i)_{z_{\kappa(i)}} + 1}{|\gamma_i| + 1}$ is "local", i.e. a vertex




receives a prefactor

$$\frac{\gamma_z + 1}{|\gamma| + 1} W^j(\gamma + e_z)$$

Reset in notation

We have decided to adopt a different starting point to (2.1) of ainfmf7 to express $H_\infty$ and $\partial_\infty$ (although the difference is small). Namely, we compute, with $p=|w|$ and $b=|f|$, as in (2.1) there

$$(\mathrm{Hd}_{\mathrm{End}})^m \beta(w\otimes f\,\Psi) \tag{10.1}$$

$$= \sum_{\underline{j},\,\underline{z}} \sum_{\underline{\gamma}\in \mathbb{Z}^n_{\geq 0}} (-1)^m C_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \partial_{x_{z_i}}\left(W^{j_i}(\gamma_i) x^{\gamma_i}\right) \prod_{i=1}^m [\psi_{j_i}, -]\,\theta_{z_i}\, \beta(w\otimes f\,\Psi)$$

Then we compute as in (6.2),

$$= (-1)^m \sum_{J,\,Z} \sum_{\sigma,\tau\in S_m} \sum_{\underline{\gamma}} C_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \partial_{x_{z_{\tau(i)}}}\left(W^{j_{\sigma(i)}}(\gamma_i) x^{\gamma_i}\right) \prod_{i=1}^m [\psi_{j_{\sigma(i)}}, -]\,\theta_{z_{\tau(i)}}\, \beta(w\otimes f\,\Psi) \tag{10.2}$$

where $J=\{j_1<\cdots<j_m\}$ and $Z=\{z_1<\cdots<z_m\}$ range over all subsets.

$$= (-1)^m \sum_{J,\,Z} \sum_{\sigma,\tau\in S_m} \sum_{\underline{\gamma}} (-1)^{|\sigma|} C_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \partial_{x_{z_{\tau(i)}}}\left(W^{j_{\sigma(i)}}(\gamma_i) x^{\gamma_i}\right) \prod_{i=1}^m [\psi_{j_i}, -]\,\theta_{z_{\tau(i)}}\, \beta(w\otimes f\,\Psi)$$

(10.3)

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(11.1)

$$= (-1)^m \sum_{J,Z} \sum_{\sigma, J \in S_m} \sum_{\underline{\gamma}} (-1)^{|\sigma|} C_{p+b}(|\underline{\gamma}|) \prod_{i=1}^{m} \partial_{x_{z_{J\sigma^{-1}(i)}}} \left( W^{j_i}(\gamma_{\sigma^{-1}(i)}) x^{\gamma_{\sigma^{-1}(i)}} \right)$$

$$\prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_{J(i)}} \beta(w \otimes f \Psi)$$

But we can absorb the $(-1)^{|\sigma|}$ to change $\prod_i \theta_{z_{J(i)}}$ to $\prod \theta_{z_{J\sigma^{-1}(i)}}$ and absorb the permutation of $\gamma$'s, by using $\tilde{\gamma}_i = \gamma_{\sigma(i)}$ in place of $\gamma$ (the sum hits all such the same # of times)

$$= (-1)^m \sum_{J,Z} \sum_{\sigma, J \in S_m} \sum_{\underline{\gamma}} C_{p+b}(|\gamma_{\sigma(1)}|, \dots, |\gamma_{\sigma(m)}|)$$

$$\prod_{i=1}^{m} \partial_{x_{z_{J\sigma^{-1}(i)}}} \left( W^{j_i}(\gamma_i) x^{\gamma_i} \right) \qquad (11.2)$$

$$\prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_{J\sigma^{-1}(i)}} \beta(w \otimes f \Psi)$$

But as we sum over $\sigma, J$ we can make $\sigma$ and $J\sigma^{-1}$ independent by writing $\kappa := J\sigma^{-1}$

$$= (-1)^m \sum_{J,Z} \sum_{\sigma, \kappa \in S_m} \sum_{\underline{\gamma}} C_{p+b}(|\gamma_{\sigma(1)}|, \dots, |\gamma_{\sigma(m)}|) \qquad (11.3)$$

$$\prod_{i=1}^{m} \partial_{x_{z_{\kappa(i)}}} \left( W^{j_i}(\gamma_i) x^{\gamma_i} \right)$$

$$\prod_{i=1}^{m} [\psi_{j_i}, -] \theta_{z_{\kappa(i)}} \beta(w \otimes f \Psi)$$

Def^N Given a sequence $l_1, \dots, l_m \geq 1$ of integers and $\alpha \geq 0$ we define

$$C^{un}_\alpha(l_1, \dots, l_m) := l_1 \cdots l_m \sum_{\sigma \in S_m} C_\alpha(l_{\sigma(1)}, \dots, l_{\sigma(m)}) \tag{12.1}$$

$$= l_1 \cdots l_m \sum_{\sigma \in S_m} \frac{1}{\alpha + l_{\sigma(m)}} \frac{1}{\alpha + l_{\sigma(m-1)} + l_{\sigma(m)}} \cdots \frac{1}{\alpha + l_{\sigma(1)} + \cdots + l_{\sigma(m)}}$$

Obviously this depends only on the set $\{l_1, \dots, l_m\}$ not the ordering.

Then we have shown

Lemma For any $W$ and $m \geq 1$ we have

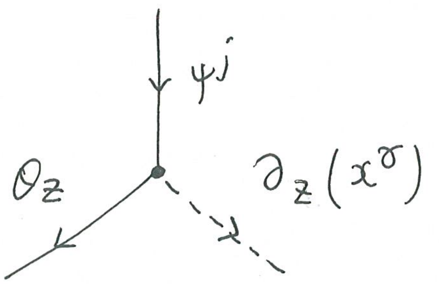
$$(H d_{End})^m \beta(w \otimes f \Psi) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J| = m}} \sum_{\substack{Z \subseteq \{1, \dots, n\} \\ |Z| = m}} \sum_{\pi \in S_m} \sum_{\underline{\gamma} \in \mathbb{Z}^n_{\geq 0}} \tag{12.2}$$

$$C^{un}_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \left( \frac{W^{j_i}(\gamma_i)}{|\gamma_i|} \partial x_{z_{\pi(i)}}(x^{\gamma_i}) \mathcal{O}_{z_{\pi(i)}} [\psi_{j_i}, -] \right)$$

$$\circ \beta(w \otimes f \Psi)$$

Note the $(-1)^m$ is swallowed into $\mathcal{O} \longleftrightarrow [\psi, -]$ reversal.

(12.3)




$(j, z, \gamma)$-interaction

prefactor $\frac{W^j(\gamma)}{|\gamma|}$

As we computed on p. 5,

$$C_0^{un}(\ell_1, .., \ell_m) = 1 \tag{13.1}$$

so (12.2) is much simpler in this case. We conclude ($\partial_i := \partial_{x_i}$)

Lemma For any $W$, (13.2)

$$\delta_\infty = \sum_{m \geq 0} \sum_{\substack{J, \\ |J|=m}} \sum_{\substack{Z, \\ |Z|=m}} \sum_{\kappa \in S_m} \sum_{\underline{\gamma} \in \mathbb{Z}^n_{\geq 0}} (-1)^m \prod_{i=1}^m \left\{ \frac{1}{|\gamma_i|} W^{j_i}(\gamma_i) \partial_{z_{\kappa(i)}} (x^{\gamma_i}) \theta_{z_{\kappa(i)}} [\psi_{j_i}, -] \right\} \delta$$

where $J = \{ j_1 < \cdots < j_m \}$ and $Z = \{ z_1 < \cdots < z_m \}$ range over all subsets of $\{1, \ldots, n\}$. For $m=0$ this is just $\delta$.

Lemma For any $W$, and $p = |w|$, $b = |f|$,

$$H_\infty(w \otimes f \Psi) = \sum_{m \geq 0} \sum_{\substack{J, \\ |J|=m}} \sum_{\substack{Z, \\ |Z|=m}} \sum_{\kappa \in S_m} \sum_{\underline{\gamma} \in \mathbb{Z}^n_{\geq 0}} \sum_{1 \leq t \leq n} \tag{13.3}$$

$$\frac{(-1)^m}{p+b} C^{un}_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \left( \frac{W^{j_i}(\gamma_i) \partial_{z_{\kappa(i)}}(x^{\gamma_i}) \theta_{z_{\kappa(i)}} [\psi_{j_i}, -]}{|\gamma_i|} \right) \circ \theta_t \partial_t (w \otimes f \Psi)$$

Finally,

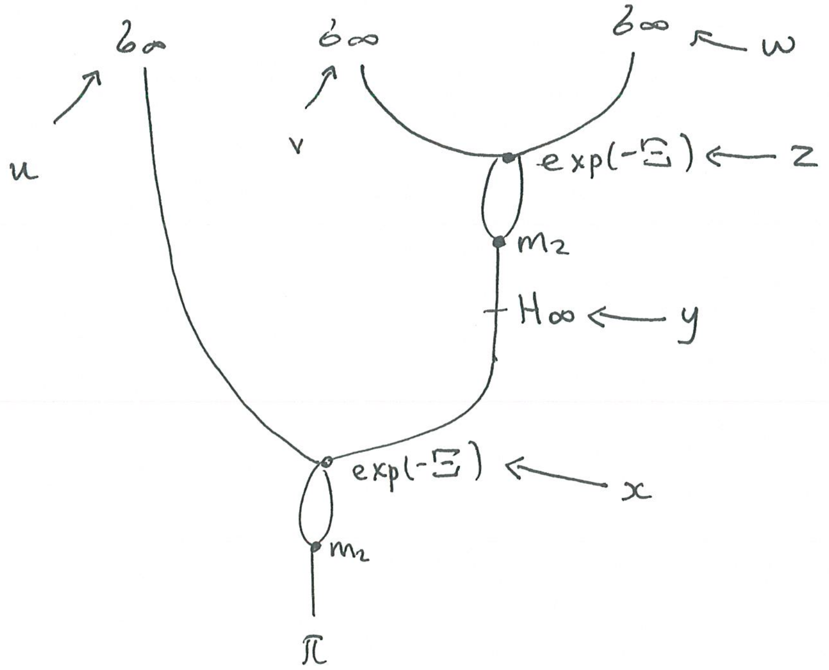
$$\Xi = \sum_j [\psi_j, -] \otimes \theta_j^*$$

$$\exp(-\Xi) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \Big\{ \sum_j [\psi_j, -] \otimes \theta_j^* \Big\}^m \tag{14.1}$$

$$= \sum_{m \geq 0} \frac{1}{m!} \sum_{\substack{J \subseteq \{1, \ldots, n\} \\ |J| = m}} (-1)^m \prod_{i=1}^m \left( [\psi_{j_i}, -] \otimes \theta_{j_i}^* \right) \cdot m!$$

$$= \sum_{m \geq 0} \sum_{\substack{J \\ |J| = m}} (-1)^m \prod_{i=1}^m \left( [\psi_{j_i}, -] \otimes \theta_{j_i}^* \right)$$

Before making a general statement of the Feynman rules let us consider an example:




(14.2)

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From (13.2), (13.3), (14.1) we read off

$(14.2)$ on $\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2$ (note no Koszul signs for moving inputs into place) (15.1)

$$= \pi m_2 \exp(-\Xi)\big(\delta_\infty(\Lambda_0) \otimes H_\infty m_2 \exp(-\Xi)\big(\delta_\infty(\Lambda_1) \otimes \delta_\infty(\Lambda_2)\big)\big)$$

$$= \pi m_2 \sum_{m_1 \geq 0} \sum_{\substack{J_1 \\ |J_1| = m_1}} (-1)^{m_1} \prod_{i_1=1}^{m_1} \big([\psi_{j_{i_1}}, -] \otimes \theta^*_{j_{i_1}}\big)\,(\cdots)$$

This will quickly get ludicrous, so we use "spacetime" labels $x, y, z, u, v, w$ as in (14.2) and e.g. $m(x)$ for the value of $m$ at $x$, $J(x)$, $\underline{\gamma}(y)$, etc.

(15.2)

$$= \pi m_2 (-1)^{m(x)} \prod_{i \in J(x)} \big([\psi_i, -] \otimes \theta_i^*\big)\Big((-1)^{m(u)} \prod_{j \in J(u)} \Big\{ \frac{1}{|\gamma_j(u)|} W^j(\gamma_j(u))$$

$$\partial_{z_{K(j)}}(x^{\gamma_j(u)})\,\theta_{z_{K(j)}}[\psi_j, -]\Big\}(\Lambda_0) \otimes$$

$$\frac{(-1)^{m(y)}}{p(y) + b(y)}\, C^{un}_{p(u)+b(y)}(|\underline{\gamma}(y)|) \prod_{k \in J(y)} \Big( \frac{W^k(\gamma_k(y))}{|\gamma_k(y)|} \partial_{z_{K(k)}}(x^{\gamma_k(y)})\,\theta_{z_{K(k)}}^{[\psi_k, -]}\Big)$$

$$\circ\, m_2 \circ (-1)^{m(z)} \prod_{l \in J(z)} \big([\psi_l, -] \otimes \theta_l^*\big)\Big((-1)^{m(v)} \prod_{j \in J(v)} \Big\{ \frac{1}{|\gamma_j(v)|} W^j(\gamma_j(v))$$

$$\partial_{z_{K(j)}}(x^{\gamma_j(v)})\,\theta_{z_{K(j)}}[\psi_j, -]\Big\}(\Lambda_1) \otimes (-1)^{m(w)} \prod_{j \in J(w)} \Big\{ \frac{1}{|\gamma_j(w)|} W^j(\gamma_j(w))$$

$$\partial_{z_{K(j)}}(x^{\gamma_j(w)})\,\theta_{z_{K(j)}}[\psi_j, -]\Big\}(\Lambda_2)\Big)\Big)$$

This equality denotes an element of $\Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$ and we can project onto $k \cdot 1$. We call this <u>imposing vacuum boundary conditions</u>. Once we have done this, $\psi_i^*$, $\theta_i$ and $x_i$ all annihilate with $\pi$ (these are our creation operators) and we compute (15.2) by commuting all such operators through to the left.

Note that unlike Feynman diagrams the motion of operators is constrained by the tree. Observe that at a junction

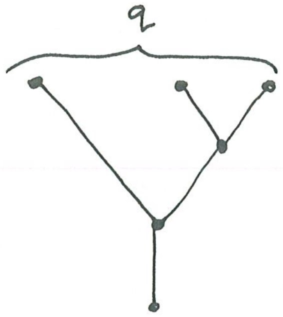
$$(S \otimes \mathrm{End})^{\otimes 2} \longrightarrow S \otimes \mathrm{End}$$

$$\begin{aligned}
& m_2([\psi_i, -] \otimes \theta_i^*)(1 \otimes \theta_j) \\
&= m_2([\psi_i, -] \otimes \theta_i^* \theta_j) \\
&= m_2([\psi_i, -] \otimes 1) - m_2([\psi_i, -] \otimes \theta_j \theta_i^*) \\
&= m_2([\psi_i, -] \otimes 1) + m_2((1 \otimes \theta_j) \circ ([\psi_i, -] \otimes \theta_i^*)) \\
&= m_2([\psi_i, -] \otimes 1) + \theta_j m_2([\psi_i, -] \otimes \theta_i^*) \cdot (-1)^?
\end{aligned} \tag{16.1}$$

Note the sign, which depends on the inputs to $m_2$. So our strategy should be to "empty left legs before right legs".

Note that in (15.2), $p(y)$ and $b(y)$ denote resp. the incoming $\theta$ and poly degree at $y$. This is determined once assignments of $m$'s, and $\sigma$'s are made at all interaction vertices.

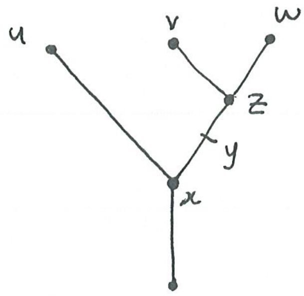
Feynman rules I  Given a planar binary rooted tree $T$




(17.1)

we denote the number of inputs by $q$. The number of internal edges is $e$.

To each input, internal vertex and internal edge we assign a non-negative integer $m \geq 0$. To help with the description we label all such locations with letters $x, y, z, \ldots$ as shown:




(17.2)

and we denote the integer at a location $x$ by $m(x)$. Next, to each labelled location $x$ with integer $m(x)$ we assign a subset (unordered)

$$J(x) \subseteq \{1, \ldots, n\} \qquad |J(x)| = m(x). \tag{17.3}$$

If further the location is an input vertex or an internal edge we have for each $j \in J(x)$ a tuple

$$\big( a_j(x), \gamma_j(x) \big), \qquad a_j(x) \in \{1, \ldots, n\}, \quad \gamma_j(x) \in \mathbb{Z}^n_{\geq 0} \tag{17.4}$$

Finally, each internal edge $\hat{x}$ is assigned an integer $t(x) \in \{1, \ldots, n\}$.

Such an assignment $\mathcal{C}$ is called a <u>configuration</u> of $T$, and given $\mathcal{C}$ we may write $m^{\mathcal{C}}(\alpha)$, $J^{\mathcal{C}}(\alpha), \ldots$ for the data or leave the superscript implicit. Let $\mathrm{Con}(T)$ denote the set of all configurations.

<u>Def$^N$</u> Given a planar binary rooted tree $T$ and $\mathcal{C} \in \mathrm{Con}(T)$, we define the associated $k$-linear operator $\mathcal{O}(T, \mathcal{C})$

$$\mathcal{O}(T, \mathcal{C}) : \mathcal{A}^{\otimes q} \longrightarrow \mathcal{A} \tag{18.1}$$

$$\mathcal{A} = \bigwedge(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$$

by assigning

- to each input vertex labelled by $m$, $J \subseteq \{1, \ldots, n\}$ and for $j \in J$ by $(a_j, \gamma_j)$ the operator on $S \otimes_k \mathrm{End}_k(k^{stab})$

$$(-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \mathcal{O}_{a_j} [\psi_j, -] \Big\} \tag{18.2}$$

Note the operators under the product are even, so the order is irrelevant.

- to each internal edge labelled with $m$, $J$, $(a_j, \gamma_j)$ and $t$ the operator on $S \otimes_k \mathrm{End}_k(k^{stab})$

$$(-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \mathcal{O}_{a_j} [\psi_j, -] \Big\} \circ \mathcal{O}_t \partial_t \tag{18.3}$$

- to each internal vertex labelled with $m, J$ the operator

$$m_2 \circ \prod_{j \in J} \{ [\psi_j, -] \otimes \Theta_j^{+} \} \cdot (-1)^m \qquad (19.1)$$

which is a map $(S \otimes \mathrm{End})^{\otimes 2} \longrightarrow (S \otimes \mathrm{End})^{\otimes 2} \xrightarrow{m_2} S \otimes \mathrm{End}$.

- to the root vertex the map $\pi$.

Together these assignments determine a $k$-linear map

$$\mathcal{O}^{pre}(T, \mathcal{C}) : \mathcal{A}^{\otimes q} \longrightarrow \mathcal{A}.$$

This linear map does <u>not</u> have signs for moving inputs to their "starting positions", e.g. how (14.2) on $\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2$ in (15.2) does not have any "initial" Koszul signs.

Then, for each internal edge $x$ we define an integer $\omega(x) \geq 0$ which is a property of the configuration $\mathcal{C}$. This is defined by

$$\omega(x) = \sum_{y < x} \sum_{j \in J(y)} |\gamma_j(y)| - \sum_{z < x} m(z) \qquad (19.3)$$

where the sum is over all internal edges and input vertices $y$ "above" $x$ in the tree, i.e. for which $x$ is on the unique path from $y$ to the root. (and $z$ ranges over internal vertices)

Then we define

$$F(x) := \frac{1}{\omega(x)} C^{un}_{\omega(x)}\left( \{ |\gamma_j(x)| \}_{j \in J(x)} \right) \qquad (19.4)$$

not really as a set, as an ordered set but whose order is irrelevant

(if $m(x) = 0$ this is just $\frac{1}{\omega(x)}$).

Then

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$$\mathcal{O}(T,\mathcal{C}) := \prod_{\text{int. edge } x} F(x) \cdot \mathcal{O}^{pre}(T,\mathcal{C}) \qquad (20.1)$$

It remains to relate this to the forward suspended higher multiplications of ainfmf2. After splitting the idempotent of p. 5 ainfmf4 (here we use $W \in \mathfrak{m}^3$) these are degree $+1$ linear maps

$$\rho : \mathcal{A}[1]^{\otimes q} \longrightarrow \mathcal{A}[1] \qquad (20.2)$$

defined for $q \geq 2$ by p. 14, 21 ainfmf2 ($T$ ranges over trees with $q$ leaves)

$$\rho_q(\Lambda_q \otimes \cdots \otimes \Lambda_1) = \sum_T \rho_T(\Lambda_q \otimes \cdots \otimes \Lambda_1) \qquad (20.3)$$

$$= \sum_T (-1)^{s(T,\underline{\Lambda})} \operatorname{eval}_{\hat{T}}(\Lambda_1 \otimes \cdots \otimes \Lambda_q) \quad \text{⌜edits 2/11/2016⌟}$$

where the sign factor is (recall $\tilde{\Lambda} = |\Lambda| + 1$)

$$s(T,\underline{\Lambda}) = \cancel{1 + \sum_{j \geq 1} \binom{M_j}{2}} + \sum_{1 \leq i < j \leq q} \tilde{\Lambda}_i \tilde{\Lambda}_j + \sum_{i=1}^{q} \tilde{\Lambda}_{q-i+1} P_i \qquad (20.4)$$

$e_i(T) + q + 1$

where

- $M_j$ is the number of internal vertices in $T$ a distance $j$ from the root
- $P_i$ is the number of times the path from the $i$th input (where ~~$\Lambda_{q-i+1}$~~ goes) to the root enters a trivalent vertex from the right, in $T$.

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Now $\mathrm{eval}_{\hat{T}}(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$ is defined on the minor tree $\hat{T}$ by assigning

- $z_\infty$ to all leaves
- $m_2 \exp(-\Xi)$ to all internal vertices
- $H_\infty$ to all internal edges (20.5.1)
- $\pi$ to the root.

and then evaluating this tree on $\Lambda_1 \otimes \cdots \otimes \Lambda_q$ (without initial Koszul signs for moving inputs into place, i.e. treating the diagram as a diagram of linear maps without attention to grading).
But we have shown for any tree $T$ that

$$\mathrm{eval}_T(\Lambda_1 \otimes \cdots \otimes \Lambda_q) \tag{20.5.2}$$
$$= \sum_{\mathcal{C} \in \mathrm{Con}(T)} \mathcal{O}(T, \mathcal{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$$

with the defn of (20.1). Hence

$$\rho_q(\Lambda_q \otimes \cdots \otimes \Lambda_1) = \sum_T (-1)^{s(T,\underline{\Lambda})} \sum_{\mathcal{C} \in \mathrm{Con}(\hat{T})} \mathcal{O}(\hat{T}, \mathcal{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$$

<u>Theorem</u> $(\mathcal{A}, \{\rho_q\}_{q \geq 2})$ is the minimal model of $\mathrm{End}_R(k^{\mathrm{stab}})$ (for $W \in \mathfrak{m}^3$).

It remains to give the Feynman rules for $\mathcal{O}(T, \mathcal{C})$.

The only potential confusion is from <u>signs</u>, because of all the pesky fermions. We rewrite (18.2) as

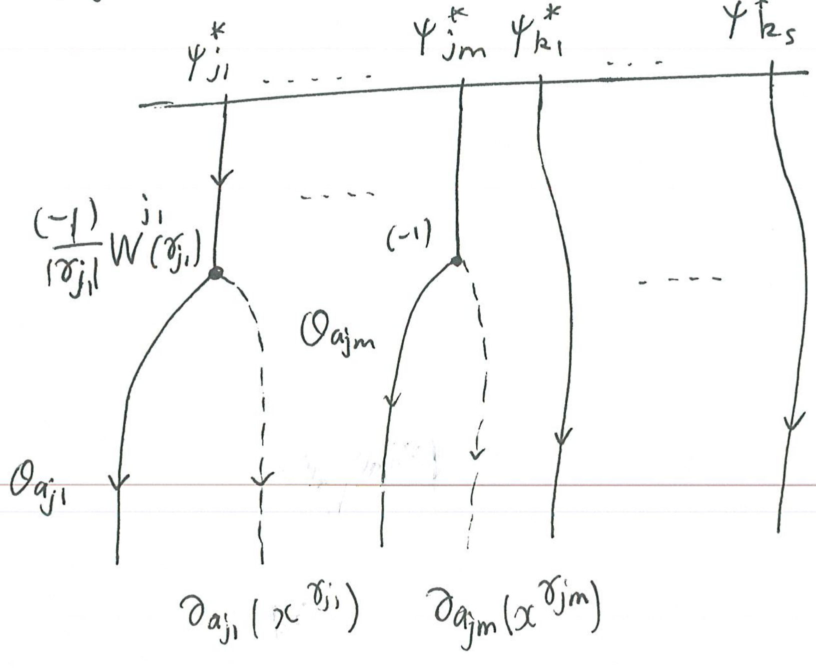
$$(-1)^m (-1)^{\binom{m}{2}} \prod_{j\in J} \mathcal{O}_{a_j} \cdot \prod_{j\in J} \left\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) [\psi_j, -] \right\} \qquad (21.1)$$

Here the order of the products must be the <u>same</u>, but is otherwise arbitrary. Then the annihilations $[\psi_j,-]$ pair with creations $\psi_i^*$ and the remainder of the $\psi_i^*$ and $\mathcal{O}$'s continue on without further intervention of signs. If we choose an ordering $J = (j_1 \cdots j_m)$ then

$$(21.1) = (-1)^m \mathcal{O}_{a_{j_1}} \cdots \mathcal{O}_{a_{j_m}} \left( \frac{1}{|\gamma_{j_m}|} W^{j_m}(\gamma_{j_m}) \partial_{a_{j_m}}(x^{\gamma_{j_m}}) [\psi_{j_m}, -] \right) \cdots \left( \frac{1}{|\gamma_{j_1}|} W^{j_1}(\gamma_{j_1}) \partial_{a_{j_1}}(x^{\gamma_{j_1}}) [\psi_{j_1}, -] \right) \qquad (21.2)$$

which, applied to inputs $\psi_{j_1}^* \cdots \psi_{j_m}^* \psi_{k_1}^* \cdots \psi_{k_s}^*$ yields

$$(-1)^m \prod_j \left\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \right\} \cdot \mathcal{O}_{a_{j_1}} \cdots \mathcal{O}_{a_{j_m}} \psi_{k_1}^* \cdots \psi_{k_s}^* \qquad (21.3)$$




(21.4)

Similarly we handle (18.3) by putting the $\theta_t \partial_t$ on the far left but otherwise copying (21.4).

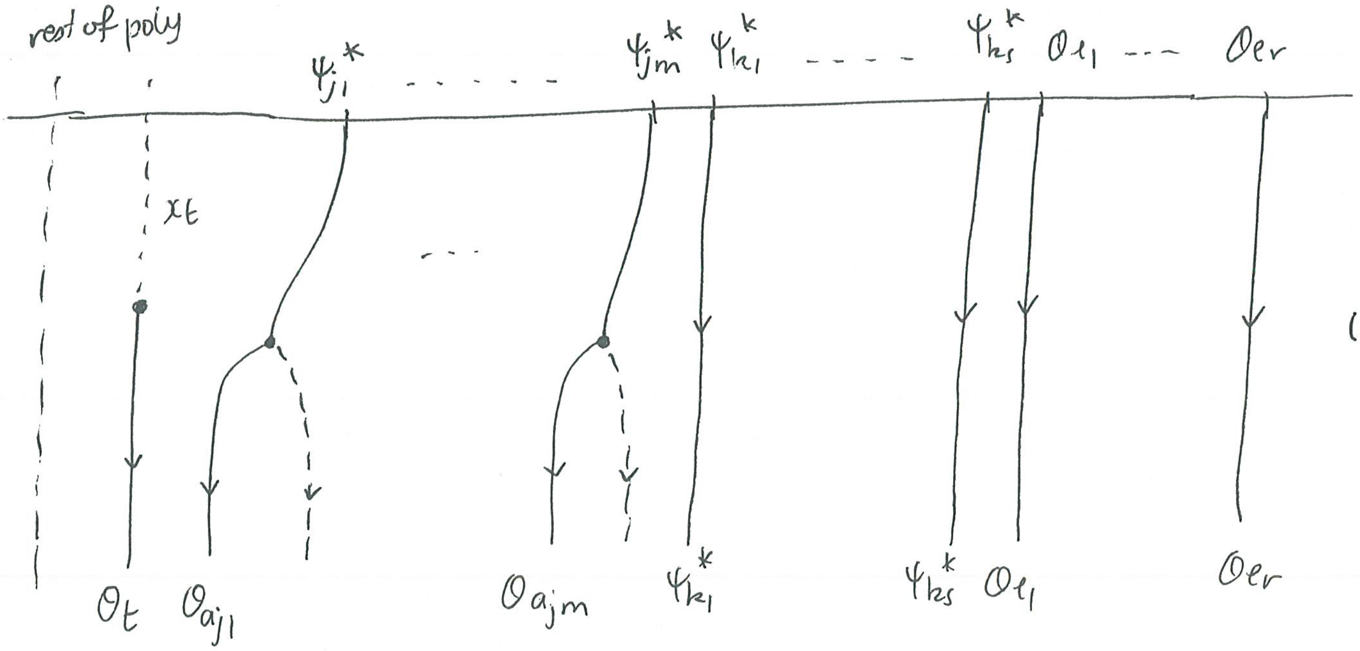
$$(18.3) \text{ on } \psi_{j_1}^* \cdots \psi_{j_m}^* \psi_{k_1}^* \cdots \psi_{k_s}^* \theta_{\ell_1} \cdots \theta_{\ell_r} \cdot f(\underline{x})$$

$$= (-1)^m \theta_t \theta_{a_{j_1}} \cdots \theta_{a_{j_m}} \Big( \frac{1}{|\gamma_{j_m}|} W^{j_m}(\gamma_{j_m}) \partial_{a_{j_m}}(x^{\gamma_{j_m}}) [\psi_{j_m}, -] \Big) \cdots \tag{22.1}$$

$$\Big( \frac{1}{|\gamma_{j_1}|} W^{j_1}(\gamma_{j_1}) \partial_{a_{j_1}}(x^{\gamma_{j_1}}) [\psi_{j_1}, -] \Big)$$

$$\partial_t(f) \cdot \big( \psi_{j_1}^* \cdots \psi_{j_m}^* \psi_{k_1}^* \cdots \psi_{k_s}^* \theta_{\ell_1} \cdots \theta_{\ell_r} \big)$$

$$= (-1)^m \prod_j \Big\{ \frac{W^j(\gamma_j)\partial_{a_j}(x^{\gamma_j})}{|\gamma_j|} \Big\} \partial_t(f)\, \theta_t \theta_{a_{j_1}} \cdots \theta_{a_{j_m}} \psi_{k_1}^* \cdots \psi_{k_s}^* \theta_{\ell_1} \cdots \theta_{\ell_r}.$$




(22.2)

We get one contribution of such a diagram for every different way of choosing an $x_t$ from $f$, i.e. the degree of $x_t$ in $f$.

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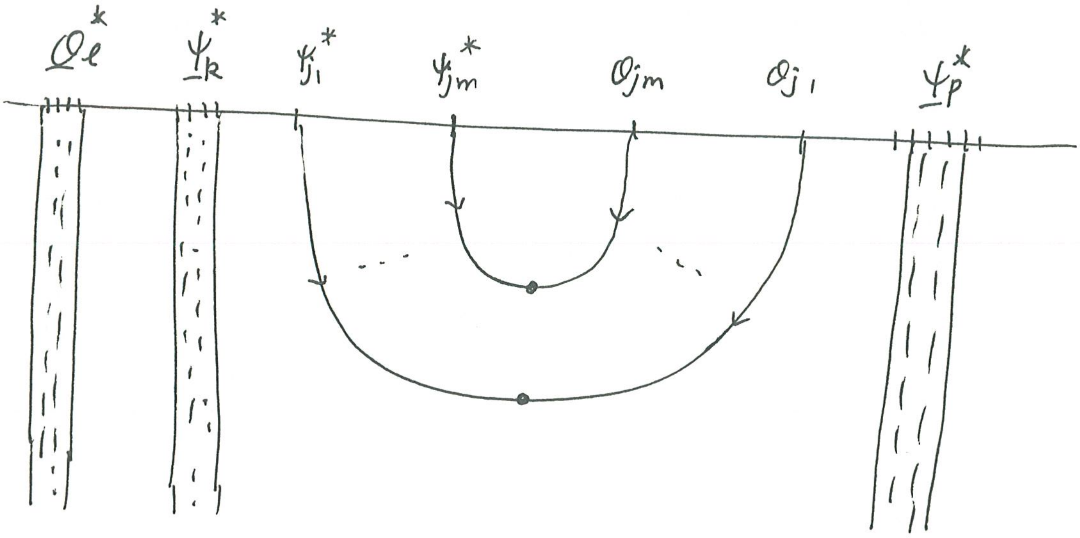
Finally, we present (19.1) on an input

$$\mathcal{O}_{\ell_1}\cdots\mathcal{O}_{\ell_r}\psi_{k_1}^*\cdots\psi_{k_s}^*\psi_{j_1}^*\cdots\psi_{j_m}^* \otimes \mathcal{O}_{j_m}\cdots\mathcal{O}_{j_1}\psi_{p_1}^*\cdots\psi_{p_u}^* \tag{23.1}$$

where we choose an (arbitrary) ordering $J=(j_1,\dots,j_m)$. We get

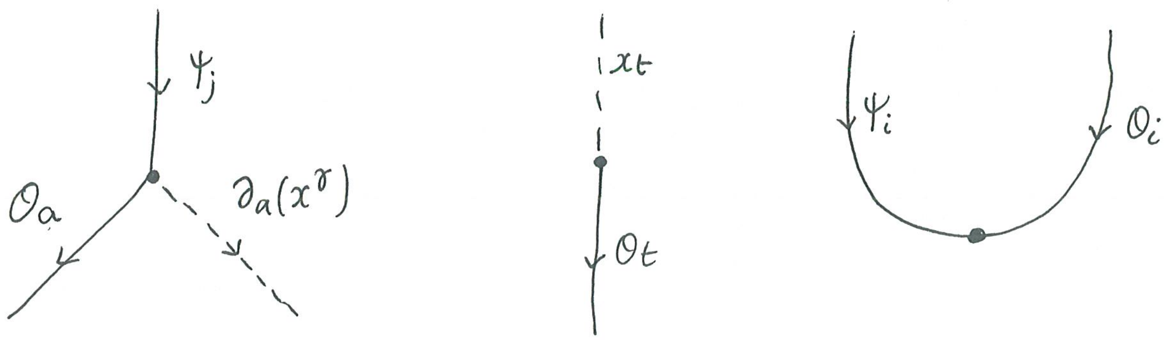
$$\begin{aligned}
(19.1) \text{ on } (23.1) &= \overset{(-1)^m}{\check{m}_2} \circ [\psi_{j_1},-]\otimes\mathcal{O}_{j_1}^* \cdots [\psi_{j_m},-]\otimes\mathcal{O}_{j_m}^* \\
&\qquad \Big( (-1)^{\binom{m}{2}} \mathcal{O}_{\ell_1}\cdots\mathcal{O}_{\ell_r}\psi_{k_1}^*\cdots\psi_{k_s}^*\psi_{j_m}^*\cdots\psi_{j_1}^* \\
&\qquad\qquad \otimes \mathcal{O}_{j_m}\cdots\mathcal{O}_{j_1}\psi_{p_1}^*\cdots\psi_{p_u}^* \Big) \\
&= (-1)^{m+\binom{m}{2}} \mathcal{O}_{\ell_1}\cdots\mathcal{O}_{\ell_r}\psi_{k_1}^*\cdots\psi_{k_s}^* \\
&\qquad m_2[\psi_{j_1},-]\otimes\mathcal{O}_{j_1}^*\cdots[\psi_{j_m},-]\otimes\mathcal{O}_{j_m}^*\big(\psi_{j_m}^*\cdots\psi_{j_1}^* \\
&\qquad\qquad \otimes \mathcal{O}_{j_m}\cdots\mathcal{O}_{j_1}\psi_{p_1}^*\cdots\psi_{p_u}^*\big) \\
&= (-1)^{m+\binom{m}{2}+\binom{m+1}{2}} \mathcal{O}_{\ell_1}\cdots\mathcal{O}_{\ell_r}\psi_{k_1}^*\cdots\psi_{k_s}^* m_2\big(1\otimes\psi_{p_1}^*\cdots\psi_{p_u}^*\big) \\
&= \mathcal{O}_{\ell_1}\cdots\mathcal{O}_{\ell_r}\psi_{k_1}^*\cdots\psi_{k_s}^*\psi_{p_1}^*\cdots\psi_{p_u}^*
\end{aligned} \tag{23.2}$$




(24.1)

The diagrams (21.4), (22.2) and (24.1) give our Feynman rules, provided we remember the signs, prefactors, and symmetry factors for the $x_t$, $\partial_t$ interactions.

<u>Summary</u> The interactions are




(24.2)

$\frac{-1}{|\gamma|} W^j(\gamma)$

given symmetry factor (any $t$)

no signs. (any $i$)

$(W = \sum_j x_j W^j \text{ fixed})$

Note With vacuum boundary conditions, we must have

$$\sum_{y} \sum_{j \in J(y)} |\gamma_j(y)| = \sum_{z} m(z) \tag{25.1}$$

where $y$ ranges over all internal edges and input vertices, and $z$ over all internal vertices.