Minimal models for MFs $\underline{\text{VIII}}$ (checked)

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5/11/15

We examine what the structure maps for $\mathbb{Z}_2$-graded A_∞-algebras mean, paying particular attention to the minimal models of x^d.

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Note: A $\mathbb{Z}_2$-graded A_∞-algebra with $m_1=0$, and only nonzero products m_2 and m_d for some particular $d>2$.

$$m_2 : A^{\otimes 2} \longrightarrow A \qquad \text{even}$$

$$m_d : A^{\otimes d} \longrightarrow A \qquad \text{degree } d$$

The constraints on these operators are

$$A^{\otimes r} \otimes A^{\otimes s} \otimes A^{\otimes t} \xrightarrow{1 \otimes m_s \otimes 1} A^{\otimes r} \otimes A \otimes A^{\otimes t} \xrightarrow{m_u} A \qquad (-1)^{r+st}$$

So the possibilities for relations are indexed by

$\boxed{u=2,\ s=2}$

$$\begin{tikzcd}
A^{\otimes 0} \otimes A^{\otimes 2} \otimes A^{\otimes 1} \arrow[r, "1\otimes m_2 \otimes 1", "r=0,\,t=1"'] \arrow[d, phantom, "\parallel_2"] & A^{\otimes 0} \otimes A \otimes A^{\otimes 1} \arrow[dr, "m_2"] & \\
 & & A \\
A^{\otimes 1} \otimes A^{\otimes 2} \otimes A^{\otimes 0} \arrow[r, "1\otimes m_2 \otimes 1"', "r=1,\,t=0"] & A^{\otimes 1} \otimes A \otimes A^{\otimes 0} \arrow[ur, "m_2"'] &
\end{tikzcd} \qquad (2.1)$$

$$\Rightarrow \quad m_2(1 \otimes m_2) = m_2(m_2 \otimes 1).$$

$\boxed{\begin{matrix} u=2,\ s=d \\ u=d,\ s=2 \end{matrix}}$

$$\begin{tikzcd}
A^{\otimes 0} \otimes A^{\otimes d} \otimes A^{\otimes 1} \arrow[r, "1\otimes m_d \otimes 1"] \arrow[d, phantom, "\parallel_2"] & A^{\otimes 0} \otimes A \otimes A^{\otimes 1} \arrow[dr, "m_2"] & (-1)^d \\
 & & A \\
A^{\otimes 1} \otimes A^{\otimes d} \otimes A^{\otimes 0} \arrow[r, "1\otimes m_d \otimes 1"'] & A^{\otimes 1} \otimes A \otimes A^{0} \arrow[ur, "m_2"'] & (-1)^1
\end{tikzcd}$$

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$$
\begin{tikzcd}
A^{\otimes 0}\otimes A^{\otimes 2}\otimes A^{\otimes d} \arrow[r, "1\otimes m_2\otimes 1"] & A^{\otimes 0}\otimes A\otimes A^{\otimes d-1} \arrow[dr, "md", "(-1)^{\delta *}"'] & \\
\vdots \arrow[r] & A^{\otimes 1}\otimes A\otimes A^{\otimes(d-2)} \arrow[r, "-", "md"'] & A \\
\vdots & \vdots & \\
 & A^{\otimes(d-2)}\otimes A\otimes A^{\otimes 1} \arrow[uur, "md"] & \\
\vdots & A^{\otimes d-1}\otimes A\otimes A^{\otimes 0} \arrow[uuur, bend right, "md"'] &
\end{tikzcd}
$$

So finally we obtain (3.1)

$$
(-1)^d m_2(md(a_0\otimes\cdots\otimes a_{d-1})\otimes a_d) - (-1)^{d|a_0|} m_2(a_0\otimes md(a_1\otimes\cdots a_d))
$$
$$
+ \; md(m_2(a_0\otimes a_1)\otimes\cdots) - md(a_0\otimes m_2(a_1\otimes a_2)\otimes\cdots)
$$
$$
+\cdots + (-1)^{d-1} md(a_0\otimes\cdots\otimes m_2(a_{d-1}\otimes a_d)) = 0.
$$

$\boxed{u=d,\ s=d}$

$r+t=d-1$

$t=d-1-r$

$$
\begin{tikzcd}
A^{\otimes 0}\otimes A^{\otimes d}\otimes A^{\otimes(d-1)} \arrow[r, "1\otimes md\otimes 1"] & A^{\otimes 0}\otimes A\otimes A^{\otimes d-1} \arrow[dr, "md"] & \\
\vdots & \vdots & A \\
A^{\otimes(d-1)}\otimes A^{\otimes d}\otimes A^{\otimes 0} \arrow[r, "1\otimes md\otimes 1"'] & A^{\otimes(d-1)}\otimes A\otimes A^{\otimes 0} \arrow[ur, "md"'] &
\end{tikzcd}
$$

$$
(-1)^{r+dt} = (-1)^{r+d(d-1-r)} = (-1)^{r+dr} = (-1)^{r(d+1)}
$$

$$
\sum_{r=0}^{d-1} md(\mathbb{1}^{\otimes r}\otimes md\otimes \mathbb{1}^{\otimes d-1-r})\cdot(-1)^{r(d+1)} = 0. \qquad (3.2)
$$

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Let $(A, \{m_n\}_{n\geq 2})$ be a $\mathbb{Z}_2$-graded A_∞-algebra as above. A $\mathbb{Z}_2$-graded A_∞-module M over A is defined by the same equations, where we assume now $m_1 = 0$. For simplicity let us write b_n, $n \geq 2$ for the multiplications on M.

$$b_n : M \otimes A^{\otimes(n-1)} \longrightarrow M \qquad (4.1)$$

This is degree $2-n$, so degree n.

- Associativity from (2.1) says the action

$$b_2 : M \otimes A \longrightarrow M$$

actually makes b_2 a right A-module (absent unitality which we do not discuss yet).

Then (3.1) says (4.2)

$$b_2(b_d(m_0 \otimes a_1 \otimes \cdots \otimes a_{d-1}) \otimes a_d) - \overset{(-1)^{d|m_0|}}{b_2(m_0 \otimes m_d(a_1 \otimes \cdots \otimes a_d))}$$

$$+ b_d(b_2(m_0 \otimes a_1) \otimes \cdots) - b_d(m_0 \otimes m_2(a_1 \otimes a_2) \otimes \cdots)$$

$$+ \cdots + (-1)^{d-1} b_d(m_0 \otimes a_1 \otimes \cdots \otimes m_2(a_{d-1} \otimes a_d)) = 0.$$

i.e.

$$b_d(m_0 \otimes \alpha) \cdot a_d - m_0 \cdot m_d(\alpha \otimes a_d) = \cdots \qquad \alpha \in A^{\otimes(d-1)}$$

Example Consider the A_∞-algebra A of p. 13 ainfmf3 for $d>2$, so with $|\alpha|=1$ ainfmf8

$$A = k \oplus k\cdot\alpha \qquad (5.1)$$

with the multiplication of $k[\alpha]/(\alpha^2)$ and

$$m_d : A^{\otimes d} \longrightarrow A$$

nonzero on all basis elements except

$$m_d(\alpha\otimes\cdots\otimes\alpha) = (-1)^{\binom{d}{2}}\cdot 1. \qquad (5.2)$$

First we check m_2, m_d satisfy the A_∞-constraints (3.1), (3.2). Now (3.2) holds trivially, while for (3.1)

$$m_2(m_d(a_0\otimes\cdots\otimes a_{d-1})\otimes a_d) = \begin{cases} 0 & \text{not all } a_0=\cdots=a_{d-1}=\alpha \\ (-1)^{\binom{d}{2}} a_d & \text{all } a_0=\cdots=a_{d-1}=\alpha \end{cases}$$

$$m_2(a_0\otimes m_d(a_1\otimes\cdots\otimes a_d)) = \begin{cases} 0 & \text{not all } a_1=\cdots=a_d=\alpha \\ (-1)^{\binom{d}{2}+d|a_0|} a_0 & \text{all } a_1=\cdots=a_d=\alpha \end{cases}$$

$$m_d(m_2(a_0\otimes a_1)\otimes\cdots) = \begin{cases} (-1)^{\binom{d}{2}} & \text{exactly one of } a_0, a_1 \text{ is } \alpha, \text{ all } a_2,\ldots \text{ are } \alpha \\ 0 & \text{else} \end{cases}$$

[Note the m's here are the μ's of ainfcat so

$$\mu_d(\alpha\otimes\cdots\otimes\alpha) = (-1)^{\binom{d}{2}} \underset{\substack{\uparrow \\ m \text{ of ainfcat}}}{M}(\alpha\otimes\cdots\otimes\alpha) = (-1)^{\binom{d}{2}} \rho_d(\alpha\otimes\cdots\otimes\alpha)$$
$$= (-1)^{\binom{d}{2}}\cdot 1$$
]

The sum of the $m_d(\cdots)$ terms in (3.1) will be zero unless the input has d α's and a single 1 somewhere. And the sum will be zero unless that 1 is at one of the extremes. (because otherwise consecutive terms cancel).

The m_2 terms are zero unless the input is of the same form (i.e. $1 \otimes \alpha \cdots \alpha$ or $\alpha \otimes \alpha \otimes \cdots \otimes 1$). On these two inputs we easily check that (3.1) works. (note the sign in (5.2) is important!). Hence (A, m_2, m_d) is an A_∞-algebra.

Modules over the A_∞-algebra A of (5.1) may, of course, have nontrivial higher multiplications in all degrees. First let us consider an A_∞-module M with only m_2 (so $m_1 = 0$, $m_n = 0$ for $n \geq 3$). So M is a $\mathbb{Z}_2$-graded k-module with even

$$m_2 : M \otimes_k A \longrightarrow M \qquad (6.1)$$

If we assume unital, this is really the data of an odd operator

$$\partial := m_2(- \otimes \alpha) : M \longrightarrow M. \qquad (6.2)$$

The A_∞-constraints involve, with sign $(-1)^{r+st}$

$$M \otimes A^{\otimes(r-1)} \otimes A^{\otimes s} \otimes A^{\otimes t} \xrightarrow{1 \otimes m_s \otimes 1} M \otimes A^{\otimes(r-1)} \otimes A \otimes A^{\otimes t} \xrightarrow{m_u} M$$

$u = r+1+t$

$r, t \geq 0.$

Here $r = 0$ means we use

$$M \otimes A^{\otimes(s-1)} \otimes A^{\otimes t} \xrightarrow{m_s \otimes 1} M \otimes A^{\otimes t} \xrightarrow{m_{t+1}} M$$

And then as usual

$$\sum (-1)^{r+st}\, m_u(\mathbb{1}^{\otimes r}\otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0 \qquad (7.1).$$

If only m_2 is nonvanishing then we have the same potential relations from (2.1), (3.1), (3.2). Clearly (2.1) expresses that M is an A-module, and this simply means

$$\partial^2 = 0.$$

So M is a $\mathbb{Z}_2$-graded complex. Now (3.1) says

$$\begin{aligned} m_2(a_0 \otimes m_d(a_1 \otimes \cdots \otimes a_d)) &= (-1)^d m_2(m_d(a_0 \otimes \cdots)\otimes a_d) \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_0 \doteq m_2(a_0 \otimes 1) &= m_2(a_0 \otimes (-1)^{d-1} m_d(\alpha \otimes \cdots \otimes \alpha)) \\ &= 0 \end{aligned}$$

well, so M is the zero module. So we have to allow at least one higher multiplication, namely $m_d : M \otimes A^{\otimes(d-1)} \to M$ of degree d. Subsituting $a_d = 1$ in (3.1) we find (for $a_0 \in M$)

$$\begin{aligned} (-1)^d m_d(a_0 \otimes \cdots \otimes a_{d-1}) = &- m_d(m_2(a_0 \otimes a_1) \otimes \cdots \otimes a_{d-1} \otimes 1) \\ &+ m_d(a_0 \otimes m_2(a_1 \otimes a_2) \otimes \cdots \otimes 1) \\ &\;\vdots \\ &+ (-1)^{d-1} m_d(a_0 \otimes \cdots \otimes m_2(a_{d-1} \otimes 1)) \end{aligned} \qquad (7.2)$$

From (7.2) we compute

$$(-1)^d \cdot 2 \cdot m_d(a_0 \otimes \cdots \otimes a_{d-1})$$

$$\begin{aligned} &= -m_d(a_0 \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{d-1} \otimes 1) \\ &\quad + m_d(a_0 \otimes a_1 \cdot a_2 \otimes \cdots \otimes a_{d-1} \otimes 1) \\ &\quad\ \vdots \\ &\quad + (-1)^d m_d(a_0 \otimes a_1 \otimes \cdots \otimes a_{d-2} \cdot a_{d-1} \otimes 1) \end{aligned} \tag{8.1}$$

which gives us a recursive formula for m_d in terms of m_d on more 1's? (no)

The relation (3.2) says

$$\begin{aligned} &m_d(m_d(a_0 \otimes \cdots \otimes a_{d-1}) \otimes a_d \otimes \cdots) \\ &+ (-1)^{d-1+d|a_0|} m_d(a_0 \otimes m_d(\cdots) \otimes \cdots) \\ &+ \\ &\ \vdots \\ &+ (-1)^{(d-1)(d+1) + d(|a_0| + \cdots + |a_{d-2}|)} m_d(a_0 \otimes \cdots \otimes a_{d-2} \otimes m_d(\cdots)) = 0. \end{aligned}$$

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Consider the case of an A_∞-module M with only m_2, m_3 nonzero for $d=3$. Then the constraints are, writing m_2 as $\cdot$

- $m_2(m \otimes m_2(a \otimes b)) = m_2(m \otimes ab)$

$$m \cdot (ab) = (m \cdot a) \cdot b. \qquad (9.1)$$

Now if we assume $1 \in A$ acts as a unit, the only data here is the odd k-linear operator

$$\partial := m_2(- \otimes \varepsilon) : M \longrightarrow M$$

and (9.1) says precisely $\partial^2 = 0$. Next, (3.1) says

$$(9.2)$$

$$- m_2(m_3(m \otimes a \otimes b) \otimes c) - (-1)^{|m|} m_2(m \otimes m_3(a \otimes b \otimes c))$$
$$+ m_3(m \cdot a \otimes b \otimes c) - m_3(m \otimes ab \otimes c)$$
$$+ m_3(m \otimes a \otimes bc) = 0$$

We define

$$\begin{aligned} h &:= m_3(- \otimes \varepsilon \otimes \varepsilon) & |h| &= 1 \\ k &:= m_3(- \otimes 1 \otimes \varepsilon) & |k| &= 0 \\ \ell &:= m_3(- \otimes \varepsilon \otimes 1) & |\ell| &= 0 \\ \alpha &:= m_3(- \otimes 1 \otimes 1) & |\alpha| &= 1 \end{aligned} \qquad (9.3)$$

Then the constraint (9.2) says precisely $\qquad (9.4)$

a	b	c	
1	1	1	$-\alpha(m) + \alpha(m) - \alpha(m) + \alpha(m) = 0$
1	1	ε	$-\partial\alpha(m) + k(m) - k(m) + k(m) = 0$

$$\Rightarrow \boxed{k = \partial\alpha} \qquad (9.5)$$

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$1 \quad \varepsilon \quad 1 \qquad -k(m) + \ell(m) - \ell(m) + k(m) = 0$ (10.1)

$\varepsilon \quad 1 \quad 1 \qquad -\ell(m) + \alpha\partial(m) - \ell(m) + \ell(m) = 0$

$$\boxed{\ell = \alpha\partial} \qquad (10.2)$$

$1 \quad \varepsilon \quad \varepsilon \qquad -\partial k(m) + h(m) - h(m) = 0$

$$\boxed{\partial k = 0} \qquad (10.3)$$

$\varepsilon \quad \varepsilon \quad 1 \qquad -h(m) + \ell\partial(m) + h(m) = 0$

$$\boxed{\ell\partial = 0} \qquad (10.4)$$

$\varepsilon \quad 1 \quad \varepsilon \qquad -\partial\ell(m) + k\partial(m) - h(m) + h(m) = 0$

$$\boxed{k\partial = \partial\ell} \qquad (10.5)$$

$\varepsilon \quad \varepsilon \quad \varepsilon \qquad -\partial h(m) - (-1)^{|m|+d-1} \overset{3}{m}$

$+ h\partial(m) = 0$

$$\boxed{(h\partial - \partial h)(m) = (-1)^{|m|} m} \qquad (10.6)$$

If we set $a(m) = (-1)^{|m|} m$: $M \circlearrowleft$ then $ah = -ha$ and (10.6) says

$$ah\partial - a\partial h = 1$$

$$\therefore\ ah\partial + \partial ah = 1$$

$$[ah, \partial] = 1.$$

Finally the relation (3.2) says

$$
\begin{aligned}
& m_3(m_3(m\otimes a\otimes b)\otimes c\otimes d) \\
&\quad + (-1)^{|m|} m_3(m\otimes m_3(a\otimes b\otimes c)\otimes d) \\
&\quad + (-1)^{|m|+|a|} m_3(m\otimes a\otimes m_3(b\otimes c\otimes d)) = 0
\end{aligned}
\tag{11.1}
$$

which reads

a	b	c	d		
1	1	1	1	$\alpha^2(m)=0$	$\boxed{\alpha^2=0}$
1	1	1	ε	$k\alpha(m)=0$	$\boxed{k\alpha=0}$
1	1	ε	1	$\ell\alpha(m)=0$	$\boxed{\ell\alpha=0}$
1	ε	1	1	$\alpha k(m)=0$	$\boxed{\alpha k=0}$
ε	1	1	1	$\alpha\ell(m)=0$	$\boxed{\alpha\ell=0}$ (11.2)
1	1	ε	ε	$h\alpha(m)=0$	$\boxed{h\alpha=0}$
1	ε	ε	1		$\boxed{\ell k=0}$
ε	ε	1	1		$\boxed{\alpha h=0}$
1	ε	1	ε		$\boxed{k^2=0}$
ε	1	1	ε		$\boxed{k\ell=0}$
ε	1	ε	1		$\boxed{\ell^2=0}$
1	ε	ε	ε	$hk(m)+(-1)^{\lvert m\rvert}\cdot 0 + (-1)^{\lvert m\rvert}\alpha(m)=0$	$\boxed{hk+\alpha a=0}$
ε	ε	ε	1	$\ell h(m)+(-1)^{\lvert m\rvert}\alpha(m)=0$	$\boxed{\ell h+\alpha a=0}$
ε	1	ε	ε	$h\ell(m)=0$	$\boxed{h\ell=0}$
ε	ε	1	ε	$kh(m)=0$	$\boxed{kh=0}$
ε	ε	ε	ε	$h^2(m)+(-1)^{\lvert m\rvert}k(m)-(-1)^{\lvert m\rvert}\ell(m)=0$	$\boxed{h^2+ka-\ell a=0}$

So an A_∞-module with only m_2, m_3 nonzero over A for $d=3$ is precisely the data of

- a $\mathbb{Z}_2$-graded vector space M
- operators $\partial, h, k, \ell, \alpha$ as in (9.3) with $\partial^2 = 0$ and the relations in (9.5), (10.1) – (10.6), (11.2).

$$(11.2) \Rightarrow \quad \alpha = -hka = -\ell h a$$

$$\therefore \quad hk = \ell h$$

$$(10.2) \Rightarrow \quad \ell = \alpha\partial = -hka\partial = hk\partial a \qquad (12.1)$$

$$\ell = \alpha\partial = -\ell h a\partial = \ell h \partial a$$

now $\quad k\partial = \partial\ell$ by (10.5)

$$\therefore \ell = h\partial\ell a$$

Let us eliminate k, ℓ via (4.5), (10.2), leaves

$$h\partial - \partial h = a \qquad (a)$$

$$\alpha^2 = 0 \qquad (b)$$

$$\alpha\partial\alpha = 0 \qquad (c)$$

$$h\alpha = 0 \qquad (d)$$

$$\alpha h = 0 \qquad (e) \qquad (12.2)$$

$$\cancel{h\partial\alpha + \alpha a = 0} \qquad (f)$$

$$\cancel{\alpha\partial h + \alpha a = 0} \qquad (g)$$

$$h^2 + \partial\alpha a - \alpha\partial a = 0 \qquad (h)$$

Now a $\Rightarrow$ $\alpha a = \alpha h\partial - \alpha\partial h \overset{e}{=} -\alpha\partial h$ so g is redundant. ainfmf8

a $\Rightarrow$ $h\partial\alpha = (\partial h + a)\alpha \overset{d}{=} a\alpha = -\alpha a$ so f is redundant.

h + d + e $\Rightarrow$ $0 = h^3 + h\partial\alpha a - h\alpha\partial a \overset{d}{=} h^3 + h\partial\alpha a$

$= h^3 - \alpha a^2 = h^3 - \alpha$

$$\boxed{h^3 = \alpha}$$

So (12.2) are equivalent to

$$h\partial - \partial h = a \qquad \text{a}$$

$$h^4 = 0 \qquad \text{ii}$$

$$\cancel{h^3\partial h^3 = 0} \qquad \text{iii}$$

$$h^2 \cancel{+ \partial h^3 a - h^3\partial a = 0} \qquad \text{iv}$$

Now (iv) $\Rightarrow$ (iii) and (iv) gives

$$h^3 + -\partial \cancel{h^4} a + h^3\partial h a = 0$$

$$h^3 + h^3(h\partial - a)a = 0 \qquad \text{nothing.}$$

$\therefore$ An A_∞-module is

- odd operators h, ∂ s.t. (i), (iii), (iv) hold.

$m_3(-\otimes 1 \otimes 1) = h^3$ $\qquad$ $m_3(-\otimes \varepsilon \otimes 1) = h^3\partial$

$m_3(-\otimes 1 \otimes \varepsilon) = \partial h^3$ $\qquad$ $m_3(-\otimes \varepsilon \otimes \varepsilon) = h.$

From ⓐ we obtain

$$\partial h^3 = \partial h h^2$$
$$= (h\partial - a)h^2$$
$$= h\partial h^2 - ah^2 = h(h\partial - a)h - ah^2$$
$$= h^2\partial h - hah - ah^2$$
$$= h^2\partial h$$

$$h^3\partial = h^2(\partial h + a)$$
$$= h^2\partial h + h^2 a = h^2\partial h + ah^2$$

$$\therefore \quad (\partial h^3 - h^3\partial)a = (h^2\partial h - h^2\partial h - ah^2)a$$
$$= -ah^2a$$
$$= -h^2$$

so in fact (iv) is redundant.

Writing $H = \cancel{ha}$ ah we can rewrite (i), (ii) as

$$H\partial + \partial H = 1$$
$$H^4 = 0.$$