Minimal models for MFs VII (checked)

ainfmf7

5/11/15

Here we explore ideas for finding closed formulas for the higher multiplications in the general case, using intuition from ainfmf5. The starting point is of course the formula on p. 16 of ainfmf4 which covers both δ_∞ and H_∞, namely

$$H_\infty = \sum_{m \geq 0} (-1)^m (H d_{End})^m H, \qquad H = J^{-1} \nabla$$

$$\delta_\infty = \sum_{m \geq 0} (-1)^m (H d_{End})^m \delta. \qquad (1.1)$$

For either $\beta = H$ or δ we have as k-linear maps, defined on $\omega \in \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n)$ of θ-weight $p = |\omega|$ and homogeneous $f \in k[x_1, \ldots, x_n]$ of degree b, and $\Psi \in \underline{\text{End}}$ a basis element

$$(H d_{End})^m \beta(\omega \otimes f \Psi) \qquad (1.2)$$

$$= \sum_{\underline{j}} \sum_{\underline{u}} \sum_{\underline{\ell}} \sum_{\underline{z}} (-1)^m C_{p+b}(\underline{\ell}) \prod_{i=1}^m \partial_{x_{z_i}} (f_{j_i, \ell_i}^{u_i})$$

$$\cdot \prod_{i=1}^m [\psi_{j_i}^{u_i}, -] \theta_{z_i} \beta(\omega \otimes f \Psi)$$

where $\underline{j} = j_1, \ldots, j_m$ ranges over $\{1, \ldots, n\}$, as does $\underline{z} = z_1, \ldots, z_m$, while $\underline{u} \in \mathbb{Z}_2^m$ and $\underline{\ell} = \ell_1, \ldots, \ell_m \geq 1$.

Here we make the simplifying assumption that $W \in \mathfrak{m}^3$ so all $u_i = 1$ and hence $f^{u_i}_{j_i, l_i} = W^{j_i}_{l_i}$ where $W = \sum_i x_i W^i$, and W^i_l is the degree l piece of W^i. In fact we will go even further and write for $g \in k[\underline{x}]$

$$g = \sum_{\gamma \in \mathbb{Z}^n_{\geq 0}} g(\gamma) x^\gamma$$

so that

(2.1)

$$\begin{aligned}
(H d_{End})^m \beta(\omega \otimes f \Upsilon) &= \sum_{\underline{j}, \underline{z}} \sum_{\gamma_1, \ldots, \gamma_m \in \mathbb{Z}^n_{\geq 0}} (-1)^m C_{p+b}(|\gamma_1|, \ldots, |\gamma_m|) \prod_{i=1}^m \partial_{x_{z_i}}\left(W^{j_i}(\gamma_i) \, x^{\gamma_i}\right) \\
&\qquad \prod_{i=1}^m [\psi_{j_i}, -] \theta_{z_i} \beta(\omega \otimes f \Upsilon) \\
&= \sum_{\underline{j}, \underline{z}} \sum_{\gamma_1, \ldots, \gamma_m} (-1)^m C_{p+b}(|\underline{\gamma}|) \prod_{i=1}^m \left(W^{j_i}(\gamma_i) \cdot (\gamma_i)_{z_i} x^{\gamma_i - e_{z_i}} \right) \\
&\qquad \prod_{i=1}^m \left([\psi_{j_i}, -] \theta_{z_i} \right) \beta(\omega \otimes f \Upsilon) \\
&= \sum_{\underline{j}, \underline{z}} \sum_{\underline{\gamma}} (-1)^m C_{p+b}(|\gamma_1| + 1, \ldots, |\gamma_m| + 1) \prod_{i=1}^m \Big(W^{j_i}(\gamma_i + e_{z_i}) \\
&\qquad \cdot [(\gamma_i)_{z_i} + 1] \, x^{\gamma_i} \Big) \prod_{i=1}^m \left([\psi_{j_i}, -] \theta_{z_i} \right) \beta(\omega \otimes f \Upsilon)
\end{aligned}$$

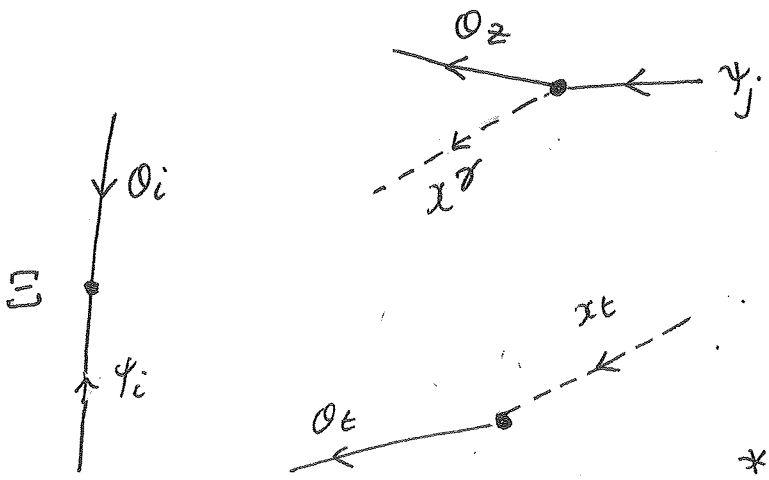
Ignoring prefactors, this is a sum of products of terms

$$W^j(\gamma + e_z)\, x^\gamma [\psi_j, -] \mathcal{O}_z \qquad (\text{for } \mathfrak{Z}_\infty, \text{ and } H_\infty)$$

$$[\psi_i, -] \otimes \mathcal{O}_i^* \quad (\Xi) \qquad \partial_t \mathcal{O}_t \quad (\text{from } H_\infty, m=0)$$

this vertex *must* occur, and only once. *

Recall that $[\psi_j, -]$ acts as an annihilation operator, and $\mathcal{O}_i$ as a creation operator (likewise we view $x_i \in k[x]$ as creation with annihilation ∂x_i). So we have vertices. (we also add Ξ vertices)



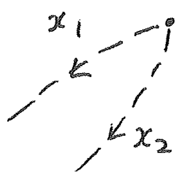
prefactor $W^j(\gamma + e_z)$
$(\mathfrak{Z}_\infty, H_\infty)$

(3.2)

* (every H_∞ region contains exactly one of these vertices)

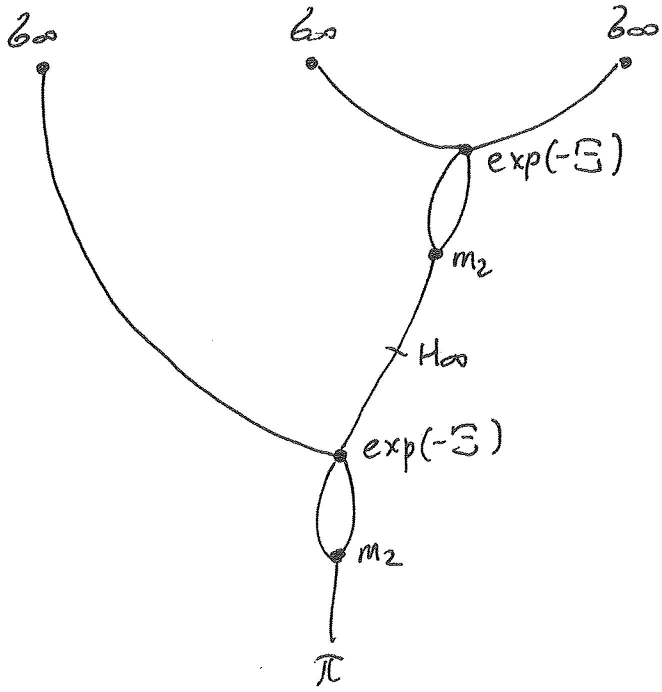
where we are supposed to imagine x^γ as $|\gamma|$ separate bosons, i.e. for $\gamma = (1,1,0,\dots)$

means



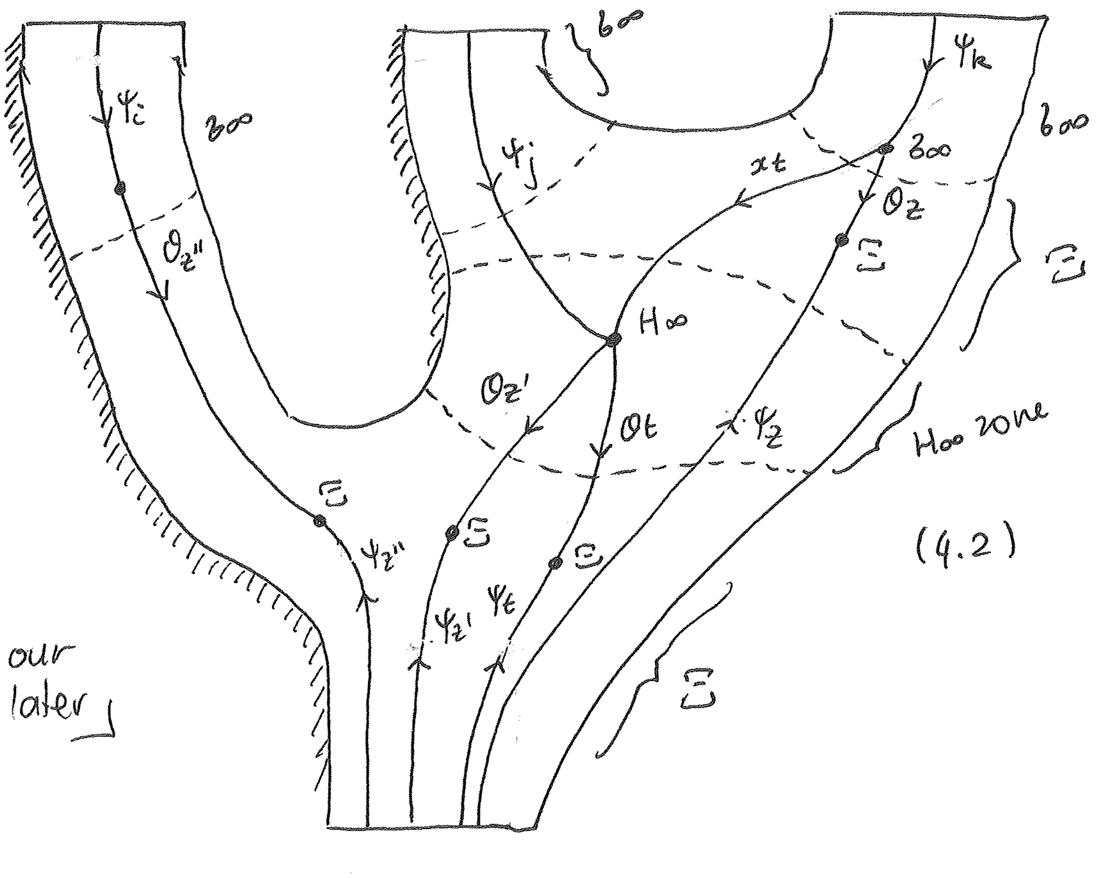
In real QFT vertices are labelled by e.g. spacetime coords, but in our case there are <u>locations</u> on trees.

Example Consider the b_3 diagram



(4.1)

The Feynman diagrams are oriented downwards, interactions happen only at vertices of (4.1), and interactions of type X only occur at vertices of type X. e.g. to compute b_3 on $\psi^* \otimes \psi^* \otimes \psi^*$ (which we view as a 3-particle input state, a contribution would be (assume $n=1$ for simplicity)



(4.2)

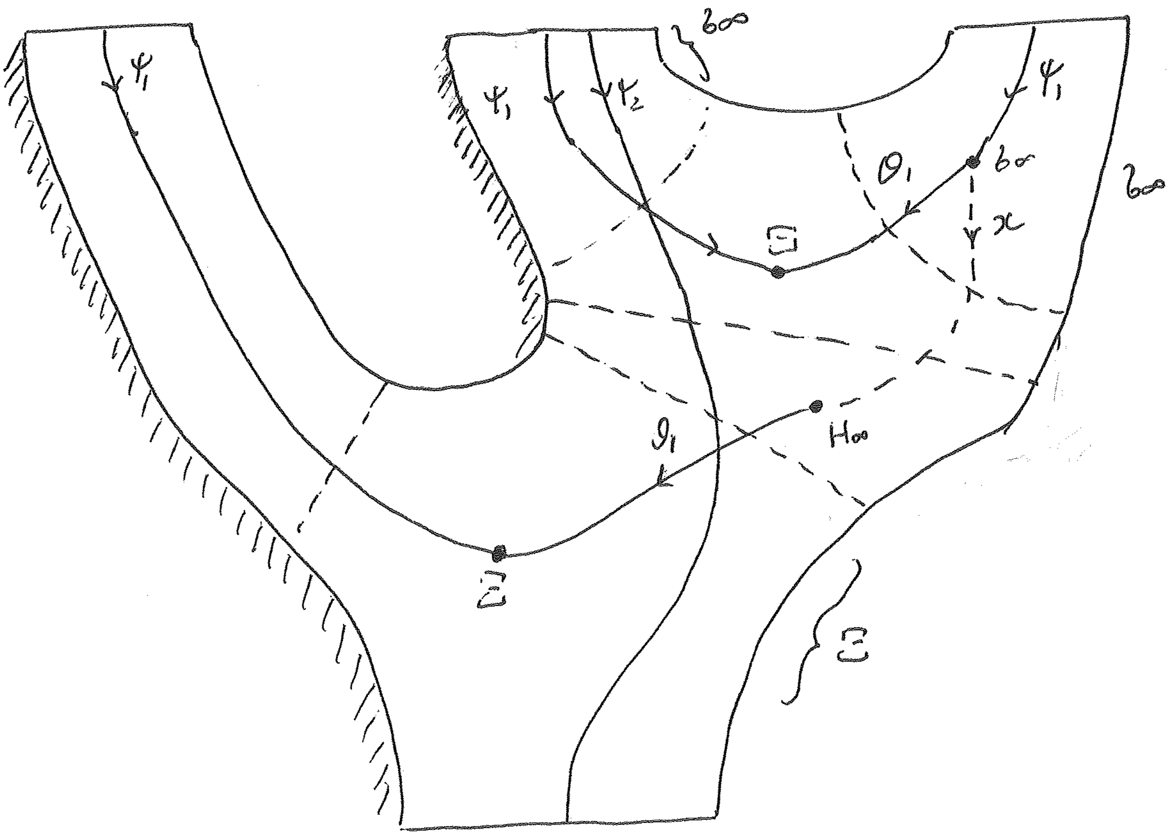
[Note we refine our understanding later]

More concretely consider the example on p. 20 of ainfmf5 where we calculated for $W = y^3 - x^3$ that

$$b_3(\psi_1^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^*) = -\psi_2^*.$$

The contribution here comes from the diagram (20.1) there, which in pictures is

(5.1)

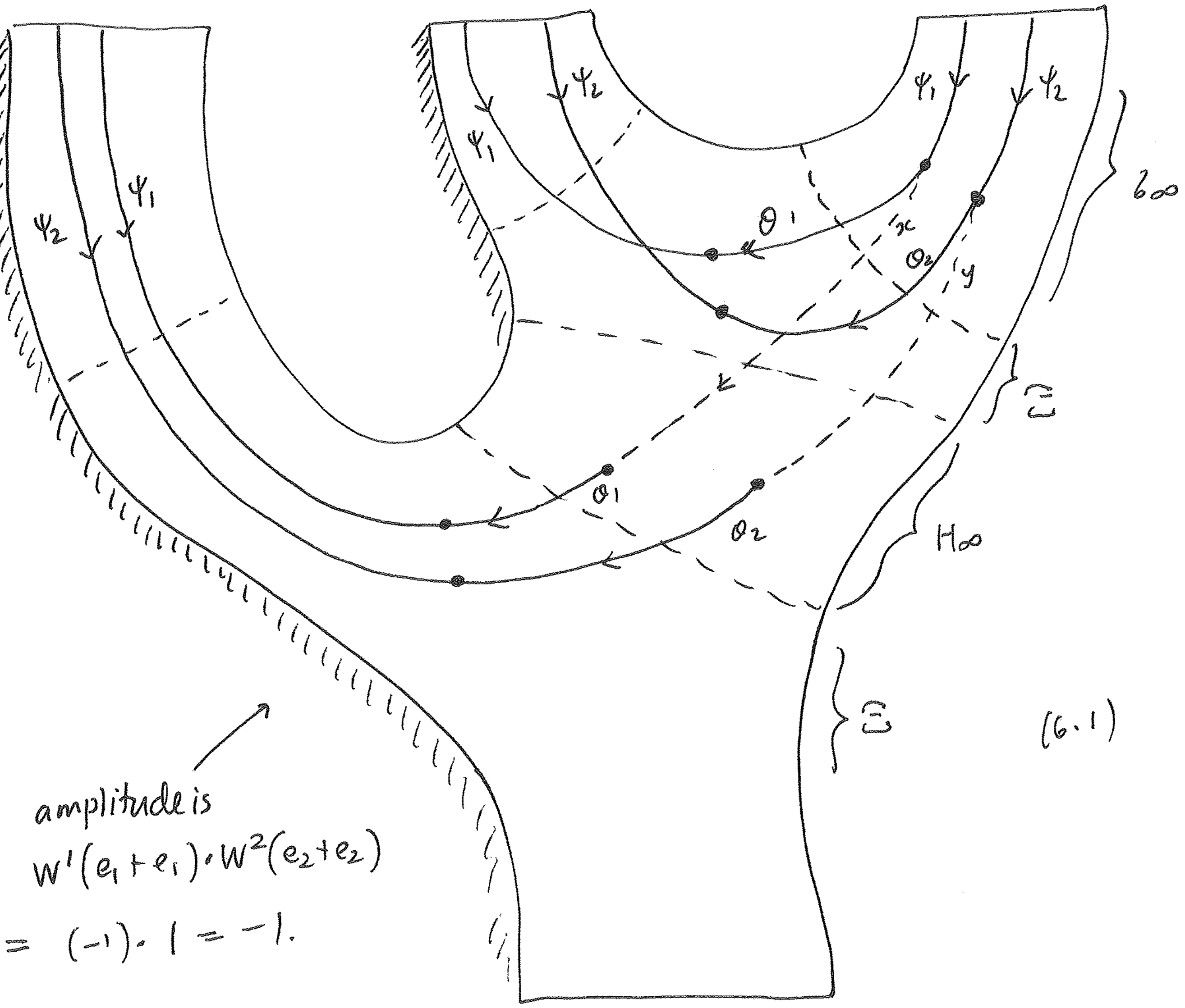


The only vertex with a prefactor is the b_∞, where it is

$$W'(e_1 + e_1) = \text{coeff of } x^2 \text{ in } W'$$

Now $W' = -x^2$ so this coeff is -1. This is the <u>unique</u> diagram contributing to this amplitude, so the coeff is -1.

Similarly



(6.1)

amplitude is
$W^1(e_1+e_1)\cdot W^2(e_2+e_2)$
$= (-1)\cdot 1 = -1.$

This computes one contribution to the amplitude of

$$b_3(\psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^*)_{\text{const term}}$$

$$= \Big(-[\psi_1, \psi_1^*\psi_2^*]\cdot[\psi_1, \psi_1^*\psi_2^*]\cdot[\psi_1, \psi_1^*\psi_2^*]$$
$$+ [\psi_2, \psi_1^*\psi_2^*]^3 \Big)_{\text{const term}}$$

$$= \left(-(\psi_2^*)^3 + (-\psi_1^*)^3\right) \underset{\text{const}}{=} 0.$$

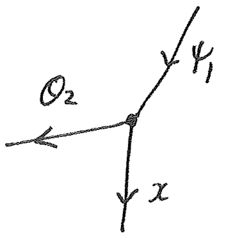
<u>NO</u> Actually (6.1) is an illegal diagram because there are <u>two</u> $\to\!\!\bullet$- - - vertices in H_∞.

Actually we can see *why* the amplitude for

$$\langle 0| \cdots |\psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \rangle$$

is zero: the ψ_1, ψ_2 on the far left can *only* be consumed by annihilating with $\mathcal{O}$'s coming from H_{∞} or the two $\mathfrak{Z}_{\infty}$'s. At most one of these $\mathcal{O}$'s can be produced from a $\mathfrak{Z}_{\infty}$ interaction (because H_{∞} can only annihilate a single x or y). But then the othe $\mathcal{O}$ must come from a H_{∞} interaction via the *same* $\mathfrak{Z}_{\infty}$ interaction producing the first $\mathcal{O}$ – this however means both $\mathcal{O}$'s are of the same type (because of the way our potential works).

Note An interaction



has prefactor $W'(e_1 + e_2) =: \lambda$

i.e. $W = x \cdot (\lambda x y + \cdots) + \cdots$

Note

- All amplitudes can be reduced to ones with vacuum output (no interactions *produce* ψ's, so if they occur in the output they must have passed through undigested). So higher multiplications are all sums of products of $[\psi_{i_1}, [\psi_{i_2}, \ldots] \ldots]$'s

- The ψ's in the rightmost channel *must* be consumed by $\mathfrak{Z}_{\infty}$ interactions (assuming vacuum output). This means for every ψ_i in the rightmost channel some oth ψ_j on its left must be responsible for annihilating with it via a Ξ interaction. (or rath the $\mathcal{O}_2$ coming from ψ_i annihilates).

- Similarly ψ's in the <u>leftmost</u> channel can only be annihilated in a Ξ-interaction. This is why an amplitude (for outgoing vacuum) with more than ψ in the leftmost channel will always be zero.

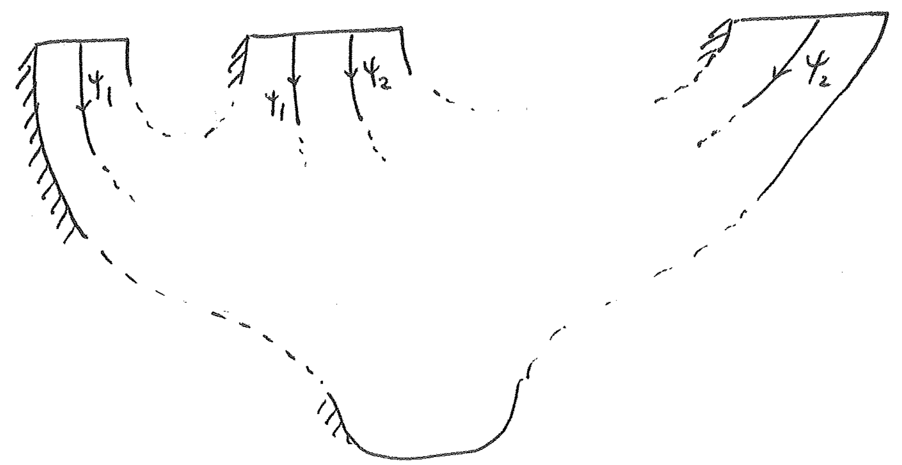
- The $\mathcal{B}_\infty$ vertices <u>can</u> convert ψ's into θ's with no bosons if W has quadratic terms (because then $\gamma = 0$ can have a prefactor which is nonzero). This is what complicates life. Also the inputs will be entangled states.

- Question: can we move all trivalent interactions into $\mathcal{B}_\infty$ zones, so that H_∞ zones only have $\partial\theta$-interactions?

- We see that $b_n : \underline{\mathrm{End}}^{\otimes n} \longrightarrow \underline{\mathrm{End}}$ (restricting in domain and codomain to products of ψ^*'s) is a linear combination of operators like

$$\underbrace{[\psi_1,-]\circ[\psi_1,[\psi_2,-]]\cdots\circ[\psi_2,-]}_{n \text{ operators multiplied.}} \qquad (8.1)$$

The coefficient of this operator is the sum over all trees of amplitudes of all possible Feynman diagrams with the shown input



(8.2)

and vacuum output.

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9

<u>Example</u> let us consider $W = y^d - x^d$, $d > 2$ and see if we can roughly calculate the higher products from the Feynman rules. We already know from p. 5 ainfmf5 the products on $\mathcal{A} = \Lambda(k\psi_1^* \oplus k\psi_2^*)$ for $n=2$ (it is just the exterior product, (5.3) there) and $n=3$ (see (19.1) there) namely

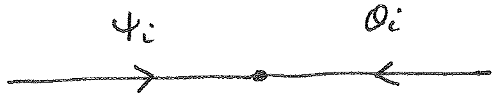
$$b_2(\Lambda_0 \otimes \Lambda_1) = \Lambda_0 \bullet \Lambda_1 \qquad (9.1)$$

and $b_3 = 0$ unless $d=3$ in which case

$$(9.2)$$

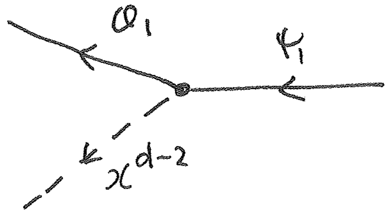
$$b_3(\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2) = -(-1)^{|\Lambda_0| + |\Lambda_1|}[\psi_1, \Lambda_0] \bullet [\psi_1, \Lambda_1] \bullet [\psi_1, \Lambda_2]$$
$$+ (-1)^{|\Lambda_0| + |\Lambda_1|}[\psi_2, \Lambda_0] \bullet [\psi_2, \Lambda_1] \bullet [\psi_2, \Lambda_2].$$

According to p. 3 the Feynman rules are

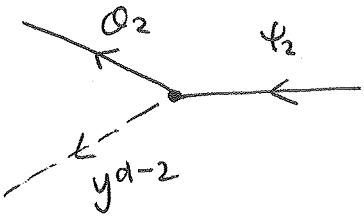


(9.3)

B∞

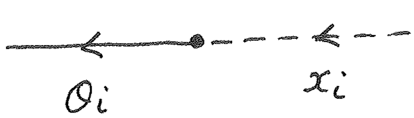


prefactor -1



prefactor 1

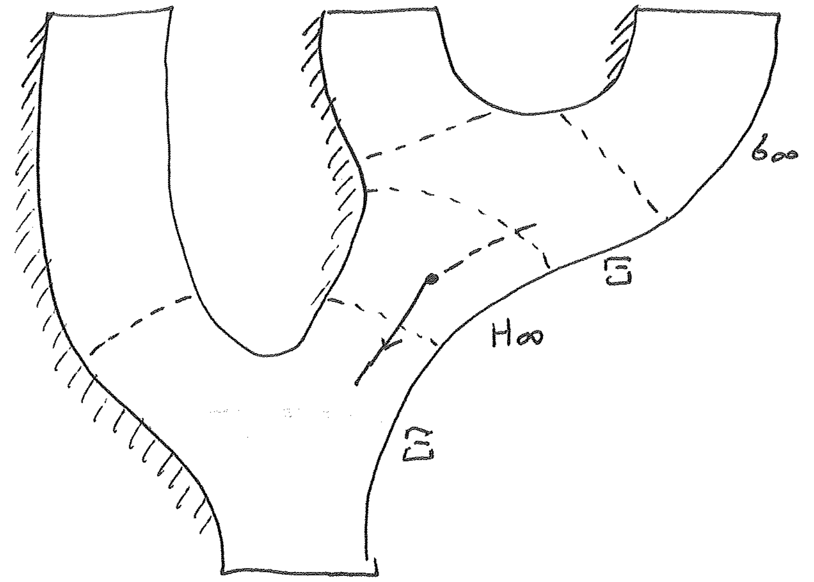
H∞



[as suggested we only put trivalent vertices in B∞ zones]

$W^1 = -x^{d-1}$, $W^2 = y^{d-1}$

For b_3 the only relevant tree is



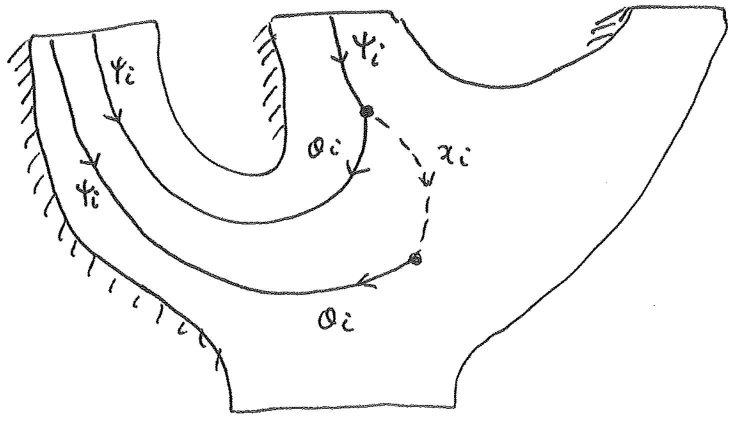
(*)

$$(10.1)$$

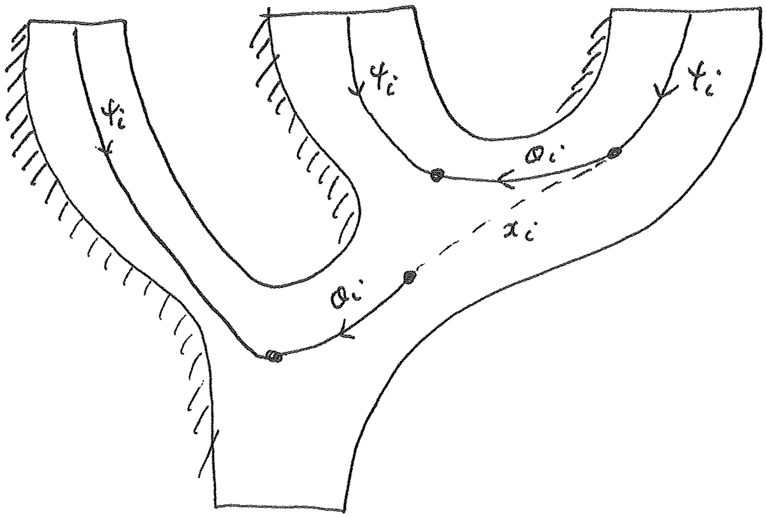
From the Feynman rules, to compute the weff of a given operation e.g. $[\psi_1,-]\circ[\psi_1,[\psi_2,-]]\circ[\psi_1,-]$ we compute amplitudes of Feynman diagrams decorating the above tree. There is exactly one H_∞ vertex so precisely one of the "antecedent" δ_∞ zones needs to contain a trivalent interaction. If $d>3$ that interaction produces more than one boson, which survives to annihilate the vacuum at (*), so $b_3 = 0$ unless $d=3$.

If $d=3$ the trivalent interactions (of which there are two) produce x or y (plus a $\mathcal{O}_1$ resp. $\mathcal{O}_2$). In principle b_3 could have contributions from e.g. ψ_1 ψ_2 i.e. $[\psi_1,-]\circ[\psi_2,-]$, but we only see two summands in b_3. Why?

e.g. if there is no ψ in the rightmost channel we get



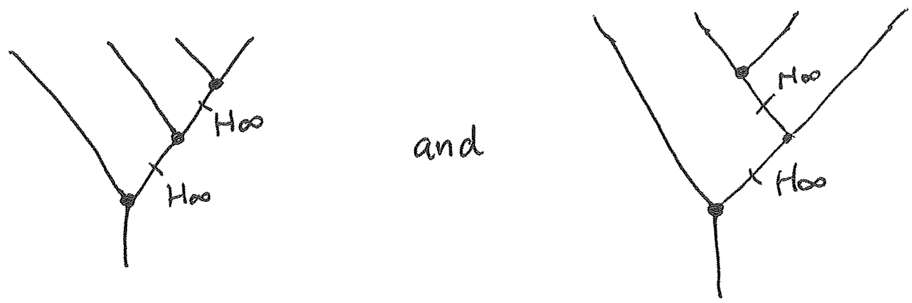
But this must be zero as $\psi_i^2 = 0$. So there is a ψ_i in the rightmost channel, it has a trivalent interaction, and that is the only trivalent interaction. The θ_i, x_i output can only be managed as shown below:



(11.2)

which explains why there are two factors of the given kind in b_3. The factors $-1, +1$ are from the trivalent interaction prefactor.

By p. 3 of ainfmf6 there are two trees to consider for b_4, namely

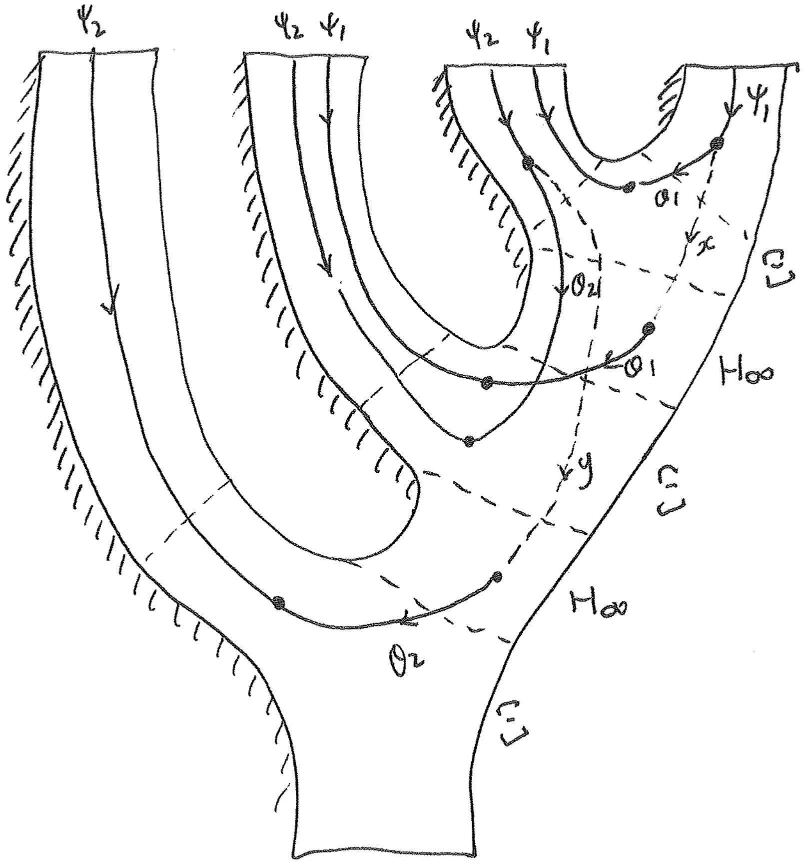


(11.3)

These have two internal edges and thus two H_{00} interactions. The number associated to a certain input configuration, e.g.

Channel 1	ψ_1
Channel 2	ψ_1, ψ_2
Channel 3	ψ_2
Channel 4	ψ_1, ψ_2

is obtained by summing over the two possible trees, and for each tree the 4 possible H_{∞} pairs ($\partial_{x_1}\mathcal{O}_1$ or $\partial_{x_2}\mathcal{O}_2$ at each) and over other configurations. For example, for $d=3$

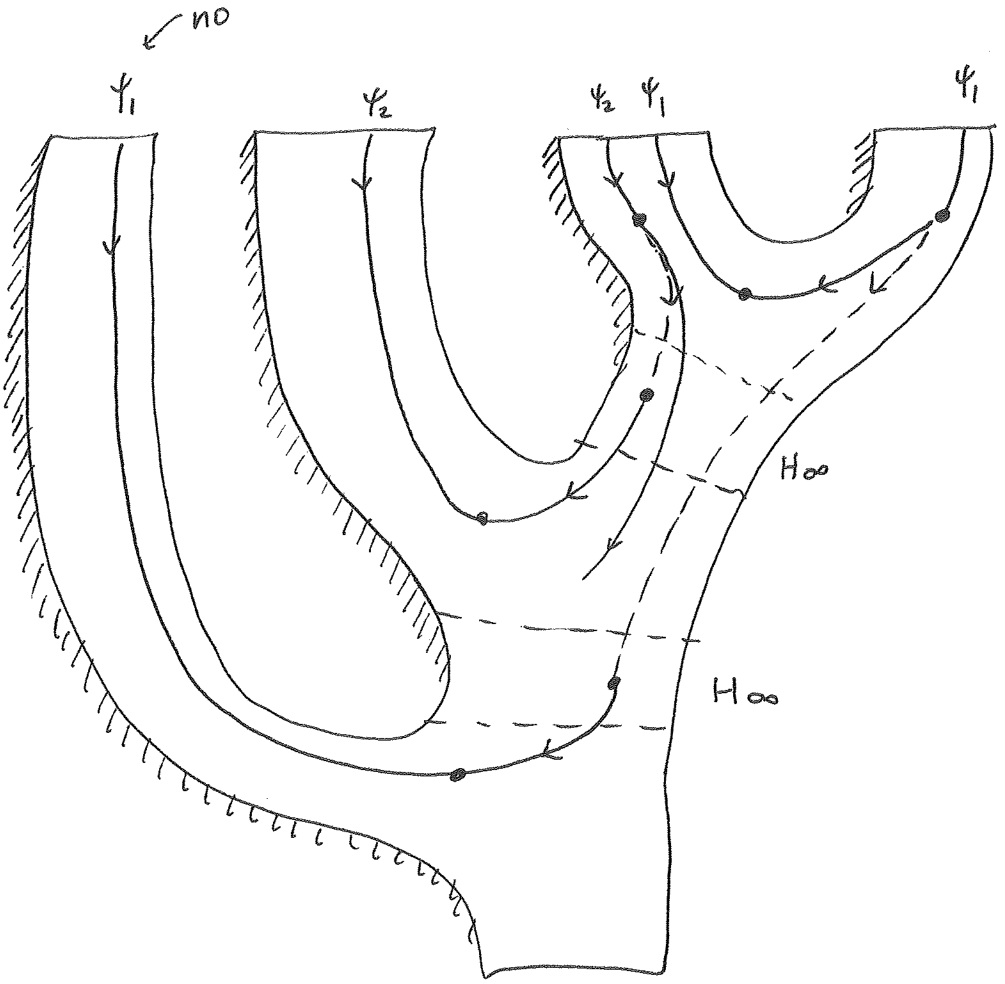


(12.1)

This contributes a factor of $(-1)\cdot 1 = -1$ to the operator

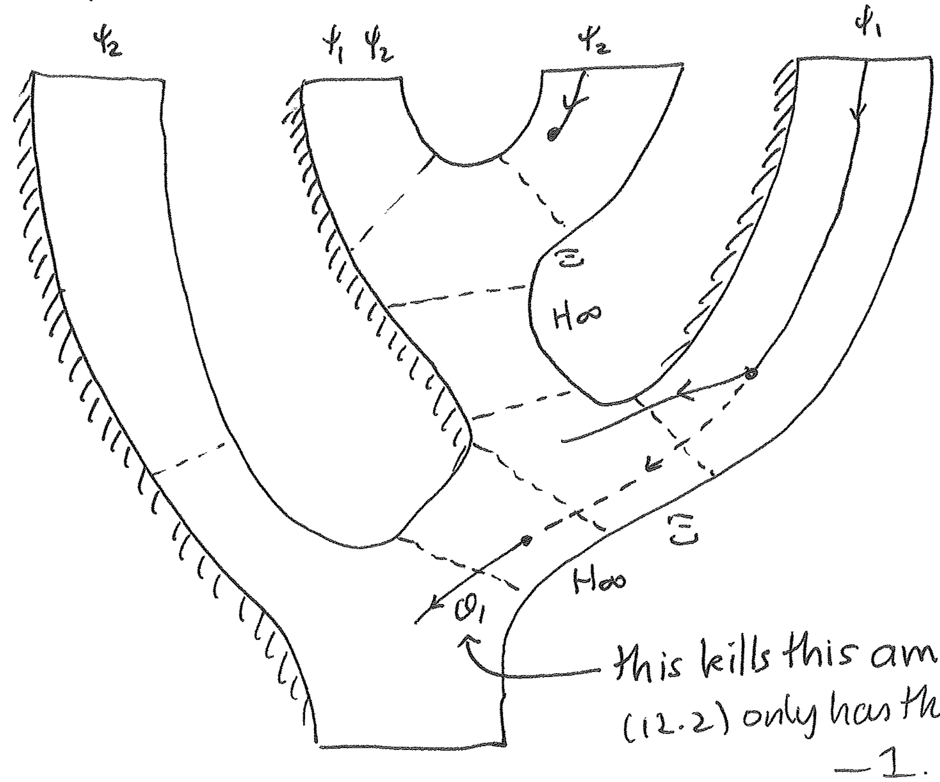
$$[\psi_2,-]\circ[\psi_1,[\psi_2,-]]\circ[\psi_1,[\psi_2,-]]\circ[\psi_1,-] \qquad (12.2)$$

Note the behaviour of the ψ_2's is forced on us: the rightmost ψ_2 must be involved in a trivalent interaction, but the ultimate destinations could switch as shown below:



(13.1)

Well no! This doesn't work (as there is <u>not</u> a ψ_1 on the leftmost channel). So the "first" H_{∞} must be a θ_1 interaction and (12.1) is forced on us. So the operator (12.2) only gets the -1 contribution from the first kind of tree. For the second:



(13.2)

this kills this amplitude, so (12.2) only has the single contribution -1.

In particular this shows $b_4 \neq 0$ for $d=3$. Can we compute b_4 fully from these diagrams?

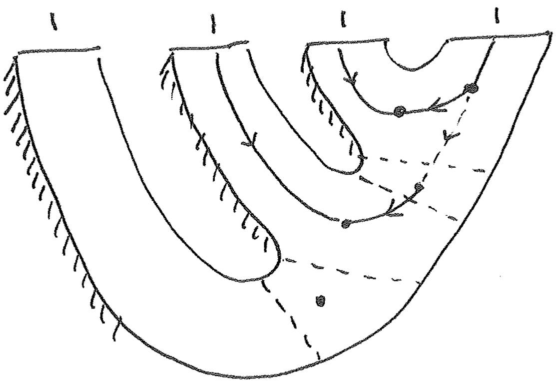
e.g. what is the amplitude for

$$\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 \longmapsto \ ? \ [\psi_1, \Lambda_0] \cdot \Lambda_1 \cdot \Lambda_2 \cdot [\psi_1, \Lambda_3] \qquad (14.1)$$

This is clearly zero. But

$$\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 \longmapsto \ ? [\psi_1, \Lambda_0] \cdot [\psi_1, \Lambda_1] \cdot [\psi_1, \Lambda_2] \cdot [\psi_1, \Lambda_3] \qquad (14.2)$$

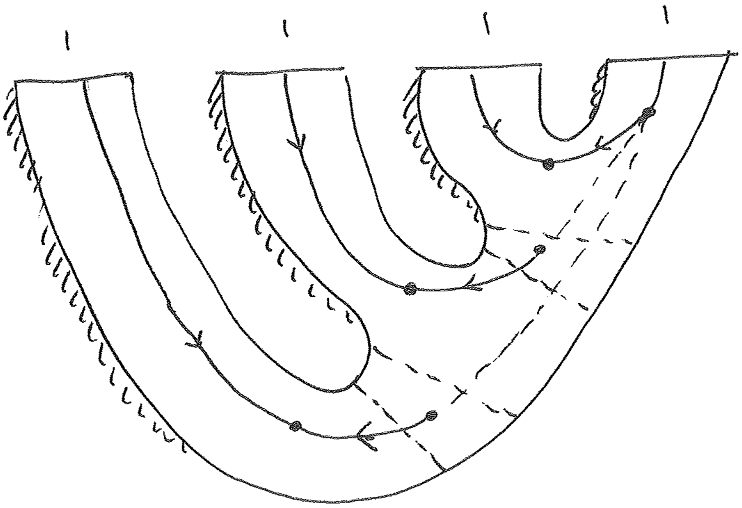
has contributions



(14.3)

---- well, no, two H_∞'s mean this also has amplitude <u>zero</u>.

<u>Note</u> If $d=4$, the diagram



(14.4)

contributes $+1$ to (14.2)