Minimal models of MFs VIB (checked)

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Ultimately we need to split an idempotent on $\underline{\mathrm{End}}(k^{stab})$ and in this note we start in on this work. For $W \in m^3$ this is genuinely easy, but if W has degree 2 components there is more to it, as we will see. A general point is that (k is a char. 0 field)

$$\underline{\mathrm{End}}(k^{stab}) = \mathrm{End}_k\big(\textstyle\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\big) \tag{1.1}$$
$$\cong \textstyle\bigwedge(k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \otimes \bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)$$

with creation and annihilation operators $\psi_i^*, [\psi_i, -]$ on the left tensor factor and $\psi_i, [\psi_i^*, -]$ on the right tensor factor. This observation plays an important role in the rest of our calculations.

We begin with a precise statement of (1.1). Set $T = k\psi_1 \oplus \cdots \oplus k\psi_n$ as a $\mathbb{Z}_2$-graded k-module. Then there is a natural iso

$$\xi : (\wedge T)^* \otimes_k \wedge T \longrightarrow \mathrm{End}_k(\wedge T) \tag{1.1}$$
$$\xi(\nu \otimes \eta)(x) = (-1)^{|\nu||\eta|} \nu(x) \cdot \eta$$

and a natural iso

$$\beta : \textstyle\bigwedge^a T^* \longrightarrow \mathrm{Hom}_k(\bigwedge^a T, k) \tag{1.2}$$
$$\nu_1 \wedge \cdots \wedge \nu_a \longmapsto \Big\{ f_1 \wedge \cdots \wedge f_a \longmapsto \sum_{\sigma \in S_a} \mathrm{sgn}(\sigma) \prod_{i=1}^a \nu_{\sigma(i)}(f_i)$$

ainfmf6

Combining which we deduce a canonical iso of $\mathbb{Z}_2$-graded k-modules

$$\bigwedge T^* \otimes_k \bigwedge T \xrightarrow{\ \beta\ } (\bigwedge T)^* \otimes_k \bigwedge T \xrightarrow{\ \xi\ } \operatorname{End}_k(\bigwedge T) \tag{2.1}$$

We denote this isomorphism by Θ. We have

$$\begin{aligned}\Theta\left(\psi_{i_1}^* \cdots \psi_{i_r}^* \otimes \psi_{j_1} \cdots \psi_{j_s} \right) &= \xi\left([\psi_{i_1} \cdots \psi_{i_r}]^* \otimes \psi_{j_1} \cdots \psi_{j_s} \right) \\ &= \left\{ x \longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(x) \cdot \psi_{j_1} \cdots \psi_{j_s} \right\}\end{aligned} \tag{2.2}$$

<u>lemma</u> There are commutative diagrams

$$\begin{tikzcd}
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"] \arrow[d, "1 \otimes \psi_i \wedge -"'] & \operatorname{End}_k(\bigwedge T) \arrow[d, "\psi_i \circ -"] \\
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"'] & \operatorname{End}_k(\bigwedge T)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"] \arrow[d, "1 \otimes \psi_i^* \lrcorner (-)"'] & \operatorname{End}_k(\bigwedge T) \arrow[d, "\psi_i^* \circ -"] \\
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"'] & \operatorname{End}_k(\bigwedge T)
\end{tikzcd}$$

$$\begin{tikzcd}
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"] \arrow[d, "\psi_i^{**} \lrcorner (-) \otimes 1"'] & \operatorname{End}_k(\bigwedge T) \arrow[d, "- \circ \psi_i"] \\
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"'] & \operatorname{End}_k(\bigwedge T)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"] \arrow[d, "\psi_i^* \wedge - \otimes 1"'] & \operatorname{End}_k(\bigwedge T) \arrow[d, "- \circ \psi_i^*"] \\
\bigwedge T^* \otimes \bigwedge T \arrow[r, "\Theta"'] & \operatorname{End}_k(\bigwedge T)
\end{tikzcd}$$

<u>Proof</u> We have on $\psi_{i_1}^* \cdots \psi_{i_r}^* \otimes \psi_{j_1} \cdots \psi_{j_s}$

$$x \longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(x) \cdot \psi_{j_1} \cdots \psi_{j_s}$$

$$\longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(x) \cdot \psi_i \psi_{j_1} \cdots \psi_{j_s}$$

which is $(-1)^r \ominus(\psi_{i_1}^* \cdots \psi_{i_r}^* \otimes \psi_i \psi_{j_1} \cdots \psi_{j_s})$, as claimed.

For the top right square

$$x \longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(x) \cdot \psi_i^*(\psi_{j_1} \cdots \psi_{j_s})$$

This leaves the bottom right square

$$x \longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(\psi_i^*(x)) \cdot \psi_{j_1} \cdots \psi_{j_s}$$

$$= (-1)^{rs} [\psi_i \psi_{i_1} \cdots \psi_{i_r}]^*(x) \cdot \psi_{j_1} \cdots \psi_{j_s}$$

$$= (-1)^s \ominus(\psi_i^* \psi_{i_1}^* \cdots \psi_{i_r}^* \otimes \psi_{j_1} \cdots \psi_{j_s})$$

So the diagram commutes as long as we understand $- \circ \psi_i^*$ to mean $\alpha \longmapsto (-1)^{|\alpha|} \alpha \circ \psi_i^*$, as it should. For the bottom left

$$x \longmapsto (-1)^{rs} [\psi_{i_1} \cdots \psi_{i_r}]^*(\psi_i x) \cdot \psi_{j_1} \cdots \psi_{j_s}$$

$$= (-1)^{rs} [\psi_i^*(\psi_{i_1} \cdots \psi_{i_r})]^*(x) \cdot \psi_{j_1} \cdots \psi_{j_s}$$

gives the result. $\square$

ainfmf6

Unfortunately the iso (2.1) is not the one useful for (1.1), where we want $[\psi_i, -]$ on $\mathrm{End}_k(\Lambda T)$ to act as annihilation on the left tensor factor. In the iso (2.1) instead it acts as

$$\Lambda T^* \otimes \Lambda T \circlearrowleft \quad 1 \otimes \psi_i \wedge (-) + \psi_i^{**} \lrcorner (-) \otimes 1 \qquad (4.1)$$

which is not what we want. Instead:

<u>Lemma</u> There is an isomorphism of $\mathbb{Z}_2$-graded k-modules

$$\Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n) \otimes \Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \xrightarrow{\rho} \mathrm{End}_k(\Lambda T)$$

$$\rho(\psi_{i_1} \cdots \psi_{i_r} \otimes \psi_{j_1}^* \cdots \psi_{j_s}^*) = \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* \qquad (4.2)$$

where on the RHS, ψ_i^* means left mult by ψ_i^*, and the same for ψ_i.

This is clear; but it is also <u>not</u> (H).

<u>Lemma</u> The isomorphism ρ makes the following identifications:

$\Lambda T \otimes \Lambda T^*$	$\mathrm{End}_k(\Lambda T)$
$1 \otimes \psi_i^{**} \lrcorner (-)$	$[\psi_i, -]$
$\psi_i^* \lrcorner (-) \otimes 1$	$[\psi_i^*, -]$

Koszul signs $\longrightarrow$ (4.3)

Remark Under the iso ρ, the action of

$$\{ \gamma_i \circ - , [\gamma_i^*, -] \}_{1 \le i \le n} \curvearrowright \mathrm{End}_k(\Lambda T)$$

becomes the standard Clifford action on ΛT, and hence

$$\begin{aligned} &\mathrm{Im}\left([\gamma_1^*, -] \circ \cdots \circ [\gamma_n^*, -] \circ \gamma_n \circ \cdots \circ \gamma_1 \right) \qquad (5.1) \\ &\quad \cong (k \cdot 1) \otimes_k \Lambda T^* \cong \Lambda T^* . \end{aligned}$$