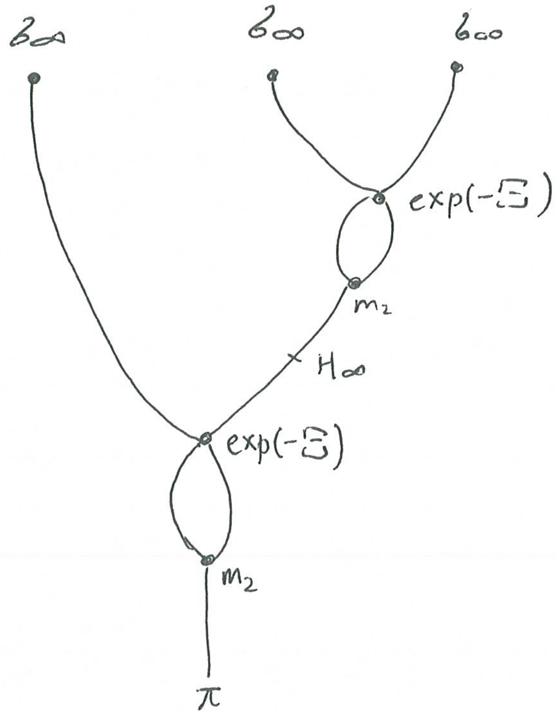
Minimal models for MFs V (checked)

We continue with some calculations in the case where $n=2$, so $k[\underline{x}] = k[x_1, x_2]$. We begin with b_3, which by p. 9 of ainfmf4 is (up to signs) given by



(1.1)

Since the left diagram in (9.1) there vanishes for the usual reasons. Now by p. 10 and 16 there

$$b_\infty = \sum_{m\geq 0} (-1)^m (H d_{End})^m b \tag{1.2}$$

$$= \sum_{m \geq 0} (-1)^m \sum_{j_1,\ldots,j_m} \sum_{\underline{u}\in \mathbb{Z}_2^m} \sum_{l_1,\ldots,l_m\geq 1} \sum_{z_1,\ldots,z_m} (-1)^m c_0(\underline{l}) \prod_{i=1}^m \partial_{x_{z_i}}(f_{j_i,l_i}^{u_i}) \prod_{i=1}^m [\psi_{j_i}^{u_i}, -]\,\theta_{z_i}\, b$$

For the moment we concentrate on b_∞, ignoring (1.1).

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$$\mathfrak{z}_\infty = \mathfrak{z} \qquad (m=0)$$

$$+ \sum_z \{ \cdots \} O_z \qquad z \text{ ranges over } 1,2.$$

$$+ \sum_{z_1, z_2} \{ \cdots \} O_{z_1} O_{z_2}$$

$$= \mathfrak{z} + \sum_z \Big\{ \sum_j \sum_{u \in \mathbb{Z}_2} \sum_{\ell \geq 1} \frac{1}{\ell} \partial x_z (f^u_{j,\ell}) [\psi^u_{j,} -] \Big\} O_z \mathfrak{z}$$

$$+ \sum_{z_1, z_2} \Big\{ \sum_{j_1, j_2} \sum_{u_1, u_2} \sum_{\ell_1, \ell_2 \geq 1} \frac{1}{\ell_1 + \ell_2} \frac{1}{\ell_2} \partial x_{z_1} (f^{u_1}_{j_1, \ell_1}) \partial x_{z_2} (f^{u_2}_{j_2, \ell_2})$$
$$[\psi^{u_1}_{j_1,} -] O_{z_1} [\psi^{u_2}_{j_2,} -] O_{z_2} \Big\} \qquad (2.1)$$

$$= \mathfrak{z} + \sum_z \Big\{ \sum_j \sum_u \sum_\ell \frac{1}{\ell} \partial x_z (f^u_{j,\ell}) [\psi^u_{j,} -] \Big\} O_z \mathfrak{z}$$
$$(2.2)$$
$$- \sum_{z_1 < z_2} \Big\{ \sum_{\underline{j}, \underline{u}, \underline{\ell}} \frac{1}{\ell_1 + \ell_2} \frac{1}{\ell_2} \Big[\partial x_{z_1} (f^{u_1}_{j_1, \ell_1}) \partial x_{z_2} (f^{u_2}_{j_2, \ell_2})$$
$$- \partial x_{z_2} (f^{u_1}_{j_1, \ell_1}) \partial x_{z_1} (f^{u_2}_{j_2, \ell_2}) \Big]$$
$$[\psi^{u_1}_{j_1,} -] [\psi^{u_2}_{j_2,} -] \Big\} O_{z_1} O_{z_2} \mathfrak{z}$$

Of course for $n=2$ there is only $O_1 O_2$ in the second summand.

Similarly by (10.1) of ainfmf4 and p. 16

$$H_\infty = \sum_{m \geq 0} (-1)^m (H d_{End})^m H$$

Hence for $\omega \in S$ of degree p and homogeneous f of degree b

$$H_\infty(\omega \otimes f \Psi) = \sum_{m\geq 0} (-1)^m (H d_{End})^m J^{-1} \nabla (\omega \otimes f \Psi)$$

$$= \sum_{z_0} \sum_{m \geq 0} (-1)^m (H d_{End})^m J^{-1} (\theta_{z_0} \omega \otimes \partial_{x_{z_0}}(f) \Psi)$$

$$= \sum_{z_0} \sum_{m \geq 0} (-1)^m (H d_{End})^m \frac{1}{p+b} (\theta_{z_0} \omega \otimes \partial_{x_{z_0}}(f) \Psi)$$

$$(3.1)$$

$$= \sum_{z_0} \sum_{m \geq 0} (-1)^m \frac{1}{p+b} \sum_{\underline{j}} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{\underline{\ell}} \sum_{\underline{z}} (-1)^m C_{p+b}(\underline{\ell}) \prod_{i=1}^m \partial_{x_{z_i}}(f_{j_i, \ell_i}^{u_i}) \prod_{i=1}^m [\psi_{j_i}^{u_i}, -] \theta_{z_i} (\theta_{z_0} \omega \otimes \partial_{x_{z_0}}(f) \Psi)$$

In fact for $n=2$ only $m=0$ and $m=1$ can contribute, so

$$(3.2)$$

$$= \sum_{z_0} \frac{1}{p+b} (\theta_{z_0} \omega \otimes \partial_{x_{z_0}}(f) \Psi)$$
$$+ \sum_{z_0} \frac{1}{p+b} \sum_{j,u,\ell,z} C_{p+b}(\ell) \partial_{x_z}(f_{j,\ell}^u) [\psi_j^u, -] \theta_z (\theta_{z_0} \omega \otimes \partial_{x_{z_0}}(f) \Psi)$$

$$= \sum_{z_0} \frac{1}{p+b} \left(\mathcal{O}_{z_0} \omega \otimes \partial x_{z_0}(f) \Psi \right) \tag{4.1}$$

$$+ \sum_{z_0} \sum_{j,u,\ell,z} \frac{1}{p+b} \frac{1}{p+b+\ell} \partial x_z (f^u_{j,\ell}) [\psi^u_j, -] \mathcal{O}_z \left(\mathcal{O}_{z_0} \omega \otimes \partial x_{z_0}(f) \Psi \right)$$

The final ingredient is

$$\Xi = [\psi_1, -] \otimes \mathcal{O}_1^* + [\psi_2, -] \otimes \mathcal{O}_2^*$$

$$\exp(-\Xi) = 1 - \Xi + \tfrac{1}{2} \Xi^2 \tag{4.2}$$

$$= 1 - [\psi_1, -] \otimes \mathcal{O}_1^* - [\psi_2, -] \otimes \mathcal{O}_2^*$$
$$+ \tfrac{1}{2} \left(\sum_i [\psi_i, -] \otimes \mathcal{O}_i^* \right) \left(\sum_i [\psi_i, -] \otimes \mathcal{O}_i^* \right)$$

$$= 1 - [\psi_1, -] \otimes \mathcal{O}_1^* - [\psi_2, -] \otimes \mathcal{O}_2^*$$
$$- \tfrac{1}{2} [\psi_1, -][\psi_2, -] \otimes \mathcal{O}_1^* \mathcal{O}_2^* .$$
$$- \tfrac{1}{2} [\psi_2, -][\psi_1, -] \otimes \mathcal{O}_2^* \mathcal{O}_1^*$$

$$= 1 - [\psi_1, -] \otimes \mathcal{O}_1^* - [\psi_2, -] \otimes \mathcal{O}_2^*$$
$$- [\psi_1, -][\psi_2, -] \otimes \mathcal{O}_1^* \mathcal{O}_2^*$$

Example $W = y^d - x^d$ for $d > 2$. Then

$$W = x \cdot \underbrace{(-x^{d-1})}_{W^1} + y \cdot \underbrace{(y^{d-1})}_{W^2} \qquad (5.1)$$

Then $S = \Lambda(k\theta_1 \oplus k\theta_2)$ and $\underline{\mathrm{End}} = \mathrm{End}_k(\Lambda(k\psi_1 \oplus k\psi_2))$.
To write down the minimal model on $\underline{\mathrm{End}}(k^{stab})$ induced by p. (i) of ainfmf we write down δ_∞, $\exp(-\Xi)$, H_∞ for this particular potential. But we can already write down b_2 from (8.1) of ainfmf4. Note that since $\partial_x(W^1) = -(d-1)x^{d-2}$

$\partial_y(W^2) = (d-1)y^{d-2}$, $d > 2$

$$At_i = -[\psi_i^*, -] \qquad (5.2)$$

Hence for $\beta_1, \beta_2 \in \underline{\mathrm{End}}$

$$\begin{aligned} b_2(\beta_1 \otimes \beta_2) = \beta_1 \bullet \beta_2 + \sum_q (-1)^{|\beta_1|} [\psi_q, \beta_1] \bullet [\psi_q^*, \beta_2] \\ - [\psi_1, [\psi_2, \beta_1]] \bullet [\psi_1^*[\psi_2^*, \beta_2]]. \end{aligned} \qquad (5.3)$$

On β_1, β_2 which are products only of ψ^*'s, this yields simply $b_2(\beta_1 \otimes \beta_2) = \beta_1 \bullet \beta_2$. This is relevant because under the k-linear h.e. of ainfmf

$$S \otimes_k \mathrm{End}_k(k^{stab}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \underline{\mathrm{End}}(k^{stab}) \qquad (5.4)$$

(ainfmf5)

The operaton on End corresponding to $\mathcal{O}_i, \mathcal{O}_i^*$ are

$$\gamma_i = -\psi_i, \quad At_i = -[\psi_i^*, -] = \gamma_i^\dagger$$

The idempotent

$$e_2 = \gamma_1^\dagger \gamma_2^\dagger \gamma_2 \gamma_1 = [\psi_1, -][\psi_2, -]\psi_2 \psi_1 \tag{6.1}$$

corresponds via the iso of p. 4 (ainfmf6)

$$\rho : \Lambda(k\psi_1 \oplus k\psi_2) \otimes \Lambda(k\psi_1^* \oplus k\psi_2^*) \xrightarrow{\cong} \mathrm{End}_k(\Lambda(k\psi_1 \oplus k\psi_2)) \tag{6.2}$$

$$\psi_{i_1} \cdots \psi_{i_r} \otimes \psi_{j_1}^* \cdots \psi_{j_s}^* \longmapsto \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^*$$

(with $\mathrm{End}_k(\Lambda(k\psi_1 \oplus k\psi_2)) = \underline{\mathrm{End}}$)

To the projection onto $(k \cdot 1) \otimes \Lambda(k\psi_1^* \oplus k\psi_2^*)$. Thus, if we define

$$\mathcal{A} := \Lambda(k\psi_1^* \oplus k\psi_2^*) \tag{6.3}$$

Then $\mathcal{A}$ is an associative algebra (the usual exterior algebra) under the product induced on $\mathcal{A}$ by b_2 and e.

To compute b_m for $m \geq 3$ we use e to mean we only need $u=1$ in p. 12 (ainfmf4)

$$f_i^u = \begin{cases} x_i & u=0 \\ W^i & u=1 \end{cases} \quad \text{so} \quad f_1^1 = -x^{d-1}, \quad f_2^1 = y^{d-1} \tag{6.4}$$

From (2.2) we read off that $n=1$, $j=2$, $\ell = d-1$

$$
\begin{aligned}
\mathfrak{z}_\infty = \mathfrak{z} &+ \frac{1}{d-1}\Big\{ \partial_x(-x^{d-1})[\psi_1,-]\theta_1 \\
&\qquad + \partial_y(y^{d-1})[\psi_2,-]\theta_2 \Big\}\mathfrak{z} \qquad (7.1)\\
&- \frac{1}{2(d-1)^2}\Big\{ \underset{j_1=1,\,j_2=2}{\Big[}\partial_x(-x^{d-1})\,\partial_y(y^{d-1}) \\
&\qquad - \partial_y(-x^{d-1})\,\partial_x(y^{d-1})\Big][\psi_1,-][\psi_2,-] \\
&\qquad \overset{j_1=2,\,j_2=1}{+} \Big[\partial_x(y^{d-1})\,\partial_y(-x^{d-1}) \\
&\qquad - \partial_y(y^{d-1})\,\partial_x(-x^{d-1})\Big][\psi_2,-][\psi_1,-]\Big\} \\
&\qquad \theta_1\theta_2\mathfrak{z}
\end{aligned}
$$

$$
\begin{aligned}
= \mathfrak{z} &- x^{d-2}[\psi_1,-]\theta_1\mathfrak{z} + y^{d-2}[\psi_2,-]\theta_2\mathfrak{z} \\
&- \frac{1}{2(d-1)^2}\Big\{ -(d-1)^2 x^{d-2}y^{d-2}[\psi_1,-][\psi_2,-] \\
&\qquad + (d-1)^2 x^{d-2}y^{d-2}[\psi_2,-][\psi_1,-]\Big\}\theta_1\theta_2\mathfrak{z}
\end{aligned}
$$

$$
\begin{aligned}
= \mathfrak{z} &- x^{d-2}[\psi_1,-]\theta_1\mathfrak{z} + y^{d-2}[\psi_2,-]\theta_2\mathfrak{z} \\
&+ x^{d-2}y^{d-2}[\psi_1,-][\psi_2,-]\theta_1\theta_2\mathfrak{z}
\end{aligned}
$$

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Next we compute H_∞ from (4.#), as above restricting to inputs which have only Ψ^*'s so we may take $u \equiv 1$.

(8.1)

$$H_\infty(\omega \otimes f\Psi) = \frac{1}{p+b}\left\{ \mathcal{O}_1\omega \otimes \partial_x(f)\Psi + \mathcal{O}_2\omega \otimes \partial_y(f)\Psi \right\}$$

$$+ \sum_{z_0, z} \frac{1}{(p+b)(p+b+d-1)} \partial_{x_z}\left(f^{1}_{z,d-1}\right)[\Psi_{z_1}, -]\mathcal{O}_z\left(\mathcal{O}_{z_0}\omega \otimes \partial_{x_{z_0}}(f)\Psi\right)$$

$$= \frac{1}{p+b}\left\{ \mathcal{O}_1\omega \otimes \partial_x(f)\Psi + \mathcal{O}_2\omega \otimes \partial_y(f)\Psi \right\}$$

$$+ \frac{1}{(p+b)(p+b+d-1)}\left\{ \partial_y(y^{d-1})[\Psi_2,-]\mathcal{O}_2\mathcal{O}_1\,\partial_x(f) + \partial_x(-x^{d-1})[\Psi_1,-]\,\mathcal{O}_1\mathcal{O}_2\,\partial_y(f) \right\}(\omega \otimes \Psi)$$

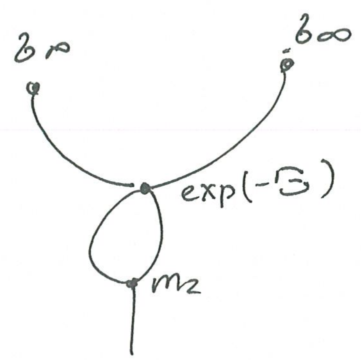
$$= \frac{1}{p+b}\left\{ \partial_x(f)\mathcal{O}_1 + \partial_y(f)\mathcal{O}_2 \right\}(\omega \otimes \Psi)$$

$$- \frac{1}{(p+b)(p+b+d-1)}\left\{ \begin{array}{l}(d-1)y^{d-2}\partial_x(f)[\Psi_2,-] \\ (d-1)x^{d-2}\partial_y(f)[\Psi_1,-]\end{array} \right\}\mathcal{O}_1\mathcal{O}_2(\omega \otimes \Psi)$$

$$= \frac{1}{p+b}\left\{ \partial_x(f)\mathcal{O}_1 + \partial_y(f)\mathcal{O}_2 \right\}(\omega \otimes \Psi)$$

$$- \frac{(d-1)}{(p+b)(p+b+d-1)}\left\{ y^{d-2}\partial_x(f)[\Psi_2,-] + x^{d-2}\partial_y(f)[\Psi_1,-] \right\} \cdot \mathcal{O}_1\mathcal{O}_2(\omega \otimes \Psi).$$

Let us return now to b_3 for $W = y^d - x^d$, firstly



Is given by (for $\Psi_i \in \Lambda(k\psi_1^* \oplus k\psi_2^*)$)

$$\Psi_1 \otimes \Psi_2 \longmapsto m_2 \exp(-\Xi)\big(\mathfrak{Z}_\infty(\Psi_1) \otimes \mathfrak{Z}_\infty(\Psi_2)\big)$$

$$= m_2\Big[1 - [\psi_1, -] \otimes \theta_1^* - [\psi_2, -] \otimes \theta_2^* - [\psi_1, -][\psi_2, -] \otimes \theta_1^*\theta_2^*\Big]$$

$$\Big(\mathfrak{Z}_\infty(\Psi_1) \otimes \mathfrak{Z}_\infty(\Psi_2)\Big) \qquad (9.1)$$

$$\begin{aligned} = m_2\Big[& \overset{ⓐ}{\mathfrak{Z}_\infty(\Psi_1) \otimes \mathfrak{Z}_\infty(\Psi_2)} - \overset{ⓑ}{(-1)^{|\Psi_1|} [\psi_1, -]\mathfrak{Z}_\infty(\Psi_1) \otimes \theta_1^* \mathfrak{Z}_\infty(\Psi_2)} \\ & - (-1)^{|\Psi_1|} [\psi_2, -]\mathfrak{Z}_\infty(\Psi_1) \otimes \theta_2^* \mathfrak{Z}_\infty(\Psi_2) \quad ⓒ \\ & \overset{ⓓ}{- [\psi_1, -][\psi_2, -]\mathfrak{Z}_\infty(\Psi_1) \otimes \theta_1^*\theta_2^* \mathfrak{Z}_\infty(\Psi_2)} \Big] \end{aligned}$$

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$$= m_2\Big[\Big(\Psi_1 + x^{d-2}\theta_1 \otimes [\psi_1, \Psi_1] - y^{d-2}\theta_2 \otimes [\psi_2, \Psi_1] + x^{d-2}y^{d-2}\theta_1\theta_2 \otimes [\psi_1,[\psi_2,\Psi_1]] \Big) \quad ⓐ$$

$$\otimes \Big(\Psi_2 + x^{d-2}\theta_1 \otimes [\psi_1,\Psi_2] - y^{d-2}\theta_2 \otimes [\psi_2,\Psi_2] + x^{d-2}y^{d-2}\theta_1\theta_2 \otimes [\psi_1,[\psi_2,\Psi_2]] \Big) \qquad (10.1)$$

$$- (-1)^{|\Psi_1|} \Big([\psi_1,\Psi_1] + y^{d-2}\theta_2 \otimes [\psi_1,[\psi_2,\Psi_1]] \Big) \quad ⓑ$$

$$\otimes \Big(x^{d-2} \otimes [\psi_1,\Psi_2] + x^{d-2}y^{d-2}\theta_2 \otimes [\psi_1,[\psi_2,\Psi_2]] \Big)$$

$$- (-1)^{|\Psi_1|} \Big([\psi_2,\Psi_1] - x^{d-2}\theta_1 \otimes [\psi_2,[\psi_1,\Psi_1]] \Big)$$

$$\otimes \Big(- y^{d-2} \otimes [\psi_2,\Psi_2] - x^{d-2}y^{d-2}\theta_1 \otimes [\psi_1,[\psi_2,\Psi_2]] \Big) \quad ⓒ$$

$$+ [\psi_1,[\psi_2,\Psi_1]] \otimes x^{d-2}y^{d-2}[\psi_1,[\psi_2,\Psi_2]] \quad ⓓ \Big]$$

$$= ⓐ\ \Psi_1 \circ \Psi_2 + (-1)^{|\Psi_1|} x^{d-2}\theta_1 \otimes \Psi_1 \circ [\psi_1,\Psi_2] - (-1)^{|\Psi_1|} y^{d-2}\theta_2 \otimes \Psi_1 \circ [\psi_2,\Psi_2] + x^{d-2}y^{d-2}\theta_1\theta_2 \otimes \Psi_1 \circ [\psi_1,[\psi_2,\Psi_2]]$$

$$
\begin{aligned}
&+ x^{d-2}\mathcal{O}_1 \otimes [\psi_1, \Psi_1] \cdot \Psi_2 \\
&\quad - (-1)^{|\Psi_1|+1} x^{d-2} y^{d-2} \mathcal{O}_1 \mathcal{O}_2 \otimes [\psi_1, \Psi_1] \cdot [\psi_2, \Psi_2] \\
&\quad - y^{d-2} \mathcal{O}_2 \otimes [\psi_2, \Psi_1] \cdot \Psi_2 \\
&\quad + (-1)^{|\Psi_1|+1} x^{d-2} y^{d-2} \mathcal{O}_1 \mathcal{O}_2 \otimes [\psi_2, \Psi_1] \cdot [\psi_1, \Psi_2] \\
&\quad + x^{d-2} y^{d-2} \mathcal{O}_1 \mathcal{O}_2 \otimes [\psi_1, [\psi_2, \Psi_1]] \cdot \Psi_2
\end{aligned}
\tag{11.1}
$$

(b)

$$
\begin{aligned}
-(-1)^{|\Psi_1|}\Big(& x^{d-2} [\psi_1, \Psi_1] \cdot [\psi_1, \Psi_2] \\
& + x^{d-2} y^{d-2} (-1)^{|\Psi_1|+1} \mathcal{O}_2 \otimes [\psi_1, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]] \\
& + \underset{x^{d-2}}{y^{d-2}} \mathcal{O}_2 \otimes [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_1, \Psi_2] \Big)
\end{aligned}
$$

(c)

$$
\begin{aligned}
-(-1)^{|\Psi_1|}\Big(& -y^{d-2} [\psi_2, \Psi_1] \cdot [\psi_2, \Psi_2] \\
& - (-1)^{|\Psi_1|+1} x^{d-2} y^{d-2} \mathcal{O}_1 \otimes [\psi_2, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]] \\
& + x^{d-2} y^{d-2} \mathcal{O}_1 \otimes [\psi_2, [\psi_1, \Psi_1]] \cdot [\psi_2, \Psi_2] \Big)
\end{aligned}
$$

(d)

$$
+ \; x^{d-2} y^{d-2} [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_1, [\psi_2, \Psi_2]].
$$

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$$Y$$

$$
\begin{aligned}
= {} & \Psi_1 \cdot \Psi_2 - (-1)^{|\Psi_1|} x^{d-2} [\psi_1, \Psi_1] \cdot [\psi_1, \Psi_2] \quad {}^{1}Y^{1} \\
& + (-1)^{|\Psi_1|} y^{d-2} [\psi_2, \Psi_1] \cdot [\psi_2, \Psi_2] \quad {}^{2}Y^{2} \\
& + x^{d-2} y^{d-2} [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_1, [\psi_2, \Psi_2]] \quad {}^{2}_{1}Y^{2}_{1}
\end{aligned}
$$

$$
\begin{aligned}
+ \ \theta_1 \otimes \Big\{ & (-1)^{|\Psi_1|} x^{d-2} \Psi_1 \cdot [\psi_1, \Psi_2] \quad Y^{1} \\
& + x^{d-2} [\psi_1, \Psi_1] \cdot \Psi_2 \quad {}^{1}Y \\
& + (-1)^{|\Psi_1| + |\Psi_1|} x^{d-2} y^{d-2} [\psi_2, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]] \quad {}^{2}Y^{2}_{1} \\
& - (-1)^{|\Psi_1|} x^{d-2} y^{d-2} [\psi_2, [\psi_1, \Psi_1]] \cdot [\psi_2, \Psi_2] \Big\} \quad {}^{1}_{2}Y^{2}
\end{aligned}
$$

$$
\begin{aligned}
+ \ \theta_2 \otimes \Big\{ & -(-1)^{|\Psi_1|} y^{d-2} \Psi_1 \cdot [\psi_2, \Psi_2] \quad Y^{2} \qquad (12.1) \\
& - y^{d-2} [\psi_2, \Psi_1] \cdot \Psi_2 \quad {}^{2}Y \\
& + (-1)^{|\Psi_1| + |\Psi_1|} x^{d-2} y^{d-2} [\psi_1, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]] \quad {}^{1}Y^{2}_{1} \\
& - (-1)^{|\Psi_1|} x^{d-2} y^{d-2} [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_1, \Psi_2] \Big\} \quad {}^{2}_{1}Y^{1}
\end{aligned}
$$

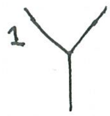
$$
\begin{aligned}
+ \ \theta_1 \theta_2 \otimes \Big\{ & x^{d-2} y^{d-2} \Psi_1 \cdot [\psi_1, [\psi_2, \Psi_2]] \quad Y^{2}_{1} \\
& + (-1)^{|\Psi_1|} x^{d-2} y^{d-2} [\psi_1, \Psi_1] \cdot [\psi_2, \Psi_2] \quad {}^{1}Y^{2} \\
& - (-1)^{|\Psi_1|} x^{d-2} y^{d-2} [\psi_2, \Psi_1] \cdot [\psi_1, \Psi_2] \quad {}^{2}Y^{1} \\
& + x^{d-2} y^{d-2} [\psi_1, [\psi_2, \Psi_1]] \cdot \Psi_2 \Big\} \quad {}^{2}_{1}Y
\end{aligned}
$$

In terms of the trees, the coefficients are:

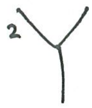
0


1

1

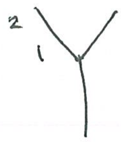

$x^{d-2}\theta_1$

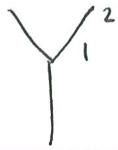

$(-1)^{|\Psi_1|}x^{d-2}\theta_1$

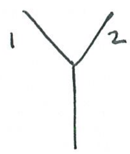

$-y^{d-2}\theta_2$

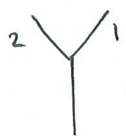

$-(-1)^{|\Psi_1|}y^{d-2}\theta_2$

2

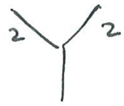

$x^{d-2}y^{d-2}\theta_1\theta_2$


$x^{d-2}y^{d-2}\theta_1\theta_2$

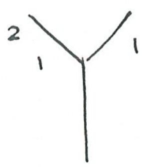

$(-1)^{|\Psi_1|}x^{d-2}y^{d-2}\theta_1\theta_2$

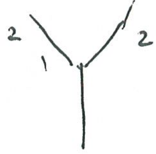

$-(-1)^{|\Psi_1|}x^{d-2}y^{d-2}\theta_1\theta_2$

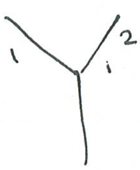

$-(-1)^{|\Psi_1|}x^{d-2}$

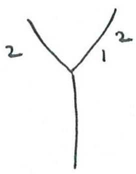

$(-1)^{|\Psi_1|}y^{d-2}$

3

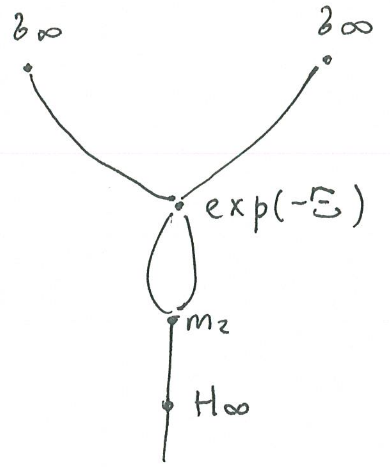

$-(-1)^{|\Psi_1|}x^{d-2}y^{d-2}\theta_2$


$(-1)^{|\Psi_1|}x^{d-2}y^{d-2}\theta_1$


$x^{d-2}y^{d-2}\theta_2$


$-x^{d-2}y^{d-2}\theta_1$

We hope this makes sense when we apply H_∞, from (8.1)



(14.1)

$$H_\infty m_2 \exp(-\Xi)\big(b_\infty(\Psi_1) \otimes b_\infty(\Psi_2)\big)$$

$$= H_\infty\big((12.1) \big)$$

- Because of the differentiation $\curlyvee$ dies

(14.2)

$$= -(-1)^{|\Psi_1|}\frac{1}{d-2}\partial_x(x^{d-2})\,\theta_1 \otimes [\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2]$$

$$+ (-1)^{|\Psi_1|}\frac{(d-1)}{(d-2)(d-2+d-1)}\left\{ y^{d-2}\partial_x(x^{d-2})\right\}\theta_1\theta_2 \otimes \big[\psi_2, [\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2]\big]$$

$$+ (-1)^{|\Psi_1|}\frac{1}{d-2}\partial_y(y^{d-2})\,\theta_2 \otimes [\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2]$$

$$- (-1)^{|\Psi_1|}\frac{(d-1)}{(d-2)(d-2+d-1)}\left\{ x^{d-2}\partial_y(y^{d-2})\right\}\theta_1\theta_2 \otimes \big[\psi_1, [\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2]\big]$$

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$$-\frac{1}{2d-3} Q_2 Q_1 \otimes x^{d-2}\partial_y(y^{d-2})[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]]$$

$$-(-1)^{|\Psi_1|}\frac{1}{2d-3} Q_2 Q_1\otimes x^{d-2}\partial_y(y^{d-2})[\psi_2,[\psi_1,\Psi_1]]\cdot[\psi_2,\Psi_2]$$

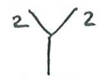
$$+\frac{1}{2d-3}Q_1 Q_2\otimes \partial_x(x^{d-2})y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]]$$

$$-(-1)^{|\Psi_1|}\frac{1}{2d-3}Q_1 Q_2\otimes\partial_x(x^{d-2})y^{d-2}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2]$$

$$= -(-1)^{|\Psi_1|} Q_1 \otimes x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2] \qquad (15.1)$$

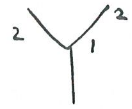
$$+(-1)^{|\Psi_1|}\frac{d-1}{2d-3}Q_1 Q_2\otimes y^{d-2}x^{d-3}[\psi_2,[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2]]$$

$$+(-1)^{|\Psi_1|}Q_2\otimes y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2]$$



$$-(-1)^{|\Psi_1|}\frac{d-1}{2d-3}Q_1 Q_2\otimes x^{d-2}y^{d-3}[\psi_1,[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2]]$$

$$+\frac{d-2}{(2d-3)}Q_1 Q_2\otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]]$$



$$+(-1)^{|\Psi_1|}\frac{d-2}{2d-3}Q_1 Q_2\otimes x^{d-2}y^{d-3}[\psi_2,[\psi_1,\Psi_1]]\cdot[\psi_2,\Psi_2]$$

$$+\frac{d-2}{2d-3}Q_1 Q_2\otimes x^{d-3}y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]]$$

$$-(-1)^{|\Psi_1|}\frac{d-2}{2d-3}Q_1 Q_2\otimes x^{d-3}y^{d-2}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2]$$

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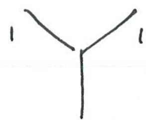
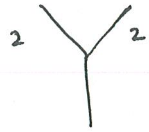
$$
\begin{aligned}
= & -(-1)^{|\Psi_1|}\alpha_1 \otimes x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2] \\
& + (-1)^{|\Psi_1|}\frac{d-1}{2d-3}\alpha_1\alpha_2 \otimes y^{d-2}x^{d-3}[\psi_2,[\psi_1,\Psi_1]]\cdot[\psi_1,\Psi_2] \\
& - \frac{d-1}{2d-3}\alpha_1\alpha_2 \otimes y^{d-2}x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_2,[\psi_1,\Psi_2]] \\
& - (-1)^{|\Psi_1|}\frac{d-2}{2d-3}\alpha_1\alpha_2 \otimes y^{d-2}x^{d-3}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2] \\
& + (-1)^{|\Psi_1|}\alpha_2 \otimes y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2] \\
& - (-1)^{|\Psi_1|}\frac{d-1}{2d-3}\alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_2,\Psi_2] \\
& + \frac{d-1}{2d-3}\alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
& + \frac{d-2}{2d-3}\alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
& (-1)^{|\Psi_1|}\frac{d-2}{2d-3}\alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_2,[\psi_1,\Psi_1]]\cdot[\psi_2,\Psi_2]. \\
& + \frac{d-2}{2d-3}\alpha_1\alpha_2 \otimes x^{d-3}y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]]
\end{aligned}
$$

$$
\begin{aligned}
= & -(-1)^{|\Psi_1|}\alpha_1 \otimes x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2] \\
& + (-1)^{|\Psi_1|}\alpha_2 \otimes y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2] \\
& - (-1)^{|\Psi_1|}\alpha_1\alpha_2 \otimes x^{d-3}y^{d-2}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2] \\
& + \alpha_1\alpha_2 \otimes x^{d-3}y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
& + \alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
& - (-1)^{|\Psi_1|}\alpha_1\alpha_2 \otimes x^{d-2}y^{d-3}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_2,\Psi_2].
\end{aligned}
$$

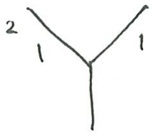
This completes the calculation of (14.1). The result is

1

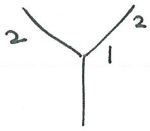

$-(-1)^{|\Psi_1|} x^{d-3} \theta_1$

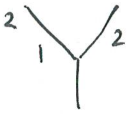

$(-1)^{|\Psi_1|} y^{d-3} \theta_2$

2


$-(-1)^{|\Psi_1|} x^{d-3} y^{d-2} \theta_1 \theta_2$


$x^{d-3} y^{d-2} \theta_1 \theta_2$


$x^{d-2} y^{d-3} \theta_1 \theta_2$


$-(-1)^{|\Psi_1|}$
$x^{d-2} y^{d-3} \theta_1 \theta_2$

(17.1)

The absence of trees like  here is due to the fact that $\partial_y(x^{d-2}) = 0$.
So really we should redo these trees with a less specific attitude.
But before we do that we will work out b_3.

17.5

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Next we compute the suspended forward multiplication

$$\rho_3 : \mathcal{A}[1]^{\otimes 3} \longrightarrow \mathcal{A}[1]$$

which is $\rho_3 = \sum_T \rho_T$ but only $T =$  with $\hat{T} =$ 

makes a nonzero contribution. Then

$$\rho_T(\Psi_2 \otimes \Psi_1 \otimes \Psi_0) = (-1)^{1 + \sum_{i<j} \tilde{\Psi}_i \tilde{\Psi}_j + \tilde{\Psi}_1 + \tilde{\Psi}_0} \operatorname{eval}_{\hat{T}}(\Psi_0 \otimes \Psi_1 \otimes \Psi_2) \qquad (17.5.1)$$

where

$$\operatorname{eval}_{\hat{T}}(\Psi_0 \otimes \Psi_1 \otimes \Psi_2) = \pi m_2 \exp(-\Xi)\big(b_\infty(\Psi_0) \otimes H_\infty m_2 \exp(-\Xi)(b_\infty(\Psi_1) \otimes b_\infty(\Psi_2))\big)$$

is the evaluation of (1.1) on inputs Ψ_0, Ψ_1, Ψ_2 (without Koszul signs).

So next we compute

$$
\begin{aligned}
&\mathrm{eval}_T(\Psi_0 \otimes \Psi_1 \otimes \Psi_2) \qquad (18.1)\\
&= \pi m_2 \exp(-\Xi)\Big(\Big[\Psi_0 + x^{d-2}\theta_1 \cancel{\otimes} [\psi_1, \Psi_0] \\
&\qquad - y^{d-2}\theta_2 \cancel{\otimes} [\psi_2, \Psi_0] \\
&\qquad + x^{d-2}y^{d-2}\cancel{\theta_1}\theta_2 \otimes [\psi_1,[\psi_2,\Psi_0]] \Big] \\
&\otimes \Big[-(-1)^{|\Psi_1|}\theta_1 \otimes x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2] \\
&\qquad + (-1)^{|\Psi_1|}\theta_2 \otimes y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2] \\
&\qquad - (-1)^{|\Psi_1|}\theta_1\theta_2 \otimes x^{d-3}y^{d-2}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2] \\
&\qquad + \theta_1\theta_2 \otimes x^{d-3}y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
&\qquad + \theta_1\theta_2 \otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
&\qquad - (-1)^{|\Psi_1|}\theta_1\theta_2 \otimes x^{d-2}y^{d-3}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_2,\Psi_2] \Big] \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi m_2 \Big(-([\psi_1,-]\otimes\theta_1^*)\big(\Psi_0 \otimes ((-1)^{|\Psi_1|+1}\theta_1 \otimes x^{d-3}[\psi_1,\Psi_1]\cdot[\psi_1,\Psi_2])\big) \\
&\qquad - ([\psi_2,-]\otimes\theta_2^*)\big(\Psi_0 \otimes ((-1)^{|\Psi_1|}\theta_2 \otimes y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_2,\Psi_2])\big) \\
&\qquad - ([\psi_1,-][\psi_2,-]\otimes\theta_1^*\theta_2^*)\Big(\Psi_0 \otimes \Big(-(-1)^{|\Psi_1|}\theta_1\theta_2 \otimes x^{d-3}y^{d-2} \\
&\qquad\qquad [\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_1,\Psi_2] \\
&\quad + \theta_1\theta_2 \otimes x^{d-3}y^{d-2}[\psi_1,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
&\quad + \theta_1\theta_2 \otimes x^{d-2}y^{d-3}[\psi_2,\Psi_1]\cdot[\psi_1,[\psi_2,\Psi_2]] \\
&\qquad - (-1)^{|\Psi_1|}\theta_1\theta_2 \otimes x^{d-2}y^{d-3}[\psi_1,[\psi_2,\Psi_1]]\cdot[\psi_2,\Psi_2] \Big)\Big)
\end{aligned}
$$

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$$= \pi m_2 \Big((-1)^{|\Psi_0|+|\Psi_1|} [\psi_1, \Psi_0] \otimes x^{d-3} [\psi_1, \Psi_1] \cdot [\psi_1, \Psi_2]$$

$$- (-1)^{|\Psi_1|+|\Psi_0|} [\psi_2, \Psi_0] \otimes y^{d-3} [\psi_2, \Psi_1] \cdot [\psi_2, \Psi_2]$$

$$- (-1)^{|\Psi_1|} [\psi_1, [\psi_2, \Psi_0]] \otimes x^{d-3} y^{d-2} [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_1, \Psi_2]$$

$$+ [\psi_1, [\psi_2, \Psi_0]] \otimes x^{d-3} y^{d-2} [\psi_1, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]]$$

$$+ [\psi_1, [\psi_2, \Psi_0]] \otimes x^{d-2} y^{d-3} [\psi_2, \Psi_1] \cdot [\psi_1, [\psi_2, \Psi_2]]$$

$$- (-1)^{|\Psi_1|} [\psi_1, [\psi_2, \Psi_0]] \otimes x^{d-2} y^{d-3} [\psi_1, [\psi_2, \Psi_1]] \cdot [\psi_2, \Psi_2] \Big)$$

Now $d > 2$ so x^{d-2} or y^{d-2} kill things after π.

$$= +(-1)^{|\Psi_0|+|\Psi_1|} [\psi_1, \Psi_0] \cdot [\psi_1, \Psi_1] \cdot [\psi_1, \Psi_2]\, x^{d-3}$$
$$- (-1)^{|\Psi_0|+|\Psi_1|} [\psi_2, \Psi_0] \cdot [\psi_2, \Psi_1] \cdot [\psi_2, \Psi_2]\, y^{d-3}. \qquad (19.1)$$

<u>Lemma</u> For $W = y^d - x^d$, $d > 2$ the product ρ_3 in the minimal model structure on $\mathcal{A} = \Lambda(k\psi_1^* \oplus k\psi_2^*)$ of (6.3) is

$$\rho_3 : \mathcal{A}[1]^{\otimes 3} \longrightarrow \mathcal{A}[1]$$

given on $\Psi_2 \otimes \Psi_1 \otimes \Psi_0$ by (19.5.1) e.g. is <u>zero</u> if $d > 3$ and for $d = 3$

$$\rho_3(\psi_1^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^*) = -[-1 \cdot \psi_2^* \cdot 1] = +\psi_2^*$$

19.5

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Hence from (17.5.1)

$$\rho_3(\Psi_2 \otimes \Psi_1 \otimes \Psi_0) = -(-1)^{\sum_{i<j} \tilde{\Psi}_i \tilde{\Psi}_j + \tilde{\Psi}_1 + \tilde{\Psi}_0} \cdot (19.1)$$

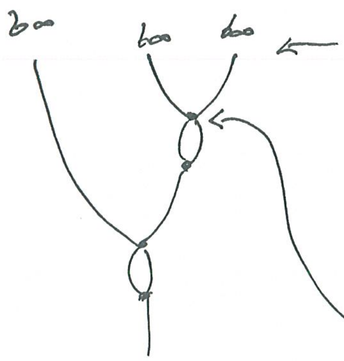
$$= (-1)^{\sum_{i<j} \tilde{\Psi}_i \tilde{\Psi}_j} \Big[-[\psi_1, \Psi_0] \cdot [\psi_1, \Psi_1] \cdot [\psi_1, \Psi_2]\, x^{d-3} + [\psi_2, \Psi_0] \cdot [\psi_2, \Psi_1] \cdot [\psi_2, \Psi_2]\, y^{d-3} \Big]$$

(19.5.1)

<u>Review</u> Now we go back and see why the answer (19.1) was "obvious". The reason comes down to the x^{d-2} and y^{d-2} introduced by both δ_∞ and H_∞, and the fact that the only way to decrease these degrees is the ∂_x, ∂_y operators in H_∞.

However ∂_x, ∂_y appear as $\partial_x \Theta_1, \partial_y \Theta_2$ in H_∞ and x^{d-2} is <u>introduced</u> with a Θ_1 (resp. y^{d-2} with Θ_2). So before H_∞ can fix the degree, Ξ needs to first remove the Θ that is protecting the x^{d-2} (resp. y^{d-2}). Since we only have one H_∞, only one x^{d-2} or y^{d-2} can be decreased (and if $d > 3$ we have no chance of reaching degree zero). So

(20.1)



← here $1 \otimes x^{d-2}[\psi_1, -]\Theta_1$

or $1 \otimes y^{d-2}[\psi_2, -]\Theta_2$

or $\left.\begin{array}{l} x^{d-2}[\psi_1, -]\Theta_1 \otimes 1 \\ y^{d-2}[\psi_2, -]\Theta_2 \otimes 1 \end{array}\right\}$ but we need this Ξ to remove Θ_i's, and it can only do this on the <u>right</u>, so in fact we only see the first two lines.

$$1 \otimes 1 \otimes x^{d-2}[\psi_1, -]\Theta_1$$
$$\downarrow \Xi$$
$$1 \otimes [\psi_1, -] \otimes x^{d-2}[\psi_1, -]$$
$$\downarrow H_\infty$$
$$1 \otimes [\psi_1, -] \otimes x^{d-3}[\psi_1, -]\Theta_1$$
$$\searrow \Xi$$
$$\underset{\psi_0}{[\psi_1, -]} \otimes \underset{\psi_1}{[\psi_1, -]}\,\underset{\psi_2}{[\psi_1, -]}\, x^{d-3}$$

$$1 \otimes 1 \otimes y^{d-2}[\psi_2, -]\Theta_2$$
$$\downarrow$$
$$\vdots$$

same.