# Minimal models for MFs IV (checked)

ainfmf4

3/11/15

In ainfmf3 we gave a description of the minimal model for a single variable. Now we generalise this. As usual we write

$$S \otimes \mathrm{End} := S \otimes_k \mathrm{End}_R(k^{stab}) \qquad (1.1)$$
$$\underline{\mathrm{End}} := \underline{\mathrm{End}}(k^{stab}).$$

Since $[\psi_i \theta_i^*, \psi_j \theta_j^*] = 0$ for all $i, j$, we have for $m \geq 1$

$$\begin{aligned} \delta^m &= \{ \psi_1 \theta_1^* + \cdots + \psi_n \theta_n^* \}^m \\ &= \sum_{i_1 + \cdots + i_n = m} \frac{m!}{i_1! \cdots i_n!} (\psi_1 \theta_1^*)^{i_1} \cdots (\psi_n \theta_n^*)^{i_n} \end{aligned} \qquad (1.2)$$

Hence as operators on $S \otimes \mathrm{End}$, writing $\delta_i = \psi_i \theta_i^*$

$$\exp(\pm \delta) = \exp(\pm \delta_1) \cdots \exp(\pm \delta_n). \qquad (1.3)$$

**Lemma** For $1 \leq i \leq n$ there is a commutative diagram

$$\begin{array}{ccc} (S \otimes \mathrm{End})^{\otimes 2} & \xrightarrow{m_2} & S \otimes \mathrm{End} \\ {\scriptstyle \delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*} \downarrow & & \downarrow {\scriptstyle \delta_i} \\ (S \otimes \mathrm{End})^{\otimes 2} & \xrightarrow{m_2} & S \otimes \mathrm{End} \end{array} \qquad (1.4)$$

**Proof** The calculation on p. (11) of (ainfmf2). □

**Def^N** We set
$$\Xi_i := [\psi_i, -] \otimes \theta_i^* \curvearrowright S \otimes \mathrm{End}.$$
$$\Xi := \sum_{i=1}^n \Xi_i$$

**Lemma** As operators on $(S \otimes \mathrm{End})^{\otimes 2}$, for all $i,j$

$$[\delta_i \otimes 1, 1 \otimes \delta_j] = 0 \tag{2.1}$$
$$[\delta_i \otimes 1, \Xi_j] = [1 \otimes \delta_i, \Xi_j] = 0.$$

**Proof** Repeating the calculation from p. (4) (ainfmf2),

$$\begin{aligned}[\delta_i \otimes 1, \Xi_j] &= (\delta_i \otimes 1)\Xi_j - \Xi_j(\delta_i \otimes 1) \\ &= (\psi_i \theta_i^* \otimes 1) \circ ([\psi_j, -] \otimes \theta_j^*) \\ &\quad - ([\psi_j, -] \otimes \theta_j^*) \circ (\psi_i \theta_i^* \otimes 1) \\ &= \psi_i \theta_i^* [\psi_j, -] \otimes \theta_j^* \\ &\quad - [\psi_j, -]\psi_i \theta_i^* \otimes \theta_j^* \\ &= -[\psi_i, [\psi_j, -]]\theta_i^* \otimes \theta_j^* = 0.\end{aligned} \tag{2.2}$$

(see p. (5.5) (ainfmf2)), and

$$\begin{aligned}[1 \otimes \delta_i, \Xi_j] &= (1 \otimes \delta_i) \circ \Xi_j - \Xi_j \circ (1 \otimes \delta_i) \\ &= (1 \otimes \psi_i \theta_i^*) \circ ([\psi_j, -] \otimes \theta_j^*) - ([\psi_j, -] \otimes \theta_j^*) \circ (1 \otimes \psi_i \theta_i^*) \\ &= [\psi_j, -] \otimes \psi_i \theta_i^* \theta_j^* \\ &\quad - [\psi_j, -] \otimes \theta_j^* \psi_i \theta_i^* \\ &= [\psi_j, -] \otimes \psi_i [\theta_i^*, \theta_j^*] = 0.\end{aligned}$$
□

<u>Proposition</u> There is a commutative diagram

$$\begin{array}{ccc}
(S\otimes \mathrm{End})\otimes(S\otimes \mathrm{End}) & \xrightarrow{\;m_2\;} & S\otimes \mathrm{End} \\
\downarrow \exp(-\delta)\otimes\exp(-\delta) & & \\
(S\otimes \mathrm{End})\otimes(S\otimes \mathrm{End}) & & \downarrow \exp(-\delta) \\
\downarrow \exp(-\Xi) & & \\
(S\otimes \mathrm{End})\otimes(S\otimes \mathrm{End}) & \xrightarrow[\;m_2\;]{} & S\otimes \mathrm{End}
\end{array}$$

<u>Proof</u> It follows from p. (2) that

$$\exp\left(-[\delta_i\otimes 1 + 1\otimes\delta_i + \Xi_i]\right) = \exp(-\Xi_i)\circ\{\exp(-\delta_i)\otimes\exp(-\delta_i)\}$$

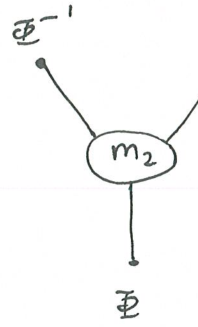
and

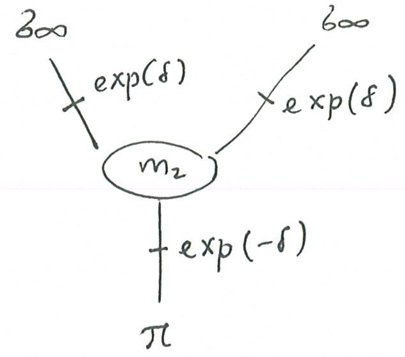
$$\exp\left(-[\delta\otimes 1 + 1\otimes\delta + \Xi]\right) = \exp(-\Xi)\circ\{\exp(-\delta)\otimes\exp(-\delta)\}$$

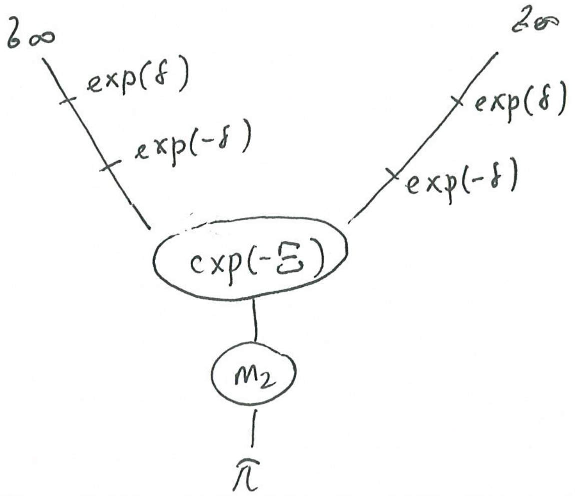
Hence by commutativity of (1.4) we have

$$\begin{aligned}
\exp(-\delta)m_2 &= \sum_{m\geq 0}(-1)^m\frac{1}{m!}\delta^m m_2 \\
&= \sum_{m\geq 0}(-1)^m\frac{1}{m!} m_2[\delta\otimes 1 + 1\otimes\delta + \Xi]^m \\
&= m_2\exp\left(-[\delta\otimes 1 + 1\otimes\delta + \Xi]\right) \\
&= m_2\exp(-\Xi)\circ\{\exp(-\delta)\otimes\exp(-\delta)\}. \quad \square
\end{aligned}$$

Hence

$b_2 =$ 


$=$ 


$=$ 


(5.1)

$$= \pi\, m_2 \exp(-\Xi)(b_\infty \otimes b_\infty)$$

Since $\Xi_i = [\psi_i, -]\otimes \theta_i^*$ can only remove $\theta$'s in the second leg, the first $b_\infty$ is just id, and $\Xi$ has the effect of converting all $\theta$'s on the right leg into commutators with $\psi$'s on the left leg. Leaving us with Atiyah classes

$$At_i = -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \qquad (5.2)$$

That is,

$$b_2(\beta_1 \otimes \beta_2) = \pi\, m_2 \exp(-\Xi)\big(\beta_1 \otimes b_\infty(\beta_2)\big)$$

ainfunf4

$$= \pi m_2 \sum_{t \geq 0} \frac{1}{t!} (-1)^t (\Xi_1 + \cdots + \Xi_n)^t \Big( \beta_1 \otimes \sum_{s \geq 0} \sum_{p_1, \ldots, p_s} (-1)^{\binom{s+1}{2}} \frac{1}{s!} At_{p_1} \cdots At_{p_s} \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_s} (\beta_2) \Big)$$

$$= \pi m_2 \exp(-\Xi_n) \cdots \exp(-\Xi_1) \Big( \beta_1 \otimes \sum_{s \geq 0} \sum_{p_1 < \cdots < p_s} (-1)^{\binom{s+1}{2}} At_{p_1} \cdots At_{p_s} \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_s} (\beta_2) \Big) \quad (5.1)$$

$$= \pi m_2 \exp(-\Xi_n) \cdots \exp(-\Xi_1) \Big( \beta_1 \otimes \sum_{s \geq 0} \sum_{p_1 < \cdots < p_s} (-1)^{\binom{s}{2}} \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_s} \circ At_{p_1} \cdots At_{p_s} (\beta_2) \Big)$$

Now since $\Xi_1 = [\Psi_1, -] \otimes \mathcal{O}_1^*$ it vanishes on a term with $p_1 \neq 1$, so since $\Xi_i^2 = 0$,

$$= \pi m_2 \exp(-\Xi_n) \cdots \exp(-\Xi_2) \Big( \beta_1 \otimes \sum_{s \geq 0} \sum_{p_1 < \cdots < p_s} (-1)^{\binom{s}{2}} \mathcal{O}_{p_1} \cdots \mathcal{O}_{p_s} At_{p_1} \cdots At_{p_s} (\beta_2) \Big)$$

$$- \pi m_2 \exp(-\Xi_n) \cdots \exp(-\Xi_2) \Big( (-1)^{|\beta_1|} [\Psi_1, \beta_1] \otimes \sum_{s \geq 0} \sum_{1 < p_2 < \cdots < p_s} (-1)^{\binom{s}{2}} \mathcal{O}_{p_2} \cdots \mathcal{O}_{p_s} At_1 At_{p_2} \cdots At_{p_s} (\beta_2) \Big) \quad (5.2)$$

The sum in the first summand also restricts (because of $\pi$) to $1 < p_1 < \cdots < p_s$.

ainfmt4

Expanding again,

$$= \pi m_2 \exp(-\Xi_n) \cdots \exp(-\Xi_3)\Big( \beta_1 \otimes \sum_{s\geq 0} \sum_{2<p_1<\cdots<p_s} (-1)^{\binom{s}{2}} \Theta_{p_1} \cdots \Theta_{p_s} At_{p_1} \cdots At_{p_s}(\beta_2) \Big)$$

$$- \pi m_2 \exp(-\Xi_m) \cdots \exp(-\Xi_3)\Big( (-1)^{|\beta_1|} [\Psi_2, \beta_1] \otimes \sum_{s\geq 0} \sum_{1<2<p_2<\cdots<p_s} (-1)^{\binom{s}{2}} \Theta_{p_2} \cdots \Theta_{p_s} At_2 At_{p_2} \cdots At_{p_s}(\beta_2) \Big)$$

$$- \pi m_2 \exp(-\Xi_m) \cdots \exp(-\Xi_3)\Big( (-1)^{|\beta_1|} [\Psi_1, \beta_1] \otimes \sum_{s\geq 0} \sum_{2<p_2<\cdots<p_s} (-1)^{\binom{s}{2}} \Theta_{p_2} \cdots \Theta_{p_s} At_1 At_{p_2} \cdots At_{p_s}(\beta_2) \Big)$$

$$+ \pi m_2 \exp(-\Xi_m) \cdots \exp(-\Xi_3)\Big( (-1)^{|\beta_1|+|\beta_1|+1} [\Psi_2, [\Psi_1, \beta_1]] \otimes \sum_{s\geq 0} \sum_{1<2<p_3<\cdots<p_s} (-1)^{\binom{s}{2}} \Theta_{p_3} \cdots \Theta_{p_s} At_1 At_2 At_{p_3} \cdots At_{p_s}(\beta_2) \Big)$$

Well this is not the right way.

We have to be careful about H. On a $\tilde{\omega}$-form of weight p with coefficient a polynomial $f(x)$ homogeneous of degree b, we have

$$[d_K, \nabla](f.\omega) = \{d_K\nabla + \nabla d_K\}(f.\omega)$$
$$= (p+b) f.\omega.$$

Since $x\partial_x(f) = b.f$. Hence

$$[d_K, \nabla]^{-1}(f.\omega) = \frac{1}{p+b} f.\omega \qquad (6.5.1)$$

Thus as an operator on $S \otimes \mathrm{End}$, using the obvious homogeneous basis of End for the extension, if $\Psi$ is one of the basis elements then for $f \in k[x]$ arbitrary

$$H(1 \otimes f\Psi) = [d_K, \nabla]^{-1}\partial_x\theta(f\Psi)$$
$$= [d_K, \nabla]^{-1}(\partial_x(f)\cdot\theta\otimes\Psi) \qquad (6.5.2)$$
$$= \Omega_1(\partial_x(f))\,\theta\otimes\Psi$$

where we define

<u>Def^n</u> The map $k[x] \to k[x]$ given by $x^b \mapsto \frac{1}{p+b}x^b$ is denoted $\Omega_p$.
(k-linear)

We also have to be wary about $\delta_\infty$, because (1.4) of (ainfmf2) only holds <u>modulo m</u> and with a H in the formula (which is not $k[x]$-linear) we cannot use this. Instead we use $J = [d_K, \nabla] : k[x] \to k[x]$,

$$\delta_\infty = 1 - J^{-1}At\theta \qquad (6.5.3)$$
$$= 1 - \Omega_1 At\theta$$

Instead we return to (5.1) and use (indices in $\{1,\dots,n\}$) ainfmf4

$$\exp(-\Xi_n)\cdots\exp(-\Xi_1)$$

$$= (1-\Xi_1)\cdots(1-\Xi_n)$$

$$= \sum_{s\geq 0}\sum_{q_1<\cdots<q_s}(-1)^s\,\Xi_{q_1}\cdots\Xi_{q_s} \tag{7.1}$$

$$= \sum_{s\geq 0}\sum_{q_1<\cdots<q_s}(-1)^s\left([\Psi_{q_1},-]\otimes \mathcal{O}_{q_1}^*\right)\cdots\left([\Psi_{q_s},-]\otimes \mathcal{O}_{q_s}^*\right)$$

$$= \sum_{s\geq 0}\sum_{q_1<\cdots<q_s}(-1)^{s+\binom{s}{2}}[\Psi_{q_1},-]\circ\cdots\circ[\Psi_{q_s},-]\otimes \mathcal{O}_{q_1}^*\cdots \mathcal{O}_{q_s}^*$$

$$= \sum_{s\geq 0}\sum_{q_1<\cdots<q_s}(-1)^s[\Psi_{q_1},-]\circ\cdots\circ[\Psi_{q_s},-]\otimes \mathcal{O}_{q_s}^*\cdots \mathcal{O}_{q_1}^*$$

Hence

$$b_2(\beta_1\otimes\beta_2) = \pi m_2 \sum_{t\geq 0}\sum_{q_1<\cdots<q_t}(-1)^t\Big([\Psi_{q_1},-]\circ\cdots\circ[\Psi_{q_t},-] \otimes \mathcal{O}_{q_t}^*\cdots \mathcal{O}_{q_1}^*\Big)$$

$$\Big(\beta_1\otimes \sum_{s\geq 0}\sum_{p_1<\cdots<p_s}(-1)^{\binom{s}{2}}\mathcal{O}_{p_1}\cdots \mathcal{O}_{p_s}\,At_{p_1}\cdots At_{p_s}(\beta_2)\Big)$$

$$= \pi m_2\Big(\sum_{t\geq 0}\sum_{q_1<\cdots<q_t}(-1)^{t+t|\beta_1|+\binom{t}{2}}[\Psi_{q_1},[\Psi_{q_2},\cdots[\Psi_{q_t},\beta_1]\cdots]] \otimes At_{q_1}\cdots At_{q_t}(\beta_2)\Big)$$

Lemma To conclude

$$b_2(\beta_1 \otimes \beta_2) = \sum_{t \geq 0} \sum_{q_1 < \cdots < q_t} (-1)^{t + t|\beta_1| + \binom{t}{2}} [\psi_{q_1}, [\psi_{q_2}, \ldots [\psi_{q_t}, \beta_1] \cdots] \bullet At_{q_1} \cdots At_{q_t}(\beta_2) \qquad (8.1)$$

where for $t=0$ we have $\beta_1 \bullet \beta_2$ and $At_i = -[\psi_i^*, -] - \sum_q \partial x_i (W^q)[\psi_q, -]$.

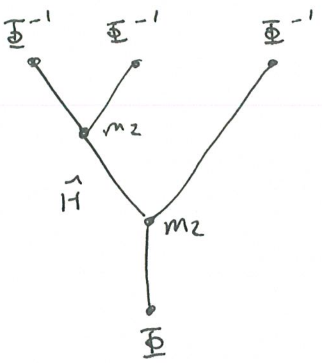
Writing $At_i = -[\psi_i^* + \sum_q \partial x_i (W^q)\psi_q, -]$,

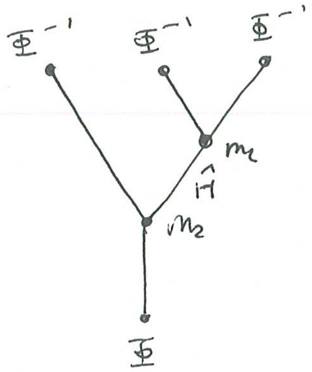
$$(8.2)$$

$$b_2(\beta_1 \otimes \beta_2) = \sum_{t \geq 0} \sum_{q_1 < \cdots < q_t} (-1)^{t|\beta_1| + \binom{t}{2}} [\psi_{q_1}, [\psi_{q_2}, \ldots [\psi_{q_t}, \beta_1] \cdots] \bullet [\psi_{q_1}^* + \sum_z \partial x_{q_1}(W^z)\psi_z, [\cdots [\psi_{q_t}^* + \sum_z \partial x_{q_t}(W^z)\psi_z, \beta_2] \cdots]$$

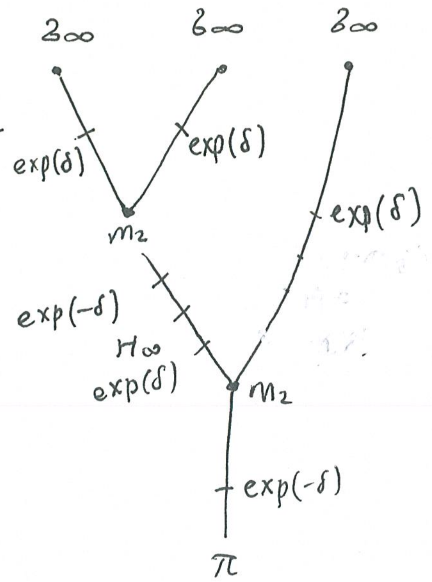
ainfmf4

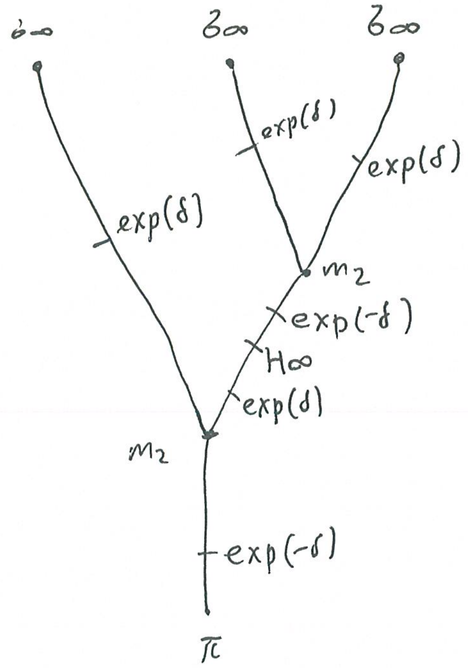
**$b_3$** There are two diagrams for $b_3$

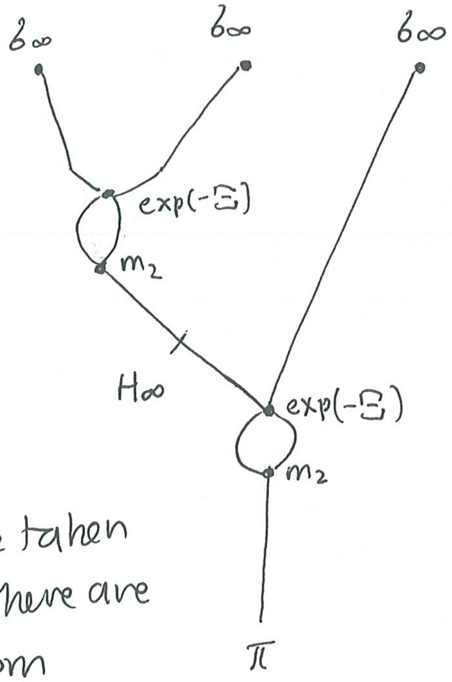
$b_3 = -$ 

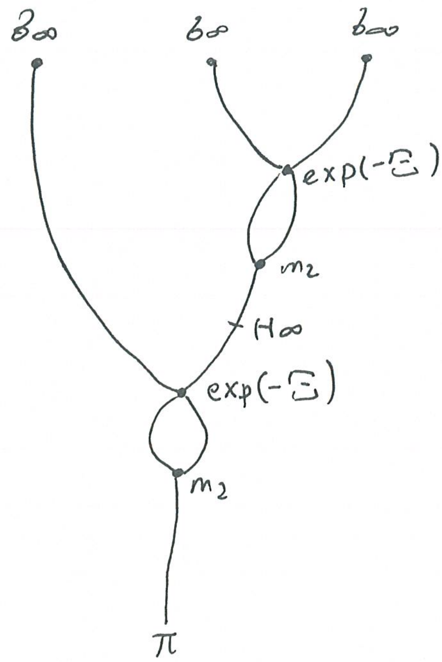

$-$ 


(9.1)

$= -$ 


$-$ 


$= -$ 


$-$ 


⌈Note: not to be taken literally as there are new signs from revised (ainfmf2) ⌋

Here both $\delta_\infty$ and $H_\infty$ need to be written out in full, without the usual simplifications which come from working modulo $\mathfrak{m}$. Thus

$$H_\infty = \sum_{m\geq 0} (-1)^m (H\, d_{End})^m H$$

$$H = J^{-1}\nabla \qquad \nabla = \textstyle\sum_i \partial x_i \theta_i \tag{10.1}$$

$$\delta_\infty = \sum_{m\geq 0} (-1)^m (H d_{End})^m \delta$$

Now $H^2 = 0$, $H\delta = 0$ so we have for $m > 0$ and $\beta \in \{H, \delta\}$

$$(H\, d_{End})^m \beta = H d_{End} H d_{End} \cdots H d_{End} \beta \tag{10.2}$$

where

$$d_{End} = \sum_j x_j [\psi_j^*, -] + \sum_j W^j [\psi_j, -] \tag{10.3}$$

Let us write $W^j = \sum_{l\geq 0} W^j_l$ where $W^j_l$ is homogeneous of degree $l$ (so $W^j_0 = 0$ as $W \in \mathfrak{m}^2$), according to the standard grading on $k[x]$. Hence

$$(H d_{End})^m \beta = \sum_{j_1,\dots,j_m} H\big( x_{j_1}[\psi^*_{j_1}, -] + W^{j_1}[\psi_{j_1}, -]\big) H \cdots H\big( x_{j_m}[\psi^*_{j_m}, -] + W^{j_m}[\psi_{j_m}, -]\big)\beta \tag{10.4}$$

(ainfmf4)

<u>Def^N</u> The $k$-linear map $\Omega_p : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ is defined on a homogeneous polynomial $f(\underline{x})$ of degree $b$ by (for $p \geq 1$)

$$\Omega_p(f) = \frac{1}{p+b} f \qquad (11.1)$$

The operator $J^{-1}$ on $S \otimes \mathrm{End}$ is defined on $\omega \in S$ of $\theta$-weight $|\omega| = p \geq 1$ ($\mathbb{Z}$-degree) and homogeneous $f \in k[\underline{x}]$ of degree $b$, multiplied by a basis element $\Psi$ of End, by

$$J^{-1}(\omega \otimes f\Psi) = \theta \otimes \Omega_p(f)\Psi \qquad (11.2)$$

To simplify the notation let us define, for $u \in \{0,1\}$

$$f_i^u = \begin{cases} x_i & u=0 \\ w^i & u=1 \end{cases}$$

$$\psi_i^u = \begin{cases} \psi_i^* & u=0 \\ \psi_i & u=1 \end{cases} \qquad (12.1)$$

and for $l \geq 0$, $(f_i^u)_l$ denotes the degree $l$ homogeneous part of $l$. Then (since $(f_i^u)_l = 0$ for $l=0$)

$$(H\,\mathrm{clEnd})^m \beta = \sum_{j_1,\dots,j_m} H\left(f_{j_1}^0[\psi_{j_1}^0,-] + f_{j_1}^1[\psi_{j_1}^1,-]\right)\cdots \qquad (12.2)$$

$$= \sum_{j_1,\dots,j_m} \sum_{\underline{u} \in \mathbb{Z}_2^m} H\left(f_{j_1}^{u_1}[\psi_{j_1}^{u_1},-]\right) H\left(f_{j_2}^{u_2}[\psi_{j_2}^{u_2},-]\right) \cdots H\left(f_{j_m}^{u_m}[\psi_{j_m}^{u_m},-]\right)\beta$$

$$= \sum_{j_1,\dots,j_m} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{l_1,\dots,l_m \geq 1} \left\{ \prod_{i=1}^m H\left(f_{j_i,l_i}^{u_i}[\psi_{j_i}^{u_i}-]\right)\right\}\beta$$

$$= \sum_{\underline{j}} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{\underline{l}} \left\{ \prod_{i=1}^m J^{-1} \circ \nabla \circ \left(f_{j_i,l_i}^{u_i}[\psi_{j_i}^{u_i},-]\right)\right\}\beta$$

ainfmf4

Suppose we feed into this contraption an element $\omega \otimes f \Psi$ of $S \otimes \mathrm{End}$, with the notation of p. (11), so $\omega$ is a $\mathcal{O}$-form of weight $p$, $f$ is homogeneous of deg $b$, and $\Psi$ is a basis element. In the case of $\beta = 3$ so we are computing $z_{\infty}$ obviously $\omega = 1$ so $p = 0$ and $f = 1$ so $b = 0$ also.

We consider what $\mathcal{O}$-degree, resp. polynomial degree, each $J$ in the last line of (11.2) sees coming in from the right:

$$\mathcal{O}\text{-degree} \qquad \underbrace{J^{-1}\nabla}_{i} \cdot\cdot \underbrace{J^{-1}\nabla}_{i+1} \cdots \underbrace{J^{-1}\nabla}_{m} \; \omega \otimes f \Psi \tag{13.1}$$

So the $J^{-1}$ in pos$^N$ $i$ takes as input a $\mathcal{O}$-form of weight

$$p + m - i + 1 \tag{13.2}$$

while the same $J^{-1}$ sees coming from the right, in terms of poly degree

$$\underbrace{J^{-1}\nabla f^{u_i}_{j_i, l_i}}_{i} \cdots \underbrace{J^{-1}\nabla f^{u_m}_{j_m, l_m}}_{m} \; \omega \otimes f \Psi \tag{13.3}$$

So the $J^{-1}$ in pos$^N$ $i$ sees a polynomial input of degree

$$b + (l_m - 1) + (l_{m-1} - 1) + \cdots + (l_i - 1) \tag{13.4}$$

$$= b + \sum_{j=i}^{m} l_j - [m - i + 1]$$

Hence the ith $J^{-1}$ acts as the scalar (for fixed $\underline{j}, \underline{u}, \underline{l}$ obviously)
in the last line of (12.2)

$$\frac{1}{(p+m-i+1)+\left(b+\sum_{j=i}^{m} \ell_j-[m-i+1]\right)} \qquad (14.1)$$

$$= \frac{1}{p+b+\sum_{j=i}^{m} \ell_j}$$

Def$^{n}$ Given a sequence $\underline{\ell}=(\ell_1,\dots,\ell_m)$ with each $\ell_i \geq 1$, we define for $1\leq i\leq m$ and $\alpha\in\mathbb{N}$ (14.2)

$$C_\alpha(\underline{\ell},i) = \frac{1}{\alpha+\sum_{j=i}^{m}\ell_j}, \quad C_\alpha(\underline{\ell}) = \prod_{i=1}^{m} C_\alpha(\underline{\ell},i)$$

Hence (12.2) and p. (13) show

(14.3)

$$(H d_{End})^m \beta(\omega\otimes f\Psi)$$

$$= \sum_{\underline{j}}\sum_{\underline{u}\in\mathbb{Z}_2^m}\sum_{\underline{\ell}}\left\{\prod_{i=1}^{m} C_{p+b}(\underline{\ell},i)\,\nabla\circ\left(f^{u_i}_{j_i,\ell_i}[\psi^{u_i}_{j_i},-]\right)\right\} \circ\beta(\omega\otimes f\Psi)$$

$$= \sum_{\underline{j}}\sum_{\underline{u}\in\mathbb{Z}_2^m}\sum_{\underline{\ell}} C_{p+b}(\underline{\ell})\prod_{i=1}^{m}\left\{\nabla\circ\left(f^{u_i}_{j_i,\ell_i}[\psi^{u_i}_{j_i},-]\right)\right\} \beta(\omega\otimes f\Psi)$$

Now since either $\beta = H$ or $\mathcal{Z}$,

$$\nabla \beta (w \otimes f \Psi)$$

$$= \begin{cases} \nabla J^{-1} \nabla (w \otimes f \Psi) & \beta = H \\ \nabla \mathcal{Z} (w \otimes f \Psi) & \beta = \mathcal{Z} \end{cases}$$

$$= \begin{cases} \frac{1}{p+b} \nabla \nabla (w \otimes f \Psi) & \beta = H \\ \nabla \mathcal{Z} (w \otimes f \Psi) & \beta = \mathcal{Z} \end{cases} \quad (15.1)$$

$$= 0$$

So we can play the usual trick on (14.3)

$$(\mathrm{Hdend})^m_{\cdot} \beta (w \otimes f \Psi) \quad (15.2)$$

$$= \sum_{j} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{\underline{\ell}} C_{p+b}(\underline{\ell}) \prod_{i=1}^{m} \left[ \nabla, f_{j_i, \ell_i}^{u_i} [\Psi_{j_i}^{u_i}, -] \right] \circ \beta (w \otimes f \Psi)$$

Now $\nabla = \sum_z \partial_{x_z} \Theta_z$ so (15.3)

$$[\nabla, f[\Psi, -]] = \sum_z \{ \partial_{x_z} \Theta_z f[\Psi, -] + f[\Psi, -] \partial_{x_z} \Theta_z \}$$

$$= \sum_z \{ -\partial_{x_z} f [\Psi, -] \Theta_z + f \partial_{x_z} [\Psi, -] \Theta_z \}$$

$$= \sum_z [f, \partial_{x_z}] \circ [\Psi, -] \Theta_z$$

$$= -\sum_z \partial_{x_z}(f) [\Psi, -] \Theta_z$$

Hence

$$(H d_{\mathrm{End}})^m \beta(w \otimes f \Psi)$$

$$= \sum_{\underline{j}} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{\underline{\ell}} \sum_{z_1,\ldots,z_m} (-1)^m C_{p+b}(\underline{\ell}) \qquad (16.1)$$

$$\circ \prod_{i=1}^{m} \partial_{x_{z_i}}\big(f^{u_i}_{j_i,\ell_i}\big)[\psi^{u_i}_{j_i}, -]\theta_{z_i}\, \beta(w \otimes f \Psi)$$

$$= \sum_{\underline{j}} \sum_{\underline{u} \in \mathbb{Z}_2^m} \sum_{\underline{\ell}} \sum_{\underline{z}} (-1)^m C_{p+b}(\underline{\ell}) \prod_{i=1}^{m} \partial_{x_{z_i}}\big(f^{u_i}_{j_i,\ell_i}\big)$$

$$\circ \prod_{i=1}^{m} [\psi^{u_i}_{j_i}, -]\theta_{z_i}\, \beta(w \otimes f \Psi)$$

where the $z_i$ range over $\{1, \ldots, n\}$. In priciple this computes both $H_\infty$ and $\delta_\infty$ as sums of products of operators (16.2)

$$f[\psi_i, -]\theta_j \quad \text{or} \quad f[\psi_i^*, -]\theta_j$$

for homogeneous polynomial $f$.