# Minimal models for MFs III (checked)

We continue the calculations at the end of ainfmf2 in the special case of $n=1$. Of particular importance is the diagram on p. 13 of ainfmf2, reproduced here:

$$\begin{tikzcd}
(S\otimes \mathrm{End})\otimes(S\otimes \mathrm{End}) \arrow[r, "m_2"] \arrow[d, "\psi\theta^*\otimes 1 + 1\otimes\psi\theta^* + [\psi,-]\otimes\theta^*"'] & S\otimes\mathrm{End} \arrow[d, "\psi\theta^*"] \\
(S\otimes \mathrm{End})\otimes(S\otimes \mathrm{End}) \arrow[r, "m_2"'] & S\otimes\mathrm{End}
\end{tikzcd} \tag{1.1}$$

We write

$$\Xi := [\psi,-]\otimes\theta^* \circlearrowright S\otimes\mathrm{End}. \tag{1.2}$$

We checked on p. 14 of ainfmf2 that $\Xi, \psi\theta^*\otimes 1, 1\otimes\psi\theta^*$ all commute with one another, and patently they are all square zero, so we have

$$\begin{aligned}
\exp(-\psi\theta^*)m_2 &= (1-\psi\theta^*)m_2 \\
&= m_2 - m_2(\psi\theta^*\otimes 1 + 1\otimes\psi\theta^* + [\psi,-]\otimes\theta^*) \\
&= m_2\big(\exp(-\Xi)\circ(\exp(-\psi\theta^*)\otimes\exp(-\psi\theta^*))\big)
\end{aligned} \tag{1.3}$$

This can be seen more generally as follow.

**Lemma** In a commutative $\mathbb{Q}$-algebra $A$, for $a, b, c \in A$ and $n \geq 1$ we have

$$(a+b+c)^n = \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!} a^i b^j c^k \qquad (2.1)$$

where $i, j, k \geq 0$.

**Proof** Using the binomial formula

$$(a+b+c)^n = \sum_{p+k=n} \frac{n!}{p!\,k!} (a+b)^p c^k$$

$$= \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!} a^i b^j c^k . \ \square$$

Hence

**Lemma** As operators on $(S \otimes \mathrm{End})^{\otimes 2}$ we have, for $m \geq 0$

$$\left[ \psi\theta^* \otimes 1 + 1 \otimes \psi\theta^* + \Xi \right]^m = \sum_{i+j+k=m} \frac{m!}{i!\,j!\,k!} (\psi\theta^* \otimes 1)^i (1 \otimes \psi\theta^*)^j \, \Xi^k \qquad (2.2)$$

$$\exp\left( \left[ \psi\theta^* \otimes 1 + 1 \otimes \psi\theta^* + \Xi \right] \right) = \exp(\psi\theta^* \otimes 1) \cdot \exp(1 \otimes \psi\theta^*) \cdot \exp(\Xi) \qquad (2.3)$$

Lemma We have

$$\exp(-\psi\theta^*)m_2 = m_2\big(\exp(-\Xi)\circ\big(\exp(-\psi\theta^*)\otimes\exp(-\psi\theta^*)\big)\big) \tag{3.1}$$

Proof Immediate from (1.1) and (2.3),

$$\frac{1}{m!}(-1)^m(\psi\theta^*)^m m_2 = \frac{1}{m!}(-1)^m m_2\left[1\otimes\psi\theta^* + \psi\theta^*\otimes 1 + \Xi\right]^m$$

$$= m_2 \sum_{i+j+k=m} \frac{(-1)^{i+j+k}}{i!\,j!\,k!}[(\psi\theta^*)^i\otimes 1][1\otimes(\psi\theta^*)^j]\,\Xi^k \quad \square$$

Example To redo the calculation on p. 15 of ainfmf2

$$b_2 = \pi\exp(-\psi\theta^*)m_2\big(\exp(\psi\theta^*)\mathfrak{z}_\infty\otimes\exp(\psi\theta^*)\mathfrak{z}_\infty\big)$$

$$\overset{(3.1)}{=} \pi m_2\big(\exp(-\Xi)\circ(\mathfrak{z}_\infty\otimes\mathfrak{z}_\infty)\big)$$

$$\therefore\ b_2(\beta_1\otimes\beta_2) = \pi m_2\big([1-\Xi]\circ\big([1-At\theta](\beta_1)\otimes[1-At\theta](\beta_2)\big)\big)$$

$$= \beta_1\cdot\beta_2 - \pi m_2\big(\Xi\circ\{[\underset{\uparrow}{1}-At\theta](\beta_1)\otimes[1-\underset{\uparrow}{At\theta}](\beta_2)\}\big)$$

only these terms survive $\pi m_2 \Xi$

$$= \beta_1\cdot\beta_2 - \pi m_2\big(\Xi(-\beta_1\otimes At\theta(\beta_2))\big)$$

$$= \beta_1\cdot\beta_2 + (-1)^{|\beta_1|}\pi m_2\big([\psi,\beta_1]\otimes\theta^* At\theta(\beta_2)\big)$$

$$= \beta_1\cdot\beta_2 - (-1)^{|\beta_1|}[\psi,\beta_1]\cdot At(\beta_2) \tag{3.2}$$

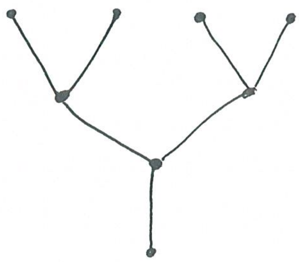
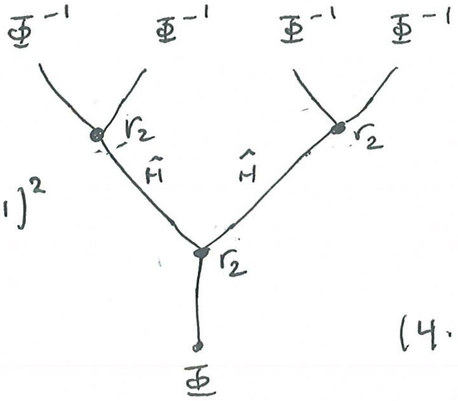
With $W = xW'$ we have, using $A_t = -[\psi^*,-] - \partial_x(W')[\psi,-]$ ainfmt3

$$= \beta_1 \cdot \beta_2 + (-1)^{|\beta_1|}[\psi, \beta_1] \cdot [\psi^*, \beta_2] \tag{4.1}$$
$$+ (-1)^{|\beta_1|} \partial_x(W')[\psi, \beta_1] \cdot [\psi, \beta_2]$$

<u>Summary</u> In the case $n=1$ the operator $b_2$ is given by

$$b_2(\beta_1 \otimes \beta_2) = \beta_1 \cdot \beta_2 + (-1)^{|\beta_1|}[\psi, \beta_1] \cdot [\psi^*, \beta_2] \tag{4.2}$$
$$+ (-1)^{|\beta_1|} \partial_x(W')[\psi, \beta_1] \cdot [\psi, \beta_2]$$
$$= m_2 \Big[ 1 + [\psi,-] \otimes [\psi^*,-] + \partial_x(W')[\psi,-] \otimes [\psi,-] \Big](\beta_1 \otimes \beta_2)$$

Now we turn to the higher multiplications. Consider the planar tree

T  $\rightsquigarrow$ $\rho_T \overset{(-1)^2}{=}$ 

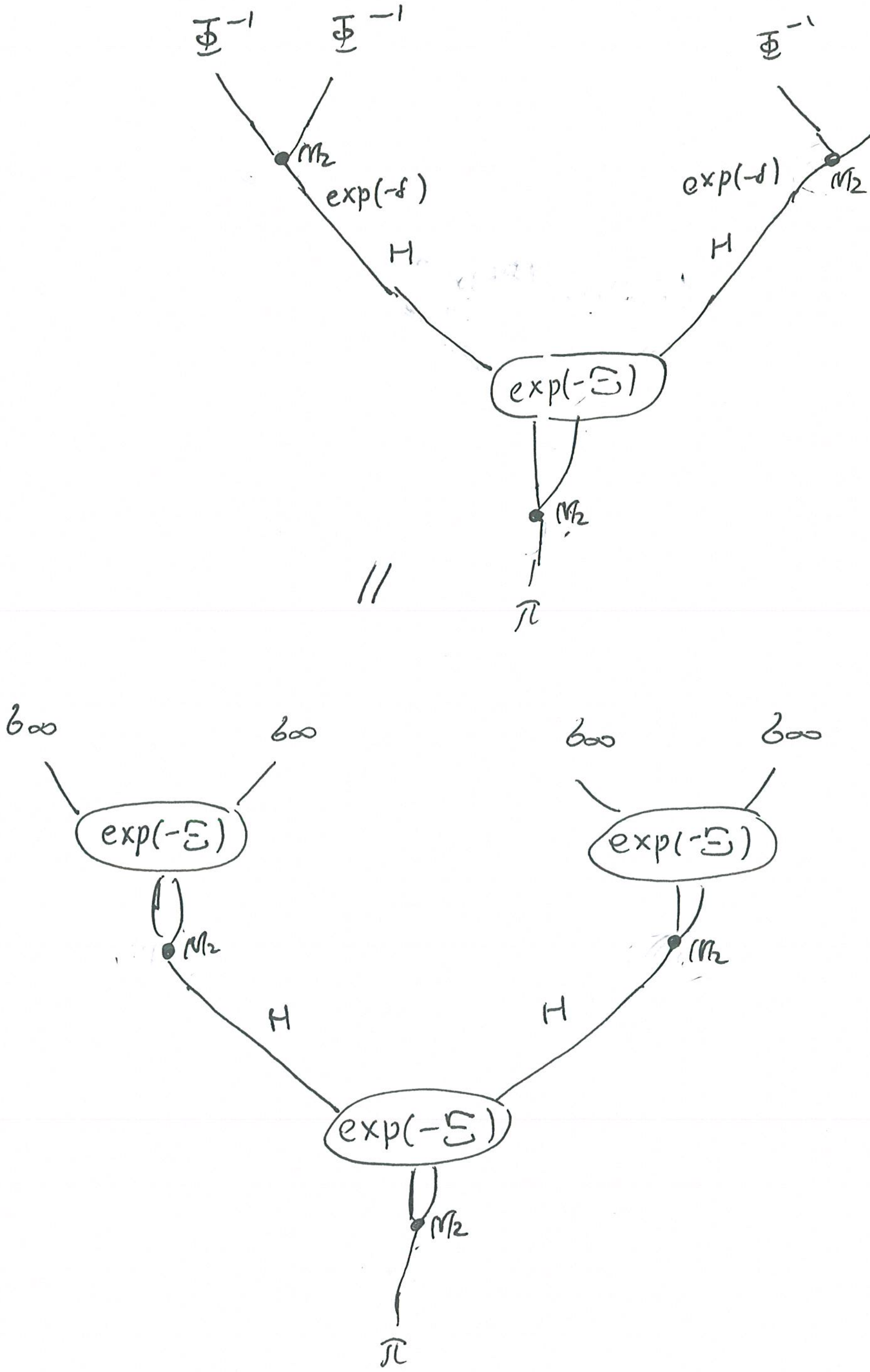

(4.3)

Now by p.20 we can compute this with $m_2$'s

$$\Phi m_2(\hat{H} \otimes \hat{H}) = \pi \exp(-\delta) m_2(\hat{H} \otimes \hat{H}) \tag{4.4}$$
$$= \pi m_2\big(\exp(-\Xi) \circ \{\exp(-\delta)\hat{H} \otimes \exp(-\delta)\hat{H}\}\big)$$

and $\hat{H} = \exp(\delta) H_\infty \exp(-\delta)$, $H = [d_K, \nabla]^{-1}\nabla$, $\nabla = \partial_x \Theta$ (ainfmf3)
so we have (in the case $n=1$, $H_\infty = H$)

$$\Phi m_2(\hat{H} \otimes \hat{H}) = \pi m_2\left(\exp(-\Xi) \circ \left(H\exp(-\delta) \otimes H\exp(-\delta)\right)\right)$$



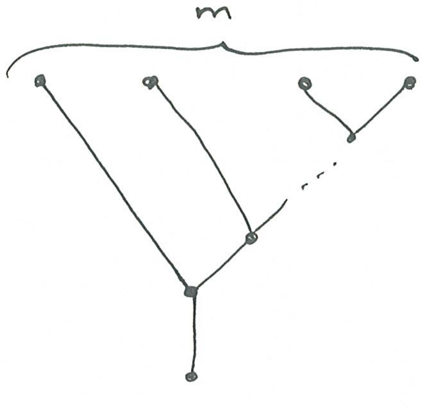

(5.1)

But $H$ contains $\Theta$'s and $\Xi = [\Psi, -] \otimes \Theta^*$ preserves a $\Theta$ in the left branch, whence the map denoted by (5.1) is zero.

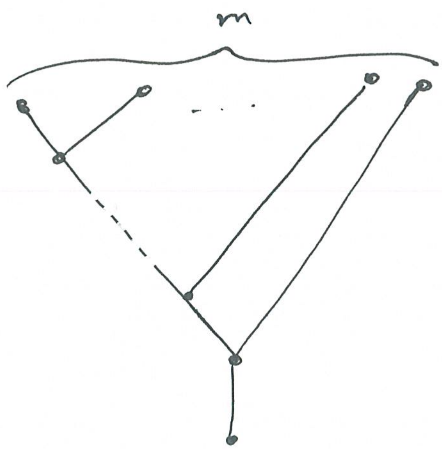
This means we get a nonzero map only if the left leg at the root node connects directly to a leaf; e.g. .

But then the same argument applies at every other node.

Hence only a tree $T$ with minor $\hat{T}$ equal to

$\hat{T} =$  (6.1)

can contribute to the final operator $\rho_m$. So consider for some $m$

$T =$  (6.2)

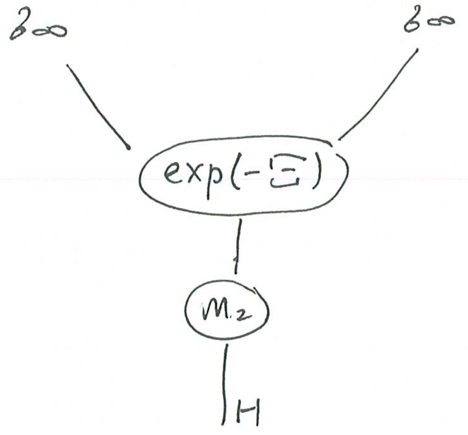
Then by ainfmf2 p. 20 we have

$$\rho_m = \rho_T = (-1)^s \operatorname{eval}_{\hat{T}}(\text{reversed}) \qquad (6.3)$$

where $\hat{T}$ is decorated with $m_2$'s as in (6.1). The sign on inputs $\rho_m(\alpha_1 \otimes \cdots \otimes \alpha_m)$ is $(M_1 = 1, M_2 = \cdots = M_{m-1} = 1)$

$$s = 1 + \sum_{i<j} \tilde{\alpha}_i \tilde{\alpha}_j + \tilde{\alpha}_2 + \cdots + \tilde{\alpha}_m \qquad (6.4)$$

Consider




(7.1)

We redo the calculation in (3.2) but more carefully (because we are not working mod $\hbar$), using $H = J^{-1}\partial_x\Theta$

$$H m_2 (1-\Xi)\Big( (1 - J^{-1}A + \Theta)(\beta_1) \otimes (1 - J^{-1}A + \Theta)(\beta_2)\Big) \tag{7.2}$$

$$= H\big( \beta_1 \cdot \beta_2 + (-1)^{|\beta_1|} m_2\big( [\Psi, \beta_1] \otimes \Theta^* J^{-1} A + \Theta | \beta_2 )\big)\big)$$

$$= (-1)^{|\beta_1|+1} H m_2 \Big( [\Psi, \beta_1] \otimes \Theta^* J^{-1} [\Psi^*, -]\,\Theta(\beta_2) + [\Psi, \beta_1] \otimes \Theta^* J^{-1}\partial_x(W')[\Psi, -]\,\Theta(\beta_2)\Big)$$

In the first summand, $J^{-1}$ acts on a scalar coeff of $\Theta$, so it is just scaling by $\frac{1}{1+0}$, i.e. it is the identity. So we get

$$= (-1)^{|\beta_1|+1} H m_2 \Big( -[\Psi, \beta_1] \otimes [\Psi^*, \beta_2] + [\Psi, \beta_1] \otimes \Theta^* \Omega_1(\partial_x(W'))[\Psi, -]\,\Theta(\beta_2)\Big)$$

$$= (-1)^{|\beta_1|} H m_2 \Big( [\Psi, \beta_1] \otimes [\Psi^*, \beta_2] + \Omega_1(\partial_x(W'))[\Psi, \beta_1] \otimes [\Psi, \beta_2]\Big)$$

$$= (-1)^{|\beta_1|}\, \Omega_1(\partial_x \Omega_1(\partial_x W'))\,{}^{\wedge}_{\Theta}\,[\Psi, \beta_1] \cdot [\Psi, \beta_2] \tag{7.3}$$

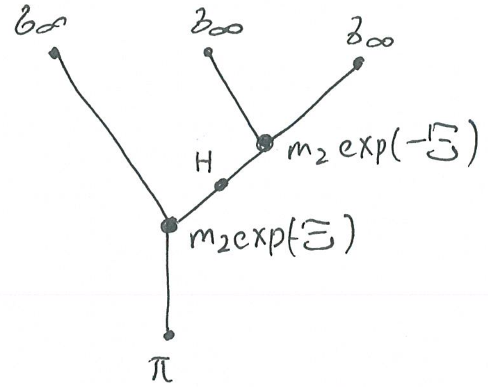
Now

$$\rho_3 : (\underline{\mathrm{End}}[1])^{\otimes 3} \longrightarrow \underline{\mathrm{End}}[1]$$

is defined by (with $T, \hat{T}$ as in (6.2), (6.1))

$$\rho_3(\beta_1 \otimes \beta_2 \otimes \beta_3) = (-1)^{1 + \sum_{i<j} \tilde{\beta_i}\tilde{\beta_j} + \tilde{\beta_2} + \tilde{\beta_3}} \operatorname{eval}_{\hat{T}}(\beta_3 \otimes \beta_2 \otimes \beta_1)$$

where

$\hat{T} =$ 


[note this diagram works on $\underline{\text{unshifted}}$ spaces $\underline{\mathrm{End}}$, $S \otimes \mathrm{End}$]

and so

$$\operatorname{eval}_{\hat{T}}(\beta_3 \otimes \beta_2 \otimes \beta_1) = \pi m_2 \exp(-\Xi)\big(b_\infty(\beta_3) \otimes H m_2 \exp(-\Xi)\,(b_\infty(\beta_2) \otimes b_\infty(\beta_1))\big)$$

$$= \pi m_2(1 - \Xi)\big([1 - T^{-1}At\theta](\beta_3) \otimes (-1)^{|\beta_2|}\theta\,\Omega_1(\partial_x \Omega_1(\partial_x W'))\,[\psi, \beta_2]\cdot[\psi, \beta_1]\big)$$

$$= -\pi m_2 \Xi(\cdots)$$

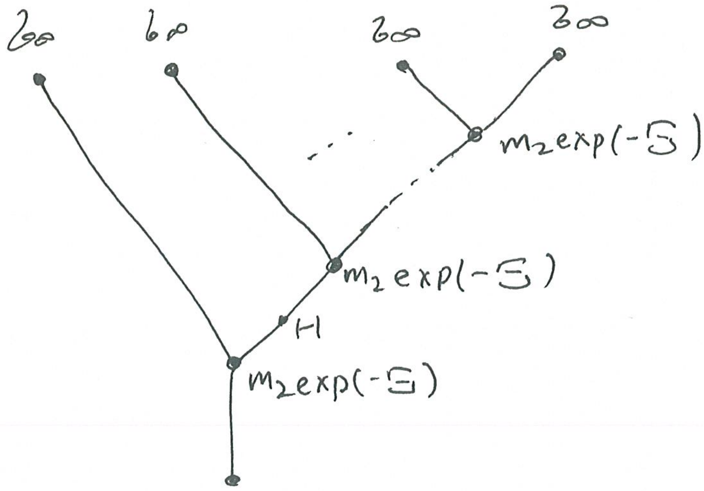
$$= (-1)^{1 + |\beta_2| + |\beta_3|}\,\pi m_2\big([\psi, \beta_3] \otimes \Omega_1(\partial_x \Omega_1(\partial_x W'))[\psi, \beta_2]\cdot[\psi, \beta_1]\big)$$

Hence

$$\rho_3(\beta_1\otimes\beta_2\otimes\beta_3) = (-1)^{\sum_{i<j}\tilde{\beta}_i\tilde{\beta}_j}\,\Omega_1(\partial_x\Omega_1(\partial_x W'))\,[\psi,\beta_3]\circ[\psi,\beta_2]\circ[\psi,\beta_1] \tag{9.1}$$

In the general case we have

$\hat{T} =$




(9.2)

and

$$\rho_m(\beta_1\otimes\cdots\otimes\beta_m) = (-1)^{1+\sum_{i<j}\tilde{\beta}_i\tilde{\beta}_j+\tilde{\beta}_2+\cdots+\tilde{\beta}_m}\,\mathrm{eval}_{\hat{T}}(\beta_m\otimes\cdots\otimes\beta_1)$$

$$= (-1)^{\substack{1+\sum_{i<j}\tilde{\beta}_i\tilde{\beta}_j+\tilde{\beta}_2+\cdots+\tilde{\beta}_m+\overbrace{(m-1)}^{\text{from } -\Xi\text{'s}}+1 \;\leftarrow\text{ from first } b_\infty \\ +|\beta_m|+\cdots+|\beta_2| \;\leftarrow\text{ from } [\psi,-]\otimes\theta^*}}\,(\Omega_1\partial_x)^{m-1}(W')\prod_{i=1}^{m}[\psi,\beta_i] \tag{9.3}$$

$$= (-1)^{\sum_{i<j}\tilde{\beta}_i\tilde{\beta}_j}\,(\Omega_1\partial_x)^{m-1}(W')\prod_{i=1}^{m}[\psi,\beta_i] \tag{9.4}$$

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Thus, if we believe the minimal model construction, $S \otimes_R \mathrm{End}_R(k^{stab})$ has as its minimal $A_\infty$-model (in the case of one variable and $W = xW'$)

$$\left( \underline{\mathrm{End}}(k^{stab}), \{\rho_m\}_{m \geq 1} \right) \tag{10.1}$$

where by (4.2) and (8.2), $b_1 = 0$ and (writing $\circ$ for product of operation in $\underline{\mathrm{End}}$)

$$\begin{aligned} b_2(\beta_1 \otimes \beta_2) = \beta_1 \circ \beta_2 &+ (-1)^{|\beta_1|} [\psi, \beta_1] \circ [\psi^*, \beta_2] \\ &+ (-1)^{|\beta_1|} \partial_x(W') [\psi, \beta_1] \circ [\psi, \beta_2] \end{aligned} \tag{10.2}$$

and for $m \geq 3$

$$\rho_m(\beta_1 \otimes \cdots \otimes \beta_m) = (-1)^{\sum_{i<j} \tilde{\beta}_i \tilde{\beta}_j} (\Omega_1 \partial_x)^{m-1}(W') \prod_{i=1}^{m} [\psi, \beta_i] \tag{10.3}$$

Since $W = xW'$ we have $\deg(W') = \deg(W) - 1$. Hence for

<u>Lemma</u> For $m > \deg(W)$ we have $\rho_m \equiv 0$.

<u>Proof</u> $\deg\left((\Omega_1 \partial_x)^{m-1}(W')\right) = \deg(W') - m + 1$

$= \deg(W) - m < 0$ if $m > \deg W$. $\square$

Example $W = x^d$ for $d \geq 2$. Then

$$W = x \cdot \underbrace{x^{d-1}}_{W'}$$

Note that $d=2$ is special (11.1)

$$b_2(\beta_1 \otimes \beta_2) = \begin{cases} \beta_1 \cdot \beta_2 + (-1)^{|\beta_1|}[\psi, \beta_1] \cdot [\psi^*, \beta_2] & d > 2 \\ \beta_1 \cdot \beta_2 + (-1)^{|\beta_1|}[\psi, \beta_1] \cdot [\psi^*, \beta_2] & d = 2 \\ \quad + (-1)^{|\beta_1|}[\psi, \beta_1] \cdot [\psi, \beta_2] & \end{cases}$$

Now by $\text{def}^N$ $\Omega_1(x^b) = \frac{1}{1+b} x^b$ hence

$$(\Omega_1 \partial_x)(x^b) = \Omega_1(b x^{b-1}) = x^{b-1} \qquad (11.2)$$

$$\therefore (\Omega_1 \partial_x)^{m-1}(W') = (\Omega_1 \partial_x)^{m-1}(x^{d-1})$$
$$= x^{d-1-(m-1)} = x^{d-m}$$

It follows that $m = 0$ for $3 \leq m < d$ and $m > d$, while

$$\rho_d(\beta_1 \otimes \cdots \otimes \beta_d) = (-1)^{\sum_{i<j} \tilde{\beta_i} \tilde{\beta_j}} \prod_{i=1} [\psi, \beta_i] \qquad (11.3).$$

Note that $\underline{\text{End}} = \underline{\text{End}}(k \oplus k\psi)$ and the products are

$$\rho_2, 0, \ldots, 0, \rho_d, 0, \ldots$$

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Clifford actions By (1.3) of ainfmf2 we have in the usual way Clifford actions

$$\underline{\mathrm{End}}(k^{stab}) \ni \gamma_i, \gamma_i^+$$

which are in the $n=1$ case

$$\gamma^+ = At = -[\psi^*, -] - \partial_x(W')[\psi, -]$$

$$\gamma = -\psi \quad \longleftarrow \text{(Note we are not totally sure about this)}$$

Now if we assume $W \in m^3$ so $W' \in m^2$ then $\partial_x(W') \in m$ and $\gamma^+ = -[\psi^*, -]$ so, rescaling, the action is

$$\gamma^+ = [\psi^*, -] \qquad \gamma = \psi.$$

Then for instance

$$e := \gamma^+\gamma = [\psi^*, -] \circ \psi$$

$$\gamma^+\gamma(\beta) = [\psi^*, \psi\beta]$$
$$= \psi^*\psi\beta - (-1)^{|\beta|+1}\psi\beta\psi^*$$

As a matrix on

$$\underline{\mathrm{End}}(\Lambda k\psi) = k \cdot \psi\psi^* \oplus k \cdot \psi^*\psi \oplus k \cdot \psi \oplus k \cdot \psi^*$$

$$e(\psi\psi^*) = \psi^*\psi\psi\psi^* + \psi\psi\psi^*\psi^* = 0$$
$$e(\psi^*\psi) = \psi^*\psi\psi^*\psi + \psi\psi^*\psi\psi^* = \psi^*\psi + \psi\psi^*$$
$$e(\psi) = 0 \qquad e(\psi^*) = \psi^*\psi\psi^* - \psi\psi^*\psi^*$$
$$= \psi^*$$

Hence

$$[e] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e. $e$ is the projector onto $k\cdot(\psi^*\psi + \psi\psi^*) \subset \underline{\mathrm{End}}$.
$\oplus\, k\cdot\psi^*$

i.e. $k\cdot 1 \oplus k\cdot\psi^* \subset \underline{\mathrm{End}}$.

The induced $A_\infty$-algebra structure on $\mathcal{A} := k\cdot 1 \oplus k\cdot\psi^*$ is

$$\overline{b_2} := e b_2 e, \quad \overline{b_2}(\beta_1 \otimes \beta_2) = e\left(\beta_1 \circ \beta_2 + (-1)^{|\beta_1|}[\psi,\beta_1]\circ[\psi,\beta_2]\,\delta_{d=2}\right)$$
$$= \beta_1\circ\beta_2 + (-1)^{|\beta_1|}[\psi,\beta_1]\circ[\psi,\beta_2]\,\delta_{d=2}$$

$$\text{e.g. } \overline{b_2}(\psi^*\otimes\psi^*) = -1\,\delta_{d=2} \qquad (13.1)$$
$$\overline{b_2}(1\otimes\psi^*) = \psi^*$$

Thus $(\mathcal{A}, \overline{b_2})$ is the Clifford algebra of $k\cdot\psi^*$ with $Q(\psi^*) = -1$, if $d=2$.
Then $\overline{\rho_m} := e\rho_m e$ are zero for $m>2$ and $m\neq d$ and

$$\overline{\rho_d}(\beta_1\otimes\cdots\otimes\beta_d) = (-1)^{\sum_{i<j}\tilde\beta_i\tilde\beta_j}\, e\left(\prod_{i=1}^{d}[\psi,\beta_i]\right) \qquad (13.2)$$

so is only nonzero on $\psi^*\otimes\cdots\otimes\psi^*$, where

$$\overline{\rho_d}(\psi^*\otimes\cdots\otimes\psi^*) = 1$$