We continue the calculations at the end of chapter 2 in the special case of \( n = 1 \). Of particular importance is the diagram on page 13 of chapter 2, reproduced here:

\[
\begin{align*}
(S \otimes \text{End}) \otimes (S \otimes \text{End}) & \xrightarrow{m_2} S \otimes \text{End} \\
& \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
(S \otimes \text{End}) \otimes (S \otimes \text{End}) & \xrightarrow{m_2} S \otimes \text{End}
\end{align*}
\]

We write

\[
E := [\gamma, -] \otimes O^* \subseteq S \otimes \text{End}. \tag{1.2}
\]

We checked on page 14 of chapter 2 that \( E, \gamma O^* \otimes 1, \otimes \gamma O^* \) all commute with one another, and patently they are all square zero, so we have

\[
\exp (\gamma O^*) m_2 = (1 - \gamma O^*) m_2
\]

\[
= m_2 - m_2 (\gamma O^* \otimes 1 + \gamma \gamma O^* + [\gamma, -] \otimes O^*)
\]

\[
= m_2 \left( \exp(-E) \circ (\exp(-\gamma O^*) \otimes \exp(-\gamma O^*)) \right)
\]

This can be seen more generally as follows.
Lemma In a commutative $A$-algebra $A$, for $a, b, c \in A$ and $n > 1$ we have

\[(a + b + c)^n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} a^i b^j c^k \quad (2.1)\]

where $i, j, k > 0$.

Proof Using the binomial formula

\[(a+b+c)^n = \sum_{p+k=n} \frac{n!}{p!k!} (a+b)^p c^k\]

\[= \sum_{i+j+k=n} \frac{n!}{i!j!k!} a^i b^j c^k. \quad \square\]

Hence

Lemma As operator on $(S \otimes \text{End})$ we have, for $m > 0$

\[\left[ Y^* \otimes 1 + 1 \otimes Y^* + \Xi \right]^m = \sum_{i+j+k=m} \frac{m!}{i!j!k!} (Y^* \otimes 1)^i (1 \otimes Y^*)^j \Xi^k \quad (2.2)\]

\[\exp \left( \left[ Y^* \otimes 1 + 1 \otimes Y^* + \Xi \right] \right) = \exp(Y^* \otimes 1) - \exp(1 \otimes Y^*) - \exp(\Xi). \quad (2.3)\]
Lemma. We have

\[
\exp(-40^*)m_2 = m_2 \left( \exp(-\Xi) \circ \left( \exp(-40^*) \circ \exp(-40^*) \right) \right)
\]

Proof. Immediate from (1.1) and (2.3),

\[
\frac{1}{m!} (-1)^m (40^*)^m m_2 = \frac{1}{m!} (-1)^m m_2 \left[ \Theta 40^* + 40^* \Theta 1 \Xi \right]^m
\]

\[
= m_2 \sum_{i+j+k=m} \frac{(-1)^{i+j+k}}{i!^j!^k!} \left( [40^*]^i \otimes 1 \right) \left[ 1 \otimes (40^*)^j \right] \Xi^k.
\]

Example. To redo the calculation on p. 13 of \textit{qinfm/2},

\[
b_2 = \Pi \exp(-40^*)m_2 \left( \exp(40^*) \Theta 0 \otimes \exp(40^*) \Theta 0 \right)
\]

(3.1)

\[
= \Pi m_2 \left( \exp(-\Xi) \circ (\Theta 0 \otimes \Theta 0) \right)
\]

\[
\therefore b_2(\beta_1 \otimes \beta_2) = \Pi m_2 \left( [I-\Xi] \circ \left( [I-\Theta 0](\beta_1) \otimes [I-\Theta 0](\beta_2) \right) \right)
\]

\[
= \beta_1 \beta_2 - \Pi m_2 \left( \Xi \circ \left\{ [I-\Theta 0](\beta_1) \otimes [I-\Theta 0](\beta_2) \right\} \right)
\]

\[
\uparrow \text{only these terms survive } \Pi m_2 \Xi
\]

\[
= \beta_1 \beta_2 - \Pi m_2 \left( \Xi (-\beta_1 \otimes \Theta 0(\beta_2) \right))
\]

\[
= \beta_1 \beta_2 + (-1)^{\beta_1} \Pi m_2 \left( [\Theta 0(\beta_1) \otimes \Theta^* \Theta 0(\beta_2) \right)
\]

\[
= \beta_1 \beta_2 + (-1)^{\beta_1} \left[ \Theta 0(\beta_1) \cdot \Theta (\beta_2) \right] \quad (3.2)
\]
With $W = \chi W'$ we have, using $A' = -[\gamma^*] - \partial_x (W')[\gamma,-]$

$$= \beta_1 \beta_2 + (-1)^{1_{P_1}} [\gamma, \beta_1][\gamma^*, \beta_2] \tag{4.1}$$

$$+ (-1)^{1_{P_1}} \partial_x (W')[\gamma, \beta_1][\gamma, \beta_2]$$

**Summary** In the case $n=1$ the operator $b_2$ is given by

$$b_2 (\beta_1 \beta_2) = \beta_1 \beta_2 + (-1)^{1_{P_1}} [\gamma, \beta_1][\gamma^*, \beta_2] \tag{4.2}$$

$$+ (-1)^{1_{P_1}} \partial_x (W')[\gamma, \beta_1][\gamma, \beta_2]$$

$$= m_2 \left[ 1 + [\gamma,-] \Theta [\gamma^*, -] + \partial_x (W') [\gamma,-] \Theta [\gamma,-] \right] (\beta_1 \beta_2)$$

Now we turn to the higher multiplications. Consider the planar tree

$$\overset\sim\longrightarrow$$

Now by Eq. (4.5) we can compute this with $m_2$'s

$$\Theta m_2 (\hat{\Lambda} \Theta \hat{\Lambda}) = \pi \exp(-\delta) m_2 (\hat{\Lambda} \Theta \hat{\Lambda}) \tag{4.5}$$

$$= \pi m_2 (\exp(-\Xi) \Theta \left\{ \exp(-\delta) \hat{\Lambda} \Theta \exp(-\delta) \hat{\Lambda} \right\})$$
and \( \hat{H} = \exp(d) H \exp(-d) \), \( H = [d \kappa, \nabla]^{-1} \nabla, \nabla = d \kappa \Theta \) (a left).

So we have (in the case \( n=1, H_\infty = H \))

\[
\Phi M_2 (\hat{H} \otimes \hat{H}) = \Pi M_2 \left( \exp(-\Sigma) \circ (H \exp(-d) \otimes H \exp(-d)) \right)
\]

But \( H \) contains 0's and \( \Sigma = \{4,-\} \otimes \Theta \) preserves a 0 in the left branch, whence the map denoted by (5.1) is zero.
This means we get a nonzero map only if the left leg at the root node connects directly to a leaf, e.g.

But then the same argument applies at every other node. Hence only a tree $T$ with minor $\hat{T}$ equal to

$$\hat{T} = \quad \text{(6.1)}$$

can contribute to the final operator $\rho_m$. So consider for some $m$

$$T = \quad \text{(6.2)}$$

Then by \textit{ainfmf2} p. 20 we have

$$\rho_m = \rho_{\hat{T}} = (-1)^{s} \text{eval}_{\hat{T}} (\text{reversed}) \quad \text{(6.3)}$$

where $\hat{T}$ is decorated with $m_2$'s as in (6.1). The sign on inputs $\rho_m (\alpha_1 \otimes \cdots \otimes \alpha_m)$ is \textit{if} $M_1 = 1$, $M_2 = \cdots = M_{m-1} = 1$.

$$s = 1 + \sum_{i < j} \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_2 + \cdots + \hat{\alpha}_m \quad \text{(6.4)}$$
Consider

\[
\begin{align*}
\exp(-E) & \quad M_{2} \quad H \\
\end{align*}
\]

\[ (7.1) \]

We redo the calculation in (3.2) but more carefully (because we are not working mod 11), using \( H = T^{-1} \partial x \mathcal{O} \)

\[
H m_{2} (1 - E) \left( (1 - T^{-1} A T)(\beta_{1}) \otimes (1 - T^{-1} A T)(\beta_{2}) \right)
\]

\[ (7.2) \]

\[
= H \left( \beta_{1} \cdot \beta_{2} + (-1)^{|\beta_{1}|} m_{2} \left( \left[ \gamma, \beta_{1} \right] \otimes \partial^{*} T^{-1} A T \left[ \gamma, \beta_{2} \right] \right) \right)
\]

\[
= (-1)^{|\beta_{1}|+1} H m_{2} \left( \left[ \gamma, \beta_{1} \right] \otimes \partial^{*} T^{-1} \left[ \gamma^{*}, - \right] \mathcal{O} \left( \beta_{2} \right) + \left[ \gamma, \beta_{1} \right] \otimes \partial^{*} T^{-1} \partial x (W') \left[ \gamma, - \right] \mathcal{O} \left( \beta_{2} \right) \right)
\]

In the first summand, \( T^{-1} \) acts on a scalar field of \( \mathcal{O} \), so it is just scaling by \( \frac{1}{1 + \mathcal{O}} \), i.e. it is the identity. So we get

\[
= (-1)^{|\beta_{1}|+1} H m_{2} \left( - \left[ \gamma, \beta_{1} \right] \otimes \left[ \gamma^{*}, \beta_{2} \right] + \left[ \gamma, \beta_{1} \right] \otimes \partial^{*} \partial_{2} \left( \partial x (W') \right) \left[ \gamma, - \right] \mathcal{O} \left( \beta_{2} \right) \right)
\]

\[
= (-1)^{|\beta_{1}|} H m_{2} \left( \left[ \gamma, \beta_{1} \right] \otimes \left[ \gamma^{*}, \beta_{2} \right] + \partial^{*} \partial_{2} \left( \partial x (W') \right) \left[ \gamma, \beta_{1} \right] \otimes \left[ \gamma, \beta_{2} \right] \right)
\]

\[
= (-1)^{|\beta_{1}|} \partial x \partial_{2} \left( \partial x \partial_{1} \left( \partial x W' \right) \right) \left[ \gamma, \beta_{1} \right] \cdot \left[ \gamma, \beta_{2} \right]
\]

\[ (7.3) \]
Now \[ \rho_3 : (\text{End}[1])^\otimes 3 \rightarrow \text{End}[1] \]
is defined by (with \( T, \tilde{T} \) as in (6.2), (6.1))
\[
\rho_3 (\beta_1 \otimes \beta_2 \otimes \beta_3) = (-1)^{1 + \sum_{i < j} \tilde{\beta}_i \tilde{\beta}_j + \tilde{\beta}_2 + \tilde{\beta}_3} \text{eval}_T (\beta_3 \otimes \beta_2 \otimes \beta_1)
\]
where
\[
\tilde{T} = \begin{array}{c}
\text{Note this diagram} \\
\text{work on unshifted} \\
\text{space End, S \otimes End}
\end{array}
\]
and so
\[
\text{eval}_T (\beta_3 \otimes \beta_2 \otimes \beta_1) = \pi m_2 \exp(-\Xi) (\beta_3 \otimes H m_2 \exp(-\Xi))
\]
\[
\big( \beta_2 \otimes \beta_1 \big)
\]
\[
= \pi m_2 (1 - \Xi) \left[ 1 - \Pi_{\text{At}(\Theta)} (\beta_3) \otimes (-1)^{1/2} \partial x \mathcal{L}_1 (\partial x W') \right] \left[ \gamma_1, \beta_2 \right] \cdot \left[ \gamma_1, \beta_1 \right]
\]
\[
= -\pi m_2 \Xi (\ldots)
\]
\[
= (-1)^{1 + |\beta_2| + |\beta_3|} \pi m_2 \left[ \gamma_1, \beta_3 \right] \otimes \mathcal{L}_1 (\partial x \mathcal{L}_1 (\partial x W')) [\gamma_1, \beta_2] \cdot [\gamma_1, \beta_1]
\]
Hence

\[ \rho^3 (\beta_1 \otimes \beta_2 \otimes \beta_3) = (-1)^{i \leq j} \sum_{i \leq j} \beta_i \beta_j \int \nu_x \nu_1 (\delta_x W') \]  

\[ [\psi, \beta_3] \cdot [\psi, \beta_2] \cdot [\psi, \beta_1] \]  

(9.1)

In the general case we have

\[ \hat{T} = \]  

\[ m_2 \exp(-S) \]  

(9.2)

and

\[ \rho^m (\beta_1 \otimes \cdots \otimes \beta_m) = (-1)^{1 + \sum_{i < j} \beta_i \beta_j + \beta_2 + \cdots + \beta_m} \]  

\[ \text{eval}_\nu (\beta_m \otimes \cdots \otimes \beta_1) \]  

\[ \text{from} - \mathcal{S}' \]  

\[ = (-1)^{1 + \sum_{i < j} \beta_i \beta_j + \beta_2 + \cdots + \beta_m + (m-1) + 2} \cdot \text{from hint} \cdot \infty \]  

\[ + |\beta_m| + \cdots + |\beta_2| \cdot \text{from} [\mathcal{L}_1] \otimes \Theta^* \]  

\[ (\mathcal{L}_1 \delta_x)^{m-1}(W') \prod_{i=1}^{\hat{m}} [\psi, \beta_i] \]  

(9.3)

\[ = (-1)^{\sum_{i < j} \beta_i \beta_j} (\mathcal{L}_1 \delta_x)^{m-1}(W') \prod_{i=1}^{\hat{m}} [\psi, \beta_i] \]  

(9.4)
Thus, if we believe the minimal moduli construction, $S \otimes_R \text{End}_R(k_{\text{stab}})$ has an its minimal $\infty$-model (in the case of one variable) and $W = xW'$

\[
\left( \text{End}_R(k_{\text{stab}}), \{\rho_m\}_{m \geq 1} \right) \tag{10.1}
\]

where by (4.2) and (8.2), $b_1 = 0$ and (uniting $*$ for product of operation in $\text{End}$)

\[
b_2(\beta_1 \otimes \beta_2) = \beta_1 \cdot \beta_2 + (-1)^{l_1} [\psi, \beta_1] \cdot [\psi, \beta_2] + (-1)^{l_1} \partial_x(W') [\psi, \beta_1] \cdot [\psi, \beta_2] \tag{10.2}
\]

and for $m \geq 3$

\[
\rho_m(\beta_1 \otimes \ldots \otimes \beta_m) = (-1)^{\sum_{i<j} \rho_i \beta_j} \prod_{i=1}^{m-1} (\psi, \beta_i) \tag{10.3}
\]

Since $W = xW'$ we have $\deg(W') = \deg(W) - 1$. Hence for

**Lemma** For $m > \deg(W)$ we have $\rho_m \equiv 0$.

**Proof**

\[
\deg((\psi, \beta) (W')) = \deg(W') - m + 1 = \deg(W) - m < 0 \text{ if } m > \deg(W). \quad \Box
\]
Example \( W = x^d \) for \( d > 2 \). Then

\[
W = x \cdot x^{d-1}
\]

Note that \( d = 2 \) is special

\[
b_2 (\beta_1 \otimes \beta_2) = \begin{cases} 
\beta_1 \cdot \beta_2 + (-1)^{p_1} [\gamma_1 \beta_1] \cdot [\gamma_2 \beta_2] & d > 2 \\
\beta_1 \cdot \beta_2 + (-1)^{p_1} [\gamma_1 \beta_2] \cdot [\gamma_1 \beta_2] & d = 2 
\end{cases}
\]

Now by def. \( \mathcal{L}_1 (x^b) = \frac{1}{1+b} x^b \) hence

\[
(\mathcal{L}_1 \partial x) (x^b) = \mathcal{L}_1 (b x^{b-1}) = x^{b-1}
\]

\[
\therefore (\mathcal{L}_1 \partial x)^{m-1} (W') = (\mathcal{L}_1 \partial x)^{m-1} (x^{d-1})
\]

\[
= x^{d-1-(m-1)} = x^{d-m}
\]

It follows that \( m = 0 \) for \( 3 \leq m < d \) and \( m > d \), while

\[
\rho_d (\beta_1 \otimes \cdots \otimes \beta_d) = (-1)^{\sum_{i<j} \tilde{\beta}_i \tilde{\beta}_j} \prod_{i=1}^{d} [\gamma_i \beta_i]
\]

Note that \( \text{End} = \text{End} (k \otimes k \gamma) \) and the products are

\[
\rho_2, 0, \ldots, 0, \rho_d, 0, \ldots
\]
Clifford actions. By (1.3) of \( \text{ainfmt2} \) we have in the usual way Clifford actions

\[
\text{End}(k^{\text{stab}}) \cong \mathcal{V}c \cdot \mathcal{T}c^{-t}
\]

which are in the \( n=1 \) case

\[
\sigma^+ = A t = -[\psi^*, -] - \partial_x (w')[\psi, -]
\]

\[
\gamma = -\gamma \quad \text{(Note we are not totally sure about this)}
\]

Now if we assume \( w \in \mathbb{R}^3 \) so \( w' \in \mathbb{R}^2 \) then \( \partial_x (w') \in \mathbb{R} \) and

\[
\sigma^+ = -[\psi^*, -] \quad \text{so, rescaling, the action is}
\]

\[
\sigma^+ = [\psi^*, -] \quad \sigma = \gamma.
\]

Then for instance

\[
e := \sigma^+ \gamma = [\psi^*, -] \cdot \gamma
\]

\[
\sigma^+ \gamma (\beta) = [\psi^*, \psi \beta]
\]

\[
= \psi^* \psi \beta - (-1)^{|\beta|+1} \psi \beta \psi^*
\]

As a matrix on

\[
\text{End}(\Lambda k \psi) = k \cdot \psi^* \otimes k \cdot \psi \otimes k \cdot \mathcal{F} \otimes k \cdot \psi^*
\]

\[
e(\psi \psi^*) = \psi^* \psi \psi^* + \psi \psi \psi^* = 0
\]

\[
e(\psi \psi) = \psi^* \psi \psi \psi + \psi \psi \psi \psi^* = \gamma^* \gamma + \gamma \gamma^* = 0
\]

\[
e(\gamma) = 0 \quad e(\psi^*) = \gamma^* \psi \psi^* - \psi \psi \gamma^* = \gamma^*
\]
\[ [e] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

i.e. \( e \) is the projector onto \( k \cdot (\mathfrak{g}^* \mathfrak{g} + \mathfrak{g} \mathfrak{g}^*) \subset \text{End} \) \( \otimes k \cdot \mathfrak{g}^* \)

i.e. \( k \cdot 1 \oplus k \cdot \mathfrak{g}^* \subset \text{End} \).

The induced \( \text{Ass} \)-algebra structure on \( \mathfrak{A} := k \cdot 1 \oplus k \cdot \mathfrak{g}^* \) is

\[
\bar{b}_2 := e b_2 e, \quad \bar{b}_2(\beta_1 \otimes \beta_2) = e\left(\beta_1 \cdot \beta_2 + (-1)^{|\beta_1|} [\gamma, \beta_1] \cdot [\gamma, \beta_2] \right) = \rho \cdot \beta_2 + (-1)^{|\beta_1|} [\gamma, \beta_1] \cdot [\gamma, \beta_2] \quad d=2
\]

\[ \text{c.q.} \quad \bar{b}_2(\gamma^* \otimes \gamma^*) = -1 \delta_{d=2} \]

\[ \bar{b}_2(1 \otimes \gamma^*) = \gamma^* \]

Thus \( (\mathfrak{A}, \bar{b}_2) \) is the Clifford algebra of \( k \cdot \mathfrak{g}^* \) with \( Q(\gamma^*) = -1 \), if \( d=2 \).

Then \( \bar{\rho}_m := e \rho m e \) are zero for \( m > 2 \) and \( m \neq d \) and

\[ \bar{\rho}_d(\beta_1 \otimes \cdots \otimes \beta_d) = (-1) e^{\sum \beta_i} \quad \text{for } d \text{ even} \]

so is only nonzero on \( \gamma^* \otimes \cdots \otimes \gamma^* \), where

\[ \bar{\rho}_d(\gamma^* \otimes \cdots \otimes \gamma^*) = 1 \]