Our aim in this note is to write down the theory we have developed in its full generality, namely as a model of the DG-category $\text{mf}(k[[x]], W)$ extended by the exterior algebra $\Lambda(k0_1 \oplus \cdots \oplus k0_n)$. This has several components.

Fixing a potential $(k[[x]], W)$ over a commutative $\mathbb{A}$-algebra $k$, and writing $\mathbb{C}$ for the DG-category $\mathbb{C} = \text{mf}(k[[x]], W)$, they are $(|x| = n)$.

1. Putting strict homotopy retracts on $\mathbb{C}(X, Y) \otimes_k \Lambda(\Theta_i, k0_i)$ for all $X, Y \in \mathbb{C}$

2. Running the minimal model construction

3. Extracting the Feynman rules

The result of this is an $A\infty$-structure on the collection of $\mathbb{Z}_2$-graded $k$-modules underlying $\mathbb{C} \otimes_{k[[x]]} k[[x]]/(\partial W)$. Note that this $A\infty$-category is not minimal, although some of its endomorphism $A\infty$-algebras will be. Nonetheless, this $A\infty$-category is quasi-isomorphic (in fact homotopy equivalent) over $k$ to $\mathbb{C} \otimes_k \Lambda(\Theta_i, k0_i)$.

**Notation**
- $R = k[x_1, \ldots, x_n]$
- $JW = R/(x_1 W, \ldots, x_n W)$
- $F_0 = \bigoplus_{i=1}^n k0_i$ as a $\mathbb{Z}_2$-graded module with $|0_i| = 1$
- $\mathbb{C}$ = DG-category of finite-rank free MFs of $W$ (note we do not add summands at this stage)

($\mathbb{C}$ can in fact be any full-sub-DG-category of $\text{mf}(R, W)$)

**Note** We will later put some restriction on the $Y_i^{\times X}$ (see p. 8) and we will see that in order to form a DG-category we are forced to make $Y_i^{\times X}$ indep. of $Y, X$ (see p. 4).
I. Strict homotopy retracts

Here we primarily refer to our cut operation paper, hereby denoted \([\text{Cut}]\). Let \(X, \mathcal{Y}\), be \((\text{rank free})\) matrix factorisations, and assume that

- \(t_i^\mathcal{Y}_X, \ldots, t_n^\mathcal{Y}_X\) is a quasi-regular sequence in \(R\), such that
- \(R/(t_i^\mathcal{Y}_X, \ldots, t_n^\mathcal{Y}_X)\) is a finitely generated free \(k\)-module,
- each \(t_i^\mathcal{Y}_X\) acts null-homotopically on \(\text{Hom}_R(\mathcal{Y}, X)\) with null-homotopy \(\gamma_i^\mathcal{Y}_X\) \((R\text{-linear})\)
- the Koszul complex over \(R\) of \(t_i^\mathcal{Y}_X, \ldots, t_n^\mathcal{Y}_X\) is exact except in degree zero

(see \text{[10-3-19 we had removed that \(K\) was noetherian, but forgot to add this]}).

Set \(I = (t_1^\mathcal{Y}_X, \ldots, t_n^\mathcal{Y}_X)\). The \(I\)-adic completion \(\hat{R}\) of \(R\) has \(t_i^\mathcal{Y}_X\) as a quasi-regular sequence and \(\hat{R}/(t_i^\mathcal{Y}_X) \cong R/(t_i^\mathcal{Y}_X)\). Then by the push forward paper there is a standard flat \(k\)-linear connection (writing \(t_i = \xi_i^\mathcal{Y}_X\))

\[
\nabla^0 : \hat{R} \rightarrow \hat{R} \otimes_{k[\xi]} \bigoplus_{j \geq 0} k[\xi]/k \quad (2.2)
\]

We run through the steps of \([\text{Cut}, \S 4.3]\), with \(\wedge F_0\), our new notation for \(Sm\), and \(\text{Hom}_R(\mathcal{Y}, X) \cong \mathcal{Y}_X / X\) for \(\mathcal{Y}_X / X\) \((\text{see } [\text{Cut}, \S 4.5])\), and \(R = k[\xi]\) in place of the \(k[\xi]\) there.

**First step** \(\nabla^0\) extends to a \(k\)-linear operator \(\nabla\) on \(\hat{R} \otimes_{k[\xi]} \bigoplus_{j \geq 0} k[\xi]/k\). Choosing a homogeneous basis for \(X, \mathcal{Y}\) and taking the induced basis on \(\text{Hom}_R(\mathcal{Y}, X)\) over \(R\), and extending \(\nabla\), we get a \(k\)-linear splitting homotopy

\[
H = [d_k, \nabla]^{-1} \nabla \subset (K \otimes_R \text{Hom}_R(\mathcal{Y}, X) \otimes_R \hat{R}, d_k), \quad (2.3)
\]

where \((K, d_k) = (\wedge(F_0) \otimes_R R, \sum_{i=1}^n t_i^\mathcal{Y}_X \otimes_k^\cdot)\), and we identify \(\partial_i\) with \(dt_i^\mathcal{Y}_X\).

*NOTE:* Constructing \(\nabla\) depends on a choice of \(k\)-basis for \(R/(t_1^\mathcal{Y}_X)\).
which corresponds to the strong cleformation retract

\[
\left( \frac{\hat{\mathcal{R}}(\hat{\gamma}_x)}{\mathcal{R}(\gamma_x)} \otimes \text{Hom}_R(\gamma, x), 0 \right) \xrightarrow{\pi} \left( K \otimes_R \text{Hom}_R(\gamma, x) \otimes_R \hat{\mathcal{R}}, d_K + d_{\text{Hom}} \right)
\]

\[
\text{Second step: We now view } d_{\text{Hom}}, \text{ the standard differential on } \text{Hom}_R(\gamma, x), \text{ as a perturbation, and we learn that}
\]

\[
\phi_\infty = \sum_{m=0} (-1)^m (Hd_{\text{Hom}})^m H
\]  

is a k-linear splitting homotopy, to which is associated the following k-linear strong cleformation retract of complexes

\[
\left( \frac{\hat{\mathcal{R}}(\hat{\gamma}_x)}{\mathcal{R}(\gamma_x)} \otimes \text{Hom}_R(\gamma, x), d_{\text{Hom}} \right) \xrightarrow{\pi} \left( K \otimes_R \text{Hom}_R(\gamma, x) \otimes_R \hat{\mathcal{R}}, d_K + d_{\text{Hom}} \right) \]

where

\[
\phi_\infty = \sum_{m=0} (-1)^m (Hd_{\text{Hom}})^m H, \quad \zeta_\infty = \sum_{m=0} (-1)^m (Hd_{\text{Hom}})^m \zeta
\]  

\[
\text{Third step: Since each } \hat{\gamma}_x \text{ acts null-homotopically on } \text{Hom}_R(\gamma, x) \text{ we have an isomorphism of complexes (over } R, \text{ in fact)}
\]

\[
\left( K \otimes_R \text{Hom}_R(\gamma, x) \otimes_R \hat{\mathcal{R}}, d_K + d_{\text{Hom}} \right) \xleftarrow{\exp(-\delta)} \left( \Lambda_F \otimes_R \text{Hom}_R(\gamma, x) \otimes_R \hat{\mathcal{R}}, d_{\text{Hom}} \right)
\]

where \( \delta = \sum_i \lambda_i \circ d_i^* \). Note here we do not assume \( \lambda_i \) is a partial derivative of, e.g., \( d \gamma \). This is not necessary.
Fourth step The canonical map \( \varepsilon : \text{Hom}_R(Y, X) \to \text{Hom}_R(Y, X) \otimes \hat{R} \) is by Remark 7.7 of the pushforward paper, a homotopy equivalence over \( k \) (see Eq. 3.3, 3.4). Hence we have homotopy equivalences of \( k \)-complexes, combining (3.3), (3.4) 

\[
\begin{align*}
(\Lambda \Phi \otimes_k \text{Hom}_R(Y, X), d_{\text{Hom}}) \\
\downarrow 1 \otimes \varepsilon \\
(\Lambda \Phi \otimes_k \text{Hom}_R(Y, X) \otimes_R \hat{R}, d_{\text{Hom}}) \\
\exp(-s) \downarrow \exp(s) \\
(\hat{K} \otimes_k \text{Hom}_R(Y, X) \otimes_R \hat{R}, d_K + d_{\text{Hom}}) \cong \phi_\infty \\
\Pi \downarrow \varepsilon_\infty \\
(\hat{R}/(\varepsilon \varepsilon X) \otimes_k \text{Hom}_R(Y, X), \overline{d_{\text{Hom}}}) \\
\cong \\
(\hat{R}/(\varepsilon \varepsilon X) \otimes_k \text{Hom}_R(Y, X), \overline{d_{\text{Hom}}})
\end{align*}
\]  

Note that \( \text{Hom}_R(Y, X) \otimes_R \hat{R}/(\varepsilon \varepsilon X) \) is by our hypotheses a \( \mathbb{Z}_2 \)-graded complex of finite free \( k \)-modules. Keeping in mind that all our \( \mathbf{DGA} \)-categories are \( \mathbb{Z}_2 \)-graded, we have the \( \mathbf{DGA} \)-category \( \mathbb{C} \) with \( \mathbb{C}(Y, X) := (\text{Hom}_R(Y, X), d_{\text{Hom}}) \) and we now introduce

Remark p.4 of Eq. 3.3, 3.4 shows there is a deformation retract over \( k \) between \( K_k(\varepsilon) \) and \( R/I \), and so we may apply p.4 to \( X = \text{Hom}_R(X, Y) \), and \( f: k \to R \) (so \( W = 0 \)), to get \( \varepsilon \) is a \( k \)-linear homotopy equivalence.
Next we define a DG-category $\mathcal{C}$ with $\mathcal{C}(Y, X) = \text{Hom}_R(Y, X) \otimes_R \hat{R}$.

The subtlety here is that the topology with respect to which we take the completion varies on the pair $Y, X$ varies. To be more precise we now write

$$I_{Y, X} = (t^Y_X, \ldots, t^n_Y X) \subseteq R$$

and for the associated completion we write

$$\hat{R}_{Y, X} := \lim_{\leftarrow} R/_{I_{Y, X}^g}.$$ 

In order for the DG-category $\mathcal{C}$ to make sense we need to have $k$-linear maps, for each triple $Z, Y, X$ of matrix factorisations

$$m_{Z, Y, X} : \hat{R}_{Z, Y} \otimes \hat{R}_{Y, X} \longrightarrow \hat{R}_{Z, X}.$$ 

For this to exist we would need, given Cauchy sequences $(a_n), (b_n)$ in $R$ for the $I_{Z, Y}$-adic and $I_{Y, X}$-adic topologies respectively, to show that $(a_n b_n)$ is Cauchy for the $I_{Z, Y}$-adic topology. We would show this by calculating

$$a_m b_m - a_n b_n = a_m b_m - a_m b_n + a_m b_n - a_n b_n = a_m (b_m - b_n) + (a_m - a_n) b_n,$$

and using that for any $N \geq 1$ there are $a, b \geq 1$ with $I_{Z, Y}^a \leq I_{Z, X}^N$ and $I_{Y, X}^b \leq I_{Y, X}^N$.

But given $Z, Y, X$ are arbitrary, this shows:

**Upshot** For products $m_{Z, Y, X}$ to exist (and thus for $\mathcal{C}$ to exist) we need the $I_{Y, X}$-adic topology on $R$ to be independent of $Y, X$.
In light of the previous page, we now fix \( t_i^{Y,X} = t_i \) to be independent of \( Y,X \) and let \( \hat{R} \) denote the \( I = (t_1, \ldots, t_n) \)-adic completion. We continue to let \( \lambda_i^{Y,X} \) be arbitrary, although see p. \( \otimes \). Using the usual \( R \)-algebra structure on \( \hat{R} \) we define

**Def.** The DG-category \( \hat{C} = C \otimes R \hat{R} \), has \( \hat{C}(Y,X) = \text{Hom}_R(Y,X) \otimes_R \hat{R} \).

As we see in (4.1), our methods produce an \( A_{\infty} \)-minimal model of \( \hat{C} \) not \( C \).

As an example of what can go wrong if we try different sequences \( \hat{R} \), we have:

**Example** Let \( K \) be an alg. closed field and \( W = x^3 - x = x(x-1)(x+1) \).

This is a potential, and in the DG-category \( \hat{C} \) we may consider objects

\[
P_{-1} = \begin{pmatrix} 0 & x+1 \\ x(x-1) & 0 \end{pmatrix}, \quad \bar{P}_{-1} = \begin{pmatrix} 0 & x \\ x^2-1 & 0 \end{pmatrix}, \quad P_i = \begin{pmatrix} 0 & x-1 \\ x(x+1) & 0 \end{pmatrix}.
\]

We know \( \mathfrak{X}_W \) acts null-homotopically on all mapping complexes, and completing along the ideal \( (\mathfrak{X}_W) \) corresponds to taking the formal scheme around the critical points i.e. since \( \mathfrak{X}_W = 3x^2 - 1 \), at \( \{ \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \} \).

Note e.g. that \( \mathfrak{X}_{P_{-1}} = (x+1)\mathfrak{X}^* + x(x-1)\mathfrak{X}, \) and \( \mathfrak{X}_x(d_{P_{-1}}) = \mathfrak{X}^* + (2x-1)\mathfrak{X} \).
which clearly satisfy \([d_{p-1}, \partial_x (d_{p-1})] = \partial_x W \cdot 1\).

Now it is true that \( \{x^i\} \) is a quasi-regular sequence in \( R = k[x] \) acting
null-homotopically on \( \text{Hom}_R (P_0, P_0) \), but the \((x)\)-adic topology on \( R \)
is certainly not equal to the \((\partial_x W)\)-adic topology (or the \((x-1)\)-adic topology
for that matter). Whereas we know \( C \) is homotopy equivalent to
\[
\hat{C} = C \otimes_R \lim_{r \to \infty} R/(\partial_x W)^r.
\]

So this completion preserves \( P_{-1}, P_0, P_1 \).

Remark: We do not want to assume \( t_i = \partial x_i W \) because we want to preserve the
freedom to replace \( C \) by a sub-D\( \Lambda \)-category (say just \( k^{\text{tab}} \)) in which
case we may be able to use a sequence \( t \) which is not the partial derivatives
(e.g. \( x_1, \ldots, x_n \)).
The DG-category $A\mathcal{O} \otimes_k \hat{\mathcal{C}}$ is the tensor product of $A\mathcal{O}$, viewed as a one-object DG-category (with zero differential) and $\hat{\mathcal{C}}$, as in $p(6) \otimes \mathbb{C}$, so

$$(A\mathcal{O} \otimes_k \hat{\mathcal{C}})(Y, X) := (A\mathcal{O} \otimes_k \text{Hom}_R(Y, X), 1 \otimes d_{\text{Hom}})$$

and the composition is (using graded swaps)

$$(A\mathcal{O} \otimes_k \hat{\mathcal{C}})(Y, X) \otimes_k (A\mathcal{O} \otimes_k \hat{\mathcal{C}})(Z, Y)$$

$$\cong A\mathcal{O} \otimes_k \hat{\mathcal{C}}(Y, X) \otimes_k A\mathcal{O} \otimes_k \hat{\mathcal{C}}(Z, Y)$$

$$\cong A\mathcal{O} \otimes_k \hat{\mathcal{C}}(Y, X) \otimes_k \hat{\mathcal{C}}(Z, Y)$$

$$\downarrow \text{mom}$$

$$A\mathcal{O} \otimes_k \hat{\mathcal{C}}(Z, X)$$

Next we argue that (4.1) gives a strict homotopy retraction of $\mathcal{A} = A\mathcal{O} \otimes_k \hat{\mathcal{C}}$ viewed as an $A\infty$-category, according to $p(5) \otimes \mathbb{C}$. To this end we ignore the bottommost step of (4.1) and simply identify $R/(t^{\lambda}Y)$ with $R/(t^{\lambda}X)$. With this in mind, the overall content of (4.1) is a $k$-linear homotopy equivalence

$$(A\mathcal{O} \otimes_k \text{Hom}_R(Y, X) \otimes_k \hat{R}, d_{\text{Hom}}) \xrightarrow{\cong} (R/(t^{\lambda}Y) \otimes_k \text{Hom}_R(Y, X), d_{\text{Hom}})$$

$$\cong H$$

$$\xleftarrow{\cong}$$

(5.2)

where (cf. $p.2 \otimes \mathbb{C}$) we have $\text{id} \otimes_{\text{id}} \text{id} = 1$ and $\text{id} \otimes_{\text{id}} \text{id} = 1 - [d_{\text{Hom}}, \hat{H}]$

$$\Phi = \pi \circ \exp(-\delta), \quad \Phi^{-1} = \exp(\delta) \circ \phi_\infty$$

$$\hat{H} = \exp(\delta) \circ \phi_\infty \circ \exp(-\delta)$$

(5.3)
Lemma The data \((P,G)\) consisting of

\[
P_{Y,X} := \Phi^{-1} \cdot \Phi \subseteq \mathcal{A}(Y, X)
\]

\[
G_{Y,X} := \hat{H} \subseteq \mathcal{A}(Y, X)
\]

(6.1)

form a strict homotopy retract on the DA-category \(\mathcal{A}\) in the sense of \(p. \S\(\text{ainHcat}^2\)\), with base ring \(R\).

Proof Clear by construction. \(\square\)

By the minimal model theorem \((p. \S\(\text{ainHcat}^2\))\) there are maps \(\{p_k\}_{k \geq 1}\) making the \(k\)-modules \(\text{Im}(P_{Y,X})\) into an \(A_\infty\)-category over \(R\). But there is an isomorphism

\[
\text{Im}(P_{Y,X}) = \text{Im}(\Phi^{-1}) \xrightarrow{\cong} R/(\pm YX) \otimes_R \text{Hom}_R(Y, X)
\]

(6.2)

and we interpret the minimal model construction as putting an \(A_\infty\)-structure on the set of objects \(\text{ob}(\mathcal{B})\) and \(\mathbb{Z}_2\)-graded free \(k\)-modules \(R/(\pm YX) \otimes_R \text{Hom}_R(Y, X)\).

Def. Let \(\mathcal{B}\) denote the \(\mathbb{Z}_2\)-graded \(A_\infty\)-category over \(R\) with

- objects \(\text{ob}(\mathcal{B}) = \text{ob}(\mathcal{C})\)
- \(\mathcal{B}(Y, X) := R/(\pm YX) \otimes_R \text{Hom}_R(Y, X)\)

and higher products \(\{f^k\}_{k \geq 1}\) as produced by the minimal model theorem applied to the above strict homotopy retract on \(\mathcal{A}\).
Calculating Feynman Rules

Following [\text{ainf}], let $\mathcal{A}$ be the commutative associative $k$-algebra denoted $R$, and let us write (recall $\mathcal{A} = \Lambda \mathcal{F}_\partial \otimes_k \hat{C}$)

$$\mathcal{H} = \bigoplus_{\gamma, x \in \text{Ob} \mathcal{A}} \text{Hom}_{\mathcal{A}}(\gamma, X) = \bigoplus_{\gamma, x \in \text{Ob} \mathcal{A}} \Lambda \mathcal{F}_\partial \otimes_k \text{Hom}_R(\gamma, X) \otimes_R \hat{R} \quad (7.1)$$

with its $Q$-bimodule structure, with forward suspended operations

$$\rho_n : \mathcal{H}^n \otimes_R n \rightarrow \mathcal{H}.$$ \hfill (7.2)

Then $p.\rbrack$ produces the $k$-linear $Q$-bilinear maps

$$\rho_n = \sum_{i \in \mathcal{F}_n} \rho_i \in \text{Hom}_{\text{Mod}_Q}(B^n \otimes_R n, B^n). \quad (7.3)$$

where $B = \text{Im}(\rho) \subseteq \mathcal{H}$ identified with the relevant direct sum of quotients as in $16.2$.

**Def.** For $1 \leq i \leq n$ we define the odd $\hat{R}$-linear, $Q$-bilinear operators

$$\lambda_i, \delta_i \subseteq \mathcal{H} \quad \lambda_i = \sum_{\gamma, x} \lambda_i^{\gamma, x}, \quad \delta_i = \sum_{\gamma, x} \delta_i^{\gamma, x}$$

and we set $\delta = \sum_i \delta_i$, with $d_i = \lambda_i \delta_i = \sum_{\gamma, x} \lambda_i^{\gamma, x} \delta_i^{\gamma, x}$.

**Remark** Given a matrix factorisation $X$, set $\lambda_i^x := \lambda_i^x(1)$, which is an odd $R$-linear operator on $X$, with $[dx, \lambda_i^x] = t_i \cdot 1_x$, since

$$t_i : \text{Hom}(X) \Rightarrow t_i \cdot 1_x = d_{\text{Hom}(X)}(\lambda_i^x(1)) = [dx, \lambda_i^x].$$

**Note:** $\rho_n$ uses the tilde grading $\tilde{\omega} \otimes x = |\omega| + |x| + 1$ for $\omega \in \Lambda \mathcal{F}_\partial$, $x \in \text{Hom}(\gamma, X)$.\]
Then let us write \( \bar{\lambda}_i^{\gamma \times} \) for the odd \( R \)-linear operator on \( \text{Hom}_R(\gamma, X) \) given by

\[
\bar{\lambda}_i^{\gamma \times}(\alpha) = \lambda_i^\times \circ \alpha
\]

Then

\[
\left[ d_{\text{Hom}}(\gamma, x), \bar{\lambda}_i^{\gamma \times} \right](\alpha) = d_x \circ \bar{\lambda}_i^{\gamma \times}(\alpha) + (-1)^{|\alpha|} \bar{\lambda}_i^{\gamma \times}(\alpha) \circ d_y
\]

\[
+ \bar{\lambda}_i^{\gamma \times} \circ d_{\text{Hom}}(\alpha)
\]

\[
= d_x \lambda_i^\times \circ \alpha + (-1)^{|\alpha|} \lambda_i^\times \circ d_y
\]

\[
+ \lambda_i^\times \circ d_x \circ \alpha - (-1)^{|\alpha|} \lambda_i^\times \circ d_y
\]

\[
= t_i \cdot \alpha
\]

Hence \( \left[ d_{\text{Hom}}(\gamma, x), \bar{\lambda}_i^{\gamma \times} \right] = t_i \cdot 1_{\text{Hom}(\gamma, x)} \), and so the operators \( \bar{\lambda}_i^{\gamma \times} \) are just as good as \( \lambda_i^{\gamma \times} \) for the purposes of the foregoing. Moreover, this type of homotopy is much better suited to the calculations we are about to do.

**Hypothesis** We have for each \( X \) a fixed choice of null-homotopy \( \lambda_i^{\times} : X \to X \) (odd \( R \)-linear) for \( t_i \), that is, with \([d_x, \lambda_i^\times] = t_i \cdot 1_x \), and for all \( \gamma, X \) we use the homotopies

\[
\lambda_i^{\gamma \times} \in \text{Hom}_R(\gamma, X) \quad \lambda_i^{\gamma \times}(\alpha) = \lambda_i^\times \circ \alpha. \tag{8.1}
\]

**Note** This is not a constraint: as we have shown, it is possible to arrange under our setup.

**Def.** For \( 1 \leq i \leq n \) we define the odd \( R \)-linear, \( \Theta \)-bilinear operators \( [\lambda_i, -] \in \mathcal{E} \)

defined on \( \omega \otimes \alpha \in \Lambda F_0 \otimes \text{Hom}_R(\gamma, X) \otimes R \) by

\[
[\lambda_i, -](\omega \otimes \alpha) = (-1)^{|\omega|} \omega \otimes \left\{ \lambda_i^\times \circ \alpha - (-1)^{|\alpha|} \alpha \circ \lambda_i^\gamma \right\} \tag{8.2}
\]
Lemma The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{H}[i] \otimes \mathcal{H}[i] & \xrightarrow{r_z} & \mathcal{H}[i] \\
\delta_i \otimes 1 + 1 \otimes \delta_i - \partial^* \otimes [\lambda_i, -] & \downarrow & \delta_i \\
\mathcal{H}[i] \otimes \mathcal{H}[i] & \xrightarrow{r_z} & \mathcal{H}[i]
\end{array}
\]

from (8.2)

Proof Note that \( r_z(x_2 \otimes x_1) = (-1)^{\hat{x}_i \hat{x}_j + \hat{x}_j} x_1 \circ x_2 \) where \( \circ \) is the composition in \( \mathcal{O} \).

To check (9.1) commutes we may choose objects \( X, Y, Z \) and check commutativity of

\[
\big[ \Lambda F_0 \otimes_k \text{Hom}_R(Z, Y) \otimes_k \hat{R} \big] [i] \otimes_k \big[ \Lambda F_0 \otimes_k \text{Hom}_R(Y, X) \otimes_k \hat{R} \big] [i]
\]

\[
\xrightarrow{r_z} \left[ \Lambda F_0 \otimes_k \text{Hom}_R(Z, X) \otimes_k \hat{R} \right] [i]
\]

\[
\big[ \Lambda F_0 \otimes_k \text{Hom}_R(Z, Y) \otimes_k \hat{R} \big] [i] \otimes_k \big[ \Lambda F_0 \otimes_k \text{Hom}_R(Y, X) \otimes_k \hat{R} \big] [i]
\]

\[
\xrightarrow{r_z} \left[ \Lambda F_0 \otimes_k \text{Hom}_R(Z, X) \otimes_k \hat{R} \right] [i]
\]

To this end let \( \omega_i, \omega_2 \in \Lambda F_0 \) be homogeneous and take \( x_1 \in \text{Hom}_R(Y, X), x_2 \in \text{Hom}_R(Z, Y) \).

Then

\[
r_z \left( \delta_i \otimes 1 + 1 \otimes \delta_i \right) \left( \omega_2 \otimes x_2 \otimes (\omega_1 \otimes x_1) \right) = r_z \left( \delta_i (\omega_2 \otimes x_2) \otimes \omega_1 \otimes x_1 + \omega_2 \otimes x_2 \otimes \delta_i (\omega_1 \otimes x_1) \right)
\]

\[
= r_z \left( \lambda_i ( \partial^* \otimes \gamma_i (x_2) \otimes \omega_1 \otimes x_1 + (-1)^{\hat{x}_i x_1} \omega_2 \otimes x_2 \otimes \partial^* \omega_1 \otimes \gamma_i (x_1) \right)
\]
\[
\begin{align*}
= (-1)^{w_1 + 1} \left( (\omega_1 \omega_2 \otimes x_1) \otimes (\omega_1 \omega_2 \otimes x_2) \right) \left( (\omega_1 \omega_2 \otimes x_1) \otimes (\omega_1 \omega_2 \otimes x_2) \right) + 1 \right) \mu \left( \omega_1 \otimes x_1 \otimes \Omega^* \omega_2 \otimes \lambda_c (x_2) \right) \\
+ (-1)^{w_1 + 1} \left( (\omega_2 \otimes x_1) \otimes (\omega_2 \otimes x_1) \right) \left( (\omega_2 \otimes x_1) \otimes (\omega_2 \otimes x_1) \right) + 1 \right) \mu \left( \Omega^* \omega_1 \otimes \lambda_i (x) \otimes \omega_2 \otimes x_2 \right)
\end{align*}
\]

Then we also compute
\[
\begin{align*}
\delta_i \left( (\omega_2 \otimes x_2) \otimes (\omega_1 \otimes x_1) \right) &= (-1)^{w_2 \otimes x_2} (\omega_1 \otimes x_1)^* + (w_1 \otimes x_1)^* + 1 \right) \delta_i \left( (\omega_1 \otimes x_1) \otimes (\omega_2 \otimes x_2) \right) \\
&= (-1)^{w_1 \otimes x_1 + 1} \left( (\omega_2 \otimes x_2) \otimes (\omega_1 \otimes x_1) \right) + (w_1 \otimes x_1)^* + 1 \right) \delta_i \left( (\omega_1 \otimes x_1) \otimes (\omega_2 \otimes x_2) \right) \\
&= (-1)^{w_2 \otimes x_2} (\omega_1 \otimes x_1)^* + (w_1 \otimes x_1)^* + 1 \right) \delta_i \left( (\omega_1 \otimes x_1) \otimes (\omega_2 \otimes x_2) \right)
\end{align*}
\]
\begin{align*}
&= (-1)^{|\omega_2||\omega_1| + |x_2||x_1| + |x_2||x_1| + |x_1||x_1|} \mathcal{O}_i^*(\omega_1\omega_2) \otimes \lambda_i(x_1 \circ x_2) \\
\text{Hence,} \\
\left\{ S_i r_2 - r_2 (1 \otimes S_i + S_i \otimes 1) \right\} \left( (\omega_1 \otimes x_2) \otimes (\omega_1 \otimes x_1) \right) \\
&= (-1)^{|\omega_1||\omega_1|} \mathcal{O}_i^*(\omega_1 \omega_2) \otimes \lambda_i(x_1 \circ x_2) \\
&\quad - (-1)^{|\omega_1||\omega_1|} \omega_1 \mathcal{O}_i^*(\omega_2) \otimes x_1 \circ \lambda_i(x_2) \\
&\quad - \mathcal{O}_i^*(\omega_1\omega_2) \otimes \lambda_i(x_1 \circ x_2)
\end{align*}

But
\begin{align*}
\mathcal{O}_i^*(\omega_1 \omega_2) \otimes \lambda_i(x_1 \circ x_2) &= (-1)^{|\omega_1||\omega_1|} \mathcal{O}_i^*(\omega_1 \omega_2) \otimes \lambda_i(x_1 \circ x_2) \\
&\quad - \mathcal{O}_i^*(\omega_1 \omega_2) \otimes \lambda_i(x_1 \circ x_2)
\end{align*}

So finally
\begin{align*}
\left\{ S_i r_2 - r_2 (1 \otimes S_i + S_i \otimes 1) \right\} \left( (\omega_1 \otimes x_2) \otimes (\omega_1 \otimes x_1) \right) \\
&= (-1)^{|\omega_1||\omega_1|} \mathcal{O}_i^*(\omega_1 \omega_2) \otimes \lambda_i(x_1 \circ x_2)
\end{align*}
Whereas

\[ r_2 \left( O_i^* \otimes \left[ \lambda_i \right] \right) \left( (w_2 \otimes x_2) \otimes (w_1 \otimes x_1) \right) = r_2 \left( (-1)^{|w_2|+|x_2|+1} O_i^* (w_2 \otimes x_2) \otimes \left[ \lambda_i \right] \right) (w_1 \otimes x_1) \]

\[ = (-1)^{|w_2|+|x_2|+|w_1|+1} r_2 \left( O_i^* (w_2) \otimes x_2 \otimes (w_1 \otimes \left[ \lambda_i \right]) \right) \]

\[ = (-1)^{|w_2|+|x_2|+|w_1|+1} \left( (-1)^{1} (O_i^* (w_2) \otimes [\lambda_i] (x_1)) \right) + \left( w_1 \otimes [\lambda_i] \right) \]

\[ \cdot \left( \lambda_i \otimes [\lambda_i] \right) (x_1) \otimes x_2 \]

\[ = (-1)^{|w_2|+|x_2|+|w_1|+1} \left( (-1)^{|x_1|+|w_1|+1} \left( w_1 \otimes O_i^* (w_2) \otimes [\lambda_i] \right) \right) \]

\[ \cdot \left( \lambda_i \otimes [\lambda_i] \right) (x_1) \otimes x_2 \]

\[ = - (-1)^{|x_1|+|w_1|+|x_1|+|x_2|} \left( w_1 \otimes O_i^* (w_2) \otimes [\lambda_i] \right) \]

\[ \cdot \left( \lambda_i \otimes [\lambda_i] \right) (x_1) \otimes x_2 \]

\[ = - \left\{ \delta_i r_2 - r_2 \left( 1 \otimes \delta_i + \delta_i \otimes 1 \right) \right\} \left( (w_2 \otimes x_2) \otimes (w_1 \otimes x_1) \right) \]

This proves that

\[ \delta_i r_2 = r_2 \left( 1 \otimes \delta_i + \delta_i \otimes 1 - O_i^* \otimes [\lambda_i] \right) \]
Def" Set $\Xi_i = \Theta_i^* \otimes [\lambda_i, -]$, an $R$-linear $Q$-bilinear operator on $\mathcal{H} \odot \mathcal{H}$, and write $\Xi = \sum_i \Xi_i$. (note the $\otimes$ in $\Xi_i$ sees the tilde grading.)

Lemma. As operators on $\mathcal{H} \odot \mathcal{H}$ we have, for $1 \leq i, j \leq n$

\begin{align*}
(a) \quad [\delta_i \otimes 1, 1 \otimes \delta_j] &= 0, \\
(b) \quad [\delta_i \otimes 1, \Xi_j] &= 0, \\
(c) \quad [\Xi_j, 1 \otimes \delta_i] &= 0^* \otimes [\lambda_j, \lambda_i] 0_i^*.
\end{align*}

(12.1)

where $[\lambda_i, \lambda_j]$ means $\lambda_i \lambda_j + \lambda_j \lambda_i$ with each $\lambda_i$ as in (8.1).

Proof. (a) is trivial, and (b) also since $\delta_i = \Theta_i^* \otimes \Theta_i$. For (c) we have to check

\[
[1 \otimes \delta_i, \Xi_j] (\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2) \quad \text{(don't forget $\otimes$ is wrt the $\sim$ grading!)}
\]

\[
= (-1)^{1|w_1|+|d_1|+1} \omega_1 |1 \otimes \delta_i| (\Theta_i^* \otimes \Theta_i^* \otimes \Theta_i^* \otimes \Theta_i^* \otimes \Theta_i^* \otimes \Theta_i^*) [\lambda_j, \alpha_i] \\
= (-1)^{|w_1|+|d_1|} \Theta_i^* (\omega_1) \otimes \alpha_1 \otimes \Theta_i^* (\omega_2) \otimes \Theta_i^* (\alpha_2) \otimes [\lambda_j, \alpha_i] \\
- (-1)^{|w_1|+|d_1|} \Theta_i^* (\omega_1) \otimes \Theta_i^* (\alpha_1) \otimes \Theta_i^* (\omega_2) \otimes [\lambda_j, \alpha_i] \\
= (-1)^{|w_1|+|d_1|} \Theta_i^* (\omega_1) \otimes \Theta_i^* (\alpha_1) \otimes [\lambda_j, \alpha_i] \\
+ (-1)^{|w_2|+1} \Theta_i^* (\omega_2) \otimes \Theta_i^* (\alpha_2) \otimes [\lambda_j, \alpha_i].
\]

\[= - (0_i^* \otimes [\lambda_i, \lambda_j] 0_i^*) (\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2). \square\]
Lemma  The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}[i] \otimes_a \mathcal{H}[i] & \xrightarrow{r_2} & \mathcal{H}[i] \\
\exp(-\delta) \otimes \exp(-\delta) \downarrow & & \exp(-\delta) \\
\mathcal{H}[i] \otimes_a \mathcal{H}[i] & \xrightarrow{r_2} & \mathcal{H}[i] \\
\exp(\Xi - 1\otimes \delta) \exp(1\otimes \delta) \downarrow & & \\
\mathcal{H}[i] \otimes_a \mathcal{H}[i] & \xrightarrow{r_2} & \mathcal{H}[i]
\end{array}
\]

Proof  By (12.1) we have

\[
\exp(- \left[ \delta \otimes I + 1 \otimes \delta - \Xi \right]) = \exp(\Xi - 1 \otimes \delta) \circ \left\{ \exp(-\delta) \otimes 1 \right\}
\]

Hence by p.9

\[
\exp(-\delta) r_2 = \sum_{m \in \mathbb{N}} (-1)^m \frac{1}{m!} \delta^m r_2 = \sum_{m \in \mathbb{N}} (-1)^m \frac{1}{m!} r_2 \left( \delta \otimes 1 + 1 \otimes \delta - \Xi \right)^m = r_2 \exp(- \left[ \delta \otimes I + 1 \otimes \delta - \Xi \right]) = r_2 \exp(\Xi - 1 \otimes \delta) \circ \left\{ \exp(-\delta) \otimes 1 \right\}
\]

\[
= r_2 \exp(\Xi - 1 \otimes \delta) \exp(1 \otimes \delta) \circ \left\{ \exp(-\delta) \otimes \exp(-\delta) \right\}
\]
We can calculate that

\[
\exp(-1 \otimes \delta + \Sigma) = \sum_{n \geq 0} \frac{1}{n!} \left( -\sum_j \left( 1 \otimes \lambda_j \vartheta_j^* \right) + \sum_i \vartheta_i \otimes \left[ \lambda_i, - \right] \right)^n
\]

\[
= \sum_{r, s \geq 0} \sum_{i_1 < \cdots < i_r} \sum_{j_1 < \cdots < j_s} \frac{1}{(r+s)!} (-1)^{s+r} \binom{r}{2} \binom{r+s}{2} \sum_{\delta \in S_{r+s}} (-1)^{r+s} \vartheta_{i_1}^* \otimes \delta \left( \lambda_{j_1}, \ldots, \lambda_{j_s}, \left[ \lambda_{i_1}, - \right], \ldots, \left[ \lambda_{i_r}, - \right] \right) \vartheta_{j_1}^*
\]

where $\delta$ means we arrange the $r+s$ operators in the order indicated by $\delta$, including an additional Koszul sign, and $\vartheta_{i_1}^* = \vartheta_{i_1} \cdots \vartheta_{i_r}^*$. For example if $\delta = (1, 2)$ then $\delta \left( \lambda_1, \lambda_2 \right) = -\lambda_2 \lambda_1$. So the overall sign of $\sum_{\delta \in S_{r+s}} (-1)^{r+s} \delta$ is the sign of separately ordering $i_1, \ldots, i_r$.
Define Let \( \mathcal{X} \) denote the total category of the \( \Lambda_0 \)-category \( \mathcal{B} \) from p. 6, just as \( \mathcal{H} \) is the total category of \( \mathcal{A} \) (see (7.1)) so that

\[
\mathcal{X} = \bigoplus_{y, x \in \text{ob}(\mathcal{B})} \text{Hom}_\mathcal{B}(y, x).
\] (14.1)

Now, returning to the definition of the operations \( \rho_n \) from (7.3), as a sum for \( n \geq 2 \) of \( \rho_T \) as \( T \) ranges over valid plane trees \( T \in \mathcal{T}_n \) with internal vertices of valency 3 only, and

\[
\rho_T = (-1)^{e(T)} \langle D \rangle_B
\]

is, up to a sign, the branch denotation of the decoration \( D \) of \( T \) which places

- \( \mathcal{X}[i] \) at every leaf, including the root,
- \( \mathcal{H}[i] \) to every edge,
- the morphism \( \Phi^{-1}: \mathcal{X}[i] \to \mathcal{H}[i] \) to each input vertex,
- the morphism \( \Phi: \mathcal{H}[i] \to \mathcal{X}[i] \) to the root vertex,
- to vertices of \( A(T) \) coming from an internal edge of \( T \), we assign \( \hat{\Delta} \)
- to internal vertices we assign \( \nu_2 \).
Lemma $\rho_T$ is equal to the branch denotation of the decoration assigning

- $\exists_\infty : \mathcal{H}^i \rightarrow \mathcal{H}^i$ to each input,
- $\pi : \mathcal{H}^i \rightarrow \mathcal{H}^i$ to the root,
- $\phi_\infty$ to each internal edge of $T$,
- $r_2 \circ \exp(\Xi - 1 \otimes \delta) \exp(1 \otimes \delta)$ to each internal vertex

We denote this by $\mathcal{T}$ in the tree.

Proof Fix a valid plane tree $T \in T_3$ with only internal vertices of valency 3. Then by the definition of branch denotation of (ainfmfz) p.6, if

$$T = \begin{array}{c}
\vdots \\
T_1 \\
\vdots \\
T_2
\end{array}$$

Then for some trees $T_1, T_2$ with induced decorations $D_1, D_2$ we have

$$\langle D \rangle_B = \exists_\infty \circ r_2 \circ (\langle D_1 \rangle_B \otimes \alpha \langle D_2 \rangle_B)$$

$$= \pi \exp(-\delta) r_2 (\langle D_1 \rangle_B \otimes \alpha \langle D_2 \rangle_B)$$

by (2.2) $$= \pi r_2 \exp(\Xi - 1 \otimes \delta) \exp(1 \otimes \delta) \circ (\exp(-\delta) \otimes \alpha \exp(-\delta)) \circ (\langle D_1 \rangle_B \otimes \langle D_2 \rangle_B)$$

$$= \pi r_2 \mathcal{T} (\exp(-\delta) \langle D_1 \rangle_B \otimes \exp(-\delta) \langle D_2 \rangle_B)$$
Now each $T_i$ is either just a leaf, or begins with an internal edge, and in the
former case

$$\exp(-\delta) \langle D_i \rangle_B = \exp(-\delta) \Phi^{-1} = \Phi_0$$

while in the latter case

$$\exp(-\delta) \langle D_i \rangle_B = \exp(-\delta) \hat{H} \langle D_i \rangle_B$$

$$= \exp(-\delta) \exp(\delta) \phi_\infty \exp(-\delta) \langle D_i \rangle_B$$

$$= \phi_\infty \exp(-\delta) \langle D_i \rangle_B$$

and so we see by induction on $n$ that the claim holds. □

**Def** Just to save on rewriting, set

$$\nabla := \exp(\Xi - 1 \odot \delta) \exp(1 \odot \delta).$$

See (13.1.1) for explicit formulas.
Aside on connections

In the context of Appendix B of the pushforward paper, and with the notation there, let $\nabla^o$ be a connection induced by a section $\mathcal{B}$, so that we get

$$\nabla^o : R \to R \otimes S[s] \cup S[t]/S$$

$$\nabla^o(r) = \sum_{j=1}^{\infty} \sum_{m \in \mathbb{N}} M_j \theta(r_m) t^M \otimes d\zeta_j.$$ 

It follows that, with $\delta = \sum_{j=1}^{\infty} t_i (dt_i)^{-1} (-)$ the Koszul operator on $R \otimes S[s] \cup S[t]/S$, we have

$$\delta \nabla^o(r) = \sum_{j=1}^{\infty} \sum_{m \in \mathbb{N}} M_j \theta(r_m) t^M$$

$$= \sum_{m \in \mathbb{N}} |M| \cdot \theta(r_m) t^M$$

So that $\delta \nabla^o$ is the $S$-linear operator acting on $R$ by sending $r \in R$ with its unique representation $\sum_m \theta(r_m) t^M$ to the element $\delta \nabla^o(r)$ with components

$$\delta \nabla^o(r)_M = |M| r_m, \quad M \in \mathbb{N}.$$ 

Example Let $R = S[[x]]$ and take $t = x^d$, so that $P = R/(t)$ is the free $S$-module on $1, x, \ldots, x^{d-1}$ and given (for $d > 1$) (we choose $\overline{\theta} : P \to R$, $\overline{\theta}(x^i) = x^i$)

$$r = \sum_{i=0}^{\infty} r^i x^i, \quad r^i \in S$$

$$= \sum_{m=0}^{\infty} \left( \sum_{i=0}^{d-1} r^{Md+i} x^i \right) t^M$$

call this $r_M$.
we see that this \( v_M = \sum_{i=0}^{d-1} (r^{M+d+i} x^i) \) gives the relevant components. Hence

\[
\delta \nabla^0 (r) = \delta \nabla^0 \left( \sum_M r_M t^M \right) = \sum_M M \cdot r_M t^M
\]

So in particular for \( M > 0 \) and \( 0 \leq i < d \)

\[
\delta \nabla^0 (x^{M+d+i}) = M \cdot x^{M+d+i}
\]
Passing to power series in $t$

By Appendix B of the pushforward paper we have a $k[[t^Y]]$-linear isomorphism for each pair $Y, X \in \text{ob}(A)$

$$\tau_{Y, X} : R/(t^Y) \otimes_R k[[t^Y]] \rightarrow \hat{R} \quad (19.1)$$

which is induced by the section of $\hat{R} \rightarrow R/(t^Y) \cong R/(t^Y)$ associated with our chosen connection, as explained above. Already in the perturbation step we have chosen homogeneous bases of $Y, X$, so that we may write

$$Y = \tilde{Y} \otimes_R R, \quad X = \tilde{X} \otimes_R R \quad (19.2)$$

for $\mathbb{Z}_2$-graded free $k$-modules $\tilde{X}, \tilde{Y}$, in which case

$$\text{Hom}_R(Y, X) \cong \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_R R \quad (19.3)$$

Combining (19.1) and (19.3) we have a $k[[t^Y]]$-linear isomorphism

$$\text{Hom}_R(Y, X) \otimes_R \hat{R} \cong \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_R R \otimes_R \hat{R}$$

$$\cong \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_R R/(t^Y) \otimes_R k[[t^Y]]$$

Hence finally (for $\mathcal{H}$ see (7.1), for $\mathcal{H}$ see (14.1)) we have isomorphisms of $Q$-bimodules

$$\mathcal{H} \xRightarrow{\sim} \bigoplus_{Y, X \in \text{ob}(A)} \wedge F_0 \otimes_R R/(t^Y) \otimes_R \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_R k[[t^Y]] \quad (19.4)$$

$$\mathcal{H} \xrightarrow{\sim} \bigoplus_{Y, X \in \text{ob}(A)} R/(t^Y) \otimes_R \text{Hom}_k(\tilde{Y}, \tilde{X})$$
Denote the RHS of (19.4) by $\hat{X}$ and $\hat{K}$ respectively. Let $n \geq 2$ and $T \in J_n$ have internal vertices of valency 3 only.

Def. $\hat{\rho}_T$ is equal to the branch denotation of the decoration assigning

- $\omega_T \circ w_\epsilon : \hat{K}[1] \to \hat{X}[1]$ to each input,
- $\omega_T \circ w_\epsilon^{-1} : \hat{K}[1] \to \hat{X}[1]$ to the root,
- $\omega_T \circ w_\epsilon^{-1}$ to each internal edge of $T$,
- $\omega_T \circ r_2 \circ \bigvee \circ (w_{\text{Th}} \circ w_\epsilon^{-1})$ to each internal vertex.

Lemma. There is a commutative diagram

\[
\begin{array}{ccc}
\hat{K}[1] & \xrightarrow{\hat{\rho}_T} & \hat{K}[1] \\
\uparrow{\omega_T} & & \uparrow{\omega_T} \\
\hat{K}[1] & \xrightarrow{\rho_T} & \hat{K}[1]
\end{array}
\]

Proof. We have e.g.

\[
\omega_T^{-1} \circ \hat{\rho}_T = \cdots = \rho_T \circ (w_{\text{Th}}^{-1}) \circ \omega_T^{-1} \cdot \square
\]
We deduce that the isomorphism of $\mathcal{Q}$-bimodules $\mathcal{N} \cong \hat{\mathcal{N}}$ identifies the $A_\infty$-algebra structure on $\mathcal{N}$ given by $\rho_n = \sum_T \rho_T$ with an $A_\infty$-structure on $\hat{\mathcal{N}}[1]$ given by $\hat{\rho}_n := \sum_T \hat{\rho}_T$.

It remains now to understand the operators

\[
\begin{align*}
\hat{\varrho}_\infty &:= \omega \circ \hat{\varrho}_\infty \circ \omega^{-1} \\
\hat{\pi} &:= \omega \circ \pi \circ \omega^{-1} \\
\hat{\phi}_\infty &:= \omega \circ \phi_\infty \circ \omega^{-1} \\
\omega \circ \nu \circ (\omega^{-1} \otimes \omega^{-1})
\end{align*}
\]

in the definition of $\hat{\rho}_T$. By (3.3.5) we have

\[
\hat{\phi}_\infty = \omega \circ \phi_\infty \circ \omega^{-1} = \sum_{m \geq 0} (-1)^m \omega \circ (\hat{H} \circ \text{Hom})^m \circ \omega \circ \omega^{-1}
\]

\[
= \sum_{m \geq 0} (-1)^m (\omega \circ H \circ \omega^{-1} \circ \omega \circ d_{\text{Hom}} \circ \omega^{-1})^m \circ \omega \circ \omega^{-1}
\]

\[
= \sum_{m \geq 0} (-1)^m (\hat{H} \circ \text{Hom})^m \circ \hat{H}
\]

where $\hat{H} := \omega \circ H \circ \omega^{-1}$, $\hat{\text{Hom}} := \omega \circ d_{\text{Hom}} \circ \omega^{-1}$. Similarly, with $\hat{\varrho} = \omega \circ \varrho \circ \omega^{-1}$,

\[
\hat{\varrho}_\infty = \sum_{m \geq 0} (-1)^m (\hat{H} \circ \text{Hom})^m \circ \hat{\varrho}
\]

Lemma. $\hat{\pi} : \hat{\mathcal{N}} \to \hat{\mathcal{N}}$ is given on components by the $k$-linear map

\[
\Lambda F_\delta \otimes_k R/(t^\gamma) \otimes_k \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[t^\gamma]]
\]

\[
\Lambda F_\delta \to 0 : \Lambda F_\delta \to k \text{ and } k[[t^\gamma]] \to k[[t^\gamma]] \xrightarrow{} k.
\]
Proof. This amounts to checking commutativity of (see (19.1) for $J$)

\[
\begin{array}{ccc}
R/(t^y x) \otimes_k k[[t^y x]] & \xrightarrow{\iota} & \hat{R} \\
\downarrow \pi & & \downarrow \pi \\
R/(t^y x) \otimes_k k & \xrightarrow{\text{can}} & \hat{R}/(t^y x)
\end{array}
\]  

(19.1)

where both vertical maps are quotients by the ideals generated by $t^y x$. But $\iota$ is $k[[t^y x]]$ linear, so $\pi \iota$ vanishes on the kernel of the other $\pi$, and thus induces a horizontal map on the bottom row, which is clearly the canonical one arising from $R \rightarrow \hat{R}$, since by construction $\iota$ restricted to $R/(t^y x) \otimes_k k$ is a section of $\pi$. \( \square \).

Lemma \( \hat{\iota} : \hat{\pi} \rightarrow \hat{\pi} \) is given on components by the $k$-linear map

\[
\begin{array}{c}
\wedge F_0 \otimes_k R/(t^y x) \otimes_k \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[t^y x]] \\
\uparrow \\
R/(t^y x) \otimes_k \text{Hom}_k(\tilde{Y}, \tilde{X})
\end{array}
\]  

(22.1)

which is induced by the inclusions $k = k \cdot 1 \hookrightarrow \wedge F_0$ and $k \hookrightarrow k[[t^y x]]$.

Proof. This amounts to commutativity of

\[
\begin{array}{ccc}
R/(t^y x) \otimes_k k[[t^y x]] & \xrightarrow{\iota} & \hat{R} \\
\uparrow \iota \circ \text{inc} & & \uparrow \hat{\iota} \\
R/(t^y x) \otimes_k k & \xrightarrow{\text{can}} & \hat{R}/(t^y x)
\end{array}
\]

which commutes by definition of $\iota$. \( \square \)
Lemma \( \hat{H} : \ast \rightarrow \ast \) is given on components by the \( k \)-linear operator

\[
\hat{H} \subset \wedge F_\mathcal{O} \otimes_k R(\ell^{\vee x}) \otimes_k \text{Hom}_k (X, X) \otimes_k k[[\ell^{\vee x}]] \quad (23.1)
\]

which is \( \hat{H} = [\hat{a}_k, \nabla_t]^{-1} \nabla_t \) where \( \hat{a}_k = \sum_i \hat{t}_i \omega_i^* \) and \( \nabla_t = \sum_i \nabla_{t_i} \omega_i \).

*Proof* Since \( \omega e \) is \( k[[\ell]] \)-linear, \( \omega_{e} \hat{a}_k \omega e^{-1} = \hat{a}_k \) as described. Moreover by definition of the connection \( \nabla^0 \) on \( \hat{H} \), the diagram

\[
\begin{array}{ccc}
\hat{R} & \xrightarrow{\nabla^0} & \hat{R} \otimes k[[\ell]] \\
\end{array}
\]

comutes, where we choose a \( k \)-basis for \( R(\ell^{\vee x}) \) and use it to extend the canonical connection \( \nabla^0 \) on \( k[[\ell^{\vee x}]] \). It follows that the operator \( \nabla \) on \( \hat{R} \otimes k \wedge (F_\mathcal{O}) \) in the definition of \( H \) on \( \mathcal{H} \) is identified under

\[
\hat{R} \otimes k \wedge F_\mathcal{O} \cong R(\ell^{\vee x}) \otimes_k k[[\ell^{\vee x}]] \otimes_k \wedge F_\mathcal{O}
\]

with the bottom row of \( (23.2) \). The operator \( \nabla \) in \( H \) is also extended using the chosen homogeneous basis for \( X, Y \). Thus, if we let \( \nabla_t \) denote the connection extended using these choices to the space in \((23.1)\), we have \( \omega e \nabla \omega e^{-1} = \nabla_t \) and hence

\[
\hat{H} = \omega e \hat{H} \omega e^{-1} = \omega e \left[ [\hat{a}_k, \nabla]^{-1} \nabla \omega e^{-1} \right] = \omega e \left[ [\hat{a}_k, \nabla]^{-1} \omega e \nabla \omega e^{-1} \right] = \left[ [\hat{a}_k, \nabla_t]^{-1} \right] \nabla_t .
\]

\( \square \)
Lemma Given $w \in \wedge F_0$ of $O$-degree $|w| \in \mathbb{Z}$ and a power series $f \in k[[t^{\frac{1}{K}}]]$ written as $f = \sum_{\alpha \in N^\times} f_\alpha t^\alpha$ with $f_\alpha \in k$, we have (assuming $|w| > 0$)

$$\left[ d_k, \nabla_t \right]^{-1}(\omega \otimes f) = \omega \otimes \sum_{\alpha \in N^\times} \frac{1}{|w| + |\alpha|} f_\alpha t^\alpha \quad (24.1)$$

Proof By Lemma 8.7 of the pushforward paper

$$\left[ d_k, \nabla_t \right](\omega \otimes \sum_{\alpha} \frac{1}{|w| + |\alpha|} f_\alpha t^\alpha)$$

$$= \omega \otimes \left[ (|w| + d_k \nabla_t)^{\alpha} \right] \left( \sum_{\alpha} \frac{1}{|w| + |\alpha|} f_\alpha t^\alpha \right)$$

$$= \omega \otimes \left[ \sum_{\alpha} \frac{|w| + |\alpha|}{|w| + |\alpha|} f_\alpha t^\alpha \right] + d_k \left( \sum_{\alpha} \frac{1}{|w| + |\alpha|} f_\alpha \alpha \cdot t^{\alpha - \beta} \otimes \partial_j \right)$$

$$= \omega \otimes \sum_{\alpha} \frac{|w| + |\alpha|}{|w| + |\alpha|} f_\alpha t^\alpha$$

$$= \omega \otimes \sum_{\alpha} f_\alpha t^\alpha. \Box$$

Def Let $[\lambda_i, -]^\wedge := \omega \otimes \left[ \lambda_i, - \right] \omega^{-1}$ denote the $Q$-bilinear operator on $\mathcal{H}$.

Def Set $\hat{\Xi}_i := \Theta_i \wedge [\lambda_i, -]$, a $k$-linear $Q$-bilinear operator on $\mathcal{H}[1] \otimes_k \mathcal{H}[1]$, and write $\hat{\Xi} = \sum \hat{\Xi}_i \hat{\Xi}_i$ and $\hat{\nabla} = \exp((\hat{\Xi} - 1 \otimes \hat{S})) \exp(1 \otimes \hat{S})$.

Def Let $\hat{\lambda}_i : \mathcal{H}[1] \otimes_k \mathcal{H}[1] \to \mathcal{H}[1]$ denote the $Q$-bilinear operator $\omega \otimes r_2 (\omega^{-1} \otimes \omega^{-1})$.

Lemma $\omega \otimes r_2 \hat{\lambda}_i (\omega^{-1} \otimes \omega^{-1})$ is equal to $\hat{\lambda}_i \hat{\nabla}$.

Proof Clear. \( \Box \)

where $\hat{S} = \sum \Theta_i \Theta_i^* \hat{\lambda}_i$, and $\hat{\lambda}_i$ is defined as an overleaf.
The ingredients remaining to be understood are

- \( \hat{\tau}_2 = w_{\sigma_2} r_2 (w_{\sigma_1} \Theta w_{\tau_1}) \).
- \( \hat{d}_{\text{Hom}} = w_{\sigma d_{\text{Hom}} w_{\tau_1}} \).
- \( [\lambda_i, -J] = w_{\sigma [\lambda_i, -J] w_{\tau_1}} \).

These all hinge on the following considerations:

**Def.** Let \( \tau^{Y\times X} \) denote the \( k \)-linear map

\[
\begin{align*}
R/(t^{Y\times X}) \otimes_k R/(t^{X\times X}) &\xrightarrow{\partial \otimes 2} \hat{R} \otimes_k \hat{R} \xrightarrow{\text{mult.}} \hat{R} \xrightarrow{\tau^{-1}} R/(t^{Y\times X}) \otimes_k k[t^{Y\times X}],
\end{align*}
\]

which we denote \( \tau \) if it will not cause confusion. Since \( R/(t^{Y\times X}) \) is a finite free \( k \)-module we may identify \( \tau \) as a tensor

\[
\gamma \in (R/(t^{Y\times X}))^* \otimes (R/(t^{X\times X}))^* \otimes (R/(t^{Y\times X})) \otimes k[t^{Y\times X}],
\]

and with respect to a \( k \)-basis \( z_1, \ldots, z_\mu \) of \( R/(t^{Y\times X}) \) we may write

\[
\gamma = \sum_{ij, k} \sum_{\alpha \in \mathbb{N}} \gamma^{ij \alpha} z_i^* \otimes z_j^* \otimes z_k \otimes t^\alpha
\]

where \( \gamma^{ij \alpha} \in k \) is the coefficient of \( t^\alpha \) in \( \tau^{-1}(\delta(z_i) \delta(z_j)) \). Note that \( \gamma^{ij \alpha} = \gamma^{ji \alpha} \), and by definition

\[
\delta(z_i) \delta(z_j) = \sum_R \sum_{\alpha \in \mathbb{N}} \gamma^{ij \alpha} \delta(z_k) t^\alpha
\]
Lemma Let \( r \in \mathbb{R} \) and denote by \( r^\# \) the \( k[\mathbb{Z}_{\geq 0}] \)-linear operator

\[
\mathbb{R}/(\mathbb{Z}_{\geq 0}) \otimes_k k[\mathbb{Z}_{\geq 0}] \xrightarrow{r} \mathbb{R} \xrightarrow{r^{-1}} \mathbb{R} / (\mathbb{Z}_{\geq 0}) \otimes_k k[\mathbb{Z}_{\geq 0}]. \tag{26.1}
\]

Write \( r = \sum_{j=1}^{M} \sum_{\alpha \in \mathbb{N}^{n}} r_{\alpha} \mathcal{B}(\mathbb{Z}_{k}) t^{\alpha} \) for \( r_{\alpha} \in \mathbb{R} \), then

\[
r^\#(z_{i} \otimes 1) = \sum_{j=1}^{M} Ze \otimes \sum_{d \in \mathbb{N}^{n}} \left[ \sum_{\alpha + \beta = d} \sum_{k=1}^{M} r_{\alpha} \sigma_{d}^{k} \right] t^{d}.
\]

Proof This is a simple calculation

\[
J_{r^\#}(z_{i} \otimes 1) = r \cdot \mathcal{B}(z_{i})
\]

\[
= \sum_{j=1}^{M} \sum_{\alpha \in \mathbb{N}^{n}} r_{\alpha} \mathcal{B}(\mathbb{Z}_{k}) t^{\alpha} \cdot \mathcal{B}(z_{i})
\]

\[
= \sum_{j, \alpha, \beta} r_{\alpha} \mathcal{B}(\mathbb{Z}_{k}) t^{\alpha + \beta}
\]

\[
= \sum_{j, \alpha, \beta} r_{\alpha} \sigma_{d}^{k} \mathcal{B}(\mathbb{Z}_{k}) t^{d}
\]

\[
= \sum_{d} \sum_{j, \alpha, \beta} \sum_{\alpha + \beta = d} r_{\alpha} \sigma_{d}^{k} \mathcal{B}(\mathbb{Z}_{k}) t^{d}.
\]

Thus as a tensor in \( (\mathbb{R}/(\mathbb{Z}_{\geq 0}))^{*} \otimes \mathbb{R}/(\mathbb{Z}_{\geq 0}) \otimes k[\mathbb{Z}_{\geq 0}] \) (ignoring the power series on the left due to linearity)

\[
(r^\#)_{\alpha \beta}^{d} = \sum_{\alpha + \beta = d} \sum_{k=1}^{M} r_{\alpha} \sigma_{d}^{k} \] \( \tag{26.2} \)

\[
\text{Aside For } d \in \mathbb{N}^{n} \text{ we write } r_{d}^\#: \mathbb{R}/(\mathbb{Z}) \rightarrow \mathbb{R}/(\mathbb{Z}) \text{ for the } k\text{-linear map}
\]

\[
r_{d}^\#(z_{i}) = \sum_{\alpha + \beta = d} \sum_{k=1}^{M} r_{\alpha} \sigma_{d}^{k} \cdot Ze \] \( \tag{26.3} \)

so that \( r^\#(z_{i}) = \sum_{d \in \mathbb{N}^{n}} r_{d}^\#(z_{i}) t^{d} \).
Let $X$ be an odd $R$-linear operator on $\text{Hom}_R(Y, X)$ (e.g. $a\text{Hom}$ or $[X, -]$). We define the $k[[t^{\frac{1}{2}}]]$-linear operator $\hat{X} := wze X wze^{-1}$, i.e. the composite

$$
R/(t^{\frac{1}{2}}) \otimes_k \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[t^{\frac{1}{2}}]]
$$

\[ \downarrow wze \]

$$
\text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k \hat{R}
$$

\[ \downarrow \chi \]

$$
\text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k \hat{R}
$$

\[ \downarrow wze \]

$$
R/(t^{\frac{1}{2}}) \otimes_k \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes_k k[[t^{\frac{1}{2}}]]
$$

Let us write $\{a_i^{(u)}\}_{u=1}^r$ for our $k$-basis of $\text{Hom}_k(\tilde{Y}, \tilde{X})$ based on the chosen $k$-basis of $X, Y$, then we have $X(a_i^{(u)}) = \sum_v X_{vu} a_v$, with $X_{vu} \in R$. Then

$$
\hat{X}(Z_i \otimes a_u \otimes 1) = wze \left( X(a_u \otimes \delta(Z_i)) \right)
$$

$$
= wze \left( X(a_u) \cdot \delta(Z_i) \right)
$$

$$
= \sum_v wze (a_v \otimes X_{vu} \delta(Z_i))
$$

$$
= \sum_v a_v \otimes wze (X_{vu} \cdot \delta(Z_i))
$$

$$
= \sum_v a_v \otimes X_{vu}(Z_i)
$$

Or, using (26.2), for $1 \leq l \leq u$, $s \in \mathbb{N}^n$,

$$
\hat{X}_{vld}^{iu} = (X_{vu})^{i}_{ld} = \sum_{\alpha+\beta=s} \sum_{k=1}^{\mu} (X_{vu})_{k\alpha} \delta_{l\beta}^{ki}
$$

So as a tensor in $(R/(t^{\frac{1}{2}}))^* \otimes \text{Hom}_k(\tilde{Y}, \tilde{X})^* \otimes (R/(t^{\frac{1}{2}})) \otimes \text{Hom}_k(\tilde{Y}, \tilde{X}) \otimes k[[t^{\frac{1}{2}}]]$, we have

$$
\hat{X} = \sum_{i, u, l, v, s} \hat{X}_{vld}^{iu} Z_i^{*} \otimes a_u^{*} \otimes e_l^{*} \otimes a_v \otimes t^s
$$
By definition $\hat{r}_2$ is the composite

$$
\hat{\mathcal{H}}[i] \otimes_k \hat{\mathcal{H}}[i] \xrightarrow{\omega_{\mathcal{E}}^{-1} \otimes \omega_{\mathcal{E}}^{-1}} \hat{\mathcal{H}}[i] \otimes_k \hat{\mathcal{H}}[i] \xrightarrow{r_2} \hat{\mathcal{H}}[i] \xrightarrow{\omega_{\mathcal{E}}} \hat{\mathcal{H}}[i]
$$

On components, say for $X, Y, Z \in \text{Ob}(\Lambda)$, this is

\[
\left( \Lambda F_0 \otimes_k R/(t^{z,Y}) \otimes_k \text{Hom}_R(\tilde{Z}, \tilde{Y}) \otimes_k k[[t^{z,Y}]] \right) [i] \\
\left( \Lambda F_0 \otimes_k R/(t^{x,X}) \otimes_k \text{Hom}_R(\tilde{Y}, \tilde{X}) \otimes_k k[[t^{x,X}]] \right) [i]
\]

\[
\xrightarrow{\omega_{\mathcal{E}}^{-1} \otimes \omega_{\mathcal{E}}^{-1}} \\
\left( \Lambda F_0 \otimes_k \text{Hom}_R(Z, Y) \otimes_k \hat{R} \right) [i] \\
\left( \Lambda F_0 \otimes_k \text{Hom}_R(Y, X) \otimes_k \hat{R} \right) [i]
\]

\[
\xrightarrow{r_2} \\
\left( \Lambda F_0 \otimes_k \text{Hom}_R(Z, X) \otimes_k \hat{R} \right) [i]
\]

\[
\xrightarrow{\omega_{\mathcal{E}}} \\
\left( \Lambda F_0 \otimes_k R/(t^{z,X}) \otimes_k \text{Hom}_R(\tilde{Z}, \tilde{X}) \otimes_k k[[t^{z,X}]] \right) [i]
\]

Since $t^{z,X} = t$ is independent of $Y, X$, the map (28.1) is actually $k[[t]]$-linear, and so it can be described easily using the tensor $\varphi$ from earlier, as follows. Given $\omega, \omega' \in \Lambda F_0$ and $\alpha \in \text{Hom}_R(\tilde{Z}, \tilde{Y})$, $\beta \in \text{Hom}_R(\tilde{Y}, \tilde{X})$, and $1 \leq i, j \leq \mu$

\[
\hat{r}_2 \left( [\omega \otimes z_i \otimes \alpha] \otimes [\omega' \otimes z_j \otimes \beta] \right) = \omega_{\mathcal{E}} r_2 \left( [\omega \otimes \varphi(z_i) \otimes \alpha] \otimes [\omega' \otimes \varphi(z_j) \otimes \beta] \right)
\]
\[
\begin{align*}
\omega \varepsilon (-1)^{\omega \delta \omega \delta + \omega \delta + 1} \left[ \omega \circ \delta (z) \otimes \beta \right] \cdot \left[ \omega \circ \delta (z) \otimes \alpha \right] \\
&= (-1)^{\omega \delta \omega \delta + \omega \delta + 1} \left( -1 \right)^{|\omega\|\beta|} \omega \varepsilon \left( \omega \omega \circ \delta (z) \otimes \beta \alpha \right) \\
&= (-1)^{|\omega\|\omega\| + |\omega\|\beta| + |\omega\| |\alpha\| |\omega\| + |\alpha\| |\beta\| + 1 + |\omega\| |\beta\| + |\omega\| |\beta\| + |\omega\| |\beta\| + |\omega\| |\beta| + |\alpha| + 1}
\end{align*}
\]

The sign here is

\[
|\omega\|\omega\| + |\omega\|\beta| + |\omega\| + |\alpha\| |\omega\| + |\alpha| |\beta| + 1 + |\omega\| |\beta| + |\omega\| |\beta| + |\omega\| |\beta| + |\alpha| + 1 = (29.2) + |\alpha| |\omega| + 1
\]

Let us state this as a lemma:

**Lemma** We have

\[
\hat{r}_2 \left( \left[ \omega \circ \delta (z) \otimes \alpha \right] \otimes \left[ \omega \circ \delta (z) \otimes \beta \right] \right) = (29.3)
\]

\[
(-1)^{|\alpha\| |\omega| + 1} \sum_{k} \sum_{\delta} \gamma^{ij}_{\delta k} r_2 (\omega, \omega') \otimes z_k \otimes r_2 (\alpha, \beta) \otimes t^j
\]

where \( r_2 : (\Lambda F_0)[i] \otimes (\Lambda F_0)[i] \rightarrow (\Lambda F_0)[i] \) and similarly on Hom spaces has the expected meaning.
Formally, we have

\[
r_2 ( r_2 (x, y), z) = r_2 ( (-1)^{x y + y + 1} y, z, x )
= (-1)^{x y + y + 1} y \cdot (y, z) (y, x)
= (-1)^{x y + y + 1} x y + 1 + (y + y) + 1 \cdot (y, z) (y, x)
= (-1)^{x y + y + 1} x y + 1 + (y + y) + 1 + (y + y) + 1 \cdot (y, z) (y, x)
= (-1)^{x y + y + 1} x y + 1 + (y + y) + 1 + (y + y) + 1 \cdot (y, z) (y, x)
\]

so that

\[
r_2 ( r_2 (x, y), z) = (-1)^{x y + y + 1} r_2 ( x, r_2 (y, z)). \quad (30.1)
\]

What the hell! Who wants to deal with that.

\section*{Conclusion} We switch to using the usual product \( \mu \), rather than \( r_2 \).

For this we need to revisit the algebra \( Q \). We can put a different \( Q \)-bimodule structure on \( \mathcal{H} \) to the one described in p.\( \text{ainfca(2).} \) where \( E_a \in Q \) act on the left as the projector onto \( \mathcal{H}_a \) (i.e. how \( E_a \) used to act on the right) and \( E_b \) acts on the right as the projector onto \( b \mathcal{H} \) (i.e. how it used to act on the left). When we write \( \otimes_Q \) we mean \( \otimes_Q \) using this bimodule structure on the tensor factors. Hence

\[
\mathcal{H} \otimes_Q^k = \bigoplus_{a_0, \ldots, a_k} \Hom_{\mathcal{H}} (a_k, a_{k-1}) \otimes \Hom_{\mathcal{H}} (a_{k-1}, a_{k-2}) \otimes \cdots \otimes \Hom_{\mathcal{H}} (a_1, a_0). \quad (30.2)
\]
Returning now to the broader point

We started with the DG-category $\mathcal{C} = \text{mf}(k[x],M)$, which we extended to the DG-category $\hat{\mathcal{C}} = \mathcal{C} \otimes_k R$ (see p. 4.73) and then to $\hat{\mathcal{A}} = \Lambda F_0 \otimes_k \hat{\mathcal{C}}$, (see p. 5). On this we defined a strict homotopy retract (p. 6) and denoted by $\hat{\mathcal{B}}$ the associated minimal model with

$$\hat{\mathcal{B}}(y, x) = R/(t) \otimes_k \text{Hom}_R(y, x)$$  \hspace{1cm} (31.1)

and higher products $\{ p_k \}_{k \geq 1}$. We write $\hat{\mathcal{C}}$ for the total $A_\infty$-algebra of $\hat{\mathcal{A}}$ and $\hat{\mathcal{K}}$ for the total $A_\infty$-algebra of $\hat{\mathcal{B}}$, and then in (19.4) we found isomorphic models of $\hat{\mathcal{C}}, \hat{\mathcal{K}}$ resp.

$$\hat{\mathcal{C}} = \bigoplus_{y, x \in \text{ob}(\mathcal{A})} \Lambda F_0 \otimes_k R/(t) \otimes_k \text{Hom}_k(\bar{y}, \bar{x}) \otimes_k k[[t]]$$

$$\hat{\mathcal{K}} = \bigoplus_{y, x \in \text{ob}(\mathcal{A})} R/(t) \otimes_k \text{Hom}_k(\bar{y}, \bar{x})$$  \hspace{1cm} (31.2)

We defined $\hat{\rho}_T$ to be the higher product on $\hat{\mathcal{K}}$ corresponding to $\rho_T$ on $\mathcal{K}$ under the iso between $\mathcal{K}$ and $\hat{\mathcal{K}}$, and p. (21) - (30) have been analysing the constituents of this higher product $\hat{\rho}_T$. Now, following (ainfm2) p. (24) and (ainfm9) p. (20) we would like to write the value of $\hat{\rho}_T$ (for $T \in \mathcal{K}$)

$$\hat{\rho}_T : \hat{\mathcal{K}}[1]^{\otimes k} \to \hat{\mathcal{K}}[1]$$  \hspace{1cm} (31.3)

on a tensor $y_1 \circ \cdots \circ y_k$ in terms of the evaluation of a tree involving no Koszul signs and with $M_2$ (meaning ordinary multiplication in $\Lambda F_0$ and $\hat{\mathcal{C}}$) replacing $r_2$ at all trivalent vertices.
Let $\hat{\mu}_2$ denote the $Q$-bilinear map $\hat{\mathcal{H}} \otimes_\alpha \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ given by

$$
\hat{\mathcal{H}} \otimes_\alpha \hat{\mathcal{H}} \xrightarrow{\mu_2} \hat{\mathcal{H}} \otimes_\alpha \hat{\mathcal{H}} \xrightarrow{\mu_2} \hat{\mathcal{H}} \xrightarrow{\mu_2} \hat{\mathcal{H}}
$$

where $\mu_2$ is the product. By p. 29 we have, for $\omega, \omega' \in \mathcal{A}_0$ and $1 \leq i, j \leq \mu$ and $\alpha, \beta$ composable morphisms,

$$
\hat{\mu}_2 \left( [\omega \otimes z \otimes \alpha] \otimes [\omega' \otimes z' \otimes \beta] \right)
\xrightarrow{(-1)^{d(\omega) + d(\omega')}} \sum_k \sum_d \gamma^{ij}_k \omega \otimes \omega' \otimes z \otimes \alpha \otimes \beta \otimes t^d
$$

Let $k \geq 2$ and $T \in \mathcal{T}_k$ have only internal vertices of valency 3. In the following we give $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}$ the modified $Q$-bimodule structure of (30.2), when interpreting diagrams.

Consider the following decoration of $A(T)$ by $Q$-bimodules

- $\hat{\delta}_\infty$ to each internal edge of $T$ and $\hat{\delta}_\infty$ to each leaf of $T$ (with the (30.2) bimodule structure)
- $\hat{\gamma}_\infty : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ to each input,
- $\hat{\gamma}_\infty : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ to the root,
- $\hat{\phi}_\infty : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ to each internal edge of $T$,
- $\hat{\mu}_2 \hat{\mathcal{V}}'$ to each internal vertex, where $\hat{\mathcal{V}}' := \exp(\hat{\delta}' - \hat{\delta} \otimes 1) \exp(\hat{\delta} \otimes 1)$
Here we take

\[ \text{Def}^\ast \text{ Set } \hat{\mathcal{E}}' = [\lambda_i : -] \otimes \mathcal{O}_i^*, \text{ a } k\text{-linear } \mathbb{Q}\text{-bilinear operator on } \hat{\mathcal{E}} \otimes_{\mathcal{A}} \hat{\mathcal{E}} \]

and write \( \hat{\mathcal{E}}' = \sum_i \hat{\mathcal{E}}_i' \).

\[ \text{Def}^\ast \text{ set } \hat{\delta}' = \hat{\mathcal{E}}_i \mathcal{O}_i^* \otimes 1 \text{ and write } \hat{\delta}' = \sum_i \hat{\delta}_i', \text{ operators on } \hat{\mathcal{E}} \otimes_{\mathcal{A}} \hat{\mathcal{E}}. \]

\[ \text{Def}^\ast \text{ Let } \text{eval}_T \text{ denote the } \mathbb{Q}\text{-bilinear map } \hat{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{E}^k \rightarrow \hat{\mathcal{E}} \text{ obtained by evaluating the above decorated tree, without Koszul signs. That is, we simply feed in input at the top of the tree and evaluate each operator in succession. More precisely, we take the branch denotation in the category of ungraded } \mathbb{Q}\text{-bimodules.} \]

Let \( \hat{T} \) be the tree obtained by mirroring \( T \), as in p.14 "afm2."
\[ W(\tilde{\chi}, \chi) = \sum_i W(\chi_i^* \otimes [\chi_i]^{-1})(\chi_i \chi) \]

\[ = \sum_i (-1)^{\tilde{\chi}_i} W(\chi_i^* \otimes [\chi_i]^{-1})(\chi_i \chi) \]

\[ = \sum_i (-1)^{\tilde{\chi}_i} \tilde{\chi}_i + (\tilde{\chi}_i + 1) + \tilde{\chi}_i + 1 \quad [\chi_i]^{-1}(\chi_i) \otimes \chi_i^* \]

\[ = \sum_i (-1)^{\tilde{\chi}_i} \tilde{\chi}_i \tilde{\chi}_i + \tilde{\chi}_i + 1 \quad [\chi_i]^{-1}(\chi_i) \otimes \chi_i^* \]

\[ = \sum_i (-1)^{\tilde{\chi}_i} \tilde{\chi}_i + \tilde{\chi}_i + 1 \quad (\chi_i \otimes \chi_i) \]

\[ = \hat{\otimes} W(\chi_1 \chi_2), \]

and

\[ W(1 \otimes \delta)(\chi_1 \chi_2) = W(\chi_1 \otimes \delta(\chi_2)) \]

\[ = (-1)^{\tilde{\chi}_1} \tilde{\chi}_1 \tilde{\chi}_1 + \tilde{\chi}_1 + 1 \quad \delta(\chi_2) \otimes \chi_1 \]

\[ = (\delta \otimes 1) W(\chi_1, \chi_2) \]

Hence

\[ \hat{r}_2 \hat{\chi} = \hat{r}_2 \exp(\hat{\otimes} - 1 \otimes \delta) \exp(1 \otimes \hat{\delta}) \]

\[ = \hat{\mu}_2 W \exp(\hat{\otimes} - 1 \otimes \hat{\delta}) \exp(1 \otimes \hat{\delta}) \]

\[ = \hat{\mu}_2 \exp(\hat{\otimes}' - \hat{\delta} \otimes 1) \exp(\hat{\delta} \otimes 1) W \]
\[ \hat{V}' := \exp(\hat{\omega}' - \hat{\delta} \otimes 1) \exp(\hat{\delta} \otimes 1) \]

Let us now consider an example:

\[ V' = \exp(\hat{\omega}' - \hat{\delta} \otimes 1) \exp(\hat{\delta} \otimes 1) \]

We have (using that with respect to the \( \sim \)-degree, \( \hat{r}_2, \hat{\phi}_\infty \) are odd, \( \sim \), \( \hat{\delta}_\infty \), \( \hat{\pi} \) are even,

\[
\rho_T (\chi_1, \chi_2, \chi_3) = \hat{\pi} \hat{r}_2 \hat{V}' \left( \hat{\omega}_\infty (\chi_1) \otimes \hat{\phi}_\infty \hat{r}_2 \hat{V}' (\hat{\omega}_\infty (\chi_2) \otimes \hat{\omega}_\infty (\chi_3)) \right) \\
\text{tilde degree } \hat{\pi} + \hat{\pi} + \hat{\pi}
\]

\[
= \hat{\pi} \hat{r}_2 \hat{V}' \left( \hat{\phi}_\infty \hat{r}_2 \hat{V}' (\hat{\omega}_\infty (\chi_2) \otimes \hat{\omega}_\infty (\chi_3)) \otimes \hat{\omega}_\infty (\chi_1) \right) \\
\text{tilde degree } \hat{\pi} + \hat{\pi} + \hat{\pi} + 1
\]

\[
= (-1) \hat{\omega}_\infty (\hat{\chi}_2 + \hat{\chi}_3) + \hat{\pi} + \hat{\pi} + 1 \cdot (-1) \hat{\omega}_\infty (\hat{\chi}_2 + \hat{\chi}_3 + 1) \hat{\pi} \hat{r}_2 \hat{V}' \left( \hat{\phi}_\infty \hat{r}_2 \hat{V}' \left( \hat{\omega}_\infty (\chi_2) \otimes \hat{\omega}_\infty (\chi_3) \right) \otimes \hat{\omega}_\infty (\chi_1) \right) \\
\text{eval}_T (\chi_3, \chi_2, \chi_1)
\]

The upshot is that the signs arise entirely from the \( W \), and the tilde degree arriving at an input to a \( \hat{r}_2 \hat{V} \) is the sum of all \( \hat{\chi}_j \) feeding into that input. We deduce easily from this that in general
\[ \hat{\rho}_T(\chi_1, \ldots, \chi_k) = (-1)^{\sum_{i<j} \tilde{\chi}_i \tilde{\chi}_j + \sum_{i=1}^{k} \tilde{\chi}_i P_i} + k + 1 + \epsilon^2(\tau) \]

eval_T(\chi_k, \ldots, \chi_1)

where \(P_i\) is the number of times the path from the \(i\)th leaf in \(T\) (counting from the left) enters a trivalent vertex as the right-hand branch (i.e., \(\gamma\)) on its way to the root, and \(k + 1\) is the number of internal vertices in \(T\). \(\Box\)

The next task is to describe the Feynman rules governing the expansion of \(\text{eval}_T\), but we leave this for a later chapter.

Note (14/11/2019) We corrected a serious error stemming from a mistaken calculation of \([S_i, 1 \otimes \delta_j]\) (see p. 12), which we used to think was zero.
Transfer of Clifford operators

We take as our starting point the homotopy equivalence of (5.2) over \( k \)

\[
(\mathcal{F}_0 \otimes R \text{Hom}(Y, X) \otimes \hat{R}, d_{\text{Hom}}) \xrightarrow{\Phi} \left( R/(t^Y, t^X) \otimes R \text{Hom}(Y, X), d_{\text{Hom}} \right)
\]

which is derived from (4.1). Recall \( \Phi = \pi \circ \exp(-\delta) \) and \( \Phi^{-1} = \exp(\delta) \circ \pi \). Set \( I = (t^Y, t^X) \) and write \( t^Y, t^X = (t_1, \ldots, t_n) \). In the cut operation paper [Cut] we studied the transferred operators \( \Phi \Theta_i \Phi^{-1} \) and \( \Phi \Theta_i^* \Phi^{-1} \), and here we recapitulate that discussion. This is necessary because in some relevant parts of [Cut] it is assumed \( t_i = \partial x_i W \) and we are not imposing such a hypothesis here. It is worth inserting a copy of (4.1) here (making the identification \( R/I \cong \hat{R}/I \hat{R} \)):

\[
\Theta_i, \Theta_i^* \subseteq (\mathcal{F}_0 \otimes R \text{Hom}(Y, X) \otimes \hat{R}, d_{\text{Hom}})
\]

\[
\exp(-\delta) \Theta_i \exp(\delta) \subseteq (K \otimes R \text{Hom}(Y, X) \otimes \hat{R}, d_K + d_{\text{Hom}})
\]

\[
\Theta_i \Theta_i^{-1} = \pi \exp(-\delta) \Theta_i \exp(\delta) \omega_\infty \subseteq \left( R/I \otimes R \text{Hom}(Y, X), d_{\text{Hom}} \right)
\]

Now [Cut, 54.2] handles the transfer from the first line to the second, and [Cut, p. 38] (ultimately the push-forward paper) handles the transfer from the second to the third.
Now [Cut, Lemma 4.17] gives \( \exp(-\delta) \Omega_i^* \exp(\delta) = \Omega_i^* \) and [Cut, Theorem 4.28] says
\[
\exp(-\delta) \Omega_i \exp(\delta) = \Omega_i - \sum_p \sum_{i_0 \cdots i_p} \frac{1}{(p+1)!} \left[ \lambda_{i_p} \cdots \lambda_{i_0} \lambda_i \right] \cdots \Omega_{i_p}^* \cdots \Omega_i^* \\
= \Omega_i - \lambda_i - \frac{1}{2} \sum_j [\lambda_j, \lambda_i] \Omega_j^* + \cdots \tag{38.1}
\]
Next we copy elements of the proof of [Cut, Prop 4.35]. This is the first time [Cut] is assuming \( \tilde{t} = \tilde{x} : W \) but we are not, so we must now take care. The first thing to note is that [Cut, Lemma 4.36] still holds, that is, there is a \( k \)-linear homotopy
\[
\pi \Omega_{j_1}^* \cdots \Omega_{j_p}^* \mathcal{E} \approx A_{j_1} \cdots A_{j_p} \tag{38.2}
\]
in reference to the second and third rows of (37.1). Here as usual \( A_{j} \) means
\[
A_{j} = [d_{\text{Hom}}, \partial_{t_j}] \subset R/I \otimes R \text{Hom}(Y, X)
\]
as defined in [Pushforward, 91] with respect to \( k[t] \to R \). The proof of (38.2) in the current generality is the same as in [Cut], as indeed there we cite the pushforward paper which is working in the present generality. That is, writing \( \tilde{t} \) for \([d_t, \nabla] \),
\[
\mathcal{E} \approx \sum_{m \geq 0} (-1)^m (H d_{\text{Hom}})^m \mathcal{E} \\
= \sum_{m \geq 0} (-1)^m \left( \mathcal{J}^{-1} \nabla d_{\text{Hom}} \right) \cdots \left( \mathcal{J}^{-1} \nabla d_{\text{Hom}} \right) \mathcal{E}
\]
Now \( \nabla \mathcal{J} = \mathcal{J} \nabla \) and so on input of nonzero \( \Omega \)-degree, \( \nabla \mathcal{J}^{-1} = \mathcal{J}^{-1} \nabla \), hence
\[\delta_\infty = \sum_{m \geq 0} (-1)^m J^{-1}[\nabla, d_{\text{Hom}}] J^{-1}[\nabla, d_{\text{Hom}}] \ldots J^{-1}[\nabla, d_{\text{Hom}}] \delta\]

\[= \sum_{m \geq 0} (-1)^m \left\{ J^{-1}[\nabla, d_{\text{Hom}}] \right\}^m \delta. \quad (39.1)\]

(see also p. 9 of ainfm12) 

By the same calculation, incidentally,

\[\phi_\infty = \sum_{m \geq 0} (-1)^m (H d_{\text{Hom}})^m H\]

\[= \sum_{m \geq 0} (-1)^m J^{-1} \nabla d_{\text{Hom}} \ldots J^{-1} \nabla d_{\text{Hom}} J^{-1} \nabla\]

\[= \sum_{m \geq 0} (-1)^m J^{-1}[\nabla, d_{\text{Hom}}] \ldots J^{-1}[\nabla, d_{\text{Hom}}] J^{-1} \nabla\]

\[= \sum_{m \geq 0} (-1)^m \left\{ J^{-1}[\nabla, d_{\text{Hom}}] \right\}^m J^{-1} \nabla \quad (\text{see also p. 17 of ainfm12})\]

Note that as an operator on \(\wedge F \otimes R/I \otimes k[\xi]\), we have modulo (\(\xi^2\) (see (24.1))

\[J^{-1}(\omega \otimes z \otimes f) = \frac{1}{|\omega|} \omega \otimes z \otimes f.\]

Since \([\nabla, d_{\text{Hom}}]\) is \(k[\xi]\)-linear it follows that

\[\pi J_{j_1}^* \ldots J_{j_p}^* \delta_\infty = \sum_{m \geq 0} (-1)^m \pi J_{j_1}^* \ldots J_{j_p}^* \left\{ J^{-1}[\nabla, d_{\text{Hom}}] \right\}^m \delta\]

\[= \sum_{m \geq 0} (-1)^m \frac{1}{m!} \pi J_{j_1}^* \ldots J_{j_p}^* \left\{ \sum_{k=1}^n \Theta_k \left[ \delta_{tk}, d_{\text{Hom}} \right] \right\}^m \delta\]

\[= \sum_{m \geq 0} (-1)^m \frac{1}{m!} \pi J_{j_1}^* \ldots J_{j_p}^* \left\{ \sum_{k=1}^n \Theta_k (-A_{tk}) \right\}^m \delta\]

\[= (-1)^{p+1} \pi J_{j_1}^* \ldots J_{j_p}^* \left\{ \sum_{\beta \in S_p} \left[ \Theta_{j_{b_1}} A_{j_{b_1}j_{b_2}} \ldots [\Theta_{j_{b_p}} A_{j_{b_p}j_{b_1}}] \right] \right\} \delta\]
\[ (-1)^{p+p} \frac{1}{p!} \pi \bigg\{ \sum_{\delta \in S_p} [O_{j_1} A_{j_1 \delta_1} \ldots O_{j_p} A_{j_p \delta_p}] \bigg\} \]

\[ = \frac{1}{p!} \pi \bigg\{ \sum_{\delta \in S_p} (-1)^{(\delta)} [O_{j_1} \ldots O_{j_p} A_{j_1 \delta_1} \ldots A_{j_p \delta_p}] \bigg\} \]

\[ = \frac{1}{p!} \bigg\{ \sum_{\delta \in S_p} (-1)^{|\delta|} O_{j_p} \ldots O_{j_1} A_{j_1 \delta_1} \ldots A_{j_p \delta_p} \bigg\} \]

\[ = \frac{1}{p!} \sum_{\delta \in S_p} (-1)^{|\delta|} A_{j_1 \delta_1} \ldots A_{j_p \delta_p} \]

So far this has been equality of operators on \((R/\mathfrak{I} \otimes_k \text{Hom}(Y, X), \overline{\partial_{\text{Hom}}})\), but now we can apply the fact that the Atiyah classes anti-commute up to homotopy (see Theorem 3.11 of [Cut], or rather its proof, where notice we are using the \([\partial_{t_j}, \text{At}_c]\) is \(k[\mathbb{L}]\)-linear so that it passes to the quotient) and as a result has a homotopy \((38.2)\). So finally

**Lemma.** We have for \(1 \leq i \leq n\)

\[ \pi O_i^* \pi^{-1} = \pi O_i^* O_\infty = \text{At}_c \]

\[ \pi O_i \pi^{-1} = -\lambda_i - \sum_{p > 0} \sum_{i_1 \ldots i_p} \frac{1}{(p+1)!} [\lambda_{i_1}, [\lambda_{i_2}, \ldots [\lambda_{i_p}, \lambda_i] \ldots \text{At}_{i_1} \ldots \text{At}_{i_p}] \]

**Proof.** Applying \((38.2)\) to \((38.1)\) we have

\[ \pi O_i \pi^{-1} = \pi O_i O_\infty \]

\[ - \sum_{p > 0} \sum_{i_1 \ldots i_p} \frac{1}{(p+1)!} [\lambda_{i_1}, [\lambda_{i_2}, \ldots [\lambda_{i_p}, \lambda_i] \ldots ] \pi O_i^* \ldots O_{i_p} O_\infty \]

\[ = (-1)^{p+p} \pi O_j^* \ldots O_{j_p}^* \sum_{\delta \in S_p} [O_{j_1} A_{j_1 \delta_1} \ldots O_{j_p} A_{j_p \delta_p}] \bigg\} \]

\[ = \frac{1}{p!} \pi O_j^* \ldots O_{j_p}^* \bigg\{ \sum_{\delta \in S_p} (-1)^{(\delta)} O_{j_1} \ldots O_{j_p} A_{j_1 \delta_1} \ldots A_{j_p \delta_p} \bigg\} \]

\[ = \frac{1}{p!} \bigg\{ \sum_{\delta \in S_p} (-1)^{|\delta|} O_{j_p} \ldots O_{j_1} A_{j_1 \delta_1} \ldots A_{j_p \delta_p} \bigg\} \]

\[ = \frac{1}{p!} \sum_{\delta \in S_p} (-1)^{|\delta|} A_{j_1 \delta_1} \ldots A_{j_p \delta_p} \]
\[-\sum_{p \geq 1} \sum_{i_1, \ldots, i_p} \frac{1}{(p+1)!} \left[ \lambda_{i_p}, [\lambda_{i_{p-1}}, \ldots [\lambda_{i_1}, \lambda_i] \ldots] \right] \cdot \frac{1}{p!} \sum_{\delta \in S_p} (-1)^{|\delta|} A_{i_{\delta_1}} \ldots A_{i_{\delta_p}} \]

\[-\sum_{p \geq 1} \sum_{i_1, \ldots, i_p} \frac{1}{(p+1)!} \left[ \lambda_{i_p}, [\lambda_{i_{p-1}}, \ldots [\lambda_{i_1}, \lambda_i] \ldots] \right] A_{i_1} \ldots A_{i_p} \]

\[-\lambda_i - \sum_{p \geq 1} \sum_{i_1, \ldots, i_p} \frac{1}{(p+1)!} \left[ \lambda_{i_p}, [\lambda_{i_{p-1}}, \ldots [\lambda_{i_1}, \lambda_i] \ldots] \right] A_{i_1} \ldots A_{i_p} \]
Appendix A: alternative homotopy

Note The $S_i = \lambda_i O_i^* = \sum \gamma \times \lambda_i^\gamma O_i^*$ are defined using the conventional homotopy

$$\lambda_i^\gamma \times (\alpha) = \lambda_i^\gamma \circ \alpha.$$ 

Suppose instead we use $\tilde{S}_i = \tilde{\lambda}_i O_i^* = \sum \gamma \times \tilde{\lambda}_i^\gamma O_i^*$ where

$$\tilde{\lambda}_i^\gamma \times (\alpha) = (-1)^{\gamma \alpha} \alpha \circ \tilde{\lambda}_i^\gamma.$$ 

Then the above calculation yields

$$\begin{align*}
r_2 (\tilde{S}_i \otimes 1 \otimes \tilde{S}_i) (\omega_2 \otimes x_2 \otimes (\omega_1 \otimes x_1)) \\
= (-1)^{t_{\omega_2} + t_{\omega_1} + t_{x_2} + t_{x_1} + t_{\omega_1} + t_{\omega_1}} \left\{ (-1)^{t_{\omega_1} + t_{x_1}} \omega_1 O_1^*(\omega_2) \otimes \alpha_i \circ \tilde{\lambda}_i^\gamma(x_2) \right. \\
&\left. + O_i^*(\omega_1) \omega_2 \otimes \tilde{\lambda}_i^\gamma(x_i) \circ x_2 \right\}
\end{align*}$$

$$\begin{align*}
\tilde{S}_i : r_2 \left( (\omega_2 \otimes x_2 \otimes (\omega_1 \otimes x_1)) \\
= (-1)^{t_{\omega_2} + t_{\omega_1} + t_{x_2} + t_{x_1} + t_{\omega_1}} \omega_i^* (\omega_1 \omega_2) \otimes \tilde{\lambda}_i^\gamma(x_i \circ x_2) \\
\end{align*}$$

$$\begin{align*}
\left\{ \tilde{S}_i r_2 - r_2 (1 \otimes \tilde{S}_i + \tilde{S}_i \otimes 1) \right\} (\omega_2 \otimes x_2 \otimes (\omega_1 \otimes x_1)) \\
= (-1)^{t_{\omega_2} + t_{\omega_1} + t_{x_2} + t_{x_1} + t_{\omega_1}} \left\{ \omega_i^* (\omega_1 \omega_2) \otimes \tilde{\lambda}_i^\gamma(x_i \circ x_2) \\
- (-1)^{t_{\omega_1} + t_{x_1}} \omega_i^* (\omega_2) \otimes \alpha_i \circ \tilde{\lambda}_i^\gamma(x_2) \\
- O_i^* (\omega_1) \omega_2 \otimes \tilde{\lambda}_i^\gamma(x_i) \circ x_2 \right\}
\end{align*}$$
\[ O^*_i(w_1,w_2) \otimes \tilde{\lambda}_i (x_1 \circ x_2) = (-1)^{|w_1| + |x_1|} \omega_1 O^*_i(w_2) \otimes x_1 \tilde{\lambda}_i (x_2) \]
\[- O^*_i(w_1)w_2 \otimes \tilde{\lambda}_i (x_1) \circ x_2 \]
\[ = O^*_i(w_1)w_2 \otimes (-1)^{|x_1| + |x_2|} x_1 \circ x_2 \circ \lambda_i - (-1)^{|w_1| + |x_1|} \omega_1 (O^*_i(w_2) \otimes (-1)^{|x_2|} x_1 \lambda_i \lambda_i) \]
\[ + (-1)^{|w_1|} \omega_1 O^*_i(w_2) \otimes (-1)^{|x_1| + |x_2|} x_1 \circ x_2 \circ \lambda_i - (-1)^{|x_1|} O^*_i(w_1)w_2 \otimes x_1 \lambda_i x_2 \]
\[ = - (-1)^{|x_1|} O^*_i(w_1)w_2 \otimes \{ x_1 \circ \lambda_i \circ x_2 - (-1)^{|x_1|} x_1 \circ x_2 \circ \lambda_i \} \]
\[ = - O^*_i(w_1)w_2 \otimes (-1)^{|x_1|} x_1 \circ [\lambda_i, -](x_2) \]

Hence
\[ \left\{ \delta_i r_2 - r_2 \left( \Theta \delta_i + \delta_i \Theta \right) \right\} ((w_1 \otimes x_1) \otimes (w_1 \otimes x_1)) \]
\[ = (-1)^{|w_1| + |w_2| + |x_1| + |x_2| + |x_1| + |w_1| + 1} O^*_i(w_1)w_2 \otimes (-1)^{|x_1|} x_1 \circ [\lambda_i, -](x_2) \]

whereas
\[ r_2 \left( [\lambda_i, -] \otimes O^*_i \right) ((w_2 \otimes x_2) \otimes (w_1 \otimes x_1)) \]
\[ = (-1)^{|w_2| + |x_2| + 1} r_2 \left( [\lambda_i, -] (w_2 \otimes x_2) \otimes O^*_i (w_1 \otimes x_1) \right) \]
\[ = (-1)^{|w_2| + |x_2| + 1} r_2 \left( (-1)^{|w_2|} w_2 \otimes \{ \lambda_i x_2 - (-1)^{|x_2|} x_2 \lambda_i \} \otimes O^*_i (w_1) \otimes x_1 \right) \]
\[ = (-1)^{|x_2| + 1} r_2 \left( w_2 \otimes \lambda_i x_2 \otimes O^*_i (w_1) \otimes x_1 \right) \]
\[ + r_2 \left( w_2 \otimes x_2 \lambda_i \otimes O^*_i (w_1) \otimes x_1 \right) \]
\[
\begin{align*}
&= (-1)^{|x_2|+1} + (|w_1| + |x_1| + |x_1| + |w_1| + |x_1| + 1) + |w_1| + |x_1| + 1 + |x_1| + 1 \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right) \\
&+ (-1)^{|x_2|+1} (|w_2| + |x_2| + 1) + (|w_1| + |x_1| + 1) + (|w_1| + |x_1| + 1) + 1 \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right)
\end{align*}
\]

\[
\begin{align*}
&= (-1)^{|x_2|+1} + (|w_2| + |x_2| + |w_3| + |x_3| + |x_3| + |w_1| + |x_1| + 1 + |x_1| + |x_1| + 1 + |x_1| + |x_1| + 1 + |x_1| + |x_1| + 1) \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right) \\
&+ (-1)^{|x_2|+1} (|w_2| + |x_2| + |w_3| + |x_3| + |x_3| + |w_1| + |x_1| + 1 + |x_1| + |x_1| + 1 + |x_1| + |x_1| + 1 + |x_1| + |x_1| + 1) \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right)
\end{align*}
\]

\[
\begin{align*}
&= (-1)^{|w_2| + |w_1| + |x_2| + |w_1| + |x_2| + |x_2| + |x_1| + 1 + |w_1| + |x_1| + 1) \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right) \\
&+ (-1)^{|w_2| + |w_1| + |x_2| + |w_1| + |x_2| + |x_2| + |x_1| + 1 + |w_1| + |x_1| + 1) \quad \mathcal{M}\left( \Theta_i^*(\omega) \otimes x_i \otimes \omega_2 \otimes \lambda_i \right)
\end{align*}
\]

This proves $\tilde{\delta}_i \circ r_2 = r_2 (| \otimes \tilde{\delta}_i + \tilde{\delta}_i \otimes 1) = - r_2 ([\lambda_i, -] \otimes \Theta_i^*)$ or written differently,

**Lemma.** The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{H}[1] \otimes \mathcal{H}[1] & \xrightarrow{r_2} & \mathcal{H}[1] \\
\tilde{\delta}_i \otimes 1 + 1 \otimes \tilde{\delta}_i - [\lambda_i, -] \otimes \Theta_i^* & \downarrow & \tilde{\delta}_i \\
\mathcal{H}[1] \otimes \mathcal{H}[1] & \xrightarrow{r_2} & \mathcal{H}[1]
\end{array}
\]

That is,

\[
\tilde{\delta}_i \circ r_2 = r_2 (| \otimes \tilde{\delta}_i + \tilde{\delta}_i \otimes 1 - [\lambda_i, -] \otimes \Theta_i^*).
\]
Appendix B: Notes on commutators

Lemma Let $\mathfrak{F}, \mathfrak{G}$ be homogeneous $k$-linear operators on $\bigwedge F$ and $\bigwedge G$ a homogeneous $k$-linear operator on $\bigoplus v_x \text{Hom}_R (Y, X)$ such that $|F| + |G| = |\Lambda|$, so

$$\mathfrak{F} \otimes \bigwedge G \subseteq \mathfrak{F}[1] \otimes \mathfrak{G}[1]$$

is homogeneous. Then

$$[\Xi_j, \mathfrak{F} \otimes \bigwedge G] = (-1)^{|\Xi|} \mathfrak{O}_j \mathfrak{F} \otimes [\Xi_j, -] \bigwedge \Gamma$$

Proof We calculate

$$[\Xi_j, \mathfrak{F} \otimes \bigwedge G] (\omega_1 \otimes \alpha_1 \otimes \omega_2 \otimes \alpha_2)$$

$$= \Xi_j \left( (-1)^{|\Xi|} (\lambda_1 \otimes \lambda_2) + \lambda_1 (\lambda_1 \otimes \lambda_2) \right) \mathfrak{F} (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes \bigwedge (\alpha_2)$$

$$- (-1)^{|\Xi|} (\bigwedge \mathfrak{G}) \left( (-1)^{|\omega_1| + |d_1| + 1} + \bigwedge \bigwedge \Gamma (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes \bigwedge (\alpha_2) \right)$$

$$= (-1)^{|\Xi|} (\lambda_1 \otimes \lambda_2) \mathfrak{O}_j \mathfrak{F} (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes [\lambda_j, \Lambda (\alpha_2)]$$

$$+ (-1)^{|\Xi| + |\omega_1| + |d_1| + 1} + \bigwedge \bigwedge \Gamma (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes \bigwedge [\lambda_j, \Lambda (\alpha_2)]$$

$$+ (-1)^{|\Xi|} \mathfrak{O}_j \mathfrak{F} (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes \bigwedge [\lambda_j, -] \bigwedge (\alpha_2)$$

$$- (-1)^{|\Xi| + |\omega_1| + |d_1| + 1} + \bigwedge \bigwedge \Gamma (\omega_1) \otimes \alpha_1 \otimes \bigwedge \bigwedge \Gamma (\omega_2) \otimes \bigwedge [\lambda_j, -] \bigwedge (\alpha_2)$$

$$= (-1)^{|\Xi|} \mathfrak{O}_j \mathfrak{F} (\omega_1) \otimes \alpha_1 \otimes \bigwedge \Gamma (\omega_2) \otimes \bigwedge [\lambda_j, -] \bigwedge (\alpha_2)$$
whereas

\[
\left\{ \mathfrak{f}^* J \otimes \left[ \lambda_j , \mathcal{M} \right] \right\} (\omega_1 \otimes x_1 \otimes \omega_2 \otimes x_2 )
\]

\[
= (-1)^{(|\omega_1| + |x_1| + 1)(|\lambda_j| + 1) + 1} \mathfrak{f}^* J (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \left[ \lambda_j , \mathcal{M} \right] (x_2 )
\]

\[
= (-1)^{|\mathcal{M}|} (|\omega_1| + |\omega_2| + |x_1| + |x_2| + 1) \mathfrak{f}^* J (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \left[ \lambda_j , \mathcal{M} \right] (x_2 )
\]

Lemma. With the same setup as the previous lemma,

\[
\left[ 1 \otimes \delta_j , \mathfrak{f} \otimes \mathcal{M} \mathcal{N} \right] = (-1)^{|\mathcal{M}|} \mathfrak{f} \otimes \left[ \lambda_j , \mathcal{M} \right] \mathfrak{f}^* \mathcal{N}
\]

Proof. We compute

\[
\left[ 1 \otimes \delta_j , \mathfrak{f} \otimes \mathcal{M} \mathcal{N} \right] (\omega_1 \otimes x_1 \otimes \omega_2 \otimes x_2 )
\]

\[
= (1 \otimes \delta_j ) \left( (-1)^{|\mathcal{M}|} (|\omega_1| + |x_1| + 1) + |\lambda_j| |\omega_2| \right) \mathfrak{f} (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \left[ \lambda_j , \mathcal{M} \right] (x_2 )
\]

\[
= (-1)^{|\mathcal{M}|} (|\omega_1| + |\mathcal{M} \mathcal{N}| + 1) \mathfrak{f} (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \mathcal{N} (x_2 )
\]

\[
\quad + (-1)^{|\mathcal{M}|} (|\omega_1| + |x_1| + 1) \mathfrak{f} (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \left[ \lambda_j , \mathcal{M} \right] (x_2 )
\]

\[
= (-1)^{|\mathcal{M}|} (|\omega_1| + |x_1| + |\lambda_j| |\omega_2| + 1) \mathfrak{f} (\omega_1) \otimes x_1 \otimes \mathcal{M} (\omega_2) \otimes \left[ \lambda_j , \mathcal{M} \right] (x_2 )
\]
whereas

\[
\{ \exists \otimes \left[ \chi_j, \Lambda \right] \theta_j^* \gamma \} (\omega_1 \otimes \omega_1 \otimes \omega_2 \otimes \omega_z)
\]

\[
= (-1)^{(1w_1 + |\chi_1| + 1) + (1w_2 + l_1 + 1)} \exists (\omega_1) \otimes \alpha_j \otimes \theta_j^* \gamma (\omega_2) \otimes \left[ \chi_j, \Lambda \right] (\omega_z)
\]

Now we can for example calculate that

\[
(\otimes \delta)(\otimes \delta) \Xi = (\otimes \delta) \{ \Xi (\otimes \delta) + [\otimes \delta, \Xi] \}
\]

\[
= (\otimes \delta) \Xi (\otimes \delta) - (\otimes \delta) \sum_{i,j} \theta_j^* \otimes \left[ \chi_j, \Lambda \right] \theta_i^*.
\]

\[
= \{ \Xi (\otimes \delta) + [\otimes \delta, \Xi] \} (\otimes \delta)
\]

\[- \sum_{i',j} (\theta_{i'}^* \otimes \left[ \chi_{i'}, \Lambda \right] \theta_i^*) (\otimes \delta)
\]

\[- \sum_{i',j} [\otimes \delta, \theta_{i'}^* \otimes \left[ \chi_{i'}, \Lambda \right] \theta_i^*]
\]

\[
= \Xi (\otimes \delta)^2 - 2 \sum_{i',j} (\theta_{i'}^* \otimes \left[ \chi_{i'}, \Lambda \right] \theta_i^*) (\otimes \delta)
\]

\[- \sum_{i',j,k} \theta_{i'}^* \otimes \left[ \chi_{i'}, \chi_k \right] \theta_i^* \theta_k^*
\]

\[
= \Xi (\otimes \delta)^2 + 2 \sum_{i',j,k} (\theta_{i'}^* \otimes \left[ \chi_{i'}, \Lambda \right] \chi_k \theta_i^* \theta_k^* + \sum_{i',j,k} (\theta_{i'}^* \otimes \left[ \chi_{i'}, \chi_k \right] \theta_i^* \theta_k^*
\]

\[
= \Xi (\otimes \delta)^2 + \sum_{i',j,k} \theta_{i'}^* \otimes \left\{ \chi_k [\chi_{i'}, \Lambda] + [\chi_{i'}, \Lambda] \chi_k \right\} \theta_i^* \theta_k^*
\[(10\delta)^2 \Xi^2 = \Xi (10\delta)^2 \Xi + [10\delta^2, \Xi] \Xi \]

\[= \Xi \{ \Xi (10\delta^2) + [10\delta^2, \Xi] \} + [10\delta^2, \Xi] \Xi \]

\[= \Xi^2 (10\delta^2) + \Xi \{ [10\delta^2, \Xi] \} \]

\[= \Xi^2 (10\delta^2) + \sum_{i,j,k,l} \Xi \{ \big[ \Xi (10, \Xi) \big[ \lambda_k[\lambda_j, \lambda_i] + [\lambda_j, \lambda_i] \lambda_k \big] \big] \} \]

\[= \Xi^2 (10\delta^2) - \sum_{i,j,k,l} \Xi \{ \big[ \lambda_k[\lambda_j, \lambda_i] + [\lambda_j, \lambda_i] \lambda_k \big] \} \Xi \]

\[\Xi^2 (10\delta^2) - \sum_{i,j,k,l} \Xi \{ \big[ \lambda_k[\lambda_j, \lambda_i] + [\lambda_j, \lambda_i] \lambda_k \big] \} \Xi \]

**Lemma** Let \( \Lambda \) be a homogeneous operator on \( \bigoplus_{Y \times} \text{Hom}_\mathcal{Y}(Y, X) \) of the form \( \Lambda(\alpha) = \lambda \circ \alpha \) for some family \( \{ \lambda^Y \}_Y \) all of the same degree. Then as operator on \( \mathcal{H} \),

\[
\left[ \left[ \lambda_j, - \right], \Lambda \right] = \left[ \lambda_j, \lambda \right]
\]

(16.1)

where the RHS means post-composition with \( \left[ \lambda_j^Y, \lambda^Y \right] \).

**Proof** We calculate

\[
\left[ \left[ \lambda_j, - \right], \Lambda \right](\alpha) = \left[ \lambda_j, - \right] (\lambda \circ \alpha) - (-1)^{[\lambda, \lambda]} \Lambda \left( \left[ \lambda_j, \alpha \right] \right)
\]

\[= \lambda_j \lambda \alpha - (-1)^{[\lambda, \lambda]} \lambda \alpha \lambda_j \]

\[= (-1)^{[\lambda, \lambda]} \lambda \lambda_j \alpha + (-1)^{[\lambda, \lambda]} \lambda \alpha \lambda_j \]

\[= \{ \lambda_j \lambda - (-1)^{[\lambda, \lambda]} \lambda \lambda_j \} \circ \alpha . \Box \]
In general for $a, b > 1$ we have by the cut systems paper, since $[-, \Xi^b]$ is a graded derivation

$$[(1 \otimes \delta)^a, \Xi^b] = \sum_{z=0}^{a-1} (1 \otimes \delta)^z [(1 \otimes \delta, \Xi^b)] (1 \otimes \delta)^{a-z-1}$$

and

$$\Xi^b = \left( \sum_{i=1}^{\hat{a}} O_i^* \otimes [\lambda_i, -] \right)^b$$

$$= \sum_{i_j \cdots i_b} (O_{i_1}^* \otimes [\lambda_{i_1}, -]) \cdots \circ (O_{i_b}^* \otimes [\lambda_{i_b}, -])$$

$$= \sum_{i_j \cdots i_b} (-1)^{b} O_{i_1}^* \cdots O_{i_b}^* \otimes [\lambda_{i_1}, -] \cdots \circ [\lambda_{i_b}, -]$$

Hence by (14.1)

$$[(1 \otimes \delta, \Xi^b)] = \sum_{j, i_j \cdots i_b} (-1)^{\binom{b}{2}} + b O_{i_1}^* \cdots O_{i_b}^* \otimes$$

$$\left[ \lambda_j, [\lambda_{i_1}, -] \cdots \circ [\lambda_{i_b}, -] \right] O_j^*$$

where $[\lambda_j, [\lambda_{i_1}, -] \cdots \circ [\lambda_{i_b}, -]]$ sends $\alpha \in \text{Hom}_R(Y, X)$ to

$$\chi_j^\gamma \circ \left[ \lambda_{i_1}, [\lambda_{i_2}, \cdots [\lambda_{i_b}, \alpha] \cdots] \right]$$

$$- (-1)^{b} \left[ \lambda_{i_1}, \cdots \circ [\lambda_{i_b}, \chi_j^\gamma \circ \alpha] \cdots \right].$$

Note that this is not $[\lambda_j, -] \circ [\lambda_{i_1}, -] \cdots \circ [\lambda_{i_b}, -]$ applied to $\alpha$. 
In fact, since the commutator is a graded differential

\[
\left[ \lambda_j, \left[ \lambda_{i_1}, - \right] \cdots \left[ \lambda_{i_b}, - \right] \right] = \sum_{q=0}^{b} (-1)^q \left[ \lambda_{i_1}, - \right] \cdots \left[ \lambda_{i_q}, - \right] \circ \left[ \lambda_j, \left[ \lambda_{i_{q+1}}, - \right] \right] \circ \ldots \circ \left[ \lambda_{i_b}, - \right].
\]

By the earlier calculations, see (16.1), we know \( \left[ \lambda_j, \left[ \lambda_{i_{q+1}}, - \right] \right] \) act by post-composition with \( \left[ \lambda_j, \lambda_{i_{q+1}} \right] \), and to avoid confusion we will write that operator as \( \left[ \lambda_j, \lambda_{i_{q+1}} \right]^* \). So

\[
\left[ 1 \otimes \delta, \square^b \right] = \sum_{j, i_1, \ldots, i_b} (-1)^{\binom{b}{2}} \delta^{i_1} \cdots \delta^{i_b} \otimes \\
\sum_{q=0}^{b} (-1)^q \left[ \lambda_{i_1}, - \right] \cdots \left[ \lambda_{i_q}, - \right] \circ \left[ \lambda_j, \left[ \lambda_{i_{q+1}}, - \right] \right] \circ \ldots \circ \left[ \lambda_{i_b}, - \right] \circ \left[ \lambda_j, \lambda_{i_{q+1}} \right]^* \\
\circ \ldots \circ \left[ \lambda_{i_{q+2}}, - \right] \circ \ldots \circ \left[ \lambda_{i_b}, - \right] \circ \left[ \lambda_j, \lambda_{i_{q+1}} \right]^* \circ \ldots \circ \left[ \lambda_{i_b}, - \right] \circ \left[ \lambda_j, \lambda_{i_{q+1}} \right]^*.
\]

\[
= \sum_{j, i_1, \ldots, i_b} (-1)^{\binom{b+1}{2}} \delta^{i_1} \cdots \delta^{i_b} \otimes \\
\sum_{q=0}^{b} (-1)^q \left[ \lambda_{i_1}, \left[ \lambda_{i_2}, \ldots, [ \lambda_{i_q}, \ldots, [ \lambda_{i_q}, \ldots, \left[ \lambda_j, \lambda_{i_{q+1}} \right] \circ \left[ \lambda_{i_{q+2}}, \ldots, \left[ \lambda_{i_b}, - \right] \cdots \left[ \lambda_{i_b}, - \right] \right] \right] \right] \right] \circ \left[ \lambda_j, \lambda_{i_{q+1}} \right]^* \circ \ldots \circ \left[ \lambda_{i_b}, - \right] \circ \left[ \lambda_j, \lambda_{i_{q+1}} \right]^*.
\]
\[
\hat{\Xi}(\chi_1, \chi_2) = \sum_i (\Omega_i^\times \otimes [\chi_i, -^\wedge]) (\chi_1, \chi_2)
\]
\[
= \sum_i (-1)^i \Omega_i^\times (\chi_1) \otimes [\chi_i, -^\wedge] (\chi_2)
\]

Hence
\[
\hat{\Xi}^\mathcal{E}(\chi_1, \chi_2) = \sum_{i; \ldots, i; \mathcal{E}} (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]

and so
\[
\hat{\Xi}^\mathcal{E}_2(\chi_1, \chi_2) = \sum_{i; \ldots, i; \mathcal{E}} (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= \sum_{i; \ldots, i; \mathcal{E}} (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= \sum_{i; \ldots, i; \mathcal{E}} (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]
\[
= (-1)^i \rho_1^\mathcal{E}^i (\chi_1) \otimes [\chi_i, -^\wedge] \ldots [\chi_i, -^\wedge] (\chi_2)
\]