Minimal models for MFs 25  (checked)

Let $A$ denote for $d > 2$ the minimal $A_{\infty}$-algebra with underlying $\mathbb{Z}_2$-graded algebra the exterior algebra $\wedge(k \mathcal{E})$ where $|\mathcal{E}| = 1$, and $A_{\infty}$-products $\{m_n\}_{n \geq 1}$ with only $m_2, m_d$ non-zero, and $m_d(\mathcal{E} \otimes \cdots \otimes \mathcal{E}) = 1$.

Given $2 \leq i \leq d-2$ with $i < d - i$ define

$$M(i) = \wedge(k \mathfrak{F})$$

$$\alpha_n : A \otimes^{(n-1)} \otimes M(i) \rightarrow M(i)$$

$$\alpha_i = 0, \quad \alpha_2(1,-) = \text{id}$$

$$\alpha_{i+1}(\mathcal{E}, \ldots, \mathcal{E}, -) = (-i)^{k} \mathfrak{F} \wedge(-)$$

$$\alpha_{d-i+1}(\mathcal{E}, \ldots, \mathcal{E}, -) = (-i)^{k} \mathfrak{F} \wedge(-)$$

We check (some of) the $A_{\infty}$-constraints making $\{\alpha_n\}_{n \geq 1}$ into an $A_{\infty}$-module (using $(-1)^{r+s+t}$ signs). The possible contributors are $m_2, m_d, \alpha_2, \alpha_{i+1}, \alpha_{d-i+1}$. The interesting one is where $\alpha_{i+1}, \alpha_{d-i+1}$ both contribute, which is the constraint for $n = d+1$. Schematically, this will have terms

$$A \otimes^d \otimes M(i) \rightarrow M(i)$$

$$\alpha_2 m_d + \alpha_{i+1} \alpha_{d-i+1} + \alpha_{d-i+1} \alpha_{i+1}$$

which we compute separately. It is clear all three terms are nonvanishing only on (among basis elfs of $\wedge(k \mathcal{E})^d$) the basis element $E_0^d$, and
\[ \alpha_d \cdot m(\varepsilon, \ldots, \varepsilon, x) = \alpha_2(1, x) = x \]  

(2.1)

\[ \alpha_{d-i+1} \alpha_{d-i+1}(\varepsilon, \ldots, \varepsilon, x) = \alpha_{d-i}(\varepsilon, \ldots, \varepsilon, j \wedge x) = (-1)^{h+k} j \wedge (j \wedge x) \]

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The signs in (1.2) are (1) \((-1)^d\), (2) \((-1)^i\), and (3) \((-1)^{d-i}\), so the constraint for \(n = d + 1\) reads (in its only nontrivial values) for \(x \in M(i)\) homogeneous

\[ [(-1)^d \alpha_2(m \otimes 1) + (-1)^i \alpha_3(i \otimes \alpha_{d-i+1}) + (-1)^{d-i} \alpha_{d-i+1}(1 \otimes d-i \otimes \alpha_{d-i+1})](\varepsilon, \ldots, \varepsilon, x) \]

(2.2)

Now the Koszul sign rule says (since \([\alpha_n] = 2-n = n\))

\[ (1 \otimes i \otimes \alpha_{d-i+1})(\varepsilon, \ldots, \varepsilon, x) = (-1)^{i(d-i+1)} \varepsilon \otimes i \otimes \alpha_{d-i+1}(\varepsilon \otimes d-i, x) = (-1)^{i(d-i+1)} \varepsilon \otimes i \otimes (j \wedge x) \cdot (-1)^{h+k} \]

\[ (1 \otimes (d-i) \otimes \alpha_{d-i+1})(\varepsilon, \ldots, \varepsilon, x) = (-1)^{(d-i)(i+1)} \varepsilon \otimes d-i \otimes (j \wedge x) \cdot (-1)^{h+k} \]

Hence (2.2) expands to

\[ 0 = (-1)^d x + (-1)^i i(d-i+1) + h+k j \wedge (j \wedge x) \]

\[ [j \wedge (-), j \wedge (-)] = \varepsilon d \]

\[ + (-1)^{d-i} (d-i)(i+1) + h+k j \wedge (j \wedge x) \]

\[ = (-1)^d x + (-1)^i (d-i) + h+k x \]

Upshot: provided we choose \(h, k\) such that \( h+k+i(d-i) = d+1 \) (mod 2), the \( n = d+1 \) A\(_\infty\)-constraint will be satisfied.