Minimal models for MFs II (checked)

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We construct the A_∞-minimal model of $\mathrm{End}(k^{stab})$. Let k be a char. 0 field and $W \in k[x_1, \ldots, x_n]$ a potential, with $W \in \mathfrak{m}^2$ so that $W = \sum_i x_i W^i$ for $W^i \in \mathfrak{m} = (x_1, \ldots, x_n)$. We set

$$k^{stab} = \Big(\wedge R\psi_1 \oplus \cdots \oplus R\psi_n, \ \textstyle\sum_i x_i \psi_i^* + \sum_i W^i \psi_i \Big) \qquad (1.1)$$

where $R = k[\underline{x}]$, and $|\psi_i| = 1$. The odd operator $\psi_j := \psi_j \wedge -$ on k^{stab} satisfies the identity

$$[\psi_j, d_{k^{stab}}] = x_j \cdot 1 \qquad (1.2)$$

with $S = \wedge(k\theta_1 \oplus \cdots \oplus k\theta_n)$ as usual, we have by the cut systems paper a diagram of k-linear homotopy equivalences

(1.3)

$$\begin{tikzcd} S \otimes_k \mathrm{End}_R(k^{stab}) \arrow[r, shift left, "\exp(-\delta)"] & K \otimes_R \mathrm{End}_R(k^{stab}) \arrow[l, shift left, "\exp(\delta)"] \arrow[r, shift left, "\pi"] \arrow[loop below, "H_\infty"] & \underline{\mathrm{End}}(k^{stab}) \arrow[l, shift left, "\sigma_\infty"] \\ & & \mathrm{End}_R(k^{stab}) \otimes_R R/\mathfrak{m} \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

where $\underline{\mathrm{End}}(k^{stab})$ has differential zero, and

$$\delta = \textstyle\sum_i \psi_i \theta_i^* \qquad (1.4)$$

$$\sigma_\infty = \sum_{s \geq 0} \sum_{p_1, \ldots, p_s} (-1)^{\binom{s+1}{2}} \frac{1}{s!} At_{p_1} \cdots At_{p_s} \theta_{p_1} \cdots \theta_{p_s} + (\mathfrak{m}\text{-terms})$$

$$\pi \sigma_\infty = 1$$

$$H = [d_K, \nabla]^{-1} \nabla \qquad \nabla = \textstyle\sum_i \partial_{x_i} \theta_i$$

$$1 - [d_K + d_{End}, H_\infty] = \sigma_\infty \pi.$$

$$H_\infty = \textstyle\sum_{m \geq 0} (-1)^m (H d_{End})^m H$$

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We define

$$\Phi := \pi \exp(-\delta) \qquad \Phi^{-1} := \exp(\delta)\sigma_\infty$$

So that

$$\Phi^{-1}\Phi = 1 - [d_{End}, \hat{H}], \quad \hat{H} = \exp(\delta) H_\infty \exp(-\delta)$$

Thus overall

<u>Lemma</u> There is a diagram of $\mathbb{Z}_2$-graded k-complexes

$$\hat{H} \circlearrowright S \otimes_k \operatorname{End}_R(k^{stab}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \underline{\operatorname{End}}(k^{stab}) \tag{2.1}$$

$$\Phi\Phi^{-1} = 1, \quad \Phi^{-1}\Phi = 1 - [d_{End}, \hat{H}].$$

<u>Lemma</u> The operator At_i on $S \otimes_k \operatorname{End}_R(k^{stab})$ is

$$At_i = -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]$$

$$d_{k^{stab}} = \textstyle\sum_j x_j \psi_j^* + \sum_j W^j \psi_j$$

<u>Proof</u> We have

$$\begin{aligned} At_i &= [d_{End}, \partial_{x_i}] \\ &= [[\textstyle\sum_j x_j \psi_j^* + \sum_j W^j \psi_j, -], \partial_{x_i}] \end{aligned}$$

Since ∂_{x_i} is a derivation on R, for any free R-module F and operator A on F,

$$\begin{aligned} [A, \partial_{x_i}](re_j) &= A(\partial_{x_i}(r)e_j) - \partial_{x_i}(\textstyle\sum_k A_{kj} r e_k) \\ &= \textstyle\sum_k A_{kj}\partial_{x_i}(r)e_k - \sum_k \partial_{x_i}(A_{kj} r)e_k \\ &= \textstyle\sum_k -\partial_{x_i}(A_{kj}) r e_k = -r\partial_{x_i}(A)(e_j) \end{aligned}$$

. $\square$

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As usual, on something of homogeneous $\mathcal{O}$-degree p, $[d_K, \nabla]^{-1}$ acts as $\frac{1}{p}$ + ($\mathfrak{m}$ terms).

Def^N We make $S \otimes_k \mathrm{End}_R(k^{stab})$ a $\mathbb{Z}_2$-graded DG-algebra with multiplication m_2, via the usual tensor product of DG-algebras where S is the exterior algebra.

i.e. ainfcat or [L]

Setting $\mu_1 = 1 \otimes d\mathrm{End}$, $\mu_2 = m_2$ (and noting our m is not the same as the m there) we can define the suspended forward A_∞-operations $\{r_n\}_{n \geq 1}$ on the DG-algebra $S \otimes_k \mathrm{End}_R(k^{stab})$ as defined in ainfcat (which follows Lazaroiu's paper). Specifically with $A := S \otimes \mathrm{End}_R(k^{stab})$

$$r_1 : A[1] \longrightarrow A[1] \qquad r_1(\alpha) = \mu_1(\alpha)$$

and

$$r_2 : A[1] \otimes A[1] \longrightarrow A[1] \qquad r_2(\alpha \otimes \beta) = (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \mu_2(\beta \otimes \alpha)$$

These are both maps of degree +1.

Lemma The data $P = \Phi^{-1}\Phi$ and $G = \hat{H}$ defines a strict homotopy retraction of A in the sense of (Lazaroiu, §3.3).

Proof Clear. $\square$

We may identify $\underline{\mathrm{End}}(k^{stab})$ with B, i with Φ^{-1}, p with Φ

For ease of reference:

[L] Lazaroiu, "Generating the superpotential"

3.5

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Then from [L] we obtain an A_∞-structure

$$\left(\underline{\mathrm{End}}(k^{stab}), \{\rho_n\}_{n\geq 2} \right) \tag{3.5.1}$$

A_∞-quasi-isomorphic to $\left(S \otimes_k \mathrm{End}_R(k^{stab}), \{\mu_1, \mu_2\} \right)$, i.e. the minimal model. The products ρ_n are degree $+1$ maps

$$\rho_n : \left(\underline{\mathrm{End}}(k^{stab})[1] \right)^{\otimes n} \longrightarrow \underline{\mathrm{End}}(k^{stab}) \tag{3.5.2}$$

satisfying the suspended forward A_∞-relations (i.e. (4.2) of ainfcat).
The map ρ_2 is for example

$$\rho_2 = \Phi \circ r_2 \circ (\Phi^{-1} \otimes \Phi^{-1}) \tag{3.5.3}$$

i.e. the composite (writing $\underline{\mathrm{End}}$ for $\underline{\mathrm{End}}(k^{stab})$)

$$\underline{\mathrm{End}}[1] \otimes \underline{\mathrm{End}}[1] \xrightarrow{\Phi^{-1} \otimes \Phi^{-1}} A[1] \otimes A[1] \xrightarrow{r_2} A[1] \xrightarrow{\Phi} \underline{\mathrm{End}}[1]$$

while for $n \geq 2$

$$\rho_n = \sum_{T \in \mathcal{T}_n} \rho_T \tag{3.5.4}$$

where $\mathcal{T}_n$ is the set of oriented and connected planar trees T with $n+1$ vertices of valency one (external vertices) and all other vertices of valency 3 (internal vertices). The edges meeting an external vertex are called external edges, others are internal. Other notation is as in [L].

3.75

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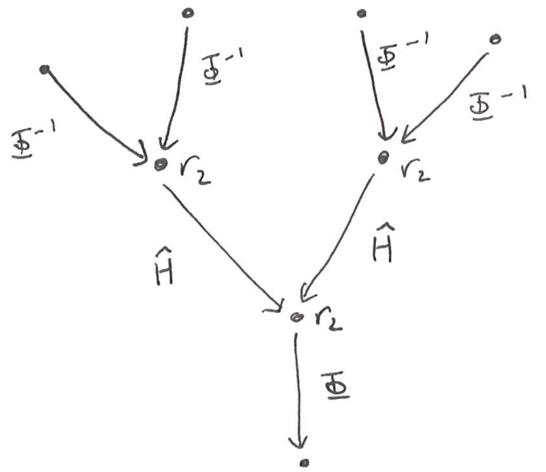
Note that ρ_T includes a sign factor $(-1)^s$ where s is the number of internal edges. For example



$s = 2$

⌈For signs see ainfcat3 and ainfcat2 p. 9⌋

(3.75.1)

$$\rho_T = (-1)^s \Phi \circ r_2 \circ (\hat{H} \otimes \hat{H}) \circ (r_2 \otimes r_2)\ (\Phi^{-1} \otimes \Phi^{-1} \otimes \Phi^{-1} \otimes \Phi^{-1})$$

no Koszul signs! Read this as a composite of ungraded maps.

Notice that these operators on $A[1]$ have degree

$$|\hat{H}| = |r_2| = 1$$

(3.75.2)

$$\rho_T(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4)$$

$$= \Phi \circ r_2 \circ (\hat{H} \otimes \hat{H}) \circ (r_2 \otimes r_2)\big(\Phi^{-1}(\alpha_1) \otimes \Phi^{-1}(\alpha_2) \otimes \Phi^{-1}(\alpha_3) \otimes \Phi^{-1}(\alpha_4) \big)$$

$$= \Phi \circ r_2 \circ (\hat{H} \otimes \hat{H}) \circ \big(r_2(\Phi^{-1}(\alpha_1) \otimes \Phi^{-1}(\alpha_2)) \otimes r_2(\Phi^{-1}(\alpha_3) \otimes \Phi^{-1}(\alpha_4)) \big)$$

$$= \Phi r_2 \big(\hat{H}(\cdot\cdot) \otimes \hat{H}(\cdot\cdot) \big).$$

<u>Example</u> The suspended forward multiplications ρ_n on <u>End</u> correspond to "backward" multiplications b_n (i.e. the kind in ainf) which are degree $2-n$ maps

$$b_n : \underline{\mathrm{End}}^{\otimes n} \longrightarrow \underline{\mathrm{End}} \qquad (4.1)$$

These are defined, for example, by

$$b_2(\beta \otimes \alpha) = (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \rho_2(\alpha \otimes \beta)$$

Now by (3.5.4),

$$\rho_2 = \begin{tikzcd}[column sep=small] \bullet \arrow[dr, "\Phi^{-1}"'] & & \bullet \arrow[dl, "\Phi^{-1}"] \\ & \bullet_{r_2} \arrow[d, "\Phi"] & \\ & {} & \end{tikzcd} = \Phi \circ r_2 \circ (\Phi^{-1} \otimes \Phi^{-1}) \qquad (4.2)$$

Hence

$$\begin{aligned} b_2(\beta \otimes \alpha) &= (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \Phi \circ r_2 \circ (\Phi^{-1} \otimes \Phi^{-1})(\alpha \otimes \beta) \\ &= (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \Phi\, r_2\big(\Phi^{-1}(\alpha) \otimes \Phi^{-1}(\beta)\big) \\ &= (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1 + \tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \Phi \mu_2\big(\Phi^{-1}(\beta) \otimes \Phi^{-1}(\alpha)\big) \\ &= \Phi \circ \mu_2 \circ (\Phi^{-1} \otimes \Phi^{-1})(\beta \otimes \alpha). \\ &= \pi \exp(-\delta)\, m_2\big(\exp(\delta) b_\infty \otimes \exp(\delta) b_\infty\big)(\beta \otimes \alpha) \end{aligned}$$

m_2 on $S \otimes \mathrm{End}_R$, i.e. usual product

So the question is how to compute $\exp(\delta) b_\infty$.

Lemma Modulo m we have

$$\mathcal{Z}_\infty = \exp(-\Sigma_j At_j \theta_j) \qquad (5.1)$$

Proof Since At_i, θ_j anticommute, as operators on $K \otimes \mathrm{End}_R(k^{stab})$,

$$\begin{aligned}
\mathcal{Z}_\infty &= \Sigma_{s \geq 0} \Sigma_{p_1,\ldots,p_s} (-1)^{\binom{s+1}{2}} \frac{1}{s!} At_{p_1} \cdots At_{p_s} \theta_{p_1} \cdots \theta_{p_s} + (m)\\
&= \Sigma_{s\geq 0} \Sigma_{p_1,\ldots,p_s} (-1)^s \frac{1}{s!} (At_{p_1} \theta_{p_1}) \cdots (At_{p_s} \theta_{p_s}) + (m)\\
&= \Sigma_{s \geq 0} (-1)^s \frac{1}{s!} \left(\Sigma_j At_j \cdot \theta_j \right)^s\\
&= \exp(-\Sigma_j At_j \theta_j). \quad \square
\end{aligned}$$

Computing $\exp(\delta)\mathcal{Z}_\infty = \exp(\delta)\exp(-\Sigma_j At_j \theta_j)$ thus reduces, by Baker-Campbell-Hausdorff, to computing commutators

$$[X, [X, \ldots [Y, [Y, \ldots [X, [X, \ldots$$

where $X = \delta$, $Y = -\Sigma_j At_j \theta_j$. We define

$$\overline{At_j} = -At_j = [\psi_j^*, -] + \Sigma_q \partial_{x_j}(W^q)[\psi_q, -]$$

and

$$\rho := -\Sigma_j At_j \theta_j = \Sigma_j \overline{At_j} \theta_j.$$

$$\therefore \quad [\overline{At_j}, \psi_\ell] = \delta_{j\ell} \cdot 1 \qquad [\overline{At_j}, \psi_\ell^*] = \partial_{x_j}(W^\ell) \cdot 1$$

(see overleaf)

As operator on $\mathrm{End}_R(k^{stab})$, (ψ_i, ψ_i^* meaning left mult)

$$[[\psi_i^*, -], \psi_j](\alpha) = [\psi_i^*, \psi_j \alpha] + \psi_j [\psi_i^*, \alpha]$$
$$= \psi_i^* \psi_j \alpha + (-1)^{|\alpha|} \cancel{\psi_j \alpha \psi_i^*}$$
$$+ \psi_j \psi_i^* \alpha - (-1)^{|\alpha|} \cancel{\psi_j \alpha \psi_i^*}$$
$$= [\psi_i^*, \psi_j]\alpha$$
$$= \delta_{ij} \alpha$$

$$[[\psi_i^*, -], \psi_j^*] = 0$$

$$[[\psi_i, -], \psi_j] = 0$$

$$[[\psi_i, -], \psi_j^*] = \delta_{ij} \alpha$$

<u>lemma</u> As operator on $\mathrm{End}_R(k^{stab})$

$$[[\psi_i^*, -], \psi_j] = [[\psi_i, -], \psi_j^*] = \delta_{ij} \cdot id$$
$$[[\psi_i^*, -], \psi_j^*] = [[\psi_i, -], \psi_j] = 0.$$

<u>lemma</u> $[At_i, \psi_j] = -[[\psi_i^*, -], \psi_j] - \sum_q \partial x_i(W^q)[[\psi_q, -], \psi_j]$

$$= -\delta_{ij} \cdot id$$

$$[At_i, \psi_j^*] = -[[\psi_i^*, -], \psi_j^*] - \sum_q \partial x_i(W^q)[[\psi_q, -], \psi_j^*]$$

$$= -\partial x_i(W^j) \cdot id$$

Example In the $n=1$ case

We have, for $\omega \in S$ and $\eta \in \mathrm{End}_R(k^{stab})$

$$\psi\theta^*([\omega\otimes\eta]\cdot[\omega'\otimes\eta'])$$
$$= \psi\theta^*((-1)^{|\eta||\omega'|}\omega\omega'\otimes\eta\eta')$$
$$= (-1)^{|\eta||\omega'| + |\psi| + |\psi||\omega| + |\psi||\omega'|}\theta^*(\omega\omega')\otimes\psi\eta\eta'$$
$$= (-1)^{|\eta||\omega'| + 1 + |\omega| + |\omega'|}\theta^*(\omega\omega')\otimes\psi\eta\eta'$$

whereas

$$\psi\theta^*(\omega\otimes\eta)\cdot[\omega'\otimes\eta']$$
$$= (-1)^{|\psi| + |\psi||\omega|}[\theta^*(\omega)\otimes\psi\eta]\cdot[\omega'\otimes\eta']$$
$$= (-1)^{|\psi| + |\psi||\omega| + |\psi||\omega'| + |\eta||\omega'|}\theta^*(\omega)\omega'\otimes\psi\eta\eta'$$
$$= (-1)^{1 + |\omega| + |\omega'| + |\eta||\omega'|}\theta^*(\omega)\omega'\otimes\psi\eta\eta'$$

$$(\omega\otimes\eta)\cdot\psi\theta^*(\omega'\otimes\eta')$$
$$= [\omega\otimes\eta]\cdot(-1)^{|\psi| + |\psi||\omega'|}\theta^*(\omega')\otimes\psi\eta'$$
$$= (-1)^{|\psi| + |\psi||\omega'| + |\eta| + |\eta||\omega'|}\omega\,\theta^*(\omega')\otimes\eta\psi\eta'$$

Now $[\psi,\eta] = \psi\eta - (-1)^{|\psi||\eta|}\eta\psi$

$$= (-1)^{|\psi| + |\psi||\omega'| + \cancel{|\eta|} + |\eta||\omega'| + |\psi||\eta| + 1}\omega\theta^*(\omega')\otimes([\psi,\eta] - \psi\eta)\eta'$$

(note $|\psi| = 1$).

Hence

$$\psi\theta^*(w\otimes\eta)[w'\otimes\eta']$$

$$+ (w\otimes\eta)\psi\theta^*(w'\otimes\eta')$$

$$= (-1)^{1+|w|+|w'|+|\eta||w'|}\theta^*(w)w'\otimes\psi\eta\eta'$$

$$+ (-1)^{|w'|+|\eta||w'|} w\theta^*(w')\otimes[\psi,\eta]\psi'$$

$$+ (-1)^{1+|w'|+|\eta||w'|} w\theta^*(w')\otimes\psi\eta\eta' \qquad (12.1)$$

$$= \psi\theta^*([w\otimes\eta][w'\otimes\eta'])$$

$$+ (-1)^{|w'|(|\eta|+1)} w\theta^*(w')\otimes[\psi,\eta]\eta'$$

Now consider the odd maps

$$\theta^*\otimes 1 : S\otimes \mathrm{End} \circlearrowright$$

$$1\otimes[\psi,-] : S\otimes\mathrm{End}\circlearrowright$$

$$\Big[(1\otimes[\psi,-])\otimes(\theta^*\otimes 1)\Big]\Big([w\otimes\eta]\otimes[w'\otimes\eta']\Big) \qquad (12.2)$$

$$= (-1)^{|w|+|\eta|}(1\otimes[\psi,-])(w\otimes\eta)\otimes(\theta^*\otimes 1)(w'\otimes\eta')$$

$$= (-1)^{|w|+|\eta|+|w|}(w\otimes[\psi,\eta])\otimes(\theta^*(w')\otimes\eta')$$

$$\therefore\ m_2((12.2)) = \underbrace{(-1)^{|\eta|+(|w'|+1)(|\eta|+1)}}_{(-1)^{1+|w'|(|\eta|+1)}} w\theta^*(w')\otimes[\psi,\eta]\eta'$$

Thus

$$
\begin{aligned}
&\psi\theta^*(\omega\otimes\eta)\cdot(\omega'\otimes\eta') \\
&\quad + (\omega\otimes\eta)\cdot\psi\theta^*(\omega'\otimes\eta') \\
&= \psi\theta^*\big([\omega\otimes\eta]\cdot[\omega'\otimes\eta']\big) \\
&\quad - m_2\Big(\big([1\otimes[\psi,-]]\otimes[\theta^*\otimes 1]\big)\big([\omega\otimes\eta]\otimes[\omega'\otimes\eta']\big)\Big)
\end{aligned}
\tag{13.1}
$$

we conclude that

$$
\begin{aligned}
&(\psi\theta^*)m_2\big((\omega\otimes\eta)\otimes(\omega'\otimes\eta')\big) \\
&= m_2(\psi\theta^*\otimes 1)\big((\omega\otimes\eta)\otimes(\omega'\otimes\eta')\big) \\
&\quad + m_2(1\otimes\psi\theta^*)\big((\omega\otimes\eta)\otimes(\omega'\otimes\eta')\big) \\
&\quad + m_2\Big(\big\{[1\otimes[\psi,-]]\otimes[\theta^*\otimes 1]\big\}\big((\omega\otimes\eta)\otimes(\omega'\otimes\eta')\big)\Big)
\end{aligned}
\tag{13.2}
$$

i.e. the following commutes

$$
\begin{tikzcd}
(S\otimes\mathrm{End})\otimes(S\otimes\mathrm{End}) \arrow[r, "m_2"] \arrow[d, "\psi\theta^*\otimes 1+1\otimes\psi\theta^*+{[1\otimes[\psi,-]]\otimes[\theta^*\otimes 1]}"'] & S\otimes\mathrm{End} \arrow[d, "\psi\theta^*"] \\
(S\otimes\mathrm{End})\otimes(S\otimes\mathrm{End}) \arrow[r, "m_2"'] & S\otimes\mathrm{End}
\end{tikzcd}
\tag{13.3}
$$

<u>Lemma</u> As operator on $(S\otimes \mathrm{End})^{\otimes 2}$

$$[\psi\theta^*\otimes 1,\ 1\otimes\psi\theta^*] = 0$$

$$[\psi\theta^*\otimes 1,\ \Xi] = [1\otimes\psi\theta^*,\ \Xi] = 0$$

where

$$\Xi = (1\otimes[\psi,-])\otimes(\theta^*\otimes 1)$$

<u>Proof</u>

$$[\psi\theta^*\otimes 1, \Xi] = -(\theta^*\otimes\psi[\psi,-])\otimes(\theta^*\otimes 1) - (\theta^*\otimes[\psi,-]\psi)\otimes(\theta^*\otimes 1) \quad (14.1)$$

$$= -(\theta^*\otimes[\psi,[\psi,-]])\otimes(\theta^*\otimes 1) = 0.$$

But $[\psi,[\psi,-]]=0$ as operator on End (ψ meaning $\psi\wedge -$), and $\psi\theta^* = (1\otimes\psi)\circ(\theta^*\otimes 1)$ on $S\otimes\mathrm{End}$. Also

$$[1\otimes\psi\theta^*, \Xi] = (1\otimes(1\otimes\psi))\circ(1\otimes(\theta^*\otimes 1))\circ((1\otimes[\psi,-])\otimes(\theta^*\otimes 1))$$

$$- ((1\otimes[\psi,-])\otimes(\theta^*\otimes 1))\circ(1\otimes(1\otimes\psi))\circ(1\otimes(\theta^*\otimes 1))$$

$$= 0$$

for more trivial reasons, since $\theta^*\theta^* = 0$. $\square$

we may compute (for $n=1$)

$$b_2 = \pi \exp(-\psi\theta^*) m_2\big(\exp(\psi\theta^*) z_\infty \otimes \exp(\psi\theta^*) z_\infty \big)$$

$$= \pi\, m_2\big(\exp(\psi\theta^*) z_\infty \otimes \exp(\psi\theta^*) z_\infty \big)$$

$$\quad - \pi(\psi\theta^*) m_2\big(\quad \text{''} \quad \big) \qquad (15.1)$$

$$\overset{(13.3)}{=} \pi m_2\big([1+\psi\theta^*][1-A^\dagger\theta] \otimes [1+\psi\theta^*][1-A^\dagger\theta] \big)$$

$$- \pi m_2\Big(\big\{ (\psi\theta^* \otimes 1) + (1 \otimes \psi\theta^*) + [1 \otimes [\psi,-]] \circ [\theta^* \otimes 1] \big\} \big([1+\psi\theta^*][1-A^\dagger\theta] \otimes [1+\psi\theta^*][1-A^\dagger\theta] \big) \Big)$$

Hence given $\beta_1, \beta_2 \in \underline{\mathrm{End}}$, (15.2)

$$b_2(\beta_1 \otimes \beta_2) = \pi m_2\big([1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_1) \otimes [1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_2) \big)$$

$$\begin{aligned} - \pi m_2\Big(& \psi\theta^*[1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_1) \otimes [1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_2) \\ + & [1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_1) \otimes \psi\theta^*[1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_2) \\ + & (-1)^{|\beta_1|} [\psi,-]\big([1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_1)\big) \\ & \otimes \theta^*[1 - A^\dagger\theta + \psi\theta^* - \psi\theta^* A^\dagger\theta](\beta_2) \Big) \end{aligned}$$

(12)

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~~Now~~ $\psi\theta^* A_t\theta = -\psi A_t \theta^*\theta$

$= -\psi A_t$ acting on something killed by θ^*

$$= \pi m_2\left([\beta_1 + \psi A_t(\beta_1)] \otimes [\beta_2 + \psi A_t(\beta_2)] \right)$$

$$- \pi m_2\Big([\cancel{\psi\theta^*} - \psi\theta^* A_t\theta](\beta_1) \otimes [\beta_2 + \psi A_t(\beta_2)]$$

$$+ [\beta_1 + \psi A_t(\beta_1)] \otimes [\cancel{\psi\theta^*} - \psi\theta^* A_t\theta](\beta_2)$$

$$+ (-1)^{|\beta_1|} [\psi, \beta_1 + \psi A_t(\beta_1)] \otimes [-\theta^* A_t\theta](\beta_2) \Big)$$

$$= (\beta_1 + \psi A_t(\beta_1)) \cdot (\beta_2 + \psi A_t(\beta_2))$$

$$- \pi m_2\Big(\psi A_t(\beta_1) \otimes [\beta_2 + \psi A_t(\beta_2)]$$

$$+ [\beta_1 + \psi A_t(\beta_1)] \otimes \psi A_t(\beta_2)$$

$$+ (-1)^{|\beta_1|} [\psi, \beta_1 + \psi A_t(\beta_1)] \otimes A_t(\beta_2) \Big)$$

$$= (\beta_1 + \psi A_t(\beta_1)) \cdot (\beta_2 + \psi A_t(\beta_2))$$

$$- \psi A_t(\beta_1) \cdot \beta_2 - \psi A_t(\beta_1) \cdot \psi A_t(\beta_2)$$

$$- \beta_1 \cdot \psi A_t(\beta_2) - \psi A_t(\beta_1) \cdot \psi A_t(\beta_2)$$

$$- (-1)^{|\beta_1|} [\psi, \beta_1 + \psi A_t(\beta_1)] \cdot A_t(\beta_2)$$

$$= \beta_1 \cdot \beta_2 - \psi At(\beta_1) \cdot \psi At(\beta_2)$$

$$- (-1)^{|\beta_1|}[\psi, \beta_1] - (-1)^{|\beta_1|}[\psi, \psi At(\beta_1)] \cdot At(\beta_2)$$
$$\cdot At(\beta_2)$$

But $[\psi, \psi At(\beta_1)] = -(-1)^{|\beta_1|} \psi At(\beta_1)\psi$ so

$$= \beta_1 \cdot \beta_2 - (-1)^{|\beta_1|}[\psi, \beta_1] \cdot At(\beta_2)$$

we can explain this as follows: by (13.3)

$$\exp(-\psi\theta^*) m_2 = m_2\big(\exp(-[1 \otimes [\psi, -]] \otimes [\theta^* \otimes 1]) \circ \exp(-\psi\theta^*) \otimes \exp(-\psi\theta^*) \big)$$

applied to (15.1) this yields

$$b_2(\beta_1 \otimes \beta_2) = \pi \exp(-\psi\theta^*) m_2\big(\exp(\psi\theta^*) b_\infty(\beta_1) \otimes \exp(\psi\theta^*) b_\infty(\beta_2)\big)$$
$$= \pi m_2\big(\exp(-[1 \otimes [\psi, -]] \otimes [\theta^* \otimes 1])(b_\infty(\beta_1) \otimes b_\infty(\beta_2))\big)$$
$$= \pi m_2\big([1 - At\theta](\beta_1) \otimes [1 - At\theta](\beta_2) \big)$$
$$- \pi m_2\big([\psi, (1 - At\theta)(\beta_1)] \otimes \theta^*(1 - At\theta)(\beta_2) \big)$$
$$(-1)^{|\beta_1|}$$
$$= \beta_1 \cdot \beta_2 - (-1)^{|\beta_1|}[\psi, \beta_1] \cdot At(\beta_2) \text{ as above.}$$

Trees and operators

The formula for the higher multiplications ρ_T involves operators r_2 where we would prefer m_2. This we may do at the cost of signs, and our aim in the following is to pay this price. We refer to ainfmf14 for details on how we convert a tree T to a map

$$\rho_T : (\underline{\mathrm{End}}[1])^{\otimes q} \longrightarrow \underline{\mathrm{End}}[1] \qquad (14.1)$$

We denote the degree of $\alpha \in \underline{\mathrm{End}}$ by $|\alpha|$ and its shifted degree as an element of $\underline{\mathrm{End}}[1]$ by $\tilde{\alpha} = |\alpha| - 1$. Given a tree T with q leaves and $\alpha_1, \ldots, \alpha_q \in \underline{\mathrm{End}}[1]$ we have a sequence of degrees

$$\underline{\ell} := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q).$$

Given T and inputs conforming to given $\underline{\ell} \in \mathbb{Z}_2^q$, we want to know the sign $S(T, \underline{\ell})$ such that

$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = (-1)^{S(T,\underline{\ell})} \mathrm{eval}_{\hat{T}}(\alpha_q \otimes \cdots \otimes \alpha_1)$$

where $\mathrm{eval}_{\hat{T}}$ is the linear map $\underline{\mathrm{End}}^{\otimes q} \to \underline{\mathrm{End}}$ obtained from the operator decorated tree $\hat{T}$ by ⌜in the terminology of ainfcat2, $A(T)^\wedge$⌟

① Replacing all r_2's with m_2's

② Computing the valuation of the diagram associated to the tree, viewing everything as <u>ungraded</u> (i.e. omitting Koszul signs).

Here $\hat{T}$ is the operator decorated tree obtained from T by swapping left and right input trees at each trivalent vertex, i.e.

$$(L, R)^\wedge = (R^\wedge, L^\wedge).$$

This element $s(T, \underline{\ell}) \in \mathbb{Z}_2$ has three contributions

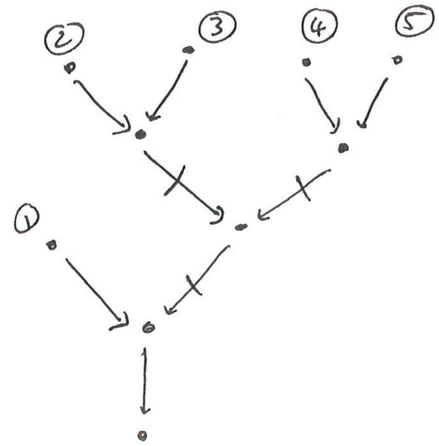
1. The number of internal edges of the tree
2. ~~Koszul signs from moving the inputs a_i to their leaf nodes and from evaluating the graded diagram.~~ as of 1/11/2016 these signs no longer contribute.
3. Converting r_2 to m_2.

Let us consider 3 first.

<u>Lemma</u> The sign generated by converting all r_2's in T to m_2's is

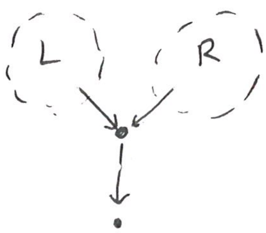
$$\sum_{1 \le i < j \le q} \ell_i \ell_j + \sum_{i=1}^{q} \ell_i P_i + q + 1 \pmod 2) \quad (15.1).$$

where the <u>parity</u> P_i of the ith leaf (counting from the left) is the number of times the path from that leaf to the root enters a trivalent vertex on the right, e.g. (all $P_i \in \mathbb{Z}_2$)



$$P_1 = 0 \quad P_2 = 1 \quad P_3 = 0 \quad P_4 = 0 \quad P_5 = 1 \qquad (15.2)$$

<u>Proof</u> At a given trivalent vertex



(15.3)

ainfmf2

(16)

Since trivalent vertices and internal edges come in pairs (with the exception of the final trivalent vertex) the sign contribution from r_2 at a vertex (15.2) is $\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1$ where

$$\tilde{\alpha} = \text{total tilde degree of all inputs in } L \qquad (16.1)$$

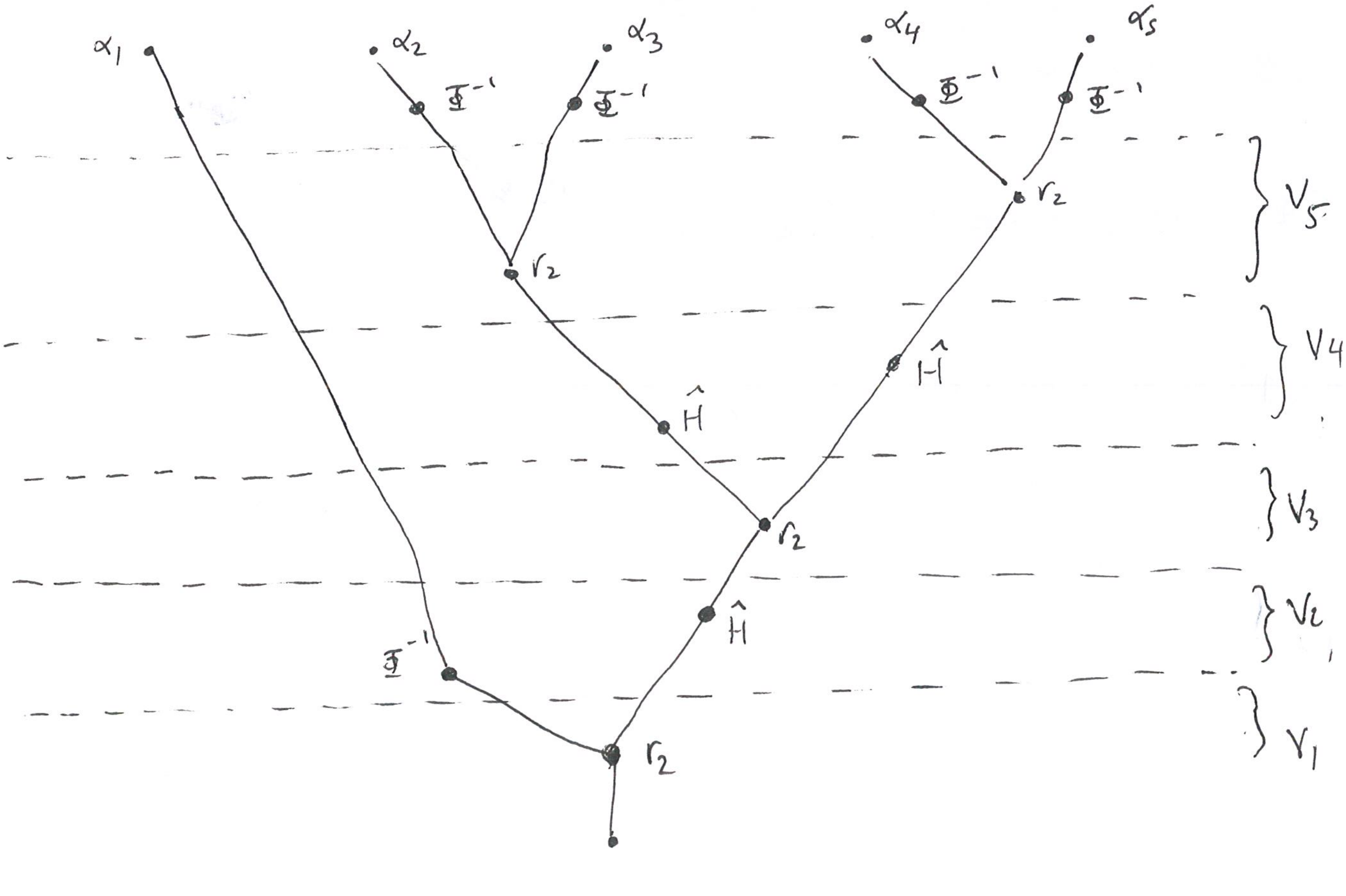
$$\tilde{\beta} = \quad " \qquad " \qquad " \quad \text{in } R$$

The $\tilde{\alpha}\tilde{\beta} + \tilde{\beta}$ term gives (15.1) while the $+1$'s count the number of r_2 vertices in T, which is # int. edges + 1, i.e. $q - 2 + 1 = q - 1$. □

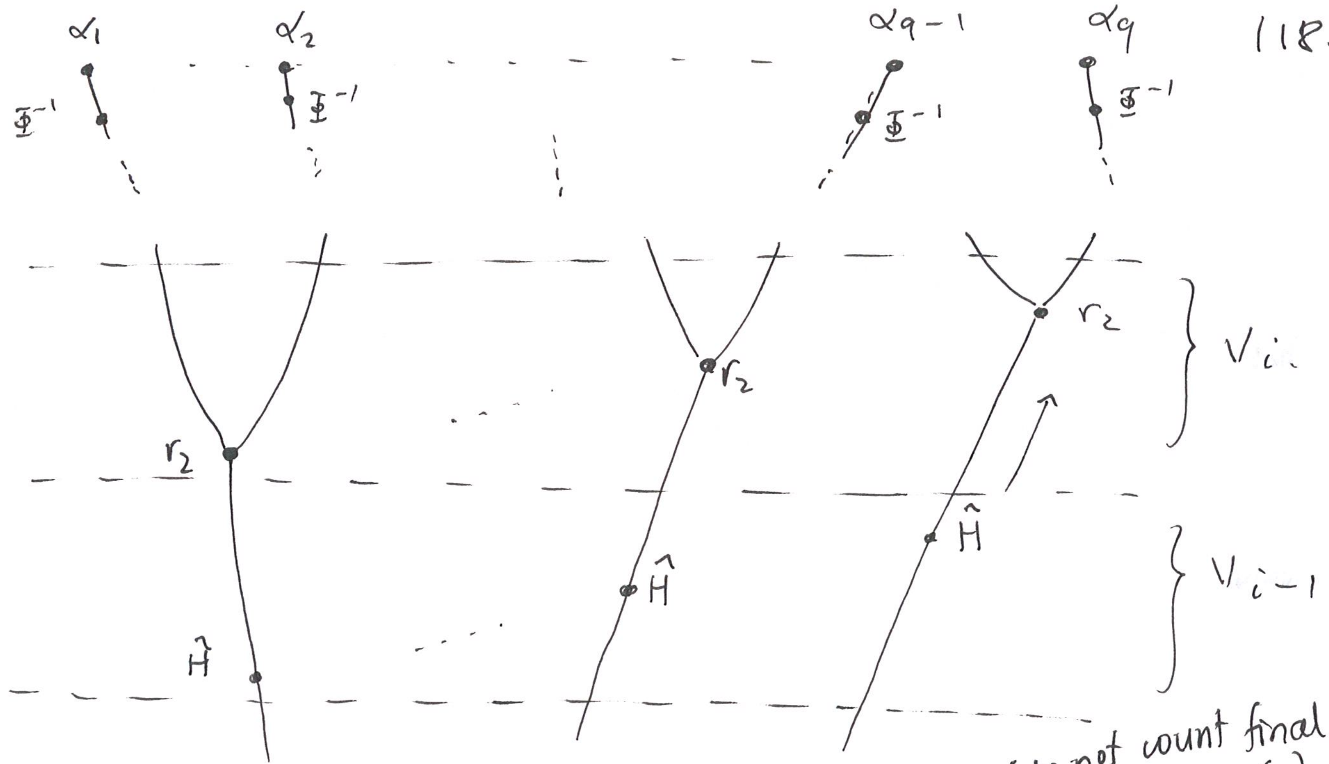
1/11/2016 ignore! ——— / plus int edges

Now we turn to (2). Partition the internal vertices by their distance from the root (counted in edges), and write V_d for vertices at distance d. Evaluating $\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q)$ means evaluating the following string diagram in the graded sense of (ainfmf14) where within each indicated stratum dots on the left are placed lower than dots on the right

(16.2)



Clearly then to resolve the general sign we just have to pair up all $\hat{H}$'s with an r_2. In a tree (15.2) we may ignore the placement of Φ^{-1}'s since they are even. With this in mind an arbitrary stratified string diagram (16.2) will have alternating layers of $\hat{H}$'s and r_2's.



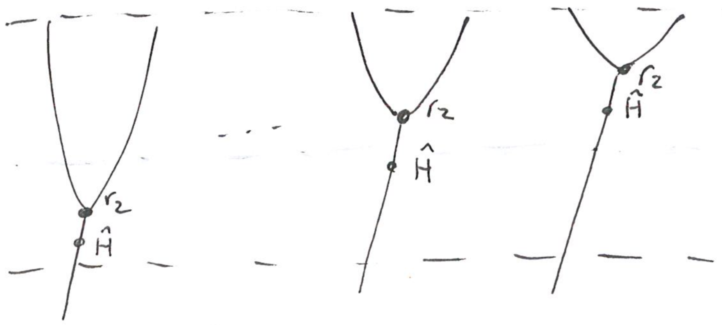
(18.1)

(do not count final r_2)

Suppose that in V_i for i odd there are N_i r_2's. So $\sum_i N_i = q-2$. The cost of joining the rightmost $\hat{H}$ with its sweetheart r_2 is $(-1)^{N_i-1}$. The cost of joining the next $\hat{H}$ to its r_2 is $(-1)^{N_i-2}$ and so on, so "fixing" all $\hat{H}r_2$ pairs in the layers V_{i-1}, V_i costs

$$(-1)^{(N_i-1)+(N_i-2)+\cdots+1} = (-1)^{\binom{N_i}{2}} \qquad (18.2)$$

This leaves



$\cdot\, (-1)^{\binom{N_i}{2}}$

Once we have done this to all layers there are no further interventions of signs in the evaluation of ρ_T. So the total contribution of signs of type ② is

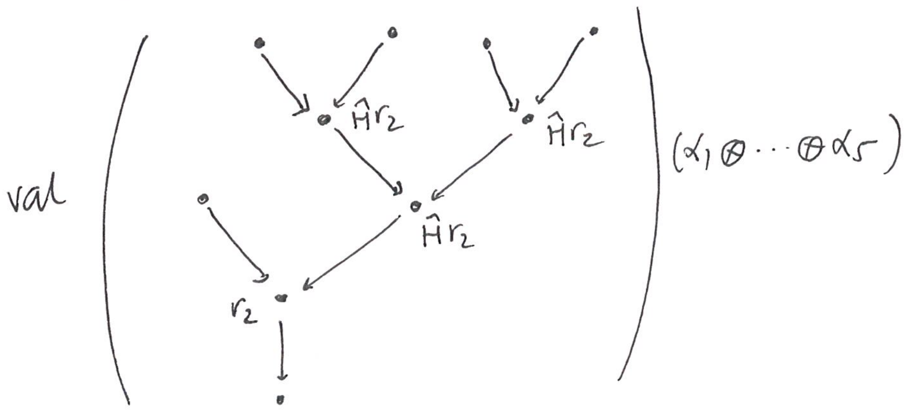
$$\sum_{\substack{i \text{ odd} \\ i>1}} \binom{N_i}{2} \qquad (\text{mod } 2)$$

To summarise: for a fixed tree T we have shown

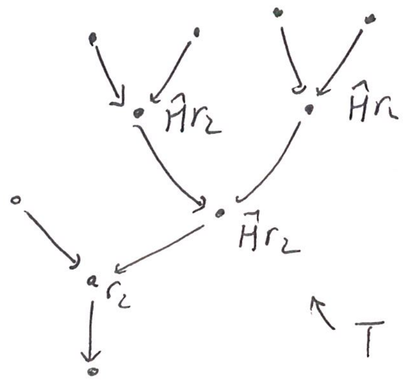
$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = (-1)^{\overset{①}{(q-2)} + \overset{②}{\sum_{\substack{i \text{ odd} \\ i>1}} \binom{N_i}{2}}}$$

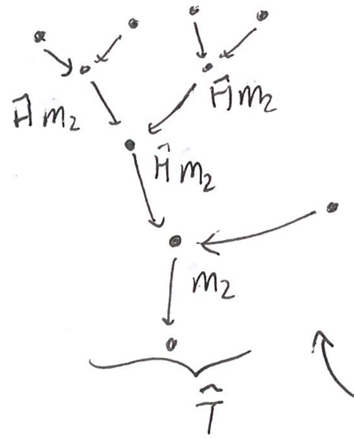
{the diagram of T evaluated with r_2's joined to $\hat{H}$'s in the form of (18.2)}
on input $\alpha_1 \otimes \cdots \otimes \alpha_q$

i.e. T with $\hat{H}r_2$ at all internal vertices, nothing on internal edges



Now when we evaluate all these r_2's we pick up the sign (15.1) and we also change T to $\hat{T}$ (the mirror) so that for example

val  $(\alpha_1 \otimes \cdots \otimes \alpha_5)$

$=$ val  $(\alpha_5 \otimes \cdots \otimes \alpha_1)$

NOTE This tree has leaves End and interior v.spaces $S \otimes$ End, i.e. all shifts have been removed.

$= \mathrm{eval}_{\hat{T}}(\alpha_5 \otimes \cdots \otimes \alpha_1)$

This shows:

1/11/2016 ignore

(20.1)

Theorem With the notation of p. 14

$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = \cancel{(-1)^{(q-2) + \sum_{\substack{i \text{ odd} \\ i > 1}} \binom{N_i}{2} + \sum_{1 \le i < j \le q} l_i l_j + \sum_{i=1}^{q} l_i P_i + q + 1}}$$

$$\mathrm{eval}_{\hat{T}}(\alpha_q \otimes \cdots \otimes \alpha_1)$$

$$(-1)^{e_i(T) + \sum_{1 \le i < j < q} l_i l_j + \sum_{i=1}^{q} l_i P_i + q + 1}.$$

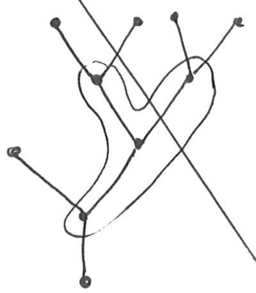
That is,

$$S(T, \underline{l}) = 1 \cancel{+} \cancel{\sum_{\substack{i \text{ odd} \\ i \geq 1}} \binom{N_i}{2}}^{\,e_i(T)} + \sum_{1 \leq i < j \leq q} l_i l_j + \sum_{i=1}^{q} l_i P_i + q + 1 \qquad (21.1)$$

where $N_i \geq 0$ is the number of r_2 vertices a distance i from the root (counting with internal edges divided into 2), and P_i is the parity of the path from the ith input to the root.

If we define more sensibly

<u>Def^{N}</u> Given a planar rooted binary tree T, let M_j be the number of internal vertices a distance j from the root, where $j \geq 1$ and we count by edge distance



$M_1 = 1$

$M_2 = 1$

$M_3 = 2$

$M_j = 0$ for $j > 3$.

$$S(T, \underline{l}) = 1 + \underbrace{\sum_{j \geq 1} \binom{M_j}{2}}_{\text{depends only on } T} + \underbrace{\sum_{1 \leq i < j \leq q} l_i l_j}_{\text{depends only on } \underline{l}} + \underbrace{\sum_{i=1}^{q} l_i P_i}_{\text{depends on } T, \underline{l}} \qquad (21.2)$$