We continue from \( \text{ainfmf7} \) in making the relationship between the Feynman rules and final \( \rho_q \)’s more detailed. This amounts to a discussion of signs (lucky me). But we begin with a survey of our notes up to now.

Let \( k \) be a characteristic 0 field. For \( W \in k[x_1, \ldots, x_n] \) with \( W \in \mathfrak{m}^2 \) and \( W = \sum_i x_i W_i \) we set up a deformation retract (\( R = k[\varepsilon] \))

\[
\hat{H} \subset S \otimes_k \text{End}_R(k^{stab}) \xrightarrow{\varphi^{-1}} \text{End}(k^{stab})
\]

\[
\left( \text{End}_k(\wedge(k y_1 \otimes \cdots \otimes k y_n)), 0 \right)
\]

in \( \text{ainfmf7} \). Since the RHS computes the cohomology of the LHS, which is a DG-algebra (exterior product on \( S = \wedge(k y_1 \otimes \cdots \otimes k y_n) \)), we can apply the standard \( A_oo \)-minimal model construction to define higher multiplications \( \rho_q \) on \( \text{End}(k^{stab}) \). The case of \( W = x^d \) was done in \( \text{ainfmf7} \). In principle these computations involve trees, e.g.

![Diagram](attachment:image.png)

where \( m_2 \) is the multiplication on \( S \otimes_k \text{End}_R(k^{stab}) \).
In (ainfmf4), we explained how this is simply (see p. 9).

\[ H_\infty = \sum_{m=0}^{\infty} (-1)^m (H a_\text{End})^m H \quad H = T^{-1} \mathcal{V} \]

\[ b_\infty = \sum_{m=0}^{\infty} (-1)^m (H a_\text{End})^m b \quad (2.2) \]

\[ \Xi = \sum_{i=1}^{n} [\psi_i, -] \otimes \partial_i^* \]

Unlike our other papers, the $T^{-1}$ symmetry factors have to be dealt with here. The case of two variables was done in (ainfmf4), and used to compute $\rho_3$ for $W = y^4 - x^4$. To proceed in the general case it turns out to simplify much greatly if we assume $W \in \mathbb{R}^3$, which we now do. Then (ainfmf6) explains some calculations relevant to showing that the Clifford action on $\text{End}(k^{\text{stable}})$ from (1.1) is (by p. 21 (ainfmf3))

\[ \text{End}(k^{\text{stable}}) \otimes \psi_i^* = A \psi_i = - [\psi_i^*, -] \]

\[ \psi_i = - \psi_i^* \]
and by p. 5 \( \text{ainfmf4} \) the idempotent

\[
\tau_1^+ \ldots \tau_n^+ \tau_n \ldots \tau_1
\]

projects onto \((k \cdot 1) \otimes (k \psi_1^* \otimes \ldots \otimes k \psi_n^*)\), i.e. we can define

\[
\mathcal{A} := \bigwedge (k \psi_1^* \otimes \ldots \otimes k \psi_n^*)
\]

and view it as a subspace of \( \text{End} (k^{\text{stab}}) \) (via \( \psi_i^* \mapsto \psi_i^* \circ - \)). The induced \( A_\infty \)-structure on \( \mathcal{A} \) is

\[
\mathcal{A}[1] \otimes q \longrightarrow \text{End}([1]) \otimes q \longrightarrow \text{End}([1]) \longrightarrow \mathcal{A}[1]
\]

and we claim (TODO: check) this \( A_\infty \)-algebra \((\mathcal{A}, \otimes q)\) is the minimal model of \( \text{End} R (k^{\text{stab}}) \). Now in practice we will find that \( p_q \) preserves the subspace \( \mathcal{A} \otimes q \) so \( p_q = p_q \big|_{\mathcal{A} \otimes q} \) (i.e. \( p_q \) sends \( \mathcal{A} \otimes q \) to \( \mathcal{A} \subset \text{End}([1]) \)), and we compute \( b_q \) by simply evaluating \( b_q \) (i.e. diagrams like \((2.1)) on inputs which are products of operators \( \psi_i^* \). In fact these are the only calculations we do.

In \( \text{ainfmf7} \) we began to sketch the Feynman rules, but this only became serious in \( \text{ainfmf9} \). The product \( b_2 \) on \( \mathcal{A} \) is just the product in the exterior algebra, by \((8.2) \) \( \text{ainfmf4} \) (where only the \( t=0 \) term survives as a consequence of our assumption that \( \text{Wen}^3 \)). So we only consider Feynman diagrams for \( q \geq 3 \).
By definition (20.3) for $\Lambda_i \in \mathcal{A}$

$$
\rho_{i_1}(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_T (-1)^{|T|} \sum_{j \in \mathcal{C}(T)} O(T, \mathcal{C})(\Lambda_q \otimes \cdots \otimes \Lambda_1). \quad (4.1)
$$

Our aim in this note is to express this in terms of the amplitudes

$$
O(T, \mathcal{C})(\Lambda_q \otimes \cdots \otimes \Lambda_1)_{\text{const}} \in \mathcal{K} \quad (4.2)
$$

for $T$, $\mathcal{C}$, and $\Lambda_i \in \mathcal{A}$. 

Note when evaluating

\[ 0(T, C)(\Lambda_9 \otimes \cdots \otimes \Lambda_1) \]

there are no Koszul signs for moving the \( \Lambda_i \) to the appropriate "holes" in \( 0(T, C) \), i.e. the relevant string diagram is evaluated as a diagram of ungraded operators.
This is just standard QFT nonsense. With $q$, $T$, $k$ fixed as in the above, let $\Psi_1, \ldots, \Psi_q \in \Lambda(k \Psi_1^* \oplus \cdots \oplus k \Psi_n^*)$ be given. Suppose these are "basis elements" i.e. product of $\Psi_i^*$'s. For another basis elt $\Psi_{out}$ we show how to write

$$O(T, \mathcal{E})(\Psi_1 \otimes \cdots \otimes \Psi_q) \Psi_{out} \in k$$

(6.1).

as a sum of amplitudes in the sense of (6.2). We identify the basis elements of $\Lambda(k \Psi_1^* \oplus \cdots \oplus k \Psi_n^*)$ with subsets of $\{1, \ldots, n\}$ in the obvious way, writing $\Psi_i^*$ for the product (in increasing order) of $\Psi_i^*$ for $i \in I$. Let us say $\Psi_i = \Psi_i^*$ for $1 \leq i \leq q$ and $\Psi_{out} = \Psi_J^*.$

**Def** A fitting of the tuple $(I_1, \ldots, I_q)$ with respect to $J$ is a collection of subsets (possibly empty) $K_i \subseteq I_i$ for $1 \leq i \leq q$ such that

(i) The $K_i$ are pairwise disjoint

(ii) $\bigcup_i K_i = J$

(6.2)

Given a fitting we write $\overline{K_i} := I_i \setminus K_i$. The sign of the fitting is the sign of the permutation required to rearrange the sequence

$$I_1, \ldots, I_q$$

(each in increasing order)

(6.3)

to

$$J, \overline{K_1}, \ldots, \overline{K_q}$$

(each in increasing).
Note In (6.3) obviously indices from \( \{1, \ldots, n\} \) may be repeated. But still the definition is unambiguous: tag an integer \( a \in \{1, \ldots, n\} \) appearing in \( T_i \) as \( a_i \). This must end up in either \( K_i \) or \( \Omega \).

We write \( F \) for a fitting and \( |F| \in \mathbb{Z}_2 \) for the sign.

The configuration \( C \) prescribes (see p. 17) integers \( m(x) \), subscripts \( T(x) \) and tuples \( (a_j(x), x_j(x)) \), integers \( t(x) \) to locations \( x \) in the tree. From these assignments is constructed an operator, e.g. (15.2) from the ingredients listed in (18.2), (18.3), (19.1) there. Omitting signs and prefactors (which we may do as they depend only on poly/\( \Omega \)-weight) we can write (15.2) schematically as

\[
\prod_i m_i \prod_j \left[ \chi_i, \chi_j \right] \left( \chi_i, \chi_j \right) \otimes \ldots
\]

The way we draw diagrams, according to p. 21, 22, 24 of (9infmfr), is arranged so that contractions are made for leftmost channels first, (where e.g. (5.3) has three channels, numbered from left to right) so that all the fermions in \( \Psi_1 \) are commuted to the left and annihilated with an interaction vertex before anything in \( \Psi_i \) for \( i > 2 \). (Assuming vacuum boundary conditions, i.e. that we are computing amplitudes in (5.2)).
We may therefore compute

$$\mathcal{O}(T, \mathcal{C})(\Psi_1 \otimes \cdots \otimes \Psi_q)$$  

(8.1)

by commuting all input fermions leftwards, and as usual we get this way a sum over all possible contractions, plus terms where \( \Psi^*_e \)'s actually make it all the way through to the left.

Now, as \( x \) ranges over all locations in \( T \), form the set

$$\bigcup_x T(x)$$  

(8.2)

which we view as a multi-set of elements of \( \{1, \ldots, n\} \) (i.e. integers in this range with multiplicity). The sum which computes (8.1) is a sum of terms \( \Psi_{j_1}^* \cdots \Psi_{j_t}^* \alpha \) where \( \alpha \) is a product of commutators.

We see that

$$t + \# \text{ commutators} = |T_1| + \cdots + |T_q|$$

(8.3)

**type**

( of \( [\Psi_e, \Psi_*] \)

input number

There are of course (many) other commutators which make up \( \alpha \).

Let us now concentrate on a particular summand of (8.1), namely the one where \( \psi_{j_1}^* \cdots \psi_{j_t}^* = \Psi_{\text{out}} \), i.e. \( T = \{ j_1, \ldots, j_t \} \).

Since the Feynman rules only consume \( \Psi \)'s and never produce them (see (24.2) of \( \text{ainfntq} \)) the number of \( \Psi \)-commutator is the same in each summand of \( \alpha \) (these summands, of course, represent individual Feynman diagrams). For this reason if we were to draw the diagrams, \( \psi_{j_1}^* \cdots \psi_{j_t}^* \) would just sail straight through from input to output.
The coefficient \( O ( T, C ) \langle \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k \rangle \) may therefore be computed by enumerating all Feynman diagrams where this happens:

1. Choose which inputs are going to survive in the output. This is precisely the data of a fitting \( F \) of \(( I_1, \ldots, I_k) \) relative to \( J \).

2. Commute these inputs all the way to the left and into the ascending order. In the operator (7.1) the only odd things we encounter in this process are from \( H_{\infty} \) interactions on internal edges, which have degree +1. The number of internal edges a fermion encounters on its voyage from its input point (say with \( \mathcal{Y}_i \) of (7.1)) to the far left, e.g.

\[
\begin{align*}
\text{is the number of internal edges (identified with "---" marks) whose path to root is either a subset of the fermions, or which meets the fermions path via a left leg of a trivalent vertex.}
\end{align*}
\]

Call this number \( V_i \) if \( \mathcal{Y}_k \) is placed at the \( i \)th vertex from the left.
If we commute the surviving fermions in $\Psi_1$, $\Psi_2$, and so on, we pick up signs from moving the surviving fermions in $\Psi_2$ ($\Psi_{k_2}^*$) past the "dying" fermions in $\Psi_1$ ($\Psi_{k_1}^*$). These signs are the sign of the permutation:

$$
\begin{array}{cccc}
I_1 & \cdots & I_2 \\
K_1 K_1 & K_2 K_2 & \cdots & K_2 K_2 \\
J & K_1 & K_2 & \cdots & K_L \\
K_1 & K_2 & \cdots & K_q \\
\end{array}
$$

i.e. the sign of the fitting $F$.

After this the calculation is identical to those in $a\text{in}f\text{m}f8$ or $a\text{in}f\text{m}f10$. 

This proves that

\[ O(T, \mathcal{C}) (\psi_1 \otimes \cdots \otimes \psi_q) \eta_{\text{out}} \]

\[ = \alpha \sum_{i=1}^{q} |K_i| \cdot V_i + 1F1 \]

\[ = \sum_{\text{fittings } F} (-1)^{|F|} O(T, \mathcal{C}) \left( \psi_{K_1}^* \otimes \cdots \otimes \psi_{K_q}^* \right)_{\text{const}} \]

upward vacuum boundary conditions.

Another way to write this is for subject \( i_1, \ldots, i_q \leq \{1, \ldots, n\} \)

\[ O(T, \mathcal{C}) \left( \psi_{i_1}^* \otimes \cdots \otimes \psi_{i_q}^* \right) \]

\[ = \sum_{J \subseteq \{1, \ldots, n\}} \sum_{\text{fittings } F \text{ of } (i_1, \ldots, i_q) \text{ rel. to } J} \left( -1 \right)^{|F|} O(T, \mathcal{C}) \left( \psi_{K_1}^* \otimes \cdots \otimes \psi_{K_q}^* \right)_{\text{const}} \cdot \psi_{J}^* \]

(note the reverse ordering, see (ext k) p. 4)

**Defn:** Given \( K \subseteq \{1, \ldots, n\} \) with \( K = \{ k_1, \ldots, k_t \} \) we write

\[ \psi_{K \setminus \{ \cdot \}} (\cdot) = [\psi_{k_t} \psi_{k_{t-1}} \cdots [\psi_{k_1}, -]] \]

as an operator on \( \wedge (k \psi_1^* \otimes \cdots \otimes k \psi_n^*) \), for \( K = \emptyset \) this is \( \text{id} \).
Let $A_1, \ldots, A_q \subseteq \{1, \ldots, n\}$ be given, and $\psi_1, \ldots, \psi_q$ in $\wedge (k\psi^*_1 \otimes \cdots \otimes k\psi^*_n)$. Since $\gamma_{A_i} \downarrow$ has Koszul degree $|A_i|$ if $\psi_i = \gamma_{\overline{A_i}}^*$ and so

$$m\left\{ (\gamma_{A_1} \downarrow \psi_i) \otimes \cdots \otimes (\gamma_{A_q} \downarrow \psi_i) \right\} (\psi_1 \otimes \cdots \otimes \psi_q)$$

$$= (-1)^{|A_q|(|I_1|+\cdots+|I_{q-1}|)+|A_{q-1}|(|I_1|+\cdots+|I_{q-2}|)+\cdots+|A_1|(|I_1|+\cdots+|I_0|)} \cdot (\gamma_{A_1} \downarrow \psi_1) \cdots \cdot (\gamma_{A_q} \downarrow \psi_q)$$

Now if we write $A_i := \overline{K_i}$ then $(K_i = I_i \setminus A_i)$ may restrict to $A_i \subseteq I_i$:

$$\gamma_{\overline{K_i}}^* \downarrow \gamma_{I_i}^* = (-1)^{s_i \cdot \gamma_{\overline{K_i}}^*}$$

where $s_i$ is the sign of $I_i \rightarrow \overline{K_i} \cdot K_i$. Hence (11.1) equals

$$m\left\{ (\gamma_{A_1} \downarrow \psi_i) \otimes \cdots \right\} (\psi_1 \otimes \cdots \otimes \psi_q)$$

$$= (-1)^{|A_q|(|I_1|+\cdots+|I_{q-1}|)+|A_{q-1}|(|I_1|+\cdots+|I_{q-2}|)+\cdots+|A_1|(|I_1|+\cdots+|I_0|)} \cdot \gamma_J^*$$

where $J = K_1 \cdots K_q$ arranged in ascending order, and $c$ the parity of the permutation which achieves this.
The sign in (11.2) is the sign of the permutation
\[
\begin{pmatrix}
I_1 & \ldots & I_q \\
\bar{K}_1 K_1 & \bar{K}_2 K_2 & \ldots & \bar{K}_q K_q \\
\bar{K}_q K_{q-1} & \bar{K}_q K_{q-1} & \ldots & \bar{K}_q K_1 \\
\bar{K}_q & \bar{K}_1 K_1 & \ldots & K_q \\
\bar{K}_1 & \ldots & K_1 & K_q
\end{pmatrix}
\] (12.1)

The fitting \( F \) determined by \( K_1, \ldots, K_q \) differs from (12.1) in its parity by \( \sum_{i<j} |A_i||A_j| \) (to arrange the \( \bar{K}_i \)) plus \( |J| \cdot \sum_i |A_i| \) (to put \( J \) on the left). Hence
\[
(11.2) = (-1)^{1|F| + \sum_{i<j} |A_i||A_j| + |J| \cdot \sum_i |A_i|} \mathcal{Y}_J. \quad (12.2)
\]

Combining this with (10.2) we obtain
\[
\mathcal{O}(T, \mathcal{C})\left(\psi_1 \otimes \cdots \otimes \psi_q\right)
= \sum_{J \subseteq \{1, \ldots, n\}} \sum_{\text{fittings } F \text{ of } (I_{y_1}, \ldots, I_{y_q}) \text{ rel. to } J} (-1)^{i} \cdot m\left\{ \left(\gamma_{k_1}^{*^J} - \right) \otimes \cdots \otimes \left(\gamma_{k_q}^{*^J} - \right) \right\} \left(\psi_1 \otimes \cdots \otimes \psi_q\right) \cdot \mathcal{O}(T, \mathcal{C})\left(\gamma_{k_1}^{\ast} \otimes \cdots \otimes \gamma_{k_q}^{\ast}\right) \text{const}
\]

We want to remove the dependence on $J$ and write this as a sum over $k_i$'s. More precisely we want no dependence on the $I_i$'s (the inputs).

\[
= \sum_{A_1, \ldots, A_q \in \{1, \ldots, n\}} (-1)^{i} \sum_{i \in A} (1_{y_i} \otimes |A_i|) \psi_i + \sum_{i \in A} |A_i| |A_i| + \sum_{i \in A} \cdot \sum_{j} (1_{y_j} \otimes |A_j|) \cdot m\left\{ \left(\gamma_{A_1}^{*^J} - \right) \otimes \cdots \otimes \left(\gamma_{A_q}^{*^J} - \right) \right\} \left(\psi_1 \otimes \cdots \otimes \psi_q\right) \cdot \mathcal{O}(T, \mathcal{C})\left(\gamma_{A_1}^{\ast} \otimes \cdots \otimes \gamma_{A_q}^{\ast}\right) \text{const}
\]

Combining this with (4.1) requires some care, e.g. $P_i$ is an invariant of $T$ while $V_i$ is associated to $T^J$ (in that context).