Minimal models 12 — NonVacuum (checked)

ainfmf12

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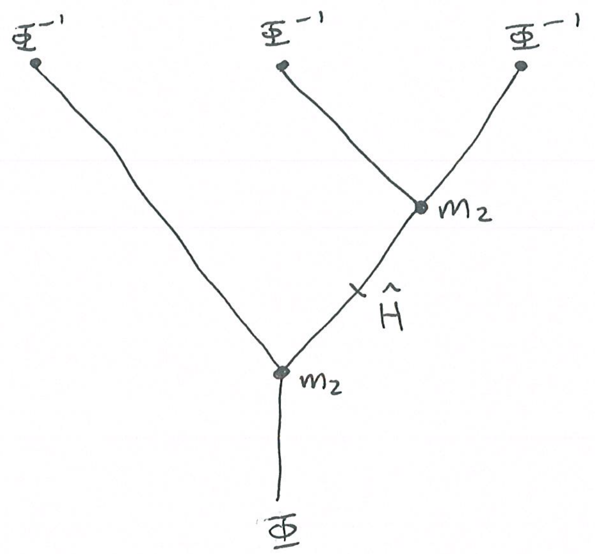
We continue from ainfmf9 in making the relationship between the Feynman rules and final ρ_q's more detailed. This amounts to a discussion of signs (lucky me). But we begin with a survey of our notes up to now.

Let k be a char. 0 field. For $W \in k[x_1, \ldots, x_n]$ with $W \in \mathfrak{m}^2$ and $W = \sum_i x_i W^i$ we set up a deformation retract ($R = k[\underline{x}]$)

$$\hat{H} \circlearrowright \; S \otimes_k \operatorname{End}_R(k^{stab}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \underline{\operatorname{End}}(k^{stab}) \qquad (1.1)$$

$$\underline{\operatorname{End}}(k^{stab}) = \left(\operatorname{End}_k\left(\Lambda(k\psi_1 \oplus \cdots \oplus k\psi_n) \right), 0 \right)$$

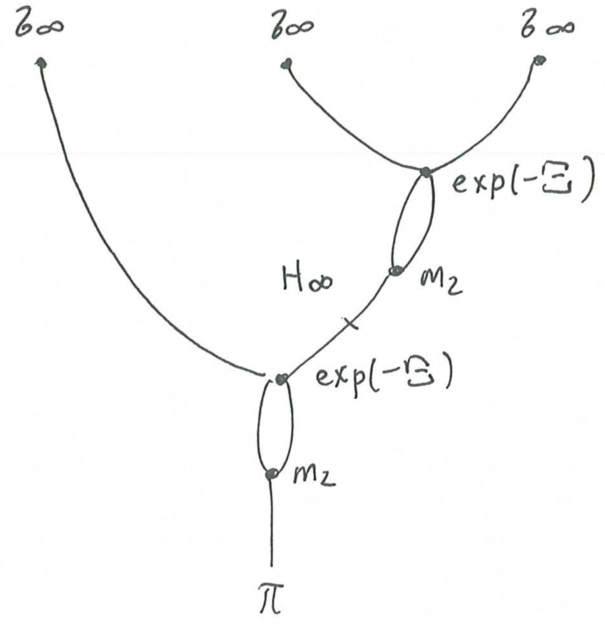
in ainfmf2. Since the RHS computes the cohomology of the LHS, which is a DG-algebra (exterior product on $S = \Lambda(k\theta_1 \oplus \cdots \oplus k\theta_n)$.), we can apply the standard A_∞-minimal model construction to define higher multiplications ρ_q on $\underline{\operatorname{End}}(k^{stab})$. The case of $W = x^d$ was done in ainfmf3. In principle these computations involve trees, e.g.



(1.2)

where m_2 is the multiplication on $S \otimes_k \operatorname{End}_R(k^{stab})$.

In ainfmf4 we explained how this is simply (see p. 9)



(2.1)

where

$$
\begin{aligned}
H_\infty &= \sum_{m \geq 0} (-1)^m (H d_{\mathrm{End}})^m H \qquad H = \mathcal{T}^{-1}\nabla \\
b_\infty &= \sum_{m \geq 0} (-1)^m (H d_{\mathrm{End}})^m b \qquad\qquad (2.2) \\
\Xi &= \sum_{i=1}^n [\psi_i, -] \otimes \theta_i^*
\end{aligned}
$$

Unlike our other papers the $\mathcal{T}^{-1}$ symmetry factors have to be dealt with here. The case of two variables was done in ainfmf5 and used to compute ρ_3 for $W = y^d - x^d$. To proceed in the general case it turns out to simplify matters greatly if we assume $W \in m^3$, which we now do. Then ainfmf6b explains some calculations relevant to showing that the Clifford action on $\underline{\mathrm{End}}(k^{stab})$ from (1.1) is (by p. 2 ainfmf3)

$$
\underline{\mathrm{End}}(k^{stab}) \circlearrowleft \quad \gamma_i^\dagger = At_i = -[\psi_i^*, -]
$$

$$
\gamma_i = -\psi_i
$$

and by p. 5 ainfmf6 the idempotent

$$\gamma_1^\dagger \cdots \gamma_n^\dagger \gamma_n \cdots \gamma_1 \qquad (3.1)$$

projects onto $(k \cdot 1) \otimes \Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$, i.e. we can define

$$\mathcal{A} := \Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \qquad (3.2)$$

and view it as a subspace of $\underline{\mathrm{End}}(k^{stab})$ (via. $\psi_i^* \mapsto \psi_i^* \circ -$). The induced A_∞-structure on $\mathcal{A}$ is

$$\mathcal{A}[1]^{\otimes q} \hookrightarrow \underline{\mathrm{End}}[1]^{\otimes q} \longrightarrow \underline{\mathrm{End}}[1] \twoheadrightarrow \mathcal{A}[1] \qquad (3.3).$$

and we claim (TODO: check) this A_∞-algebra $(\mathcal{A}, \{\ _q\})$ is the minimal model of $\mathrm{End}_R(k^{stab})$. Now in practice we will find that ρ_q preserves the subspace $\mathcal{A}^{\otimes q}$ so $\rho_q = \rho_q|_{\mathcal{A}^{\otimes q}}$ (i.e. ρ_q sends $\mathcal{A}^{\otimes q}$ to $\mathcal{A} \subset \underline{\mathrm{End}}$), and we compute b_q by simply evaluating ρ_q (i.e. diagrams like (2.1)) on inputs which are products of operators ψ_i^*. In fact these are the only calculations we do.

In ainfmf7 we began to sketch the Feynman rules, but this only became serious in ainfmf9. The product b_2 on $\mathcal{A}$ is just the product in the exterior algebra, by (8.2) ainfmf4 (where only the $t=0$ term survives as a consequence of our assumption that $W \in m^3$). So we only consider Feynman diagrams for $q \geq 3$.

By def^N (20.3) ainfmf9 for $\Lambda_i \in \mathcal{A}$

$$\rho_q(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_T (-1)^{\cancel{1+\sum_{j\geq 1}\binom{M_j}{2}} \; e_i(T)+1+q + \sum_{i<j} \tilde{\Lambda}_j \tilde{\Lambda}_j + \sum_i \tilde{\Lambda}_i P_i} \tag{4.1}$$

$$\cdot \sum_{\mathcal{C}\in \mathrm{Con}(\hat{T})} \mathcal{O}(\hat{T}, \mathcal{C})(\Lambda_q \otimes \cdots \otimes \Lambda_1).$$

Our aim in this note is to express this in terms of the amplitudes

$$\mathcal{O}(T, \mathcal{C})(\Lambda_q \otimes \cdots \otimes \Lambda_1)_{const} \in k \tag{4.2}$$

for T, $\mathcal{C}$, and $\Lambda_i \in \mathcal{A}$.

Note when evaluating

$$\mathcal{O}(T,\mathcal{C})(\Lambda_q \otimes \cdots \otimes \Lambda_1)$$

there are no Koszul signs for moving the Λ_i to the appropriate "holes" in $\mathcal{O}(T,\mathcal{C})$, i.e. the relevant string diagram is evaluated as a diagram of ungraded operators.

This is just standard QFT nonsense. With $q, T, \mathcal{C}$ fixed as in the above, let $\Psi_1, \ldots, \Psi_q \in \Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$ be given. Suppose these are "basis elements" i.e. products of ψ_i^*'s. For another basis elt Ψ_{out} we show how to write

$$\mathcal{O}(T, \mathcal{C})(\Psi_1 \otimes \cdots \otimes \Psi_q)_{\Psi_{out}} \in k \qquad (6.1).$$

as a sum of amplitudes in the sense of (4.2). We identify the basis elements of $\Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$ with subsets of $\{1, \ldots, n\}$ in the obvious way, writing ψ_I^* for the product (in increasing order) of ψ_i^* for $i \in I$. Let us say $\Psi_i = \psi_{I_i}^*$ for $1 \le i \le q$ and $\Psi_{out} = \psi_J^*$.

<u>Def^N</u> A <u>fitting</u> of the tuple $(I_1, \ldots, I_q)$ with respect to J is a collection of subsets (possibly empty) $K_i \subseteq I_i$ for $1 \le i \le q$ such that

$$\begin{aligned} &\text{(i) The } K_i \text{ are pairwise disjoint} \\ &\text{(ii) } \cup_i K_i = J \end{aligned} \qquad (6.2)$$

Given a fitting we write $\overline{K_i} := I_i \setminus K_i$. The <u>sign</u> of the fitting is the sign of the permutation required to rearrange the sequence

$$I_1, \ldots, I_q \qquad \text{(each in increasing order)} \qquad (6.3)$$

to

$$J, \overline{K_1}, \ldots, \overline{K_q}. \qquad \text{(each increasing).}$$

<u>Note</u> In (6.3) obviously indices from $\{1,..,n\}$ may be repeated. But still the def^{n} is unambiguous: tag an integer $a\in\{1,...,n\}$ appearing in I_i as a^i. This must end up in either $\overline{K_i}$ or J.

We write F for a fitting and $|F|\in\mathbb{Z}_2$ for the sign.

The configuration $\mathcal{C}$ proscribes (see p. 17 ainfmf8) integers $m(x)$, subsets $J(x)$ and tuples $(a_j(x),\gamma_j(x))$, integers $t(x)$ to locations x in the tree. From these assignments is constructed an operator, e.g. (15.2) from the ingredients listed in (18.2), (18.3), (19.1) there. Omitting signs and prefactors (which we may do as they depend only on poly/$\mathcal{O}$ weight) we can write (15.2) schematically as (7.1)

$$\pi\, m_2 \prod_i ([\psi_i,-]\otimes \mathcal{O}_i^{\,t}) \Big(\prod_j \partial_a(x^{\gamma})\,\mathcal{O}_a[\psi_j,-]\,(\Upsilon_1) \otimes \cdots$$

The way we draw diagrams, according to p. 21, 22, 24 of ainfmf9, is arranged so that contraction are made for <u>leftmost channels first</u>, (where e.g. (5.3) has three channels, numbered from left to right) so that all the fermions in Υ_1 are commuted to the left and annihilated with an interaction vertex before anything in Υ_i for $i>2$. (assuming vacuum boundary conditions, i.e. that we are computing amplitudes in (5.2)).

We may therefore compute

$$\mathcal{O}(T,\mathcal{B})(\Psi_1 \otimes \cdots \otimes \Psi_q) \qquad (8.1)$$

by commuting all input fermions leftwards, and as usual we get this way a sum over all possible contractions, plus terms where ψ_i^*'s actually make it all the way through to the left.

Now, as x ranges over all locations in T, form the set

$$\bigsqcup_x J(x) \qquad (8.2)$$

which we view as a multi-set of elements of $\{1, \dots, n\}$ (i.e. integers in this range with multiplicity). The sum which computes (8.1) is a sum of terms $\psi_{j_1}^* \cdots \psi_{j_t}^* \alpha$ where α is a sum of products of commutators. We see that

$$\underset{\substack{\text{(of } [\psi_i, \psi_j^*] \\ \text{type)}}}{t + \#\text{commutators}} = \underbrace{|I_1| + \cdots + |I_q|}_{\text{input number}} \qquad (8.3)$$

There are of course (many) other commutators which make up α. Let us now concentrate on a particular summand of (8.1), namely the one where $\psi_{j_1}^* \cdots \psi_{j_t}^* = \Psi_{out}$, i.e. $J = \{j_1, \dots, j_t\}$.

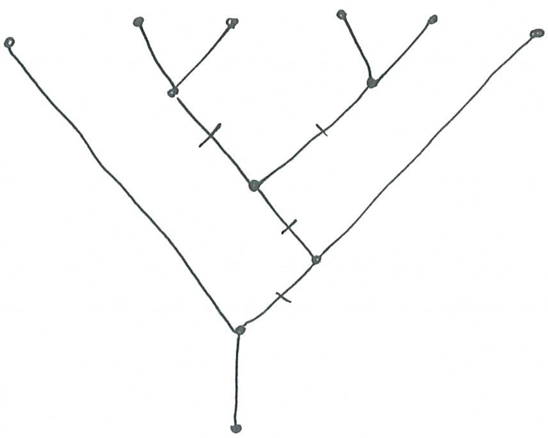
Since the Feynman rules only <u>consume</u> ψ's and never produce them (see (24.2) of ainfmf9) the number of ψ-commutators is the same in each summand of α (these summands, of course, represent individual Feynman diagrams). For this reason if we were to draw the diagrams, $\psi_{j_1}^*, \dots \psi_{j_t}^*$ would just sail straight through from input to output.

The coefficient $\mathcal{O}(T, \mathcal{E})(\Psi_1 \otimes \cdots \otimes \Psi_q)_{\Psi_{out}}$ may therefore be computed by enumerating all Feynman diagrams where this happens:

1. Choose <u>which</u> inputs are going to survive in the output. This is precisely the data of a <u>fitting</u> F of $(I_1, \ldots, I_q)$ relative to J.

2. Commute these inputs all the way to the left and into the ascending order. In the operator (7.1) the only odd things we encounter in this process are from H_{∞} interactions on internal edges, which have degree $+1$. The number of internal edges a fermion encounters on its voyage from its input pos^N (say with Ψ_1 of (7.1)) to the far left, e.g. in

ψ^*



(9.1)

$$H_\infty \Big(\cdots H_\infty \cdots (\cdots H_\infty(\cdots) \cdot \otimes \cdot\cdot H_\infty(\psi^* \otimes \cdots) \;) \otimes \cdots \Big)$$

is the number of internal edges (identified with "$+$" marks) whose path to root is either a <u>subset</u> of the fermions, or which meets the fermions path via a <u>left</u> leg of a trivalent vertex.

Call this number V_i if ψ^* is placed at the ith vertex from the left.

(3) If we commute the surviving fermions in Ψ_1 first, then Ψ_2 and so on, we pick up signs from moving the surviving fermions in Ψ_2 ($\psi^*_{K_2}$) past the "dying" fermions in Ψ_1 ($\psi^*_{\bar{K}_1}$). These signs are the sign of the permutation

$$\begin{array}{c}
I_1 \quad \cdots \quad I_q \\
\downarrow \\
K_1\bar{K}_1 \quad K_2\bar{K}_2 \quad \cdots \quad K_q\bar{K}_q \\
\downarrow \\
K_1 \cdots K_q \quad \bar{K}_1 \quad \bar{K}_2 \cdots \bar{K}_q \\
\downarrow \\
J \quad \bar{K}_1 \cdots \bar{K}_q
\end{array}$$

i.e. the sign of the fitting F.

(4) After this the calculation is identical to those in ainfmf9 or ainfmf10

This proves that

$$\mathcal{O}(T,\mathcal{C})(\Psi_1 \otimes \cdots \otimes \Psi_q)\Psi_{out} \tag{10.1}$$

$$= \alpha$$

$$= \sum_{\substack{\text{fitting } F \\ F=(K_1,\ldots,K_q) \\ \text{for } (I_1,\ldots,I_q) \\ \text{rel. to } J \text{ where} \\ \Psi_i = \psi^*_{I_i} \text{ and } \Psi_{out} = \psi^*_J.}} (-1)^{\sum_{i=1}^q |K_i|\cdot V_i + |F|} \mathcal{O}(T,\mathcal{C})(\psi^*_{\bar{K}_1} \otimes \cdots \otimes \psi^*_{\bar{K}_q})_{const}$$

vacuum boundary conditions.

Another way to write this is for subsets $I_1,\ldots,I_q \subseteq \{1,\ldots,n\}$

$$\mathcal{O}(T,\mathcal{C})(\psi^*_{I_1} \otimes \cdots \otimes \psi^*_{I_q}) \tag{10.2}$$

$$= \sum_{J \subseteq \{1,\ldots,n\}} \sum_{\substack{\text{fittings } F \\ \text{of } (I_1,\ldots,I_q) \\ \text{rel. to } J}} (-1)^{|F| + \sum_i |K_i| V_i} \mathcal{O}(T,\mathcal{C})(\psi^*_{\bar{K}_1} \otimes \cdots \otimes \psi^*_{\bar{K}_q})_{const} \cdot \psi^*_J$$

(note the reverse ordering, see extk p. 4)

<u>Defⁿ</u> Given $K \subseteq \{1,\ldots,n\}$ with $K = \{k_1 < \cdots < k_t\}$ we write

$$\psi_K \lrcorner (-) := [\psi_{k_t}, [\psi_{k_{t-1}}, [\cdots [\psi_{k_1}, -]\cdots] \tag{10.3}$$

as an operator on $\Lambda(k\psi_1^* \oplus \cdots \oplus k\psi_n^*)$, for $K = \phi$ this is id.

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Let $A_1, \dots, A_q \subseteq \{1, \dots, n\}$ be given, and $\Psi_1, \dots, \Psi_q$ in $\Lambda(k\psi_1^* \oplus \dots \oplus k\psi_n^*)$. since $\psi_{A_i} \lrcorner -$ has Koszul degree $|A_i|$ if $\Psi_i = \psi_{I_i}^*$ and so

$$m\{(\psi_{A_1} \lrcorner -) \otimes \dots \otimes (\psi_{A_q} \lrcorner -)\}(\Psi_1 \otimes \dots \otimes \Psi_q) \tag{11.1}$$

$$= (-1)^{|A_q|(|I_1| + \dots + |I_{q-1}|) + |A_{q-1}|(|I_1| + \dots + |I_{q-2}|) + \dots} (\psi_{A_1} \lrcorner \Psi_1) \cdot \dots \cdot (\psi_{A_q} \lrcorner \Psi_q)$$

Now if we write $A_i := \overline{K_i}$ then $(K_i = I_i \setminus A_i)$
may restrict to $A_i \subseteq I_i$

$$\psi_{\overline{K_i}}^* \lrcorner \psi_{I_i}^* = (-1)^{s_i} \psi_{K_i}^*$$

where s_i is the sign of $I_i \longrightarrow \overline{K_i} K_i$. Hence (11.1) equals

$$m\{(\psi_{A_1} \lrcorner -) \otimes \dots\}(\Psi_1 \otimes \dots \otimes \Psi_q) \tag{11.2}$$

$$= (-1)^{|A_q|(|I_1| + \dots) + |A_{q-1}|(|I_1| + \dots) + s_1 + \dots + s_q + c} \cdot \psi_J^*$$

where J is $K_1 \cdots K_q$ arranged in ascending order and c the parity of the permutation which achieves this.

The sign in (11.2) is the sign of the permutation

$$
\begin{array}{lllll}
I_1 & & \cdots & & I_q \\
\bar{K}_1 K_1 & \bar{K}_2 K_2 & \cdots & & \bar{K}_q K_q \\
\bar{K}_q & \bar{K}_1 K_1 \ \bar{K}_2 K_2 & \cdots & & K_q \\
\bar{K}_q \bar{K}_{q-1} & \bar{K}_1 K_1 & \cdots & & K_{q-1} K_q \\
 & & \vdots & & \\
\bar{K}_q \cdots \bar{K}_1 & K_1 \cdots K_q & & & \\
\bar{K}_q \cdots \bar{K}_1 & J & & &
\end{array}
\qquad (12.1)
$$

The fitting F determined by $K_1, \ldots, K_q$ differs from (12.1) in its parity by $\sum_{i<j} |A_i||A_j|$ (to arrange the $\bar{K}_i$) plus $|J| \cdot \sum_i |A_i|$ (to put J on the left). Hence

$$(11.2) = (-1)^{|F| + \sum_{i<j} |A_i||A_j| + |J| \cdot \sum_i |A_i|} \gamma_J^* . \qquad (12.2)$$

Combining this with (10.2) we obtain

(13.1)

$$\mathcal{O}(T,\mathcal{C})(\Psi_1 \otimes \cdots \otimes \Psi_q)$$

$$= \sum_{J \subseteq \{1,\dots,n\}} \sum_{\substack{\text{fittings } F \\ \text{of } (I_1,\dots,I_q) \\ \text{rel. to } J}} (-1)^{\sum_i |K_i| V_i + \sum_{i<j} |\bar{K}_i||\bar{K}_j| + |J| \sum_i |\bar{K}_i|} \cdot m\{(\psi^*_{\bar{K}_1} \lrcorner -) \otimes \cdots \otimes (\psi^*_{\bar{K}_q} \lrcorner -)\} (\Psi_1 \otimes \cdots \otimes \Psi_q) \cdot \mathcal{O}(T,\mathcal{C})(\psi^*_{\bar{K}_1} \otimes \cdots \otimes \psi^*_{\bar{K}_q})_{const}$$

We want to remove the dependence on J and write this as a sum over $\bar{K}_i$'s. More precisely we want no dependence on the I_i (the inputs).

$$= \sum_{A_1,\dots,A_q \subseteq \{1,\dots,n\}} (-1)^{\sum_i (|\Psi_i| + |A_i|) V_i + \sum_{i<j} |A_i||A_j| + \sum_i |A_i| \cdot \sum_j (|\Psi_j| + |A_j|)} m\{(\psi^*_{A_1} \lrcorner -) \otimes \cdots \otimes (\psi^*_{A_q} \lrcorner -)\}(\Psi_1 \otimes \cdots \otimes \Psi_q) \cdot \mathcal{O}(T,\mathcal{C})(\psi^*_{A_1} \otimes \cdots \otimes \psi^*_{A_q})_{const} \qquad (13.2)$$

Combining this with (4.1) requires some care, as e.g. P_i is an invariant of T while V_i is associated to $\bar{T}$ (in that context).