Minimal models for MFs 10 (checked)

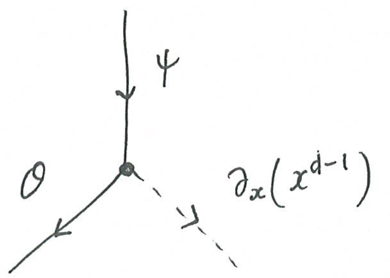
ainfmf10

①

14/11/15

We revisit the old examples of x^d and $y^d - x^d$ with the final Feynman rules from ainfmf9. Let us begin with

<u>Example</u> $W = x^d$ for $d \geq 3$ (so that $W \in m^3$), and $W' = x^{d-1}$. Then the only possible trivalent interaction is (writing $\psi = \psi_1$, $\theta = \theta_1$, $x = x_1$)



prefactor $\frac{-1}{d-1}$ (1.1)

Then we seek to compute the minimal model

$$(\mathcal{A} = k \cdot 1 \oplus k \cdot \psi^*, \{b_q\}_{q \geq 2}).$$

The product is, by p.⑬ ainfmf3 the product on $k[\varepsilon]/\varepsilon^2$, $\varepsilon = \psi^*$. For $q \geq 3$ we compute some example diagrams. Recall that the Feynman rules are for vacuum boundary conditions. (we know all the answers, see p. ⑬ ainfmf3).

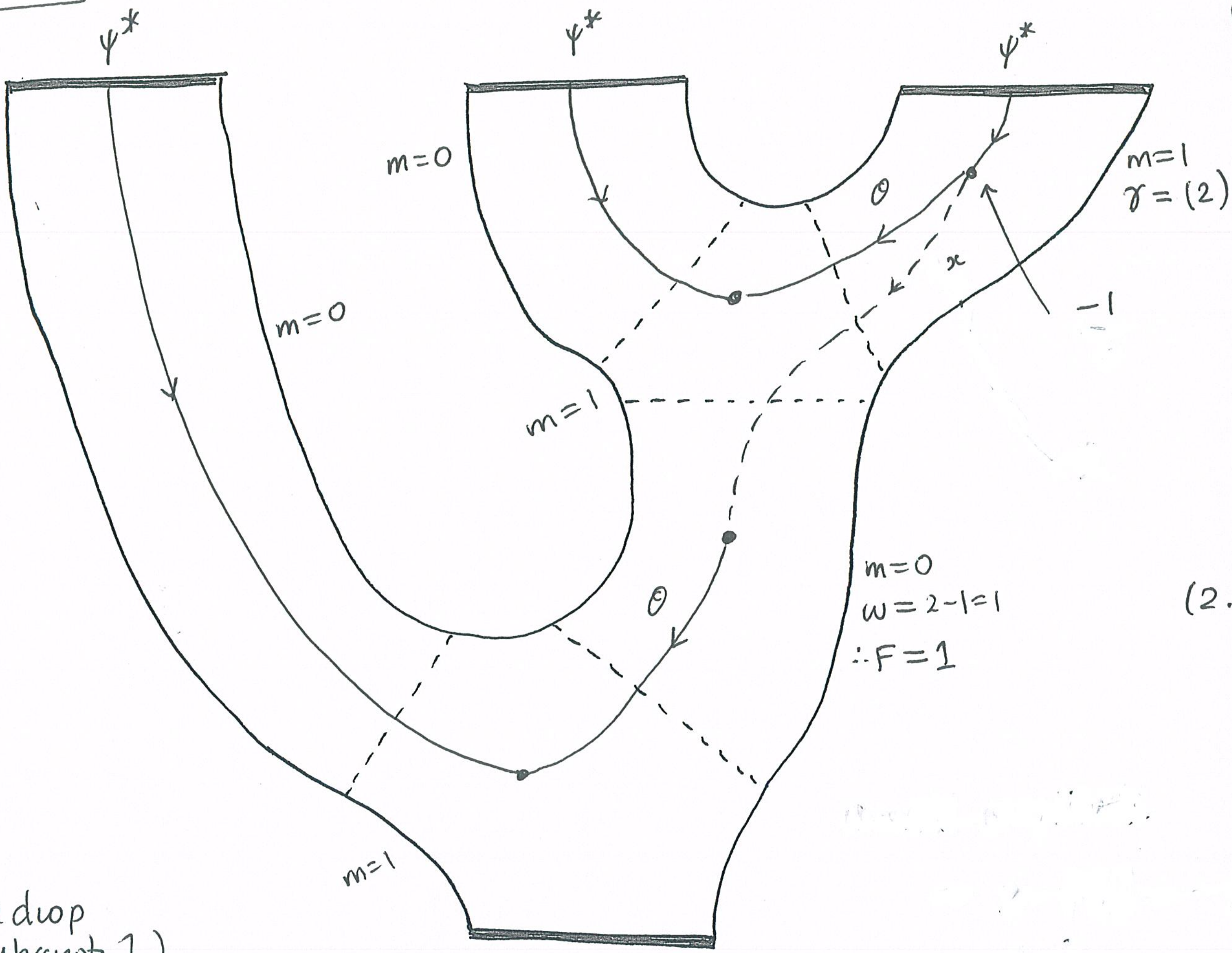
<u>Example</u> From p. ④ onwards we recover the formula of p. ⑮ ainfmf5

$$\rho_3 : (\mathcal{A}[1])^{\otimes 3} \longrightarrow \mathcal{A}[1] \quad \mathcal{A} = \Lambda(k\psi_1^* \oplus k\psi_2^*)$$

$$\rho_3(\Lambda_2 \otimes \Lambda_1 \otimes \Lambda_0) = (-1)^{\sum_{i<j} \tilde{\Lambda}_i \tilde{\Lambda}_j} \Big(-[\psi_1, \Lambda_0] \cdot [\psi_1, \Lambda_1] \cdot [\psi_1, \Lambda_2] + [\psi_2, \Lambda_0] \cdot [\psi_2, \Lambda_1] \cdot [\psi_2, \Lambda_2] \Big) \quad (1.2)$$

for $W = y^3 - x^3$,

$\boxed{W = x^3}$



$$(2.1)$$

(we drop subscripts 1)

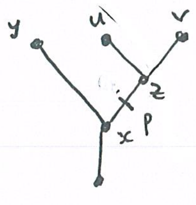
On p. 20.5 of ainfmf9 we express the higher multiplications ρ_q in terms of the $\mathcal{O}(T, \mathcal{C})$ which we may compute by Feynman diagrams. Now, for a tree T and configuration $\mathcal{C}$

$$\mathcal{O}(T, \mathcal{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_q) \in \mathcal{A}$$

has a constant term

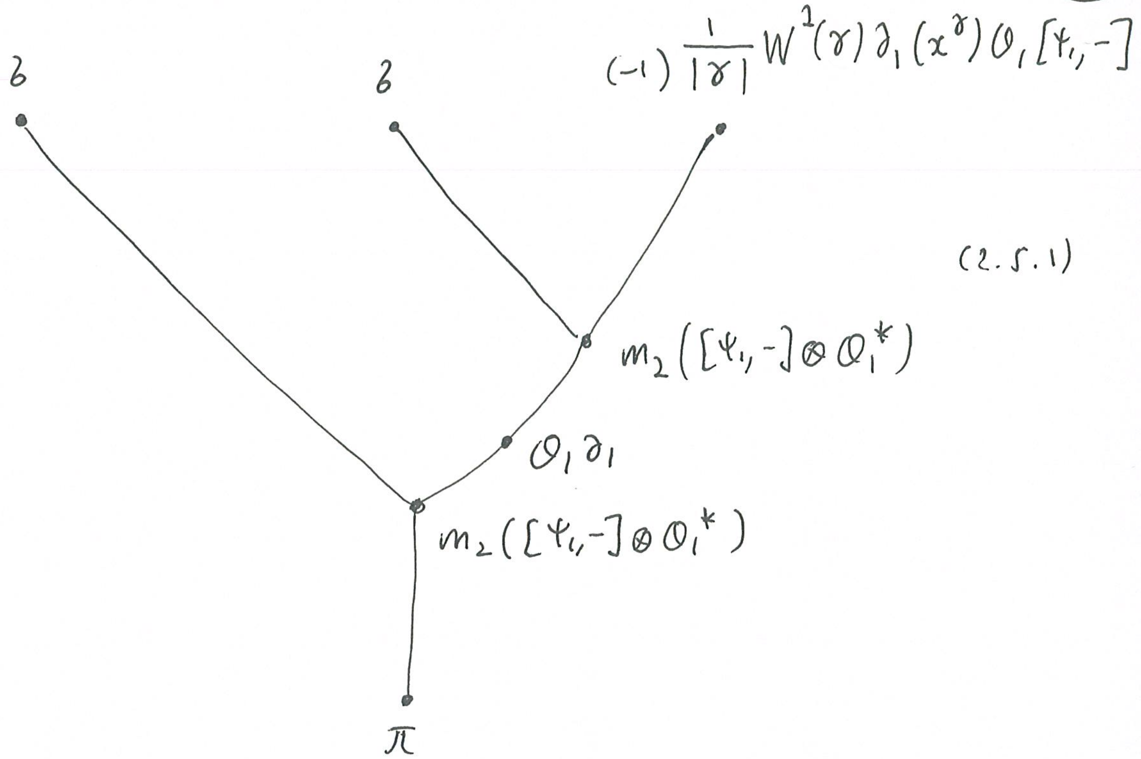
$$\mathcal{O}(T, \mathcal{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_{const} \in k.$$

This is computed by diagrams with vacuum outgoing boundary condition, as in (2.1) above, which is a config of

$T =$ 

with $\mathcal{C} = \left\{ \begin{array}{lll} m(x) = 1 & m(y) = 0 & m(z) = 1 \\ m(u) = 0 & m(v) = 1 & m(\;) = 0 \\ J(x) = J(z) = J(v) = \{1\} & & \\ \gamma(v) = (2) & & t(p) = 1 \\ a(v) = 1 & & \end{array} \right\}$

Then by def^N $\mathcal{O}(T,\theta)$ is associated to the tree



(2.5.1)

And so (2.5.2)

$$\mathcal{O}(T,\theta)(\psi_1^*\otimes\psi_1^*\otimes\psi_1^*) = \pi m_2([\psi_1,-]\otimes\theta_1^*)\Big(\psi_1^*\otimes\theta_1\partial_1 m_2([\psi_1,-]\otimes\theta_1^*)\Big(\psi_1^*\otimes(-1)\frac{1}{|\gamma|}W^1(\gamma)\partial_1(x^\gamma)\theta_1[\psi_1,\psi_1^*]\Big)\Big)$$

Hence since $|\gamma| = 2$ and $W^1(2) = 1$

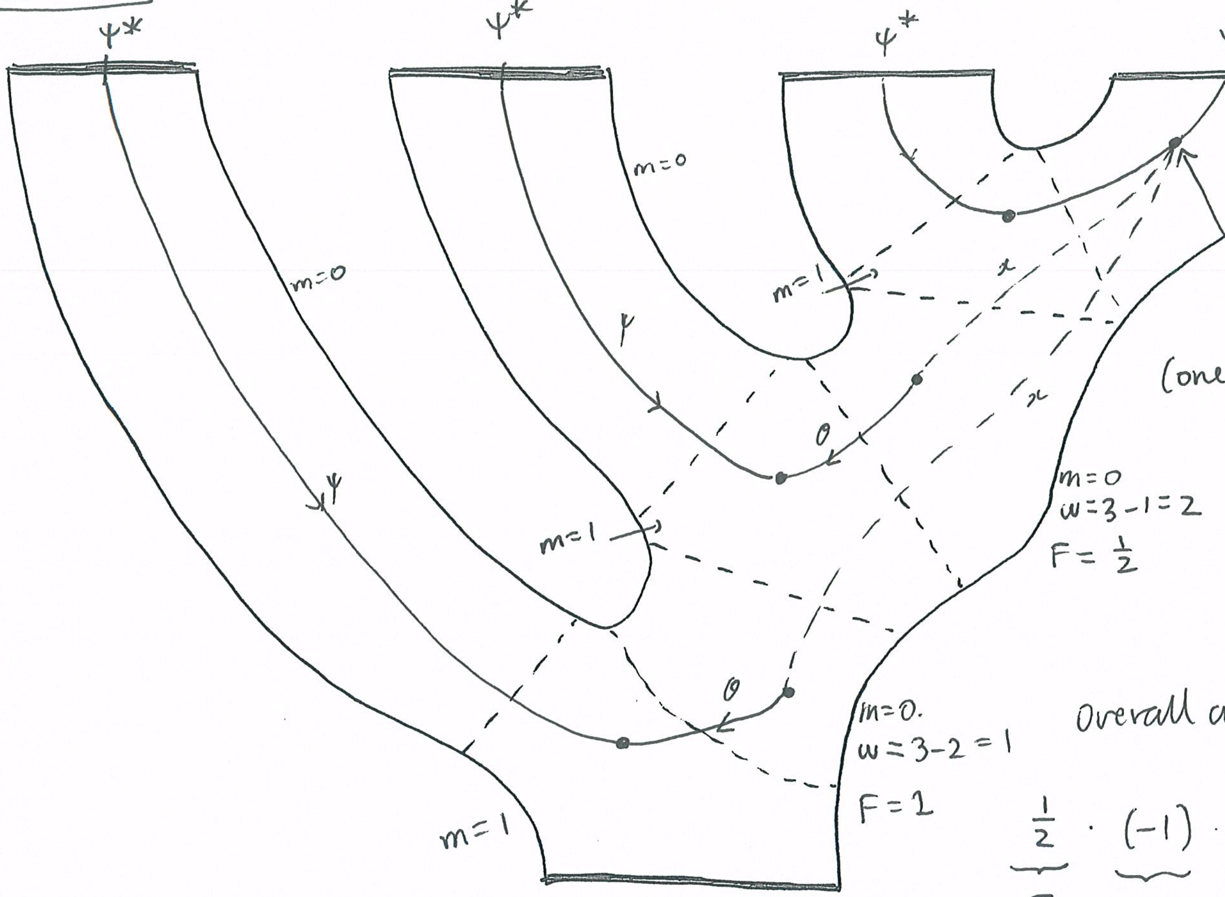
$$\mathcal{O}(T,\theta)(\psi_1^*\otimes\psi_1^*\otimes\psi_1^*)_{\text{const}}$$

$$= \Big[\pi m_2[\psi_1,-]\otimes\theta_1^*\Big(\psi_1^*\otimes\theta_1\partial_1 m_2[\psi_1,-]\otimes\theta_1^*\big(\psi_1^*\otimes x\theta_1[\psi_1,-](\psi_1^*)\big)\Big)\Big]_{\text{const}}$$

The contractions indicated are the only possible ones, so

$$\mathcal{O}(T,\theta)(\psi_1^*\otimes\psi_1^*\otimes\psi_1^*) = \boxed{-1} \qquad (2.5.3)$$

$\boxed{W = x^4}$



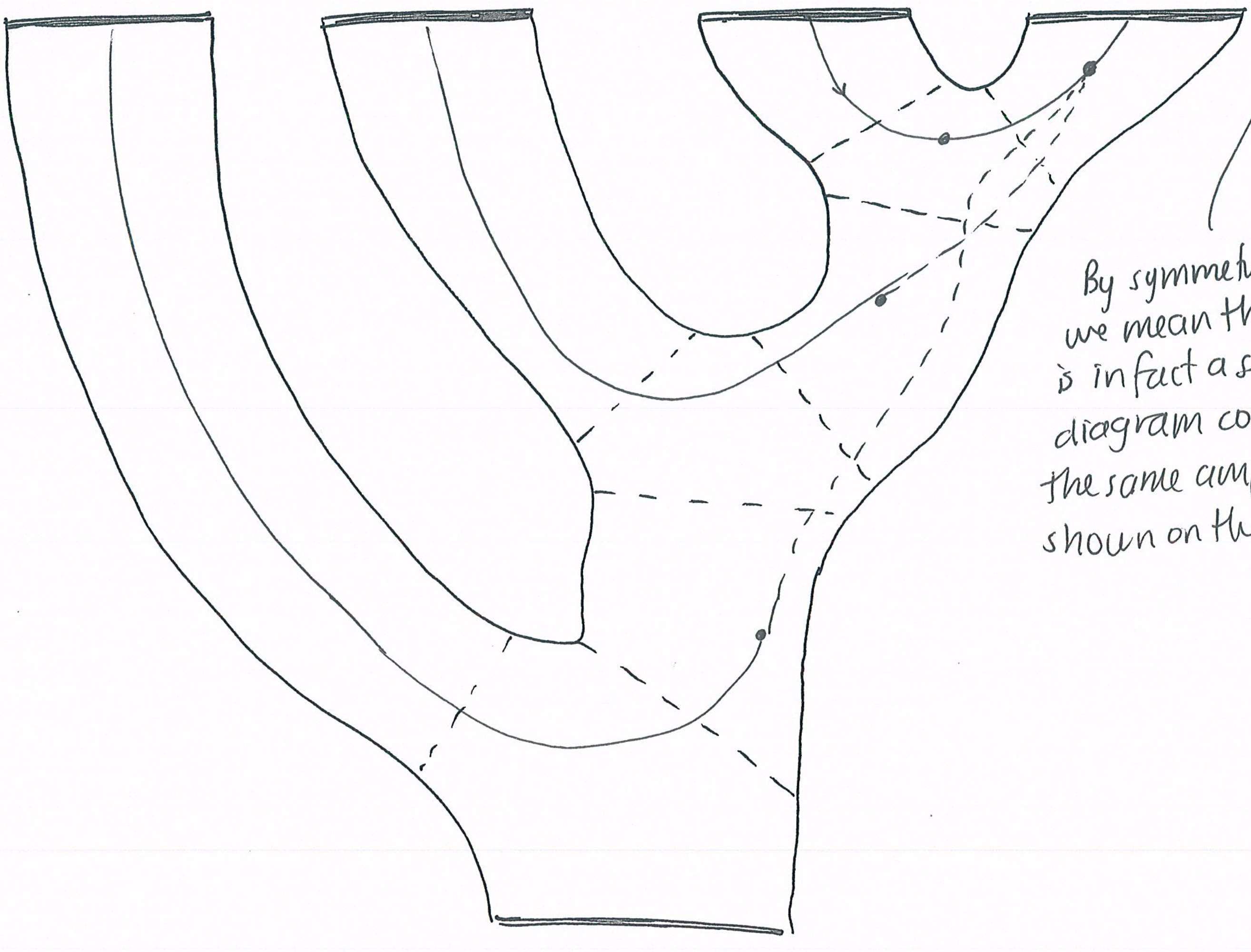
m=1 γ=(3)

$\frac{-3}{3} = -1$

(one 3 from $\partial_x(x^3)$)

Overall amplitude

$$\underbrace{\tfrac{1}{2}}_{F} \cdot \underbrace{(-1)}_{\substack{b_{\infty} \\ \text{vertex}}} \cdot \underbrace{2}_{\substack{\text{symmetry} \\ \text{factor} \\ \text{for } x \cdot x}} = -1$$

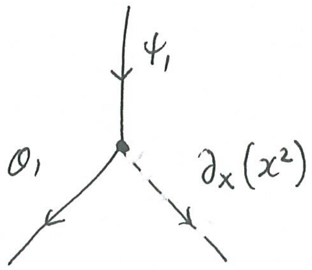


By symmetry factor we mean that there is in fact a second diagram contributing the same amplitude, shown on the left.

Now we move onto $W = y^3 - x^3$

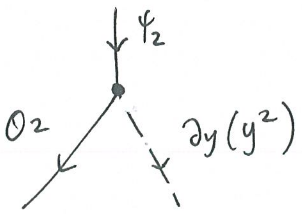
$$W = x \cdot \underbrace{(-x^2)}_{W^1} + y \cdot \underbrace{y^2}_{W^2}$$

which has two kinds of triple interactions



prefactor $-\frac{1}{2} W^1(2,0)$

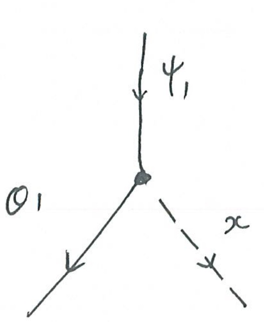
$= -\frac{1}{2} \cdot (-1) = \frac{1}{2}.$



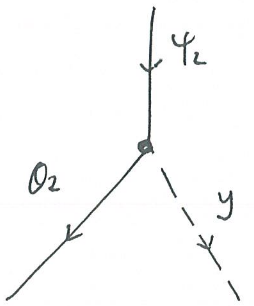
prefactor $-\frac{1}{2} \cdot W^2(0,2)$

$= -\frac{1}{2} \cdot 1 = -\frac{1}{2}.$

Thus we have



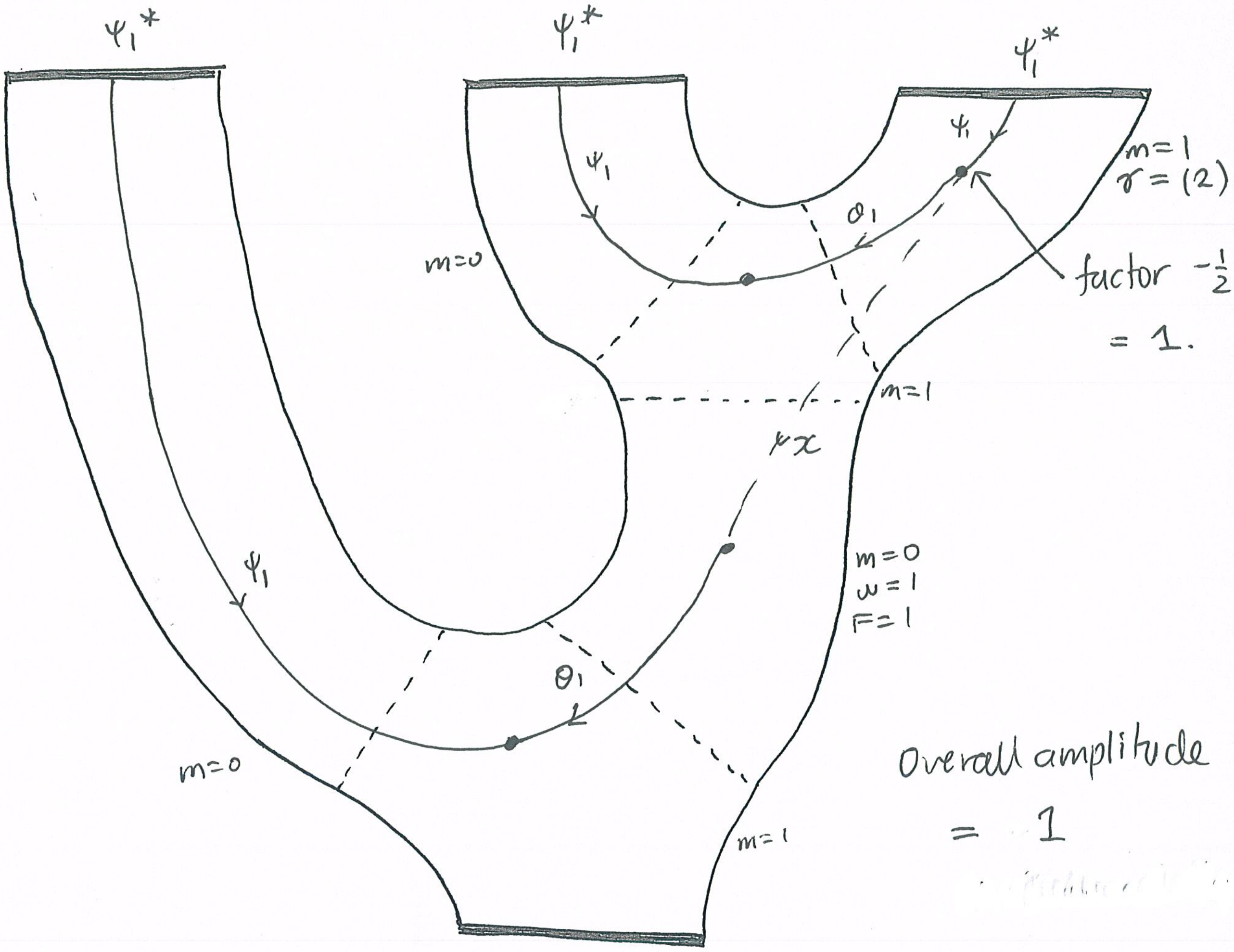
prefactor 1



prefactor -1

<u>Note</u> When we say "amplitude" we mean the contribution of a particular diagram to $\theta(T, \mathcal{E})(\Lambda_1 \otimes \cdots \otimes \Lambda_q)$ const (in particular this does <u>not</u> include a sign for # internal edges).

$\boxed{W = y^3 - x^3}$

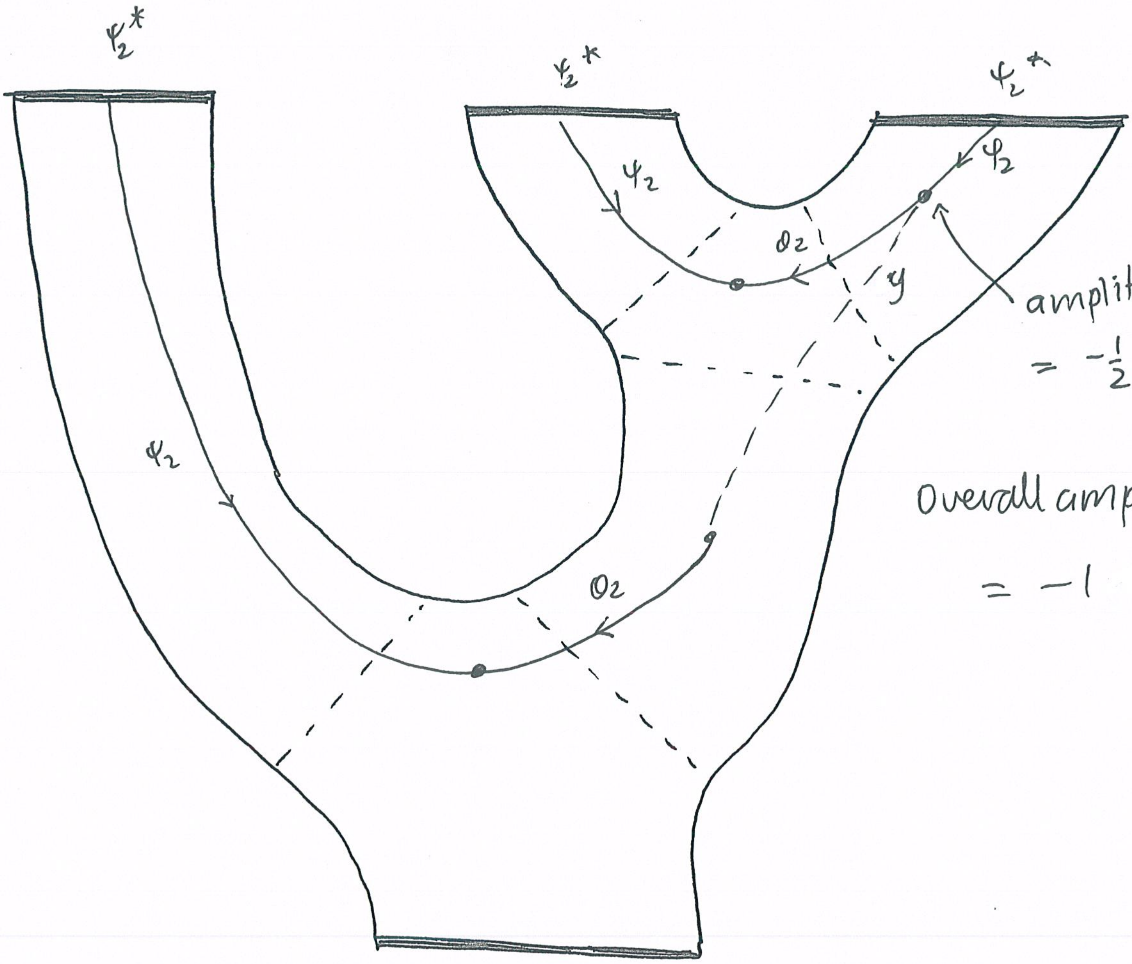


factor $-\frac{1}{2}\cdot(-2)$

$= 1.$

Overall amplitude

$= 1$



amplitude

$= -\frac{1}{2}\cdot 2 = -1$

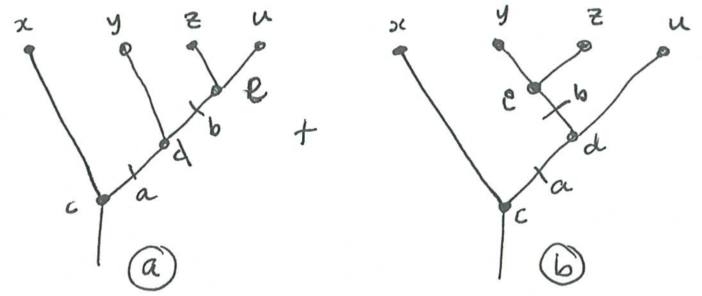
Overall amplitude

$= -1$

Later we will show how to recover (1.2) from p. 4.

Next let us consider b_4 for $W = y^3 - x^3$. Work on this was already done in ainfmf6 where we found

b_4 =



(5.1)

and we started into some calculations. There are, for each channel, possible inputs $1, \psi_1^*, \psi_2^*, \psi_1^*\psi_2^*$ and so in total $(2^2)^4 = 2^8 = 256$ possible inputs to b_4. Ouch!

All amplitudes have two internal vertices, so

- in (a) either $m(z) > 0$ or $m(u) > 0$
  in (b) either $m(y) > 0$ or $m(z) > 0$

- By ainfmf9 (25.1), since at a vertex y we have (for this particular W)

$$\sum_{j \in J(y)} |\gamma_j(y)| = 2m(y)$$

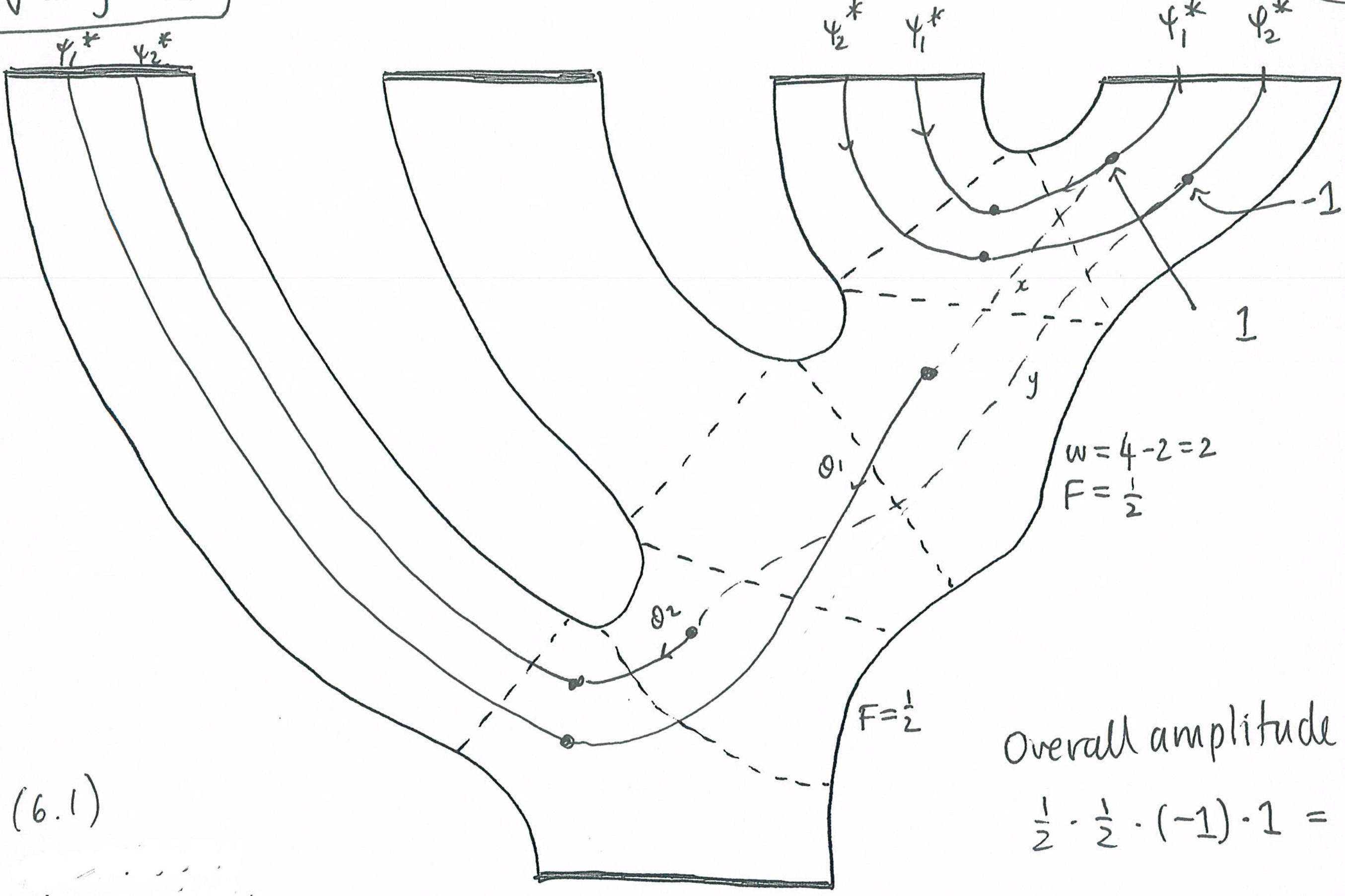
We deduce that

$$\sum_{\substack{y \\ \text{input or int. edge}}} 2m(y) = m(c) + m(d) + m(e)$$

- Because of a, we have the x-channel nonempty
- The y-channel can be empty, as (6.1) shows.

Let us compute $\mathcal{O}(T, \mathcal{C})(\psi_1^*\psi_2^* \otimes 1 \otimes \psi_2^*\psi_1^* \otimes \psi_1^*\psi_2^*)_{const}$, for some $T, \mathcal{C}$.

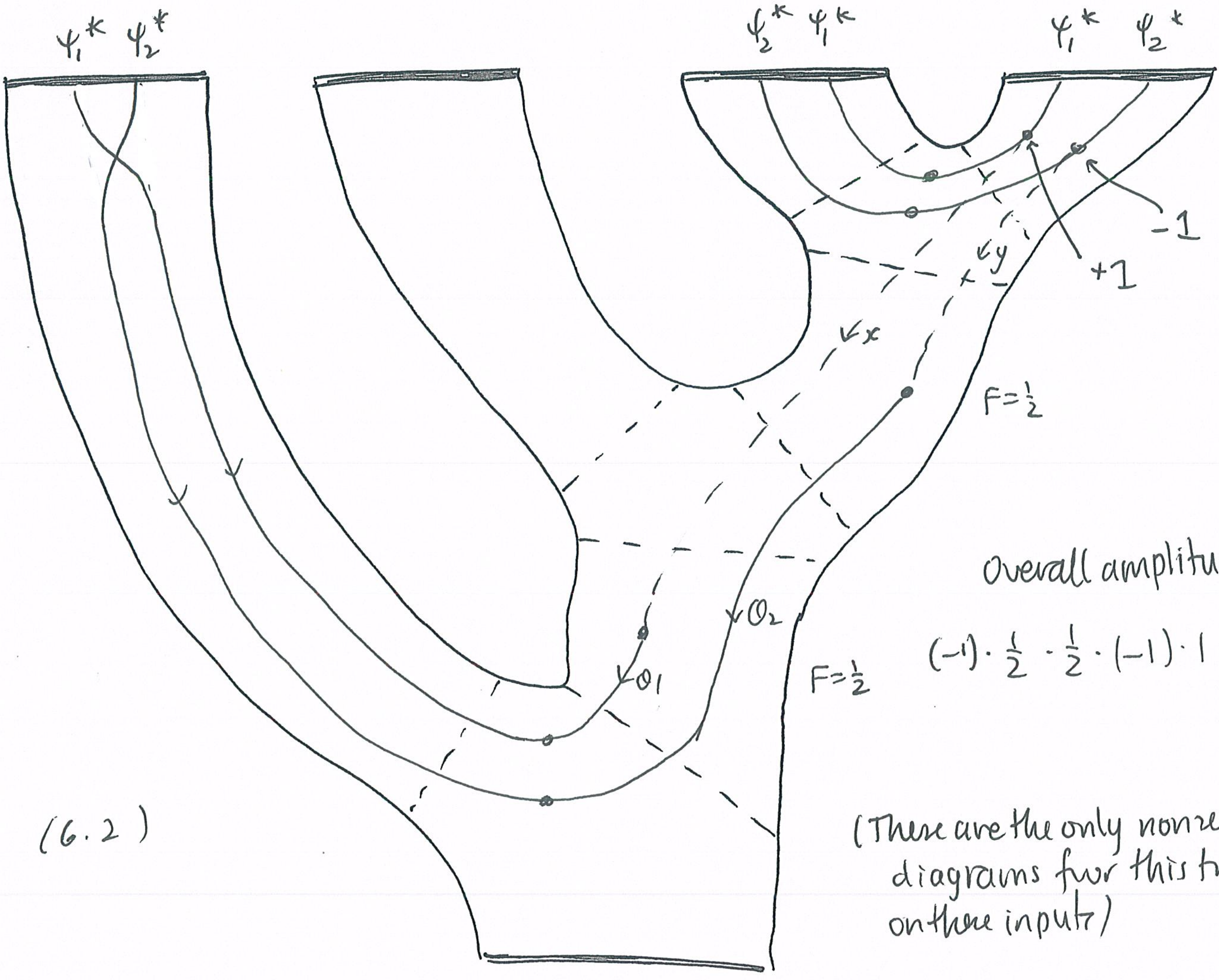
$\boxed{W = y^3 - x^3}$



Overall amplitude

$$\frac{1}{2} \cdot \frac{1}{2} \cdot (-1) \cdot 1 = -\frac{1}{4}$$

(6.1)



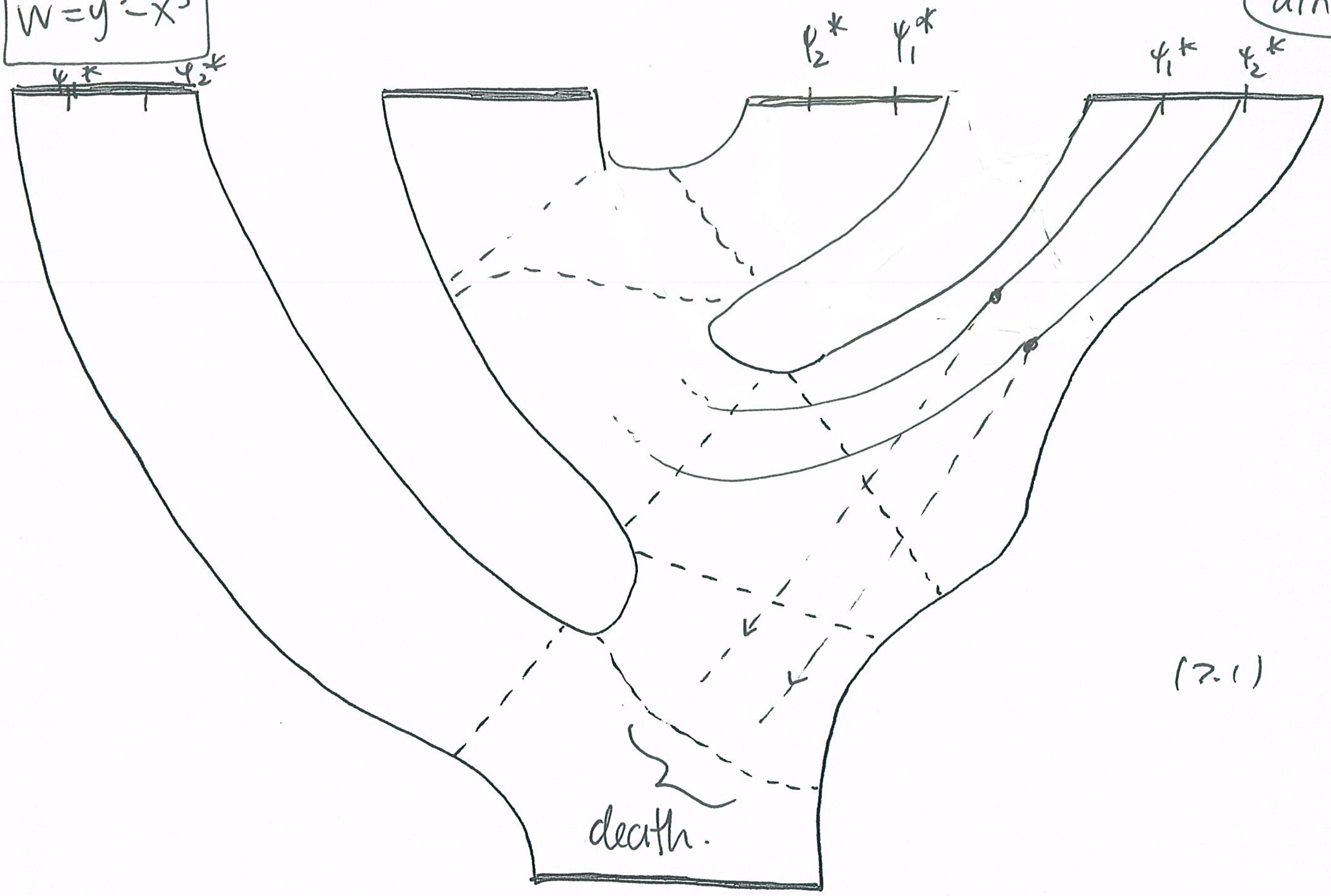
Overall amplitude

$$(-1) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (-1) \cdot 1 = +\frac{1}{4}$$

(6.2)

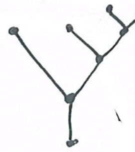
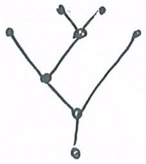
(These are the only nonzero diagrams for this tree on these inputs)

$\boxed{W = y^3 - x^3}$



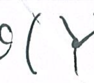
(7.1)

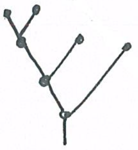
This tree does not contribute to the amplitude.

For the tree $T =$  the configurations in (6.1), (6.2) are the only ones with a nonzero amplitude for this input. For the tree $T =$  there are no configs with nonzero amplitudes.

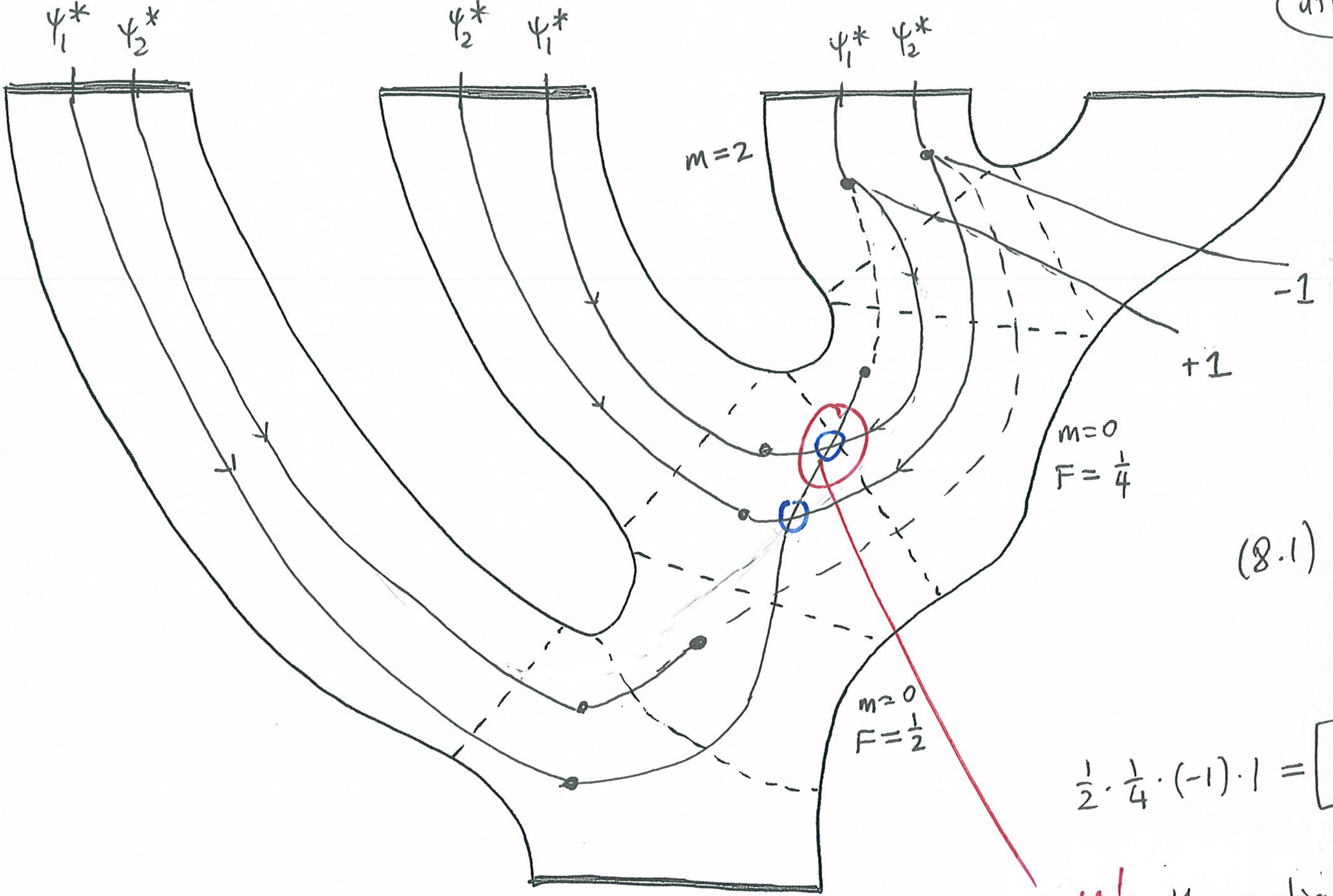
on $\psi_1^*\psi_2^* \otimes 1 \otimes \psi_2^*\psi_1^* \otimes \psi_1^*\psi_2^*$

$$\therefore\ \mathcal{O}(\;\cdot\;, \mathcal{C}_{(6.1)}) = -\tfrac{1}{4} \qquad \mathcal{O}(\;\cdot\;, \mathcal{C}_{(6.2)}) = 1/4 \tag{7.2}$$

Now for $T =$  with inputs $\Lambda_1 = \psi_1\psi_2^*$, $\Lambda_2 = 1$, $\Lambda_3 = \psi_2^*\psi_1^*$, $\Lambda_4 = \psi_1^*\psi_2^*$

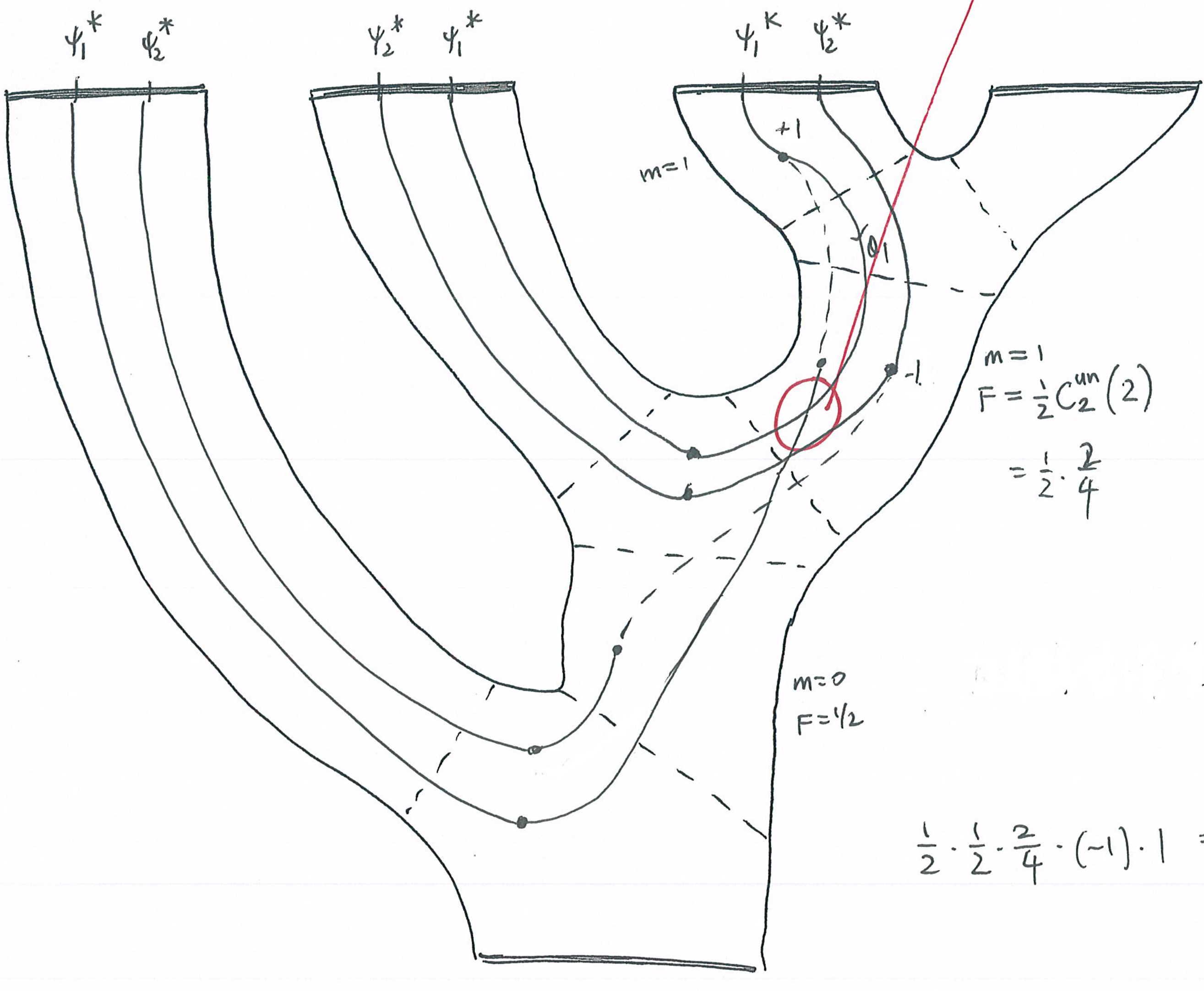
$$\rho_T(\Lambda_4 \otimes \cdots \otimes \Lambda_1) = (-1)^s \sum_{\mathcal{C}} \mathcal{O}(\hat{T}, \mathcal{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_4)$$
$$= (-1)^s (1/4 + (-1/4)) = 0.$$



(8.1)

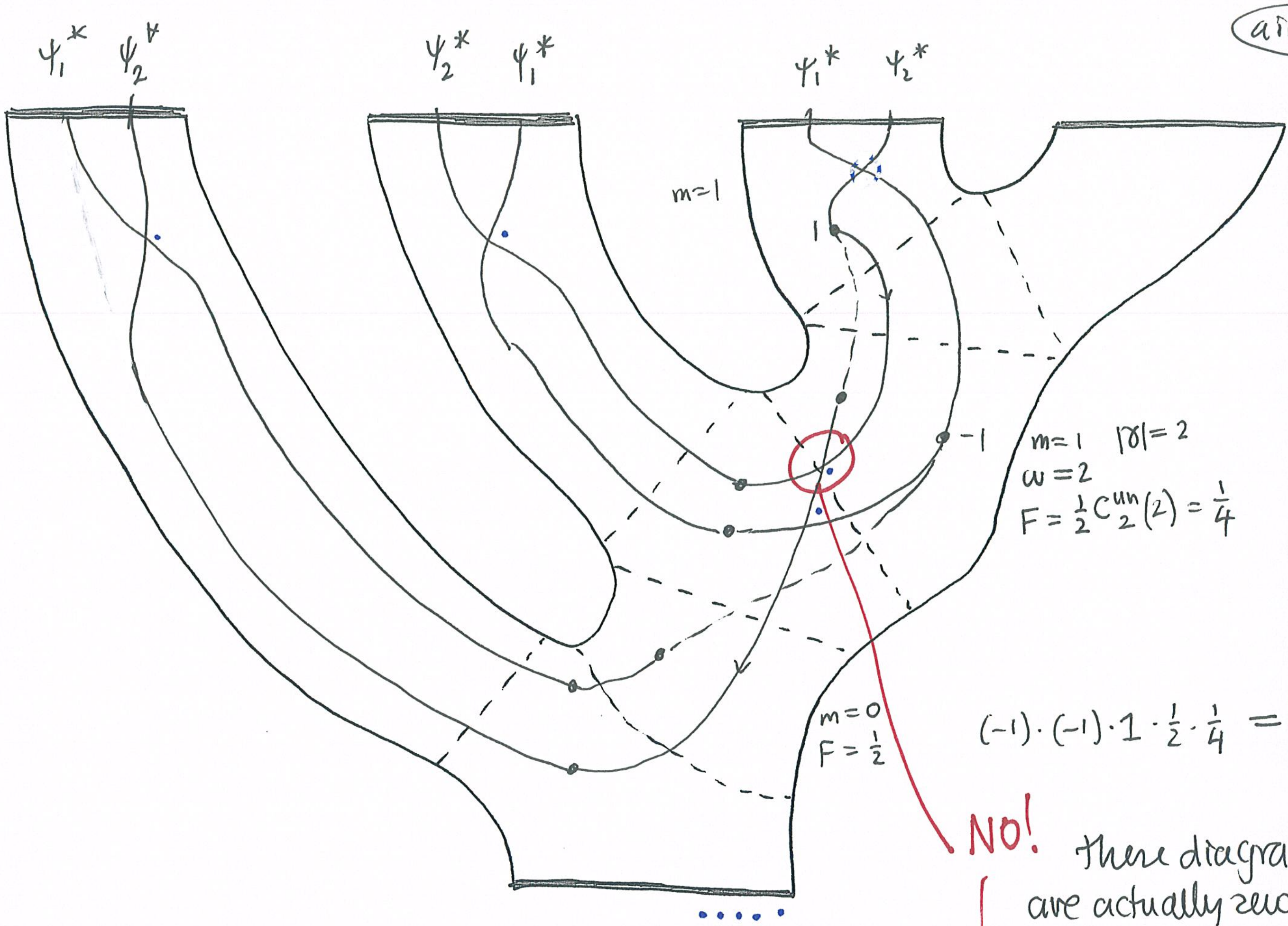
$$\frac{1}{2}\cdot\frac{1}{4}\cdot(-1)\cdot 1 = \boxed{-\frac{1}{8}}$$

NO! these diagrams are actually zero.



(8.2)

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{4}\cdot(-1)\cdot 1 = \boxed{\frac{-1}{8}}$$



(9.1)

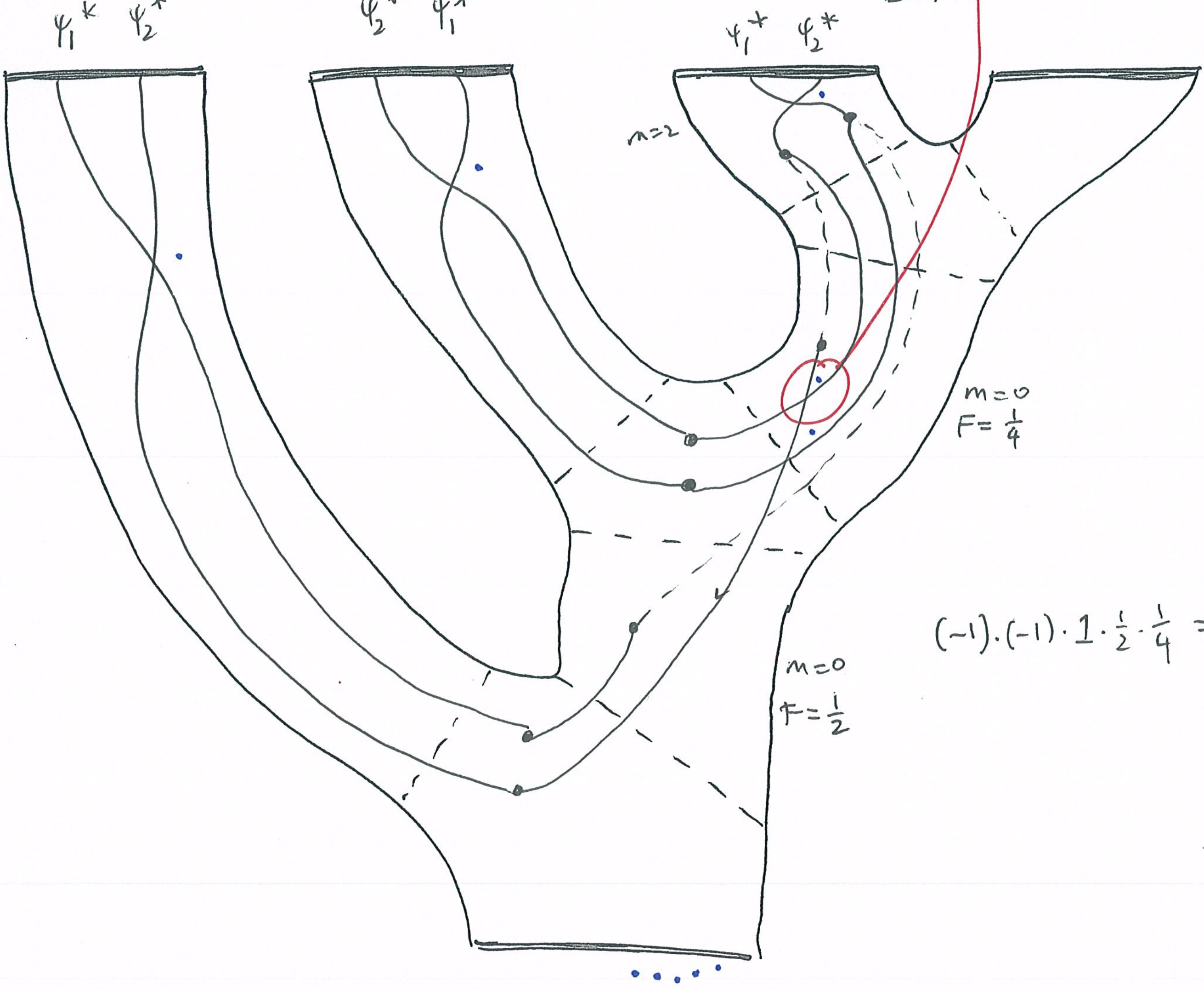
$$(-1)\cdot(-1)\cdot 1\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}=\boxed{\tfrac{1}{8}}$$

NO!

there diagrams are actually zero, as they contain O_2^2!

-1 from fermion interchanges

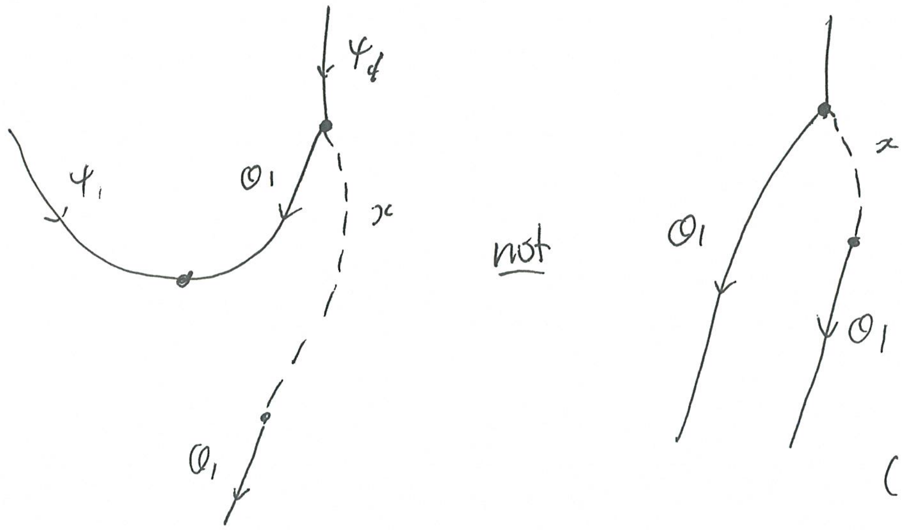
-1 from ψ_2 interaction



$$(-1)\cdot(-1)\cdot 1\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}=\boxed{\tfrac{1}{8}}$$

What we learn from the previous two pages is a constraint for nonzero diagrams

- The outputs of a trivalent vertex must have $\mathcal{O}_1$ annihilated with a ψ_1 before x becomes $\mathcal{O}_1$, i.e.



(10.1)

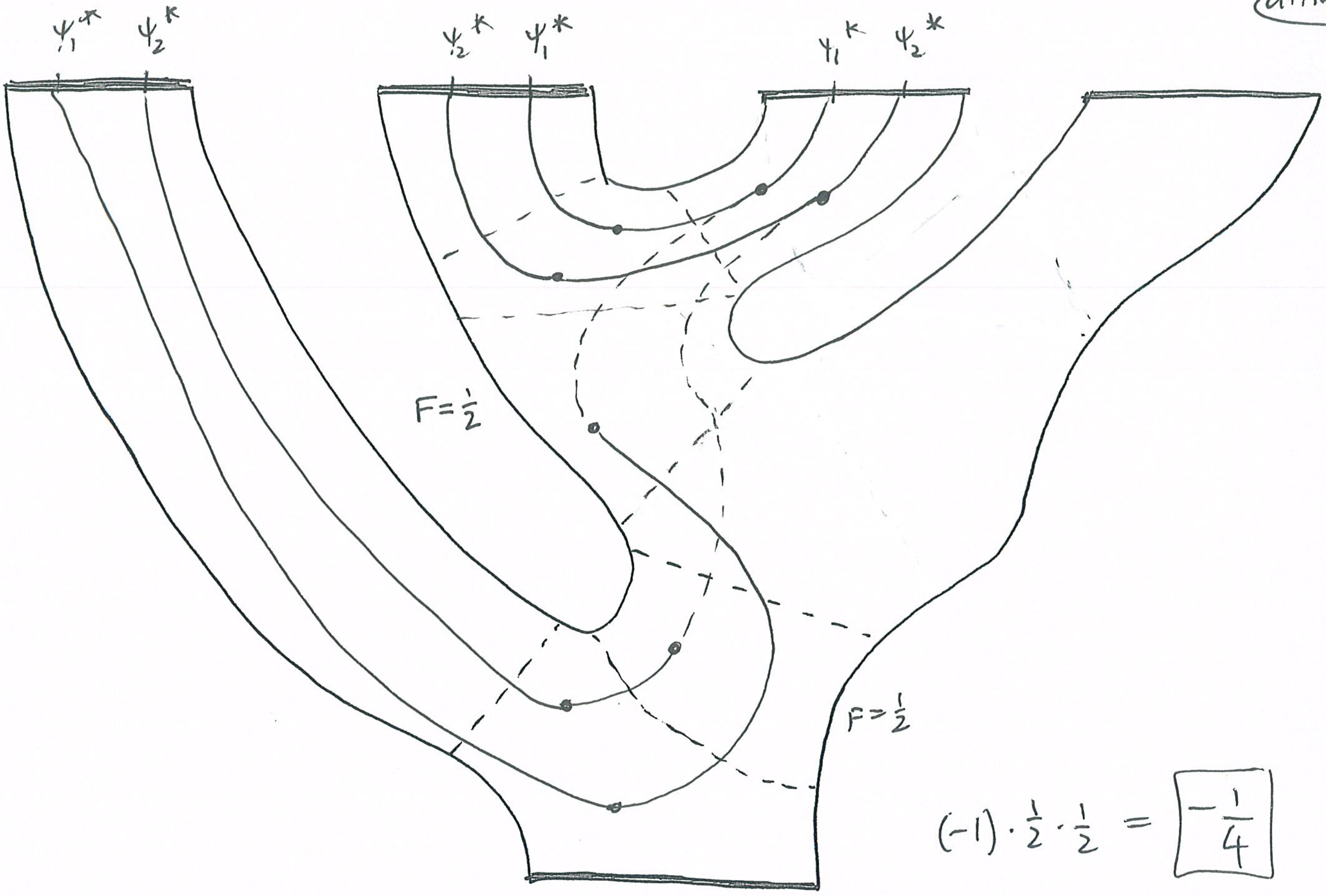
And similarly for y.

- Considering the diagrams, this means one of the ψ_i's must have its trivalent vertex in the $\mathfrak{Z}_\infty$ zone, but then that same interaction produces the bad situation on the RHS of (10.1). That is:

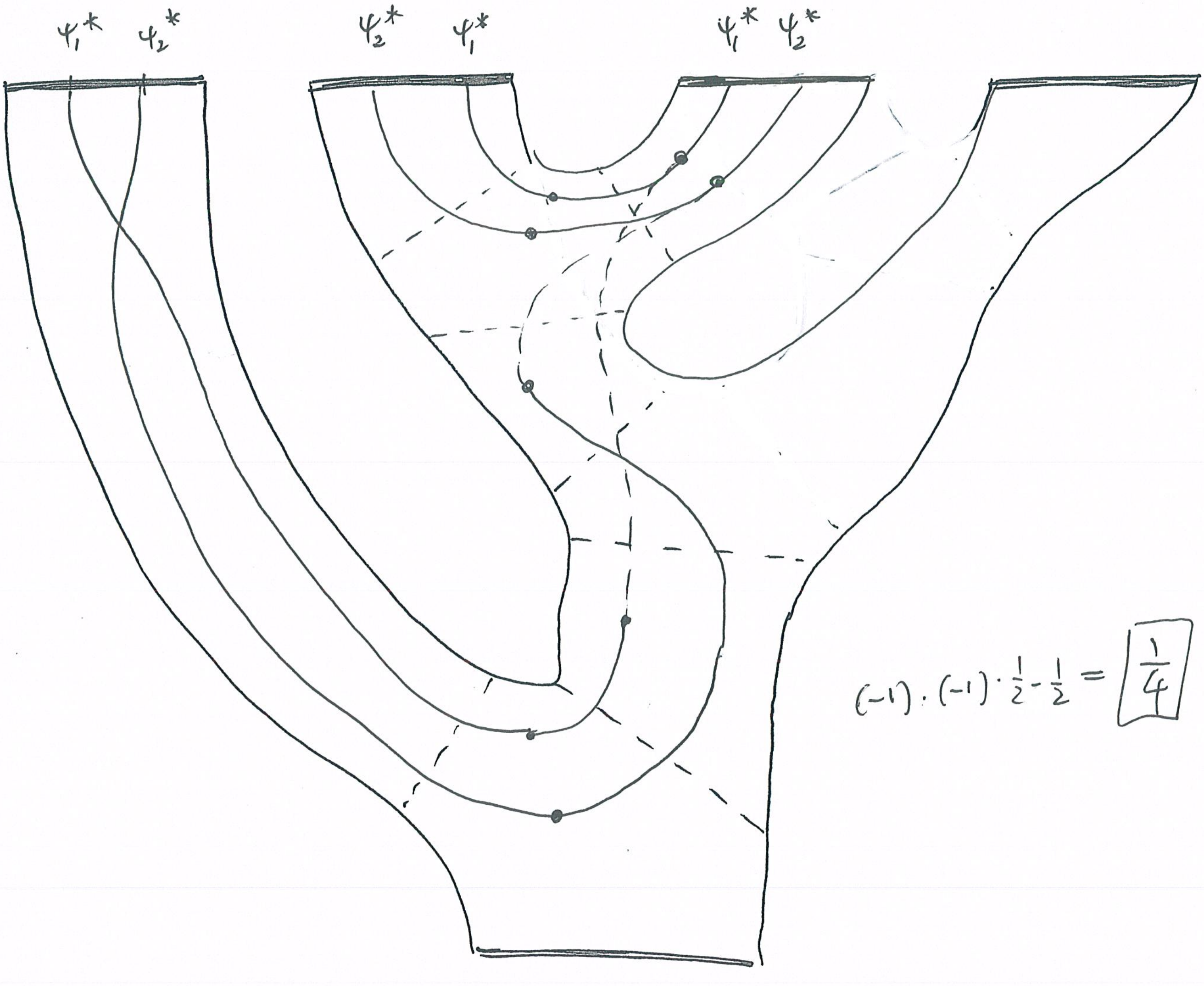
Conclusion The input $\psi_1^*\psi_2^* \otimes \psi_2^*\psi_1^* \otimes \psi_1^*\psi_2^* \otimes 1$ has no nonzero diagrams on the tree

Def^n Given a tree T and inputs $\Lambda_1, \ldots, \Lambda_q$ the total amplitude is

$$\sum_{\mathscr{C} \in \mathrm{Con}(T)} \mathcal{O}(T, \mathscr{C})(\Lambda_1 \otimes \cdots \otimes \Lambda_q)_{\mathrm{const}}. \qquad (10.2).$$



$$(-1)\cdot\frac{1}{2}\cdot\frac{1}{2} = \boxed{-\frac{1}{4}}$$



$$(-1)\cdot(-1)\cdot\frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}$$

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(12)

<u>Conclusion</u> The total amplitude for input $\psi_1^*\psi_2^* \otimes \psi_2^*\psi_1^* \otimes \psi_1^*\psi_2^* \otimes 1$ on this tree is 0.

The underlying mechanism is (12.1)

$$\left([\psi_1,-]\otimes \theta_1^*\right)\left([\psi_2,-]\otimes \theta_2^*\right)\left(\psi_1^*\psi_2^* \otimes \theta_1 \partial_x(x\theta_2)\right)$$

vs.

$$\left([\psi_1,-]\otimes \theta_1^*\right)\left([\psi_2,-]\otimes \theta_2^*\right)\left(\psi_1^*\psi_2^* \otimes \theta_2 \partial_y(y\theta_1)\right)$$

We <u>conclude</u>

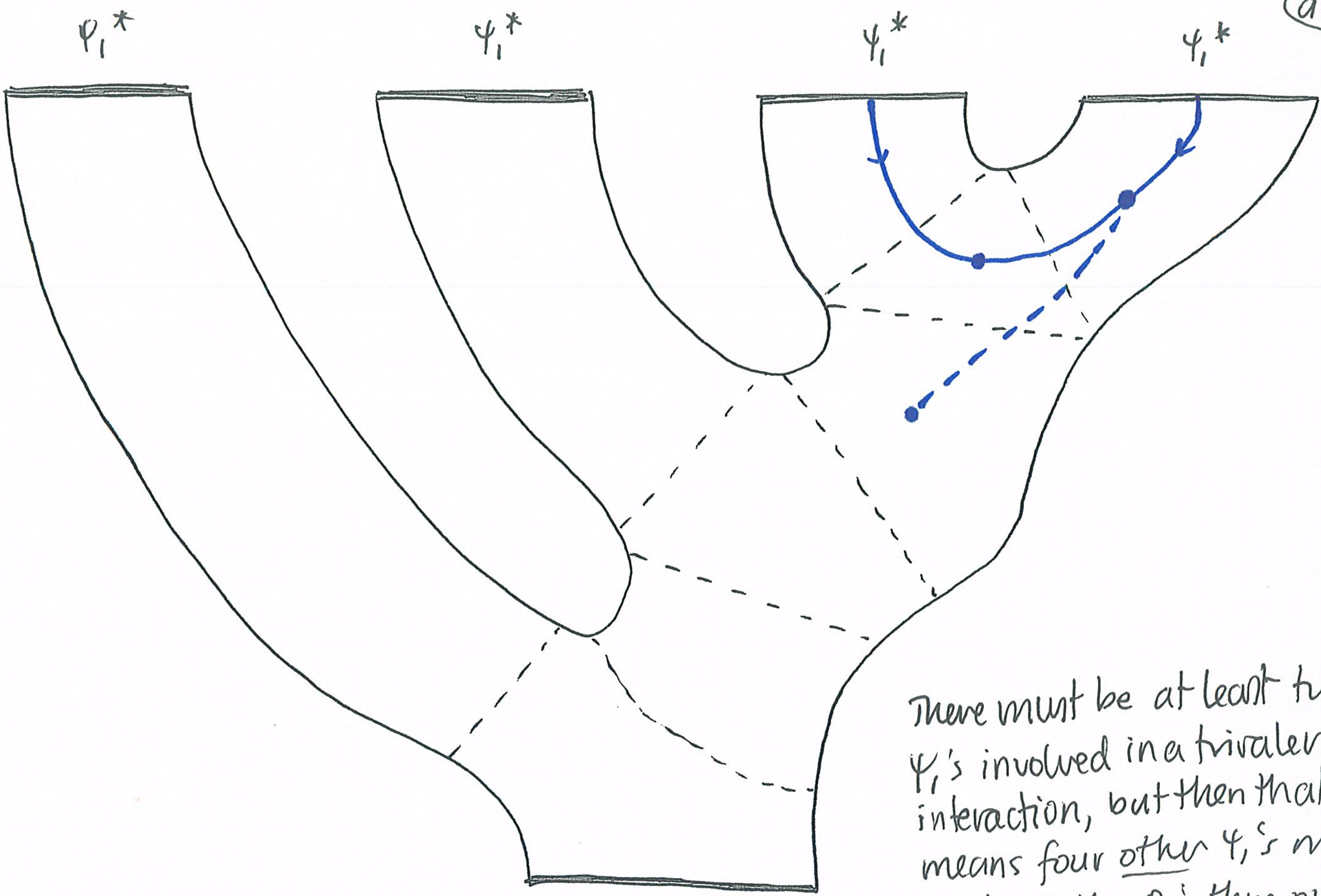
$$\rho_4(1\otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^*) = 0 \qquad (12.2)$$

The <u>general phenomenon</u> we have observed in the past two examples is $H^2=0$. If two consecutive H_∞ zones are fed directly into one another (with no Ξ interactions in between) as happens in p. 6, p. 11 then we get zero (in the form of multiple diagrams cancelling) as a consequence of:

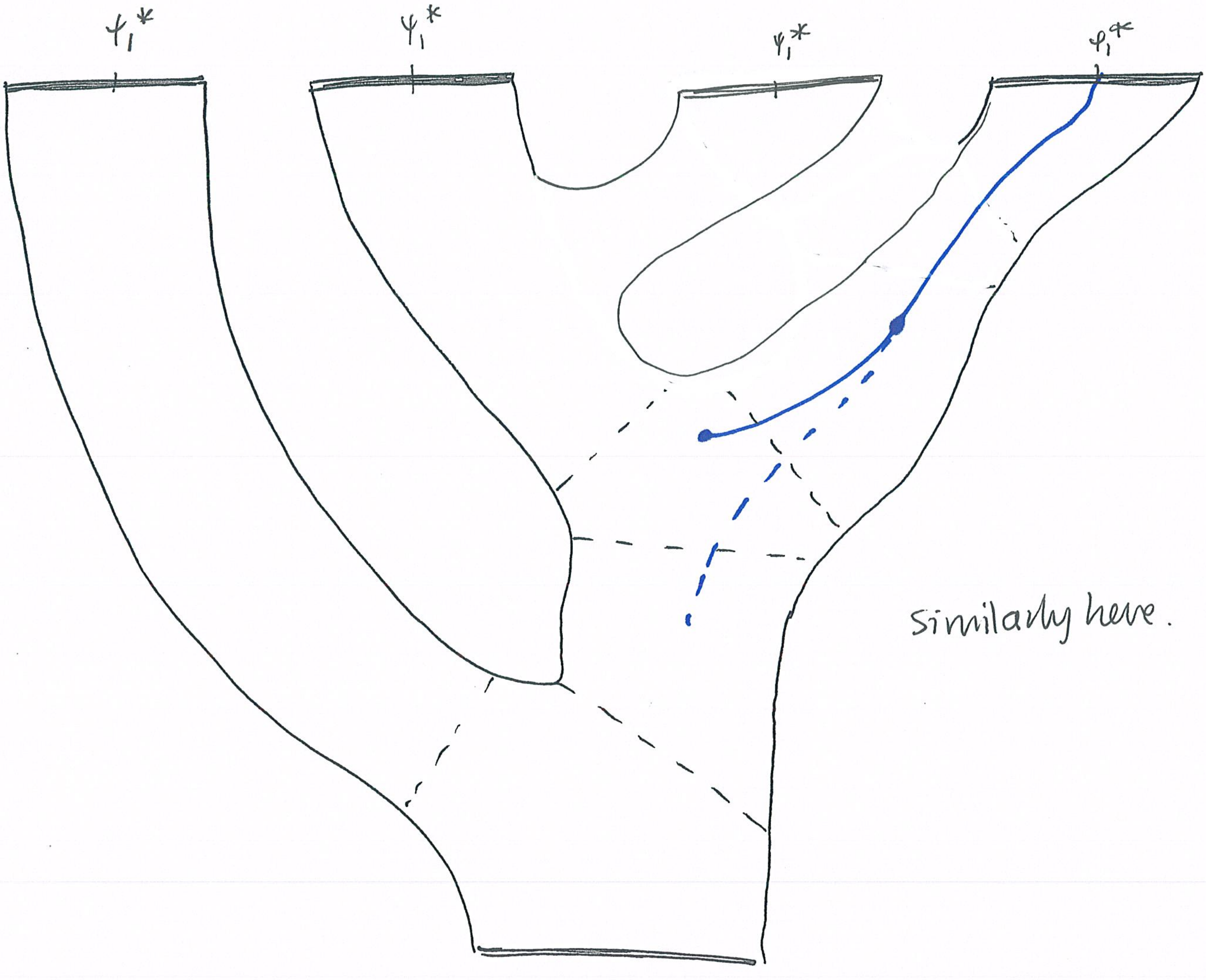
$$H_\infty H_\infty = 0, \quad H_\infty \delta_\infty = 0.$$

This is a <u>general fact</u>, not just for $W=y^3-x^3$, which says:

- The amplitudes of diagrams with empty channels do not contribute to the overall amplitude (as a consequence of cancelling with one another). (12.3)



There must be at least two ψ_1's involved in a trivalent interaction, but then that means four <u>other</u> ψ_1's need to absorb the θ_1's thus produced, so there is <u>no</u> diagram.



Similarly here.

We conclude

$$\rho_4(\psi_i^* \otimes \psi_i^* \otimes \psi_i^* \otimes \psi_i^*) = 0 \qquad i \in \{1,2\}.$$

For similar reasons; ρ_4 is zero on <u>any</u> inputs where each channel has a single occupant. (precisely one).

<u>Note</u> The total number of ψ_i's (for $i \in \{1,2\}$ fixed) in the input must be divisible by 3 (for any diagram). Thus the possible configurations are

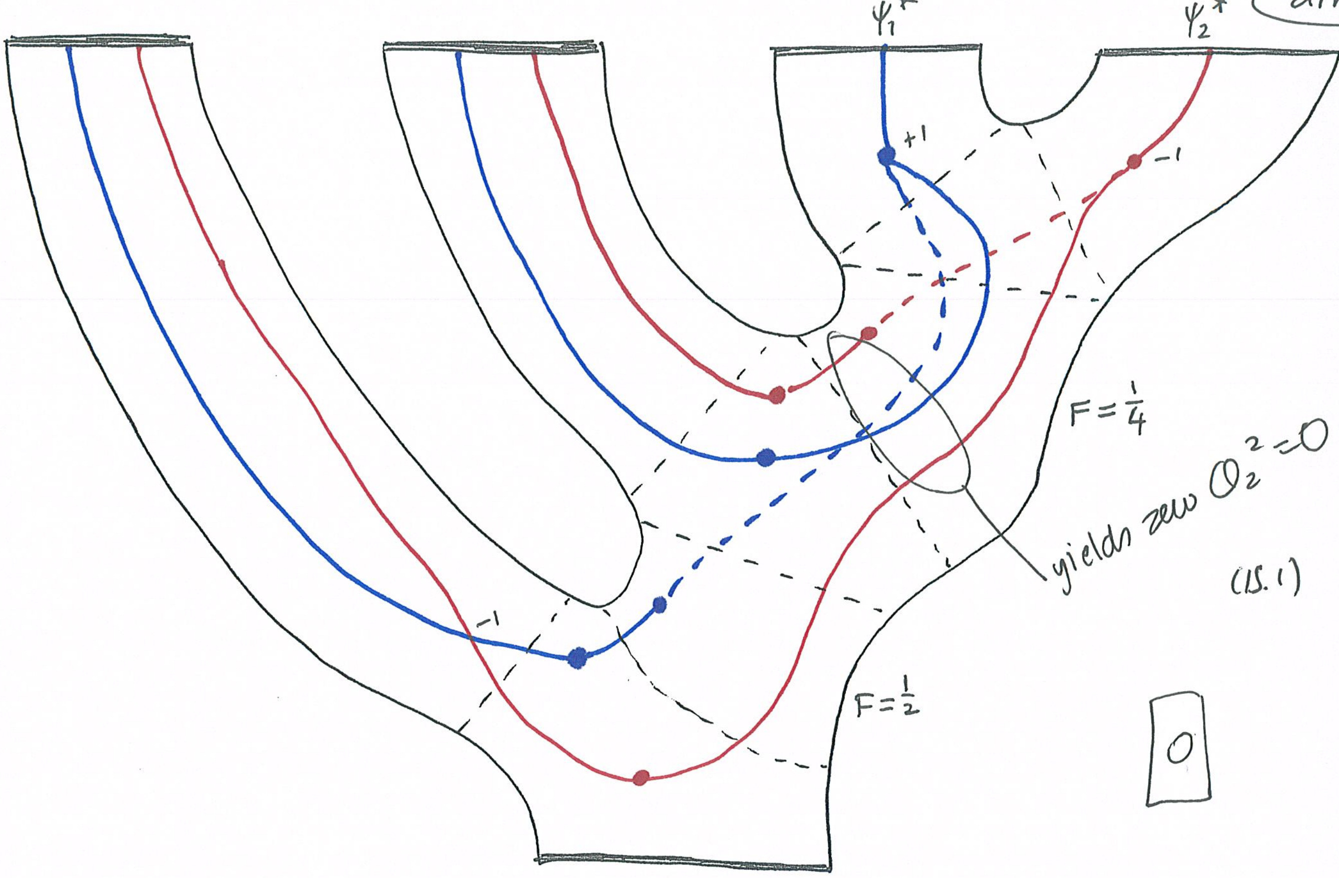
ψ_1	ψ_2
~~0~~	~~3~~
~~0~~	~~6~~
3	3
~~3~~	~~0~~
~~6~~	~~0~~

as explained above, zero channels do not contribute. and we cannot have 6 on four channels.

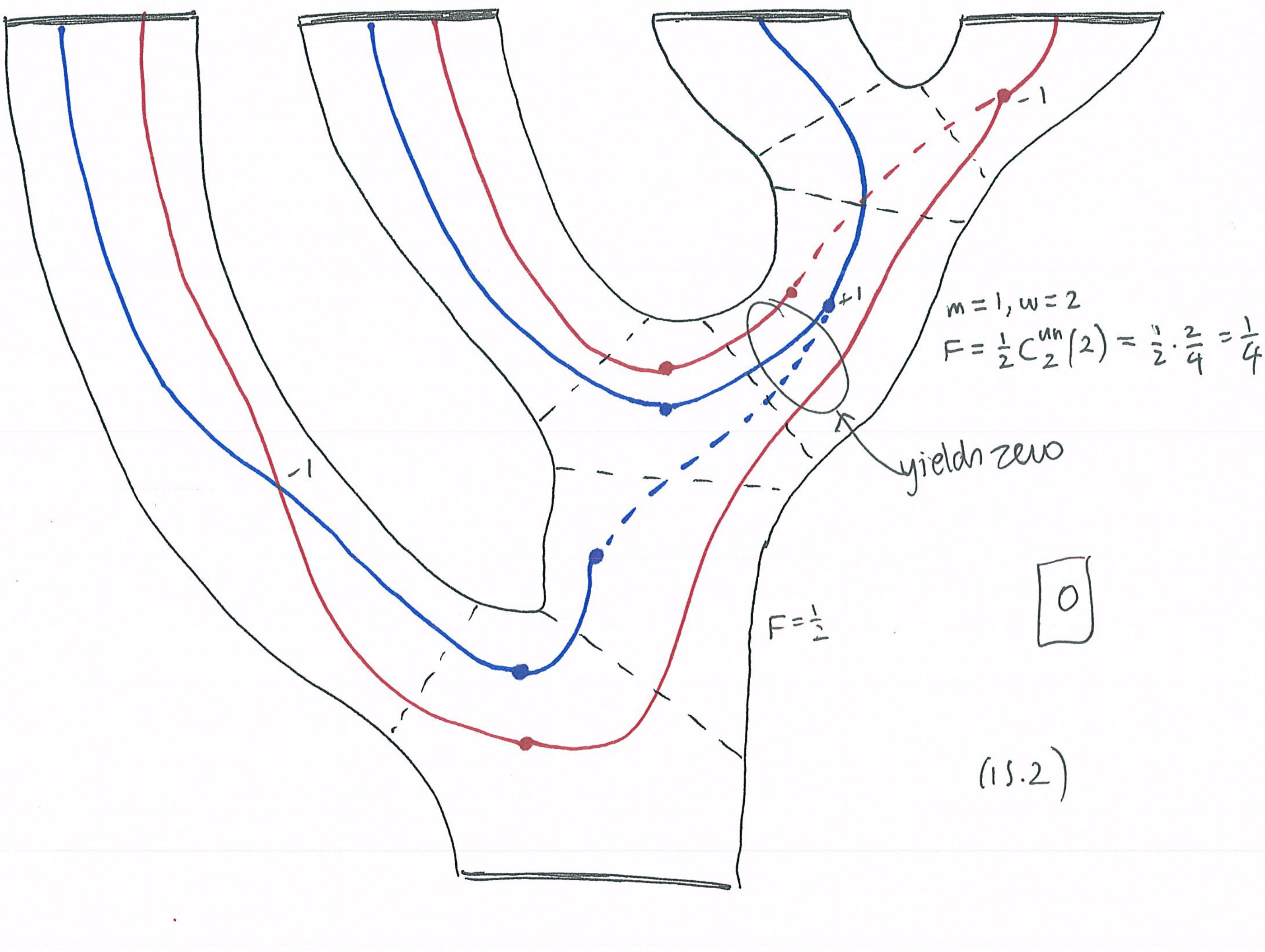
(14.1)

So the possibilities are indexed by choosing two places: where <u>doesn't have</u> a ψ_i; so $\binom{4}{1}\cdot\binom{4}{1} = 16$ diagrams, minus those where the <u>same</u> place is chosen, so 12; e.g.

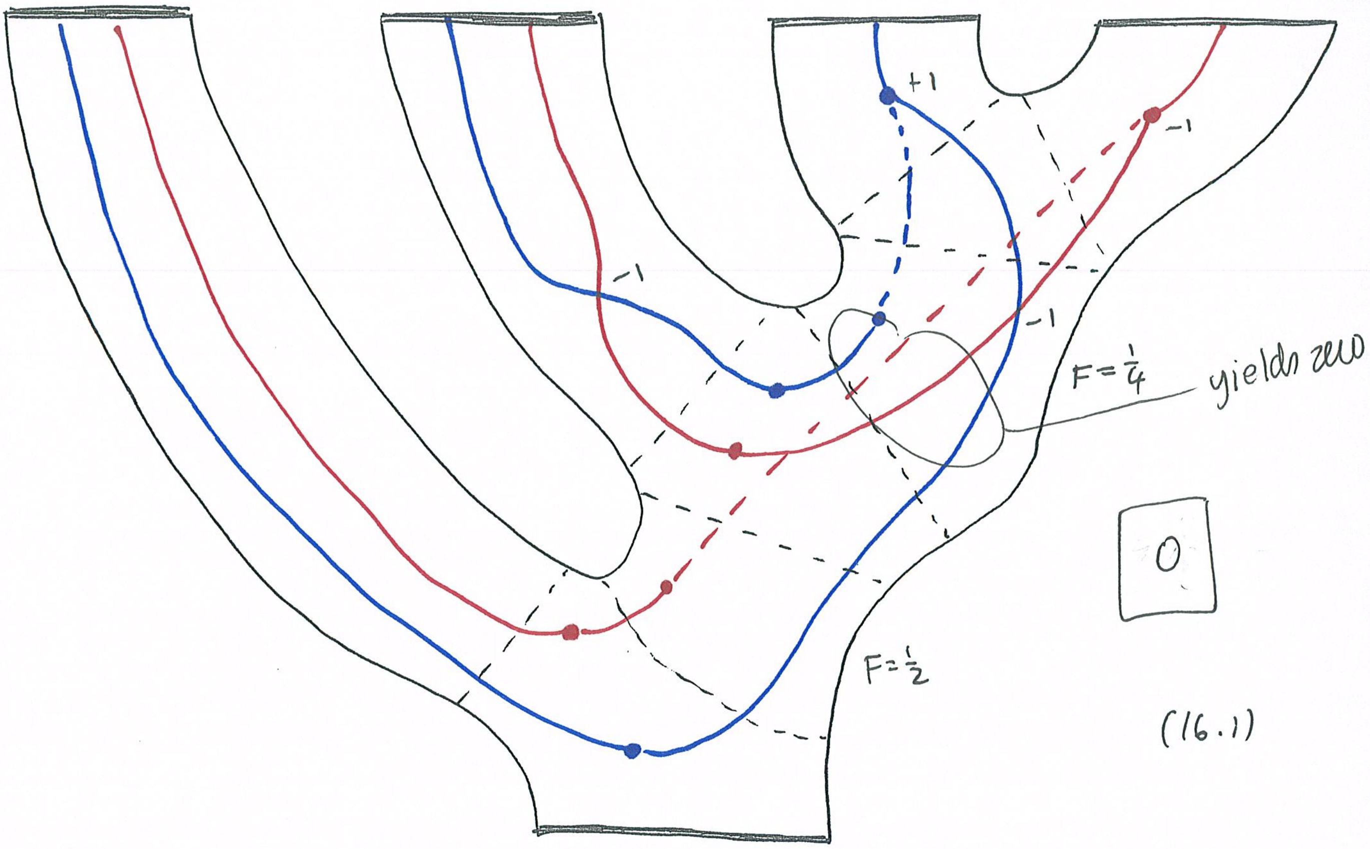
$$\psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^*$$


ψ_1^*
ψ_2^*
+1
-1
-1
$F = \frac{1}{4}$
yields zero $O_2^2 = 0$
$F = \frac{1}{2}$
0

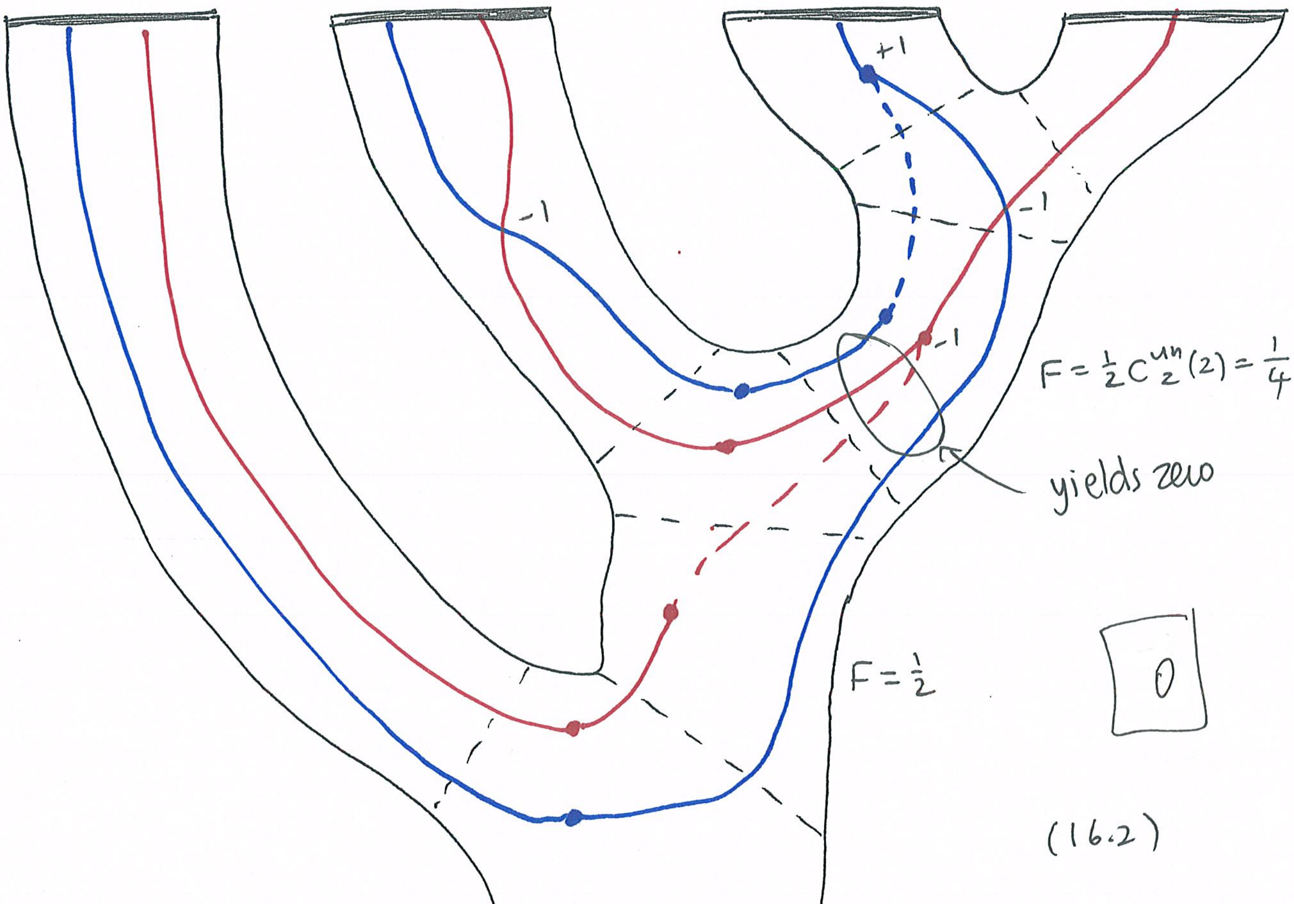
(15.1)


+1
-1
-1
$m = 1, w = 2$
$F = \frac{1}{2} C_2^{un}(2) = \frac{1}{2} \cdot \frac{2}{4} = \frac{1}{4}$
yields zero
$F = \frac{1}{2}$
0

(15.2)

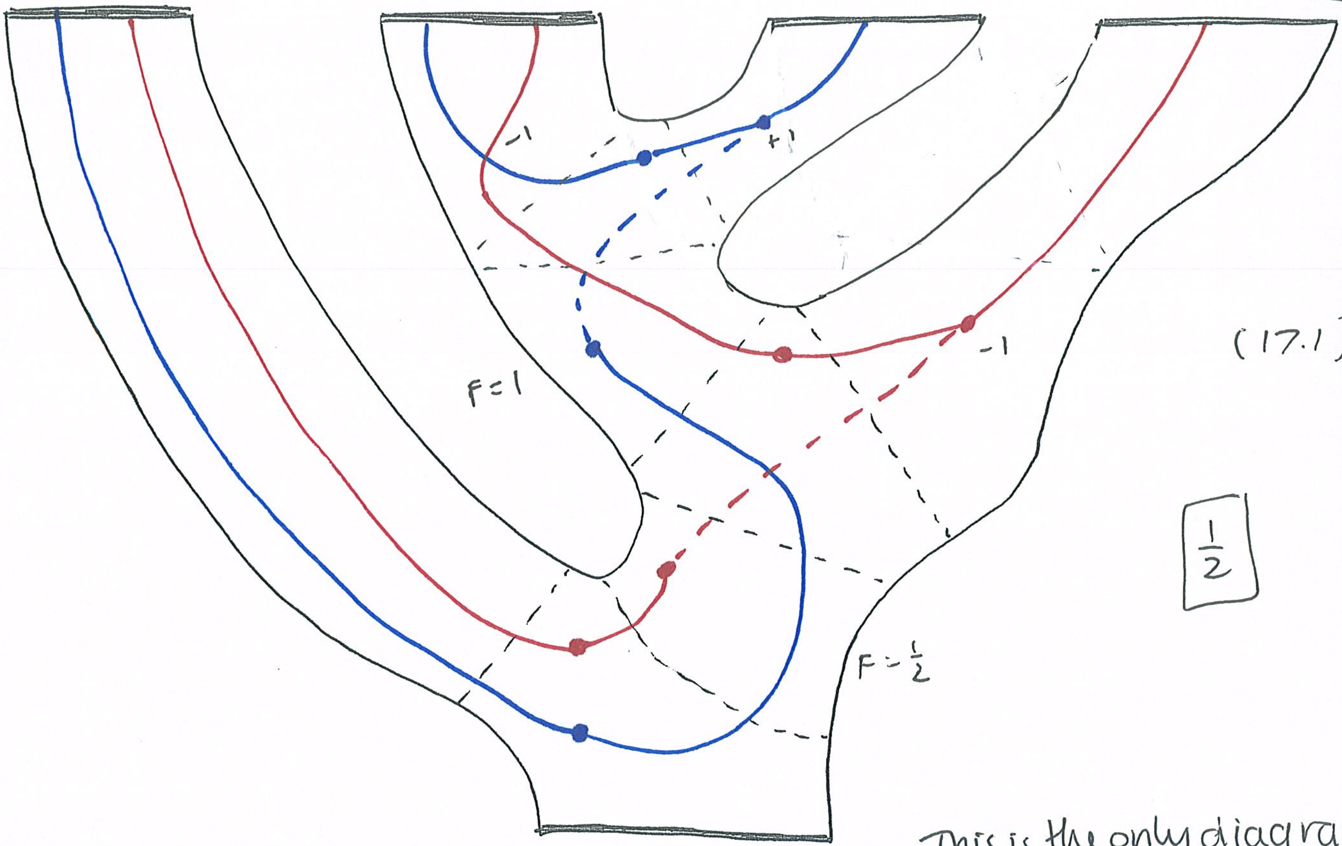


(16.1)



(16.2)

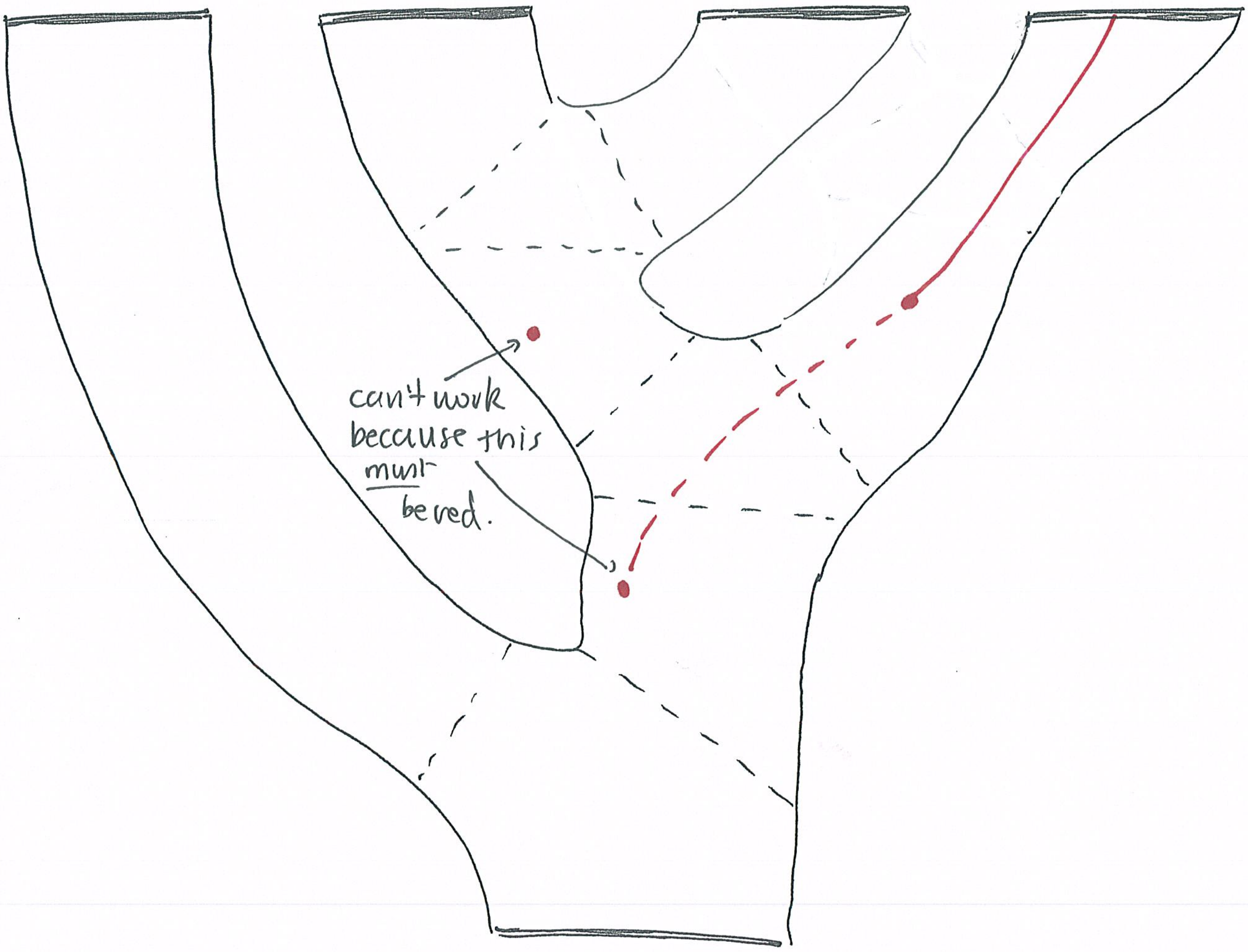
Note (15.1), (15.2), (16.1), (16.2) are the only diagrams for this tree.



(17.1)

$\boxed{\frac{1}{2}}$

This is the only diagram for this tree.



Note (Added later)

We believe the total amplitudes, for various trees and inputs, computed in the following pages to be correct. But the way these are combined to form by we are not sure about.

The conclusion is rather amusing: the amplitudes for  are zero, leaving (17.1) as the only contribution.

$$b_4(\psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^*)_{const} = 1/2. \qquad (18.1)$$

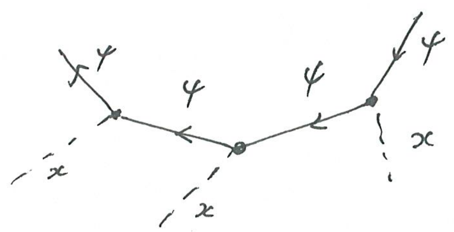
Similarly

$$b_4(\psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^* \otimes \psi_1^*)_{const} = 1/2 \qquad (18.2)$$

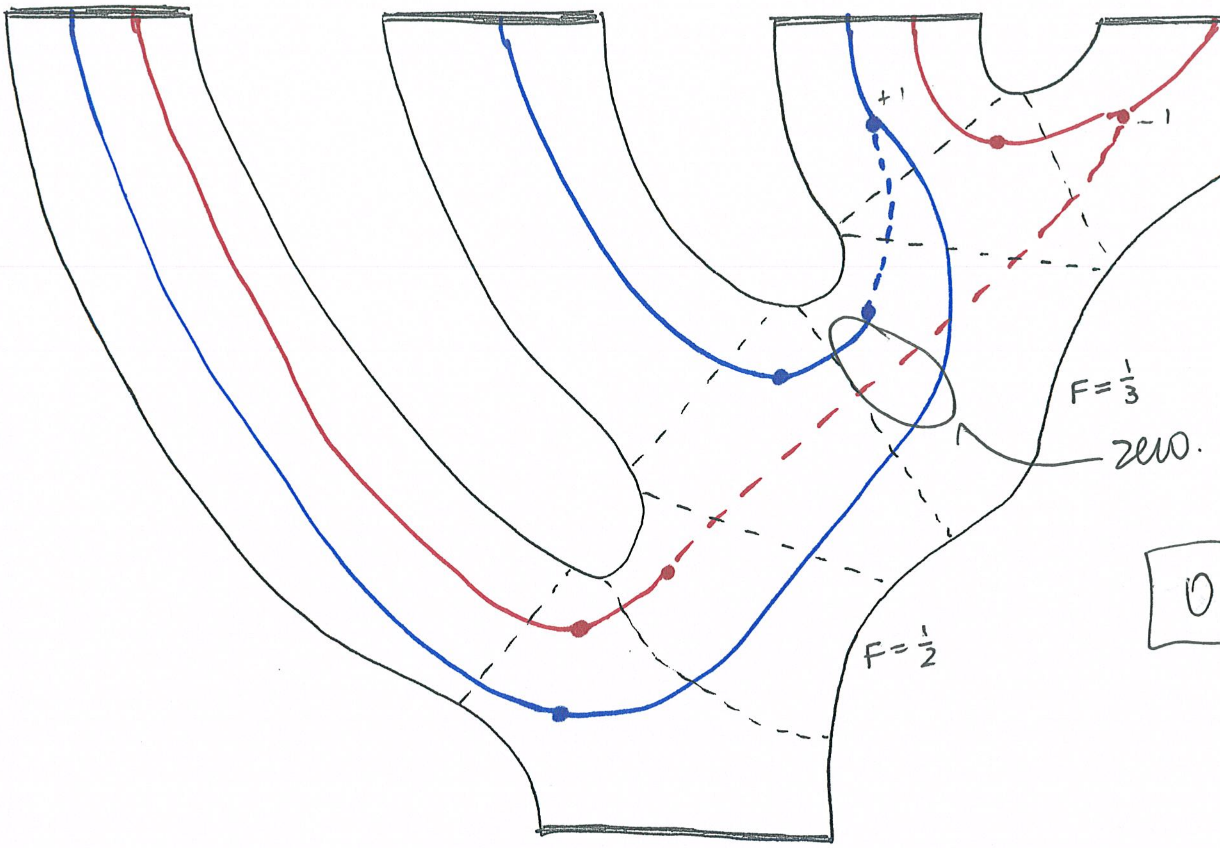
Note that since the diagrams of (14.1)-type inputs involve one Z_∞ vertex per ψ_i, there is an invariance under the $\psi_1 \longleftrightarrow \psi_2$ exchange. So it only remains to compute

$\psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^*$	$\boxed{1/2}$	(18.1) above
$\psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^*$	$\boxed{-1}$	(p. 21)
$\psi_1^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^*$	$\boxed{-1}$	(p. 23)
$\psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^*$	$\boxed{1/2}$	(p. 25)
$\psi_1^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^*$	$\boxed{1}$	(p. 26)
$\psi_1^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^*$	$\boxed{1/2}$	(p. 27)

Note A general advantage of our formalism over Efimov's is that by separating ψ's and O's we avoid crazy oscillations i.e.

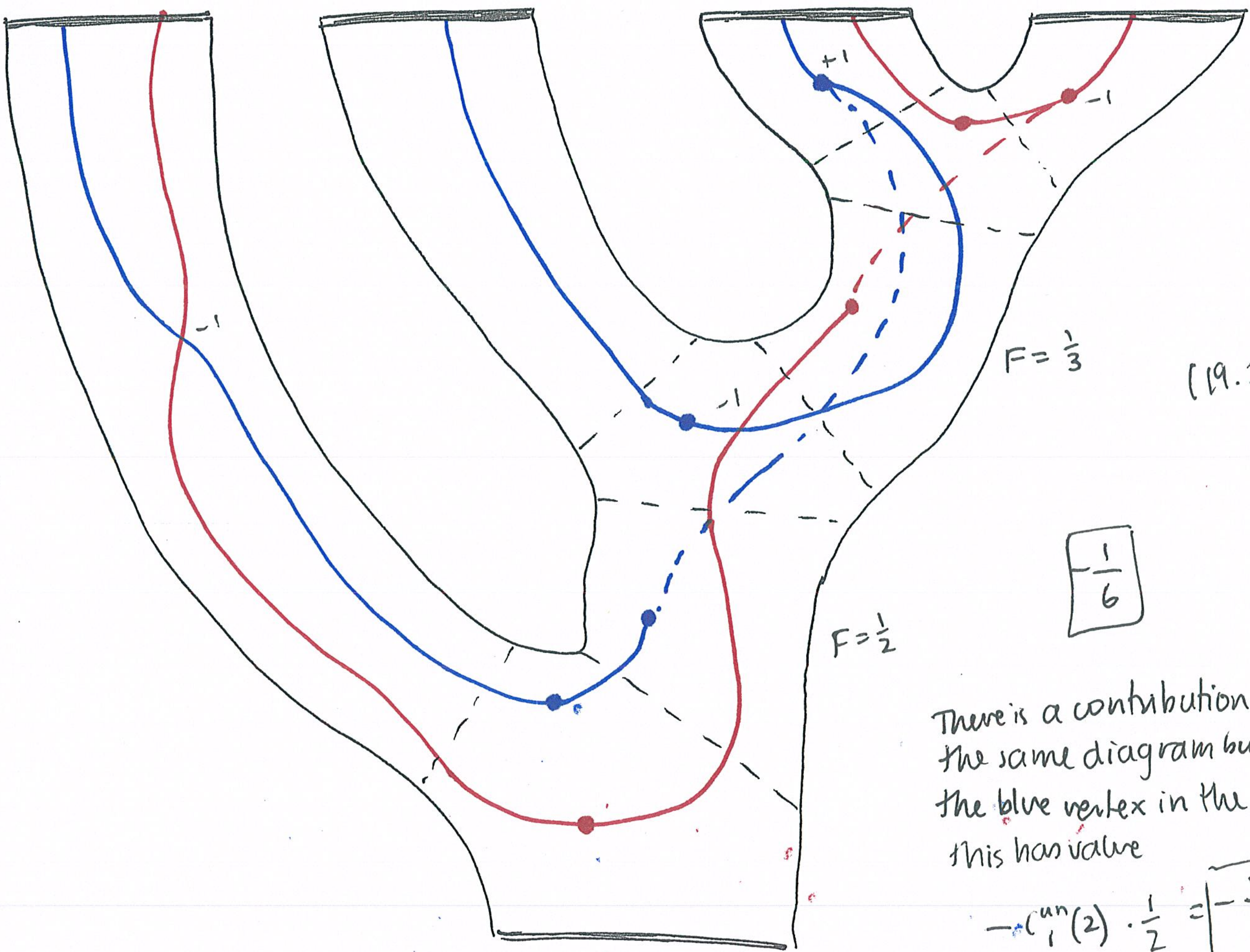


(18.3)



$\boxed{0}$

(19.1)

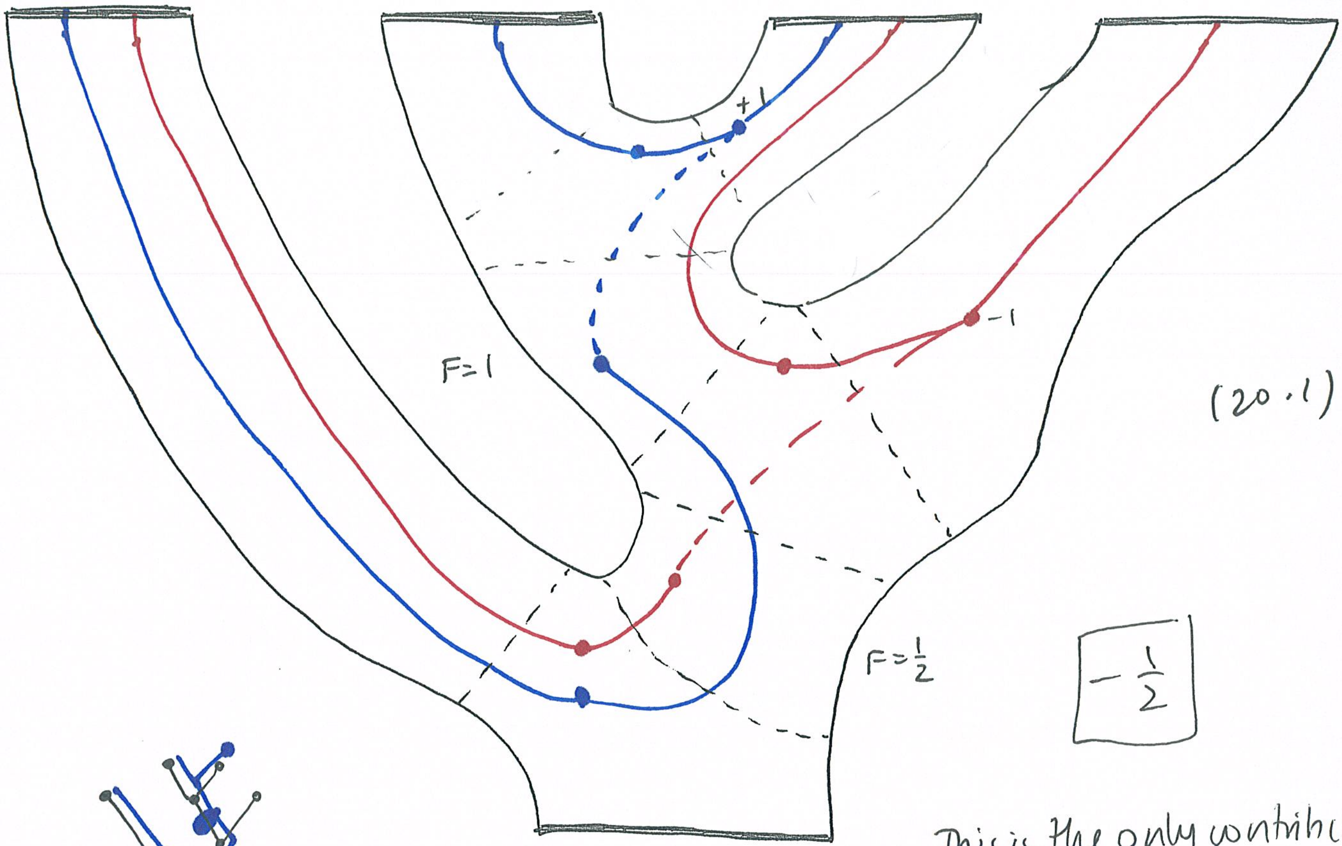


(19.2)

$\boxed{-\frac{1}{6}}$

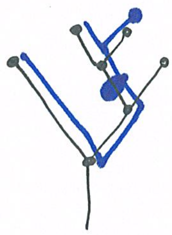
There is a contribution from the same diagram but with the blue vertex in the H_∞ zone this has value

$$-c_1^{un}(2) \cdot \frac{1}{2} = \boxed{-\frac{1}{3}}$$

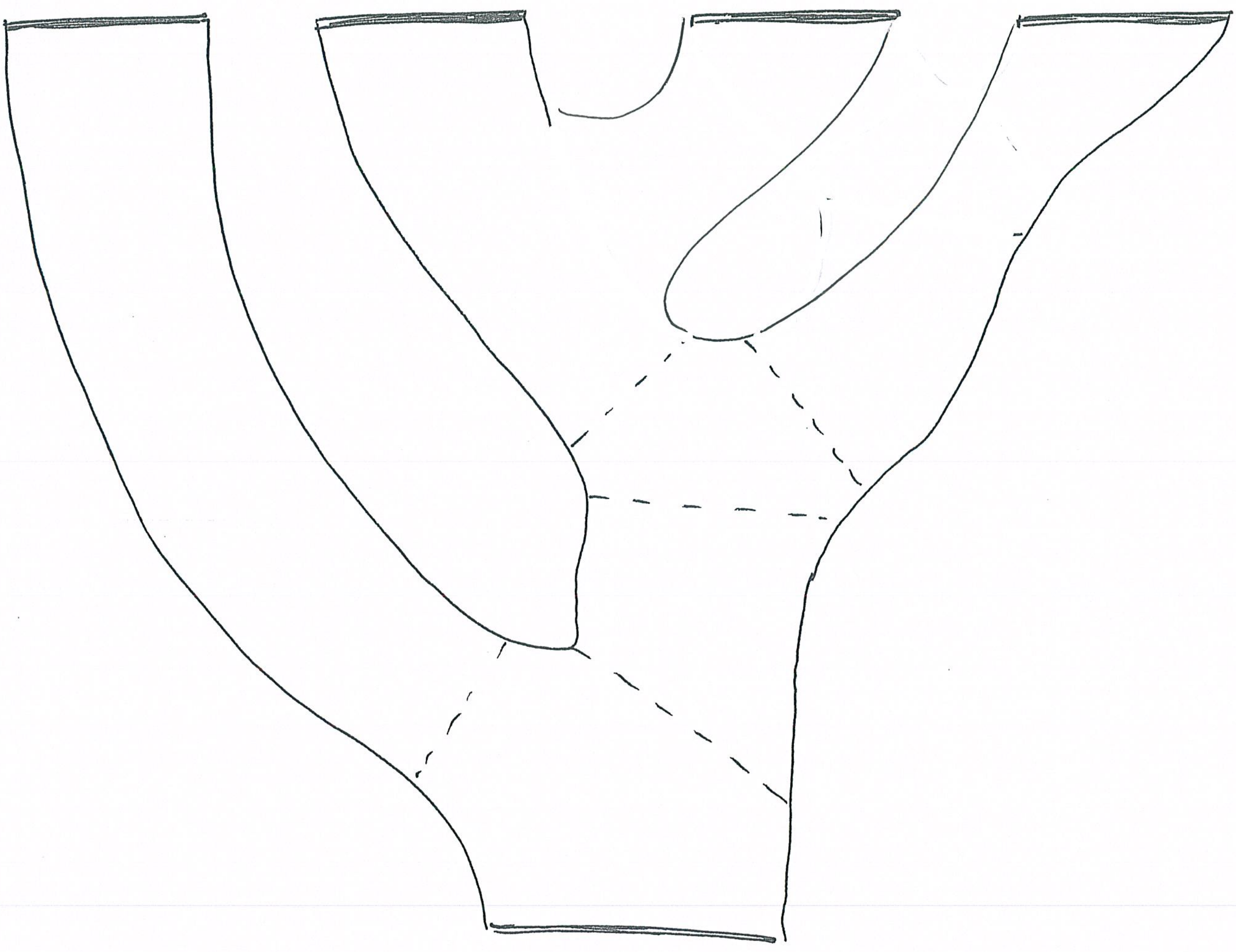


(20.1)

$\boxed{-\frac{1}{2}}$



This is the only contribution for this tree.



We conclude

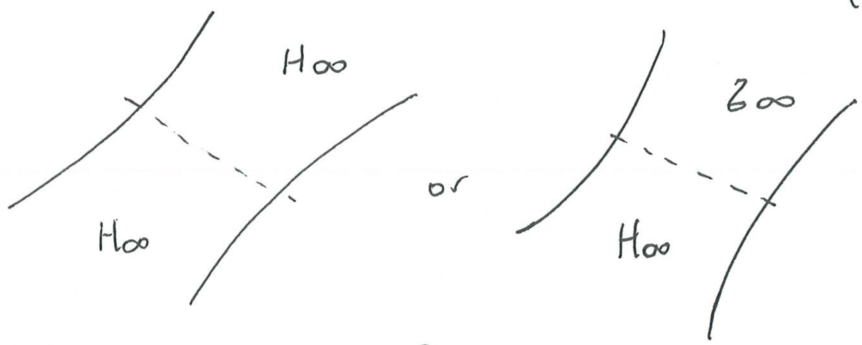
$$b_4(\psi_1^*\psi_2^*\otimes\psi_1^*\otimes\psi_1^*\psi_2^*\otimes\psi_2^*)_{const} = -\frac{1}{6}-\frac{1}{3}-\frac{1}{2} = -1$$

(well, this was corrected but the principle is valid)

From the cancelling of (15.2) and (16.1) we deduce another useful general fact. Since $H^2=0$ if we can view our diagrams as having a subdiagram where we are summing over all ways to have two H's consecutatively (with no Ξ in between) the result must be zero.

- Diagram combinations which "emulate"

(21.1)

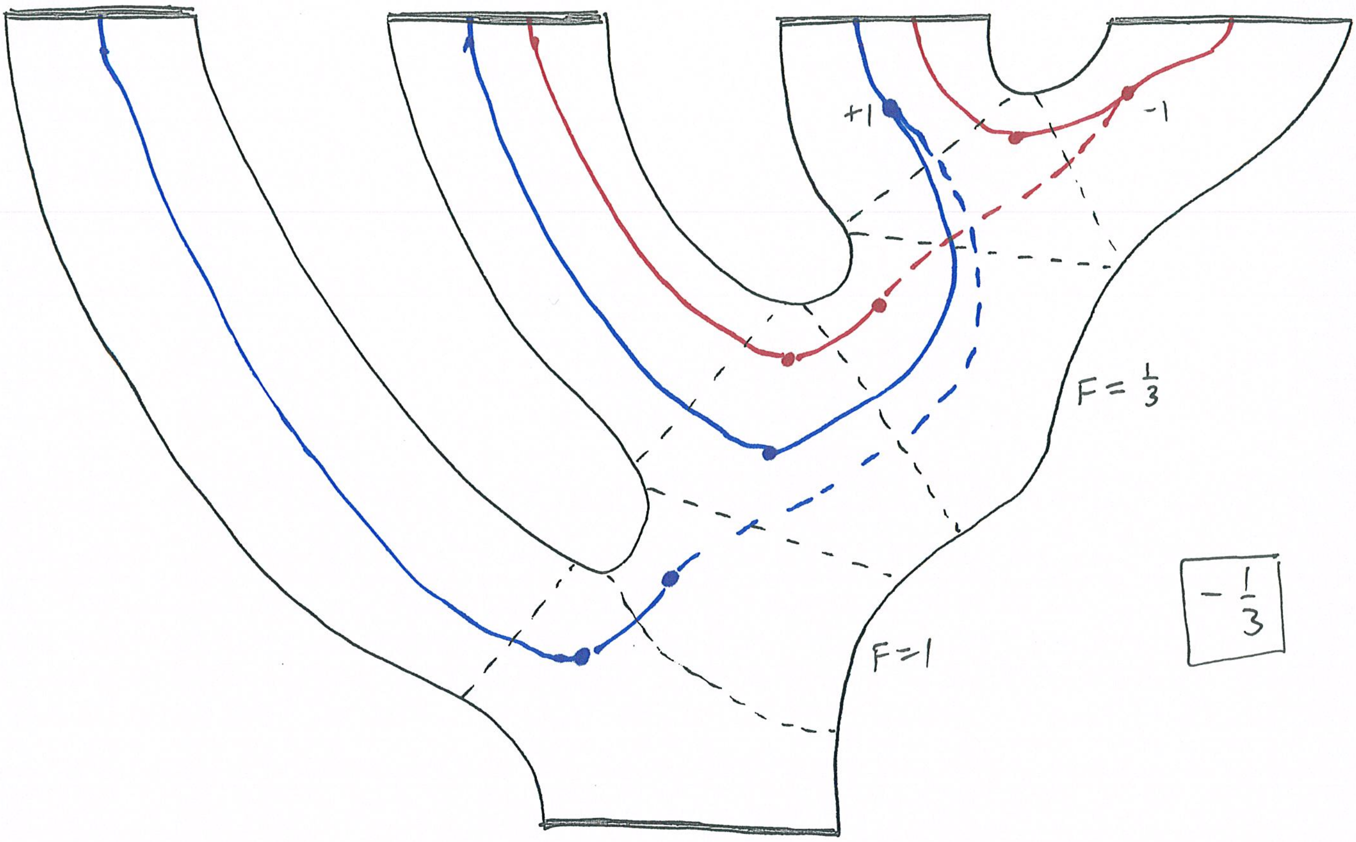


collectively contribute zero since $H^2=0$.

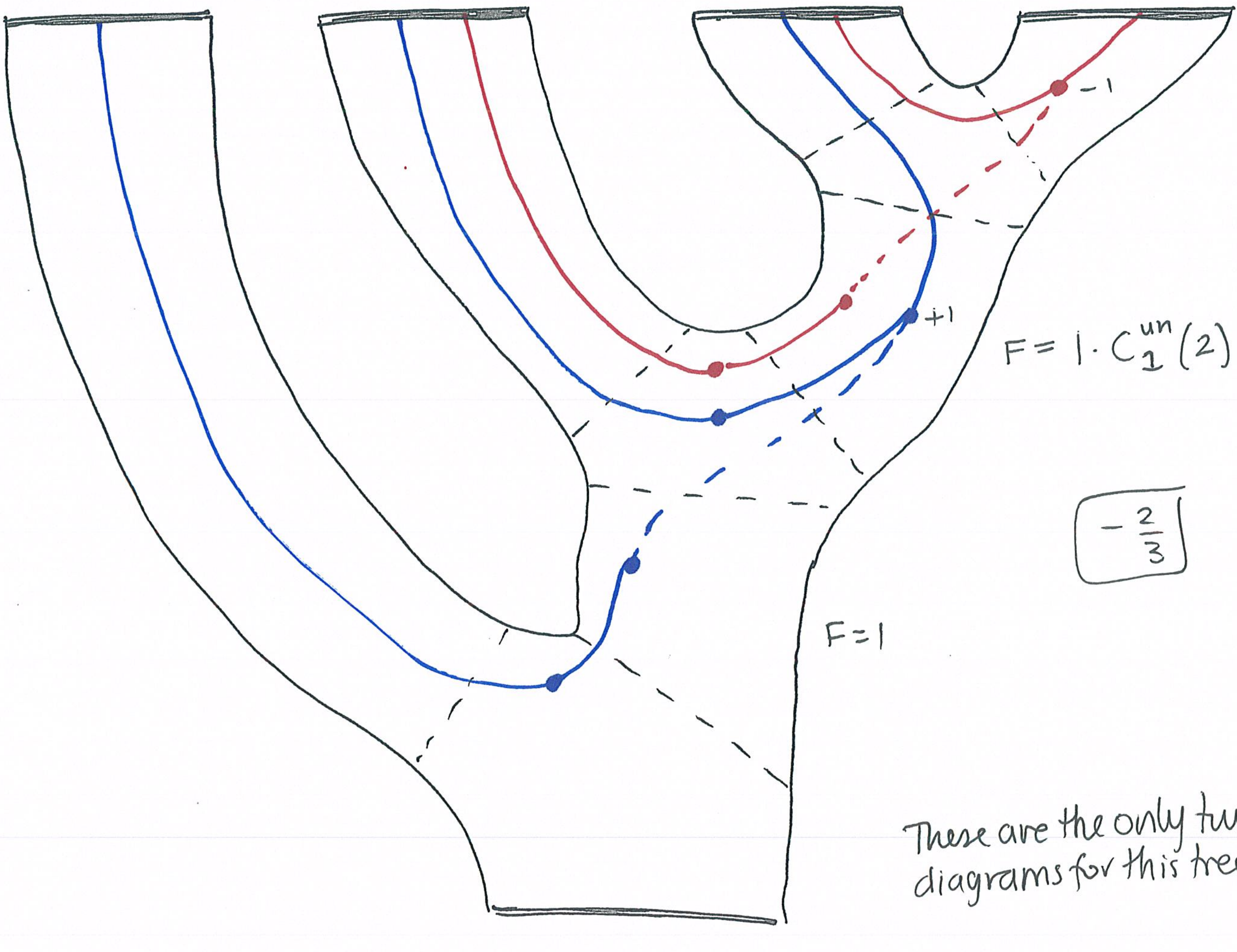
<u>Example</u> From this rule we deduce immediately that on inputs $\psi_1^*\psi_2^*\otimes\psi_1^*\psi_2^*\otimes\psi_1^*\otimes\psi_2^*$ the tree  must not contribute, because

(a) both ψ_1, ψ_2 in upper right could be moved to one channel without changing the amplitude

(b) then either by (12.3) or (21.1) we get zero.

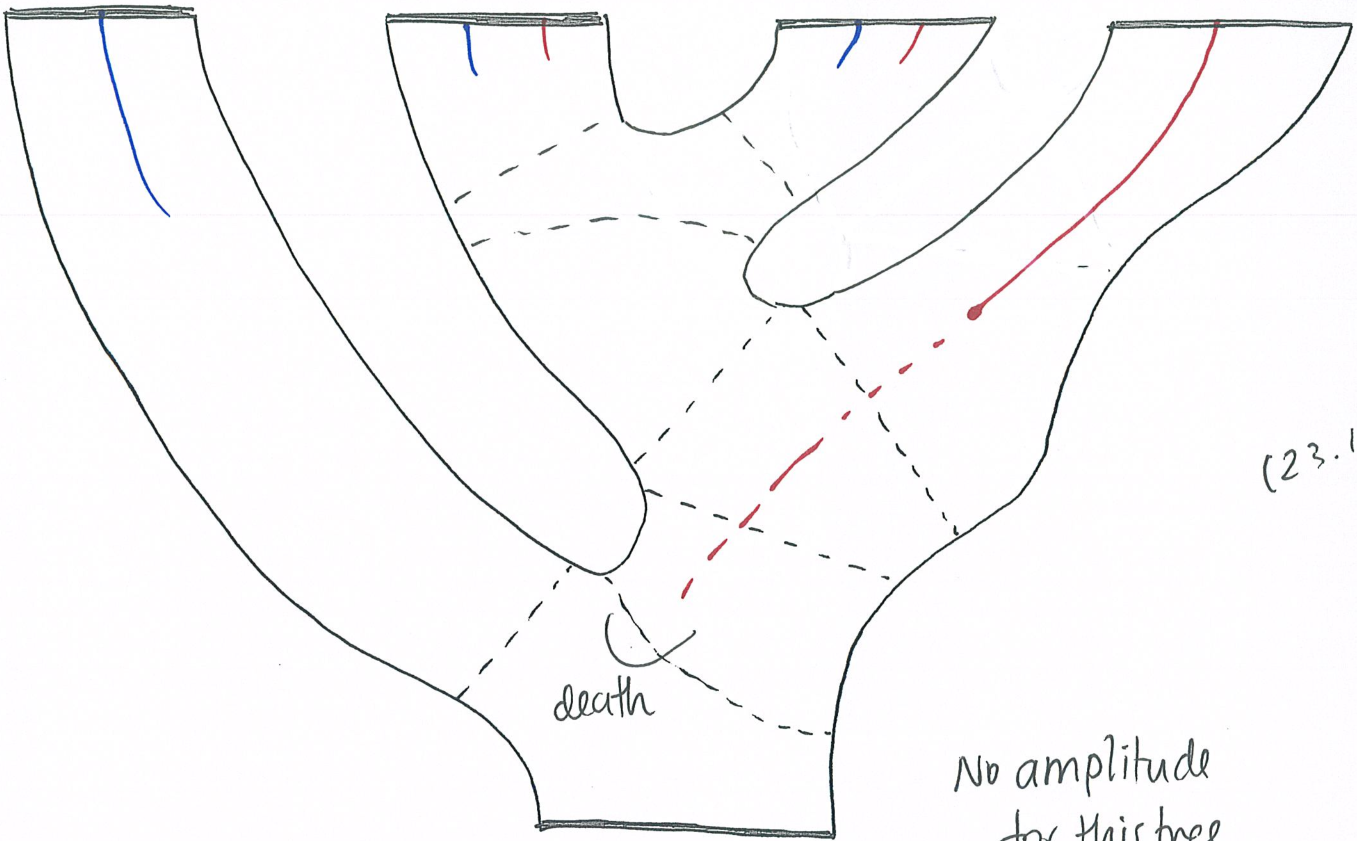


$\boxed{-\frac{1}{3}}$



$\boxed{-\frac{2}{3}}$

These are the only two diagrams for this tree



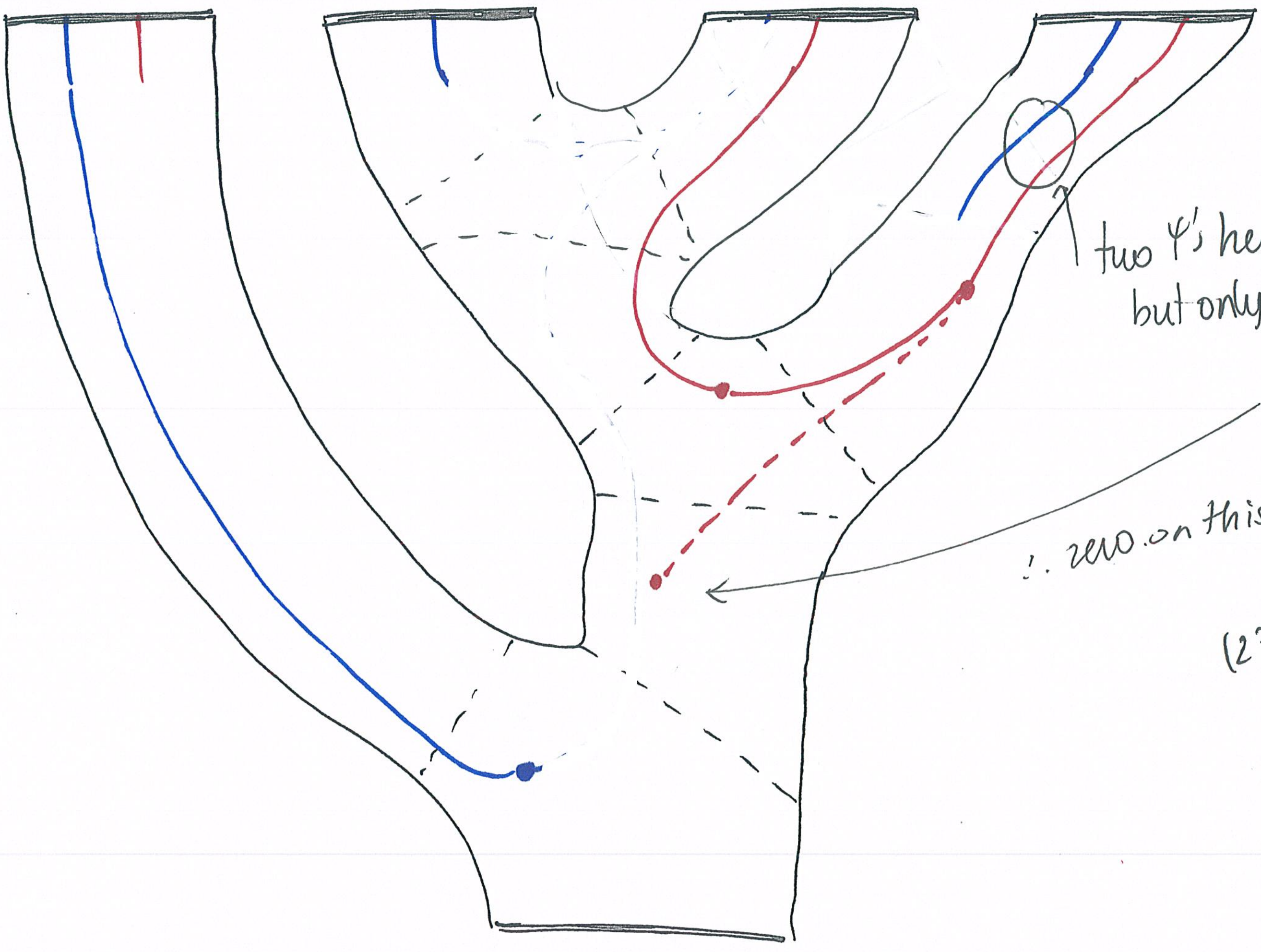
(23.1)

No amplitude for this tree

Hence $b_4(\psi_1^* \otimes \psi_1^*\psi_2^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^*)_{\text{const}} = -1$

Now we move onto

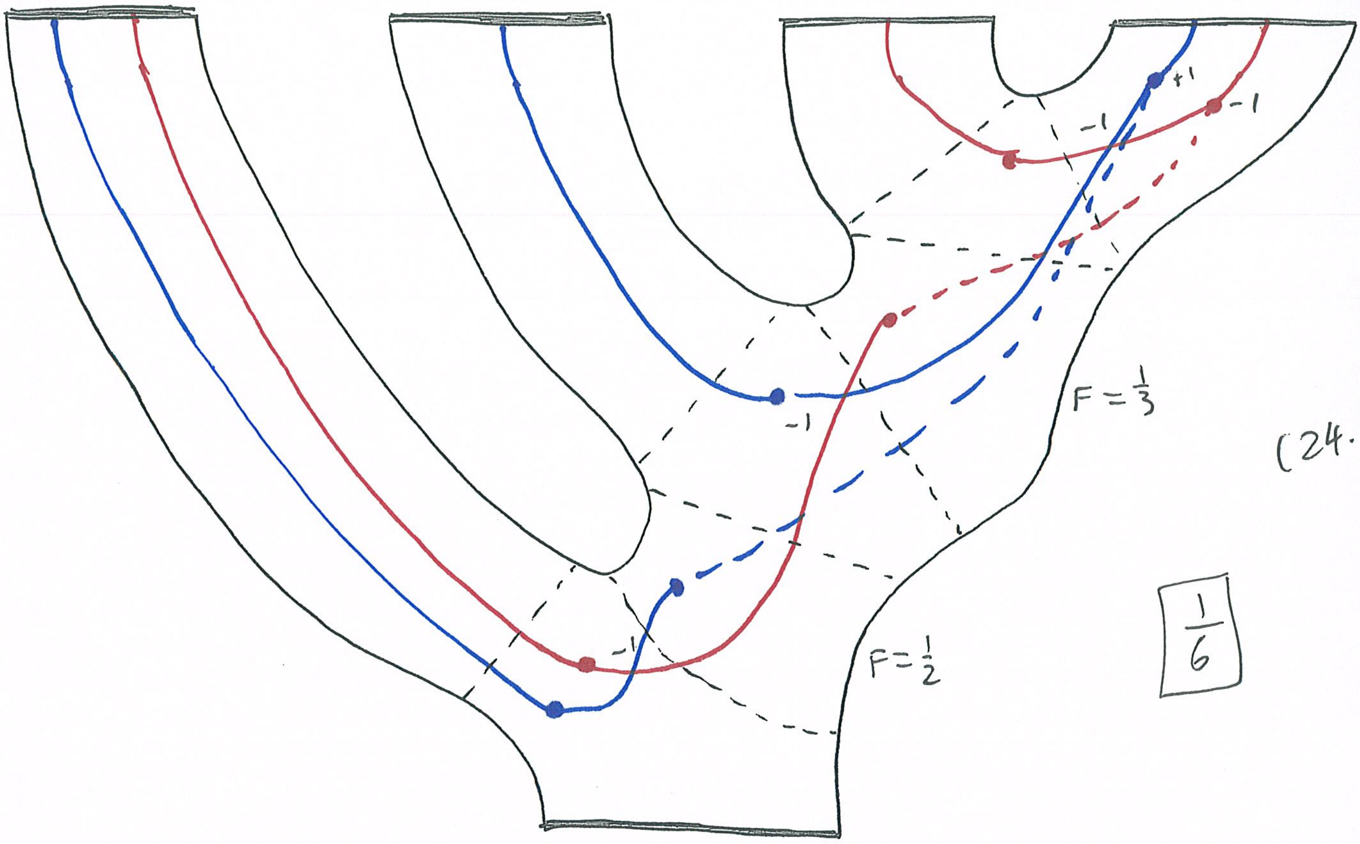
$\downarrow \quad \psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^*$



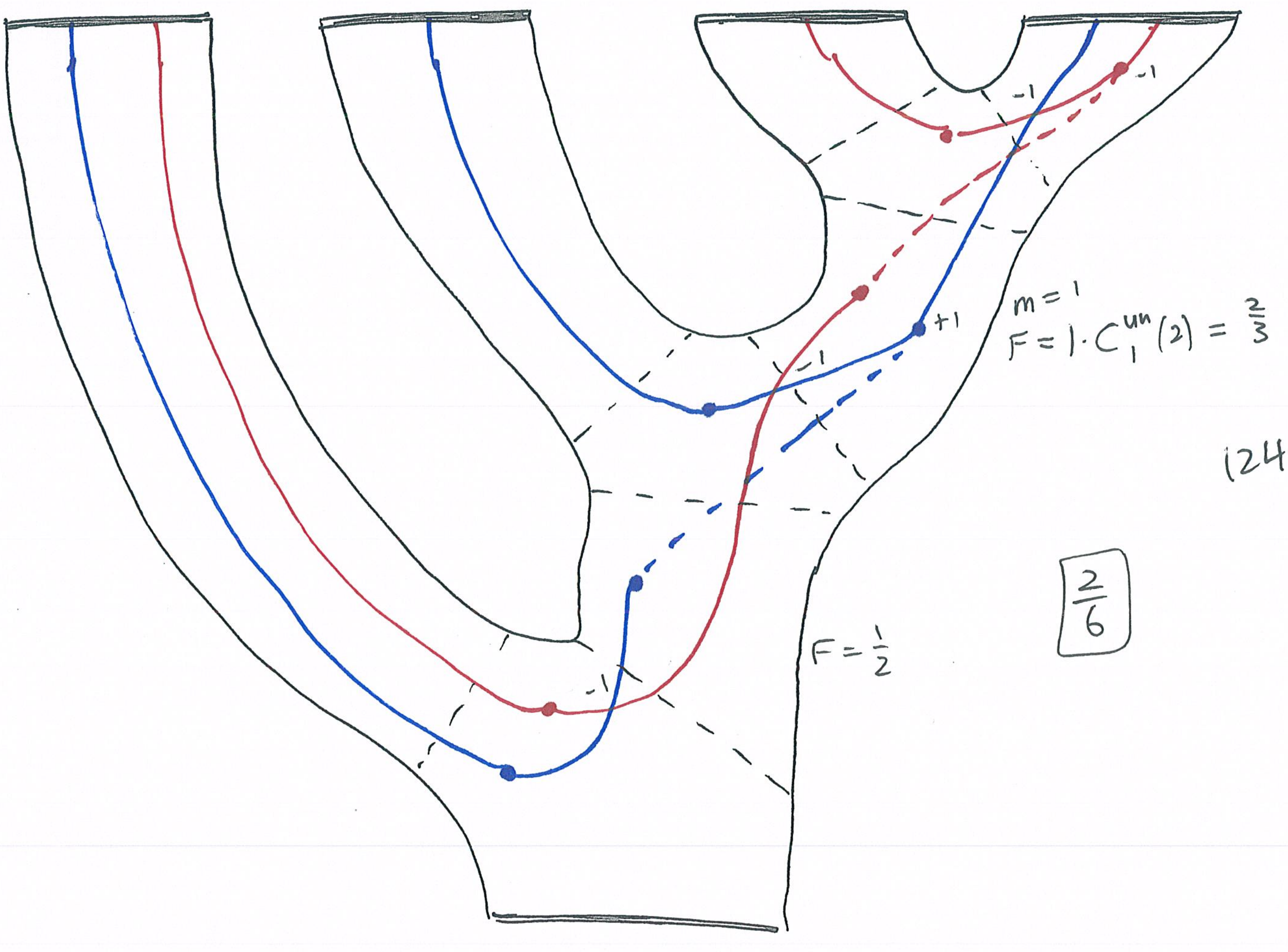
two ψ's here but only one H_∞ spot

$\therefore$ zero on this tree

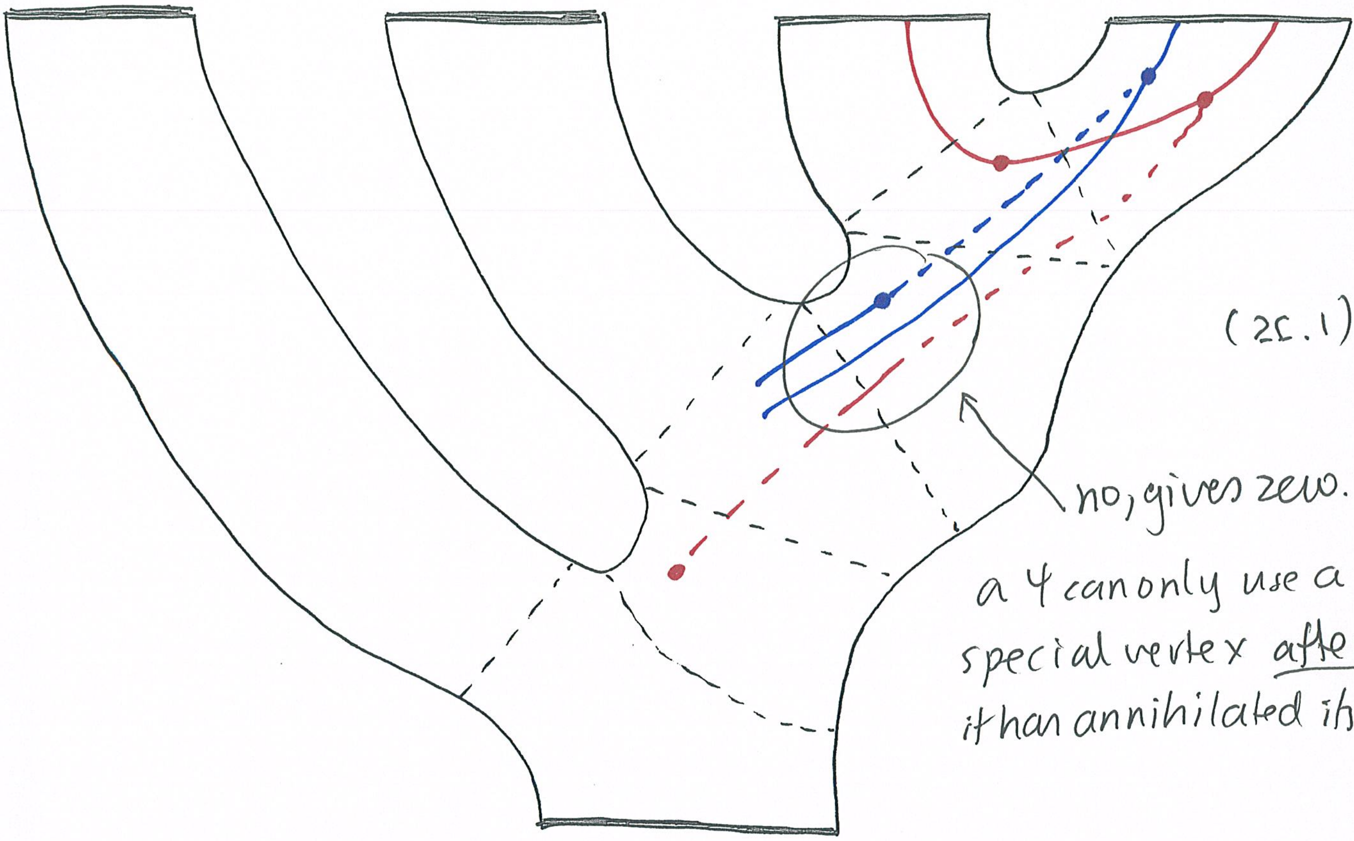
(23.2)


+1
-1
-1
-1
-1
$F=\frac{1}{3}$
-1
$F=\frac{1}{2}$
$\boxed{\frac{1}{6}}$

(24.1)


-1
-1
+1
-1
$m=1$
$F = 1\cdot C_1^{un}(2) = \frac{2}{3}$
$F=\frac{1}{2}$
-1
$\boxed{\frac{2}{6}}$

(24.2)



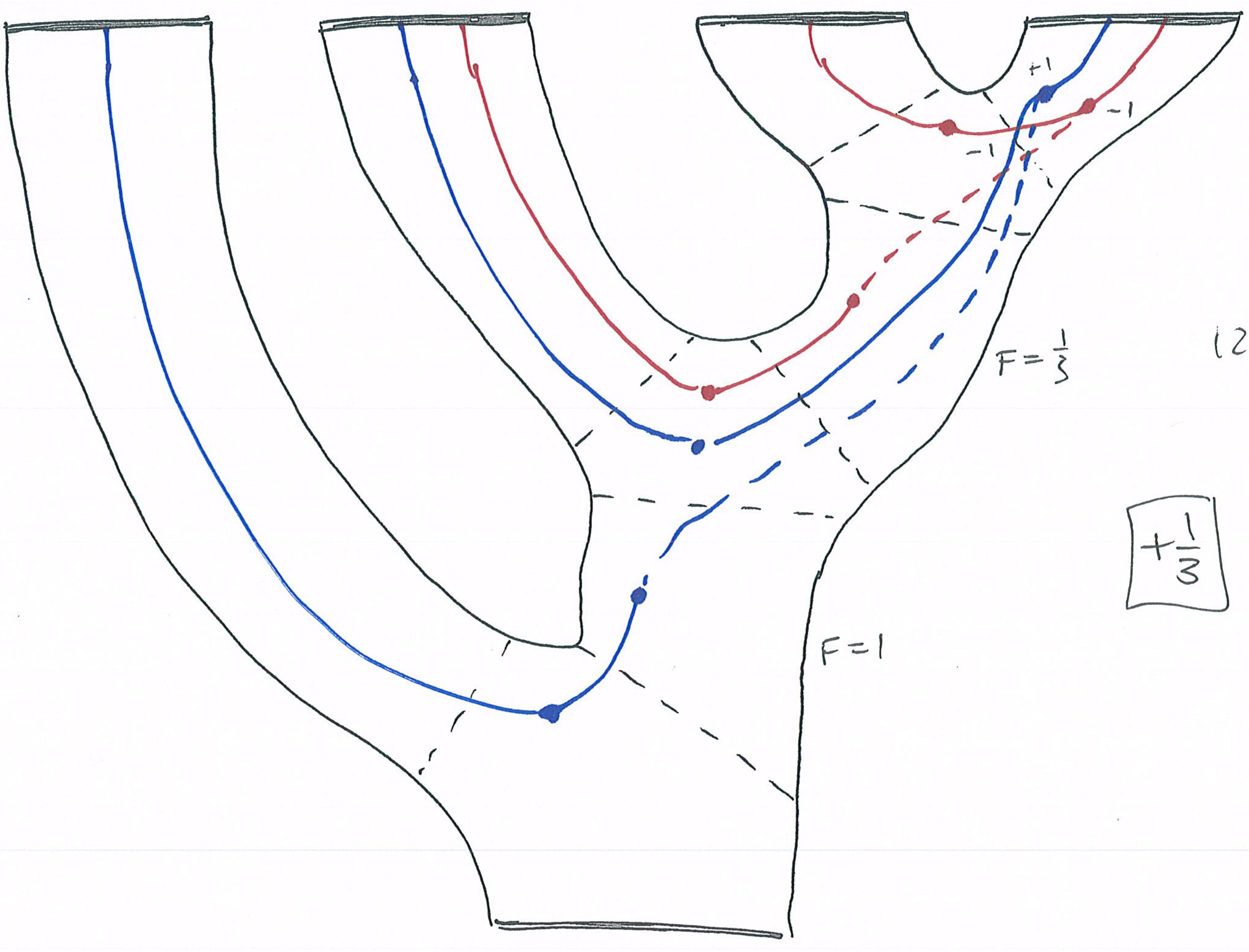
(25.1)

no, gives zero.

a ψ can only use a special vertex <u>after</u> it has annihilated its θ.

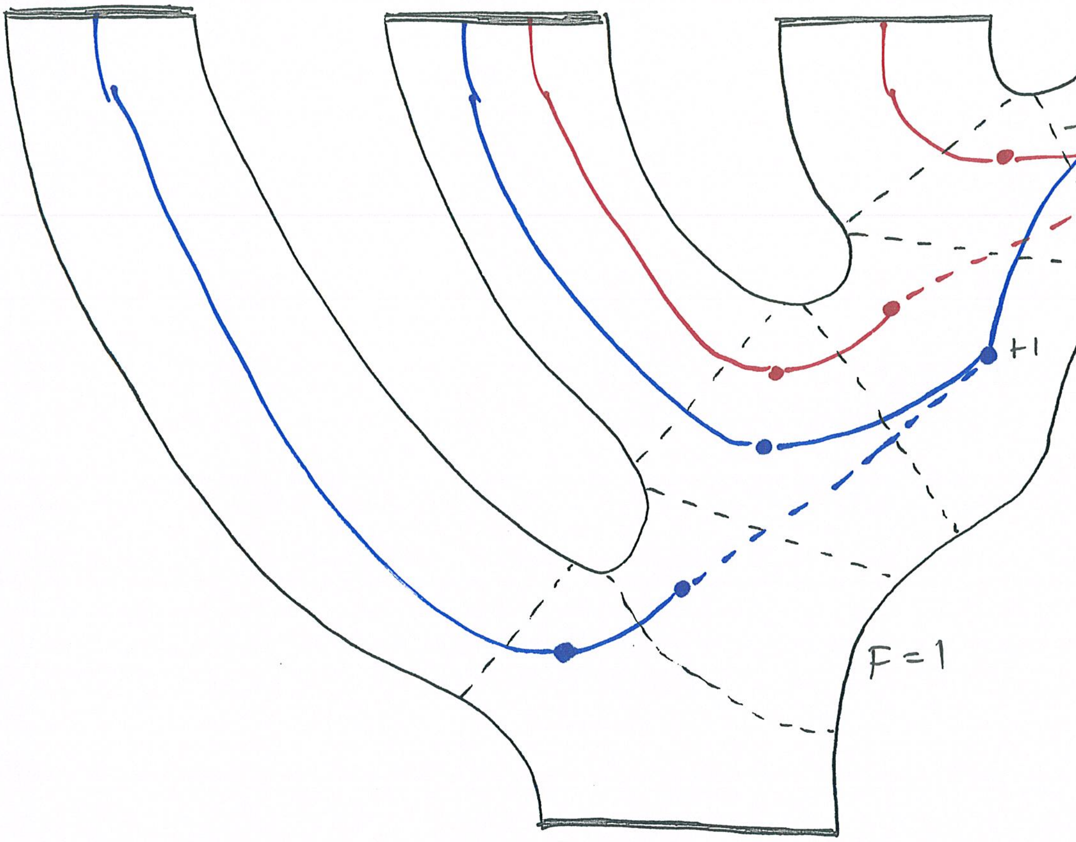
We conclude $b_4(\psi_1^*\psi_2^* \otimes \psi_1^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^*) = \frac{1}{2}$

For $\psi_1^* \otimes \psi_1^*\psi_2^* \otimes \psi_2^* \otimes \psi_1^*\psi_2^*$:



(25.2)

$\boxed{+\frac{1}{3}}$

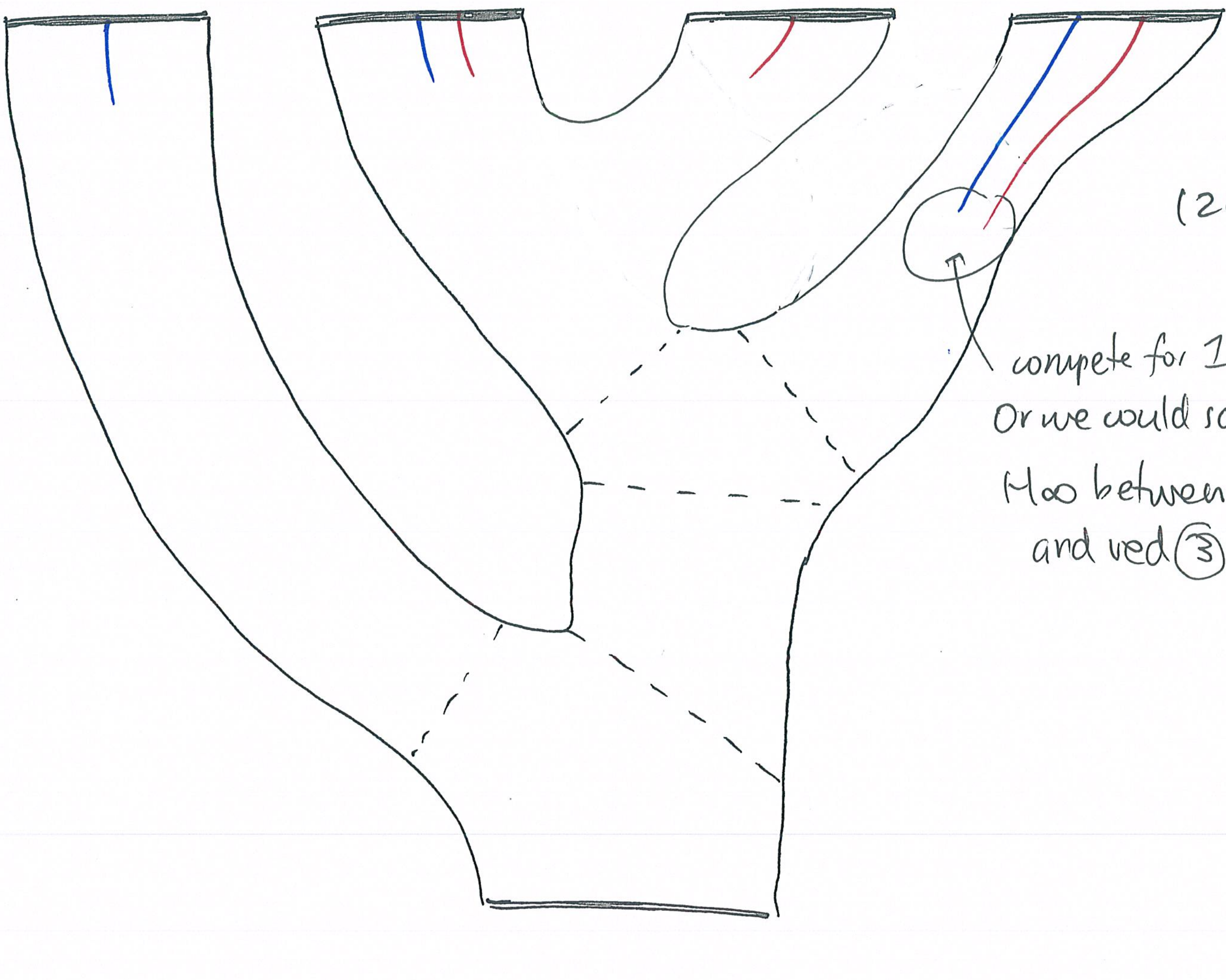


$F = 1 . C_1^{un}(2) = \frac{2}{3}$

(26.1)

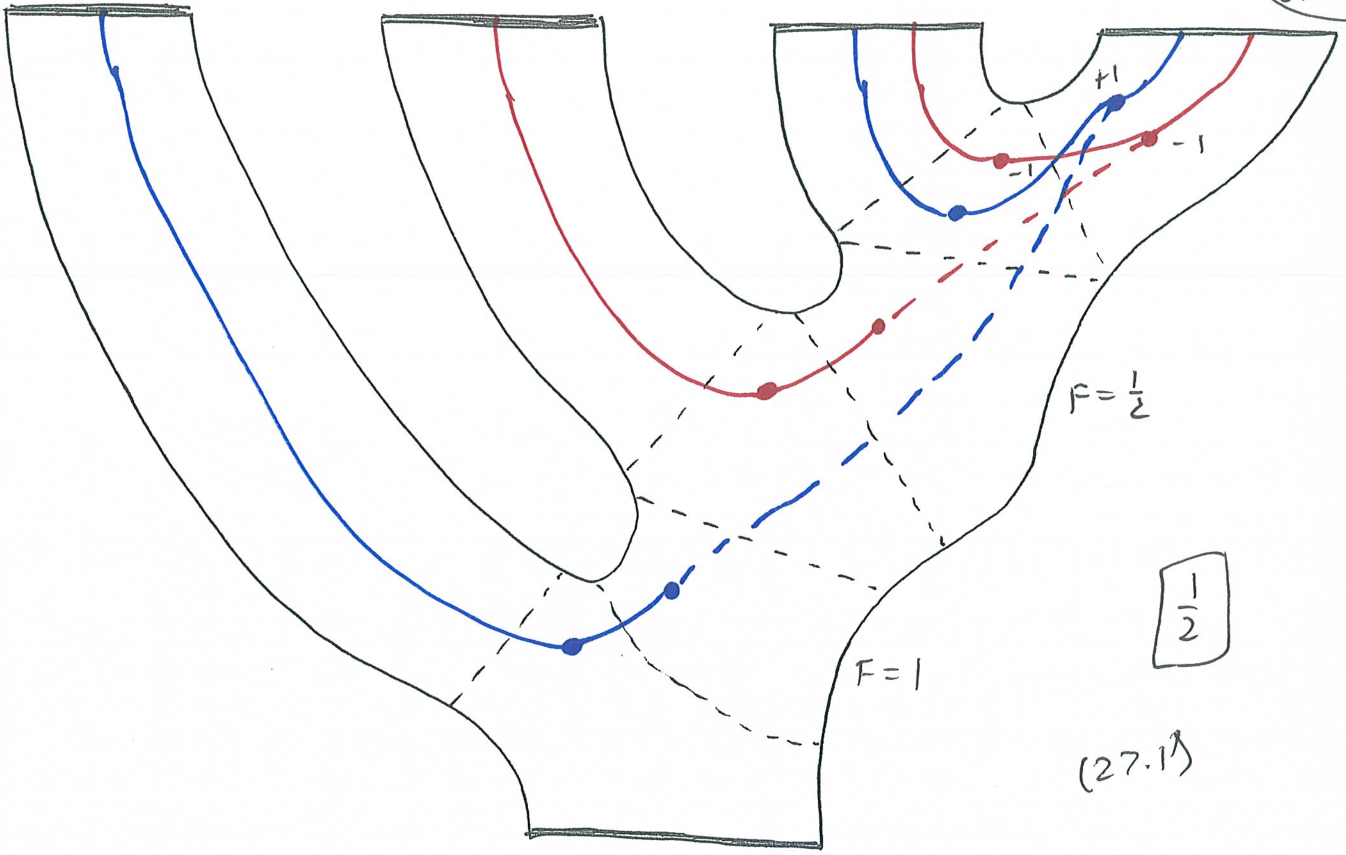
$\boxed{+\frac{2}{3}}$

There is no contrib. from the other tree so $b_4(\psi_1^* \otimes \psi_1^* \psi_2^* \otimes \psi_2^* \otimes \psi_1^* \psi_2^*)_{const} = 1$



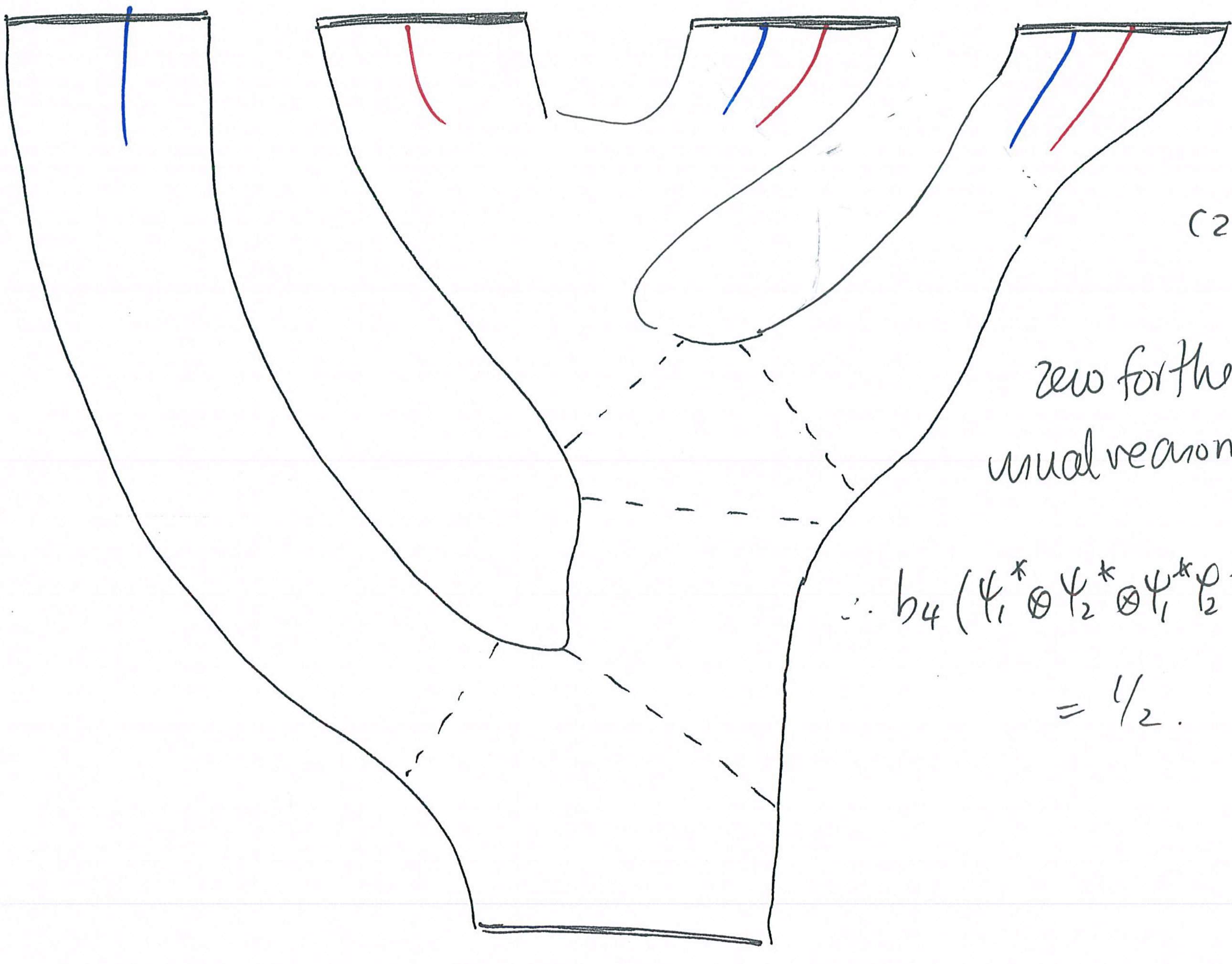
(26.2)

compete for 1 $H\infty$.
Or we could say: no
$H\infty$ between red 2
and red 3.



$\boxed{\frac{1}{2}}$

(27.1)



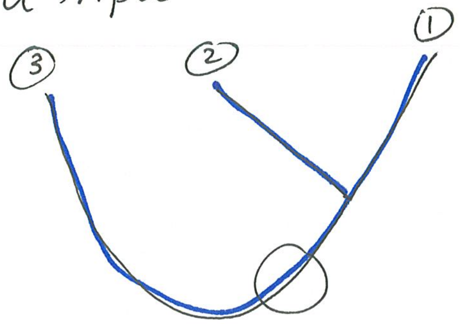
(27.2)

zero for the usual reasons

$$\therefore\ b_4(\psi_1^* \otimes \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^*)_{\text{const.}} = 1/2.$$

A <u>general fact</u> we observe from (25.1) is that

- given a triple



(28.1)

The special vertex generating the final $\mathcal{O}$ must occur in the marked zone, i.e. on the path from the 2nd to 3rd φ. Otherwise we get $\mathcal{O}^2 = 0$.