We revisit the old examples of $x^d$ and $y^d-x^d$ with the final Feynman rules from \(\text{ainfmr}^\circ\). Let us begin with

**Example** \(W = x^d\) for \(d \geq 3\) (so that \(W \in \mathbb{R}^3\)), and \(W = x^{d-1}\).

Then the only possible trivalent interaction is (writing \(\varphi = \varphi_1, \theta = \varphi_1, x = x_1,\))

\[
\text{prefactor } \frac{-1}{\alpha - 1}
\]

\[\psi
\]
\[\theta
\]
\[\exists \bar{x}(x^{d-1})
\]

Then we seek to compute the minimal model

\[
\mathcal{A} = k \cdot 1 \oplus k \cdot \psi^* \oplus \{ b_\varphi \}_{\varphi \geq 2}.
\]

The product is, by \(\text{p(13) ainfmr}\) the product on \(k[\epsilon]/\epsilon_1, \epsilon = \psi^*\).

For \(q \geq 3\) we compute some example diagrams. Recall that the Feynman rules are for vacuum boundary conditions.

(we know all the answers; see \(\text{p. (10) ainfmr}\)).

**Example** From \(\text{p. (4)}\) onwards we recover the formula of \(\text{p. (11) ainfmr}\)

\[
\beta_3 : (\mathcal{A}(1))^3 \to \mathcal{A}(1) \quad \mathcal{A} = \Lambda(k \psi^*_1 \oplus k \psi^*_2)
\]

\[
\beta_3(\Lambda_2 \oplus \Lambda_1 \oplus \Lambda_0) = (-1)^{ij} \left( \sum_{i,j} \Lambda_i \Lambda_j^* \right) \left( \begin{array}{c} -[\psi_i, \Lambda_0] \cdot [\psi_i, \Lambda_1] \cdot [\psi_i, \Lambda_2] \\ + [\psi_2, \Lambda_0] \cdot [\psi_2, \Lambda_1] \cdot [\psi_2, \Lambda_2] \end{array} \right)
\]

for \(W = y^3 - x^3\),
On p. 20.5 of [reference] we express the higher multiplications $\rho_q$ in terms of the $O(T, C)$ which we may compute by Feynman diagrams. Now, for a tree $T$ and configuration $C$

\[ O(T, C)(T_1 \otimes \cdots \otimes T_e) \in A \]

has a constant term

\[ O(T, C)(T_1 \otimes \cdots \otimes T_e)_{\text{const}} \in k. \]

This is computed by diagrams with vacuum outgoing boundary condition, as in (2.1) above, which is a config of

\[ T = \]

\[ \gamma(v) = (2) \]

\[ \tau(v) = \{1\} \]

\[ \alpha(v) = 1 \]

\[ m(x) = 1 \quad m(y) = 0 \quad m(z) = 1 \]

\[ m(u) = 0 \quad m(v) = 1 \quad m(\ldots) = 0 \]

\[ J(x) = J(z) = J(u) = \{1\} \]

\[ \psi = \{1\} \]

\[ t(p) = 1 \]
Then by def$^N$ $O(T, \Theta)$ is associated to the tree

\[ (-1) \frac{1}{|\Theta|} W^2(\gamma) \vartheta_1(\chi^\gamma) O_1(\Psi_i^{-1}) \]

(2.5.1)

\[ m_2([\Psi_i^-] \otimes O_i^*) \]

(2.5.2)

\[ \vartheta_1 \otimes \partial_1 \]

\[ m_2([\Psi_i^-] \otimes O_i^*) \]

And so

\[ O(T, \Theta)(\Psi_i^* \otimes \Psi_i^* \otimes \Psi_i^*) = \pi m_2([\Psi_i^-] \otimes O_i^*)(\Psi_i^* \otimes \vartheta_1 \otimes m_2([\Psi_i^-] \otimes O_i^*) \vartheta_1 \otimes (-1) \frac{1}{|\Theta|} W^1(\gamma) \vartheta_1(\chi^\gamma) O_1(\Psi_i, \Psi_i^*)) \]

Hence since $|\Theta|=2$ and $W^1(2)=1$

\[ O(T, \Theta)(\Psi_i^* \otimes \Psi_i^* \otimes \Psi_i^*) \text{ const} \]

\[ = \left[ \pi m_2([\Psi_i^-] \otimes O_i^*)(\Psi_i^* \otimes \vartheta_1 \otimes m_2([\Psi_i^-] \otimes O_i^*)(\Psi_i^* \otimes \vartheta_1 \otimes [\Psi_i, -](\Psi_i^*))) \right] \text{ const} \]

\[ \text{The contractions indicated are the only possible ones, so} \]

\[ O(T, \Theta)(\Psi_i^* \otimes \Psi_i^* \otimes \Psi_i^*) = -1 \]
$W = x^4$

$\psi^*$

$m=0$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=1$

$m=0$

$w=3-1=2$

$F = \frac{1}{2}$

Overall amplitude

$\frac{1}{2} \cdot (\frac{-1}{2}) \cdot 2 = -1$

By symmetry factor we mean that there is in fact a second diagram contributing the same amplitude, shown on the left.
Now we move onto \( W = y^2 - x^3 \)

\[
W = x \cdot \left(-x^2\right) + y \cdot \frac{y^2}{W^1} + \frac{y^2}{W^2}
\]

which has two kinds of triple interactions

\[
\begin{align*}
\text{prefactor} & \quad \frac{-1}{2} \cdot W^1(2,0) \\
& \quad = \frac{-1}{2} \cdot (-1) = \frac{1}{2}.
\end{align*}
\]

\[
\begin{align*}
\text{prefactor} & \quad \frac{-1}{2} \cdot W^2(0,2) \\
& \quad = \frac{-1}{2} \cdot 1 = -\frac{1}{2}.
\end{align*}
\]

Thus we have

Note When we say "amplitude" we mean the contribution of a particular diagram to \( O(T, \mathcal{C})(A_1 \otimes \cdots \otimes A_q) \) const
(in particular this does not include a sign for the internal edges).
\[ W = y^3 - x^3 \]

Overall amplitude

\[ \text{amplitude} = -\frac{1}{2} \cdot 2 = -1 \]

Overall amplitude

\[ = -1 \]
Later we will show how to recover (1.2) from p. 47.

Next let us consider \( b_4 \) for \( W = y^3 - x^3 \). Work on this was already done in (5.1) where we found

\[
b_4 = \begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array}
\]

and we started into some calculations. There are, for each channel, possible inputs \( 1, 4_1^*, 4_2^* \), \( 4_1^* 4_2^* \) and so in total \( (2^8)^4 = 2^8 = 256 \) possible inputs to \( b_4 \). Ouch!

All amplitudes have two internal vertices, so

- in (a) either \( m(2) > 0 \) or \( m(u) > 0 \)
  - in (b) either \( m(y) > 0 \) or \( m(2) > 0 \)

- By (5.1), since at a vertex \( y \) we have (for this particular \( W \))
  \[
  \sum \left| \mathcal{R}_f(y) \right| = 2m(y)
  \]

We deduce that

\[
\sum_{\text{input or int. edge}} 2m(y) = m(c) + m(d) + m(e)
\]

- Because of a, we have the x-channel nonempty
- The y-channel can be empty, as (6.1) shows.

Let us compute \( \mathcal{O}(T_i C)(4_1^* 4_2^* \otimes 4_1^* 4_2^* \otimes 4_1^* 4_2^*) \) const. for some \( T_i C \).
\( W = y^3 - x^3 \)

\[
\begin{align*}
\psi_1^* & \quad \psi_2^* \\
\psi_1 & \quad \psi_2 \\
\end{align*}
\]

Overall amplitude
\[
\frac{1}{2} \cdot \frac{1}{2} \cdot (-1) \cdot 1 = -\frac{1}{4}
\]

\[
\begin{align*}
\psi_1^* & \quad \psi_2^* \\
\psi_1 & \quad \psi_2 \\
\end{align*}
\]

Overall amplitude
\[
(-1) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (-1) \cdot 1 = +\frac{1}{4}
\]

(These are the only non-zero diagrams for this tree on their inputs.)
This tree does not contribute to the amplitude.

For the tree $T = \quad \quad \quad \quad \quad $ the configurations in (6.1), (6.2) are the only ones with a nonzero amplitude for this input. For the tree $T = \quad \quad \quad \quad \quad $ there are no configs with nonzero amplitudes.

$$\mathcal{O} (Y, \xi_{(6.1)}) = \frac{1}{4} \quad \mathcal{O} (Y, \xi_{(6.2)}) = \frac{1}{4} $$

(7.2)

Now for $T = \quad \quad \quad \quad \quad $ with inputs $\Lambda_1 = \psi_1 \psi_2^*, \Lambda_2 = 1, \Lambda_3 = \psi_2^* \psi_1^*, \Lambda_4 = \psi_1^* \psi_2^*$

$$\rho_T (\Lambda_4 \otimes \ldots \otimes \Lambda_1) = (-1)^s \sum \mathcal{O} (T, \xi) (\Lambda_1 \otimes \ldots \otimes \Lambda_4)$$

$$= (-1)^s \left( \frac{1}{4} \cdot 1 \cdot \frac{1}{4} \right) = 0.$$
\[ \frac{1}{2} \cdot \frac{1}{4} \cdot (-1) \cdot 1 = -\frac{1}{8} \]

**NO!** these diagrams are actually zero.

\[ \frac{1}{2} \cdot \frac{1}{4} \cdot (-1) \cdot 1 = -\frac{1}{8} \]
There are actually zero, as they contain $Q_2^0$!
what we learn from the previous two pages is a constraint for nonzero diagrams.

- The outputs of a trivalent vertex must have $\psi_1$ annihilated with the $\psi_1$ before $x$ becomes $\psi_1$, i.e.

And similarly for $y$.

- Considering the diagrams, this means one of the $\psi_i$'s must have its trivalent vertex in the $\mathcal{B}_\infty$ zone, but then that same interaction produces the bad situation on the RHS of (10.1). That is:

Conclusion: The input $\psi_i \psi_1 \otimes \psi_i \otimes \psi_i \psi_i \otimes 1$ has no nonzero diagrams on the tree.

Def: Given a tree $T$ and inputs $1_1, \cdots, 1_9$ the total amplitude is

$$\sum_{\mathcal{B} \in \text{Con}(T)} \mathcal{O}(T, \mathcal{B})(1_1 \otimes \cdots \otimes 1_9) \text{ const.} \quad (10.2).$$
\[(\psi_1^* \psi_2^*) \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{-1}{4}\]

\[(\psi_1^* \psi_2^*) \cdot (\psi_1^* \psi_2^*) = \frac{1}{4}\]
Conclusion

The total amplitude for input

\[ \psi_1^* \psi_2^* \otimes \psi_2^* \psi_1^* \otimes \psi_1^* \psi_2^* \otimes \mathbf{1} \text{ on this tree is } 0. \]

The underlying mechanism is

\[
( [\psi_1^* \otimes A_1^* ] \otimes [\psi_2^* \otimes A_2^* ] ) ( \psi_1^* \psi_2^* \otimes \theta_1 \partial x ( x A_2 ) )
\]

vs.

\[
( [\psi_1^* \otimes A_1^* ] \otimes [\psi_2^* \otimes A_2^* ] ) ( \psi_1^* \psi_2^* \otimes A_2 \partial y ( y A_1 ) )
\]

We conclude

\[
\rho_4 ( \mathbf{1} \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* ) = 0 \quad (12.2)
\]

The general phenomenon we have observed in the past two examples is \( H^2 = 0 \). If two consecutive \( H_0 \) zones are fed directly into one another (with no \( \Sigma \) interactions in between) as happens in p. 6, p. 10, then we get zero (in the form of multiple diagrams cancelling) as a consequence of:

\[ H_0 H_0 = 0, \quad H_0 H = 0. \]

This is a general fact, not just for \( w = y^2 - x^2 \), which says:

- The amplitudes of diagrams with empty channels do not contribute to the overall amplitude (as a consequence of cancelling with one another).
There must be at least two $\psi$'s involved in a trivalent interaction, but then that means four other $\psi$'s need to absorb the $\theta$'s thus produced, so there is no diagram.

Similarly here.
We conclude

\[ \rho_4(\psi_i^* \otimes \psi_i^* \otimes \psi_i^* \otimes \psi_i^*) = 0 \quad i \in \{1, 2\} \].

For similar reasons, \( \rho_4 \) is zero on any inputs where each channel has a single occupant. (precisely one).

**Note** The total number of \( \psi_i \)'s (for \( i \in \{1, 2\} \) fixed) in the input must be divisible by 3 (for any diagram). Thus the possible configurations are

<table>
<thead>
<tr>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

\{ as explained above, zero channels do not contribute. \\
and we cannot have 6 on four channels. \}

(14.1)

So the possibilities are indexed by choosing two places, where doesn't have a \( \psi_i \); so \( (4) \cdot (4) = 16 \) diagrams, minus those where the same place is chosen, so \( (12) \), e.g.

\[ \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \otimes \psi_2^* \]
$F = \frac{1}{2}$

$0$

$m = 1, w = 2$

$F = \frac{1}{2} C_2^{un}(2) = \frac{3}{2} \cdot \frac{2}{4} = \frac{1}{4}$

yield zero

$0$

(15.2)
$\text{Vol}^+ (15.1), (15.2), (16.1), (16.2)$

are the only diagrams for this tree.
This is the only diagram for this tree.

can't work because this must beved.
Note (Added later)

We believe the total amplitudes, for various trees and inputs, computed in the following pages to be correct. But the way these are combined to form by we are not sure about.
The conclusion is rather amusing: the amplitudes for \( \psi \) are zero, leaving (17.1) as the only contribution.

\[
b_4 \left( \psi_1^* \psi_2^* \otimes \psi_1 \psi_2 \otimes \psi_1^* \otimes \psi_2^* \right) = \frac{1}{2}. \tag{18.1}
\]

Similarly

\[
b_4 \left( \psi_1^* \psi_2^* \otimes \psi_1 \psi_2 \otimes \psi_1^* \otimes \psi_2^* \right) = \frac{1}{2} \tag{18.2}
\]

Note that since the diagrams of (14.1) type input involve one \( \infty \) vertex per \( \psi_i \), there is an invariance under the \( \psi_i \leftrightarrow \psi_j \) exchange. So it only remains to compute

\[
\begin{array}{c}
\psi_1^* \psi_2^* \otimes \psi_1 \psi_2 \otimes \psi_1^* \otimes \psi_2^* \\
\psi_1 \psi_2^* \otimes \psi_1^* \otimes \psi_1 \psi_2 \otimes \psi_2^* \\
\psi_1^* \otimes \psi_1 \psi_2 \otimes \psi_1 \psi_2^* \\
\psi_1 \otimes \psi_1^* \psi_2 \otimes \psi_1 \psi_2^* \\
\psi_1 \otimes \psi_1^* \otimes \psi_1 \psi_2 \otimes \psi_2^* \\
\psi_1 \otimes \psi_1^* \psi_2 \otimes \psi_1 \psi_2^* \\
\end{array}
\]

Note: A general advantage of our formalism over Efimov's is that by separating \( \psi \)'s and \( \phi \)'s, we avoid crazy oscillations, i.e.

\[
\begin{array}{c}
\frac{1}{2} \quad \text{(18.1) above} \\
-1 \quad \text{(p. 23)} \\
-1 \quad \text{(p. 23)} \\
\frac{1}{2} \quad \text{(p. 25)} \\
1 \quad \text{(p. 26)} \\
\frac{1}{2} \quad \text{(p. 27)}
\end{array}
\]
There is a contribution from the same diagram but with the blue vortex in the Hao zone. This Hao value is:
\[ -\alpha_1^{un}(2) \cdot \frac{1}{2} = \frac{-1}{3} \]
This is the only contribution for this tree.
We conclude

\[ b_4 \left( \psi_1^* \psi_2^* \otimes \psi_1^* \otimes \psi_1^* \psi_2^* \right) \text{const} = -\frac{1}{6} - \frac{1}{3} - \frac{1}{2} = -1 \]

(Well, this was corrected but the principle is valid)

From the cancelling of (15:2) and (16:1) we deduce another useful general fact. Since \( H^2 = 0 \), if we can view our diagrams as having a subdiagram where we are summing over all ways to have two \( H \)'s consecutively (with no \( \Sigma \) in between) the result must be zero.

- **Diagram combinations which "emulate"**

\[ \begin{array}{c}
\text{or} \\
\cdot \text{collectively contribute zero since } H^2 = 0.
\end{array} \]

---

**Example**

From this rule we deduce immediately that on

inputs \( \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \otimes \psi_1^* \psi_2^* \) the tree

must not contribute, because

(a) both \( \psi_1, \psi_2 \) in upper right could be moved to one channel without changing the amplitude

(b) then either by (12:3) or (21:1) we get zero.
These are the only two diagrams for this tree.
Now we move onto
\[ \psi_{1,2}^* \otimes \psi_{1,2}^* \otimes \psi_{1,2}^* \otimes \psi_{1,2}^* \]

Hence by \((\psi_{1,2}^* \otimes \psi_{1,2}^* \otimes \psi_{1,2}^* \otimes \psi_{1,2}^*)_{\text{cont}} = -1\)

"two \(\psi\)'s here, but only one Hoo spot"

"zero on this tree"
We conclude by $\chi(F_1^{k*} \otimes \chi_1^{k*} \otimes \chi_2^{k*} \otimes \chi_1^{k*}) = \frac{1}{2}$.

For $\chi_1 \otimes \chi_1^{k*} \otimes \chi_2 \otimes \chi_1^{k*}$:

\[ F = \frac{1}{3} \]
There is no contrib. from the other tree so \[ \mathcal{B}_1^C (\mathcal{L}) = \frac{2}{3} \]

\[ (26.1) \]

\[ + \frac{2}{3} \]

\[ (26.2) \]

compare for 1 Hoo. Or we could say: no Hoo between red(2) and red(3).
\[ (27.1) \]

\[ F = \frac{1}{2} \]

\[ F = 1 \]

\[ (27.2) \]

\[ \text{zero for the usual reasons} \]

\[ b_4 (\chi_1^* \otimes \chi_2^* \otimes \chi_4^* \otimes \chi_7^* \otimes \chi_1^* \otimes \chi_2^*) \]

\[ \text{const.} \]

\[ = \frac{1}{2} \]
A general fact we observe from (25.1) is that

- given a triple

\[ (28.1) \]

The special vertex generating the final 0 must occur in the marked zone, i.e. on the path from the 2nd to 3rd 4. Otherwise we get \( O^2 = 0 \).