Notes on $A\infty$-categories II (checked)

Our aim in this note is to check the proof of the $A\infty$ minimal model theorem in the papers (the former being an elaboration of the latter)


Our notation is as in $\text{aint4}$ and $\text{aint2}$, with one exception: we allow $k$ to be any commutative ring, $\mathcal{O} = \mathcal{O}_k$ (*i.e.* not necessarily a field, and no assumption on the characteristic). In $\text{aint} - \text{aint4}$ and $\text{aint4}$ we made more restrictive assumptions, but everything said there holds in the present generality, as observed also at the beginning of $\text{aint2}$ (with “vector space” replaced by “$k$-module”).

Cohomology Let $\mathcal{A}$ be an $A\infty$-category. The cohomology category $H(\mathcal{A})$ is the (possibly non-unital) associative graded category with the same objects as $\mathcal{A}$, and morphism spaces

$$\text{Hom}_{H(\mathcal{A})}(a,b) := H^{\ast}_{K_{ab}}(\text{Hom}_{\mathcal{A}}(a,b))$$

and morphism compositions

$$\text{Hom}_{H(\mathcal{A})}(b,c) \otimes \text{Hom}_{H(\mathcal{A})}(a,b) \rightarrow \text{Hom}_{H(\mathcal{A})}(a,c)$$

given by $[x] \ast [y] = [M_{ab}(x \otimes y)]$. We denote by $H^0(\mathcal{A})$ the full subcategory of degree zero maps.
Functors. Given $A_{\infty}$-categories $A, B$ an $A_{\infty}$-functor $F: A \to B$ is a map $F: \text{Ob}A \to \text{Ob}B$ together with linear maps

$$F_{a_0\cdots a_n}: \text{Hom}_A(a_0, a_1) \otimes \cdots \otimes \text{Hom}_A(a_{n-1}, a_n) \to \text{Hom}_B(F(a_0), F(a_n))$$

of degree $1 - n$ (here $n \geq 1$) such that the suspended maps

$$F^s_{a_0\cdots a_n} = S^B_{F(a_0)F(a_n)} \circ F_{a_0\cdots a_n} \circ \left( S^{-1}_{a_0a_1} \otimes \cdots \otimes S^{-1}_{a_{n-1}a_n} \right)$$

$$\text{Hom}_A(a_0, a_1)[i] \otimes \cdots \otimes \text{Hom}_A(a_{n-1}, a_n)[i] \to \text{Hom}_B(F(a_0), F(a_n))[i]$$

which are homogeneous of degree zero, satisfy (for $n \geq 1$)

$$\sum_{p=1}^{n} \sum_{0 < i_1 < \cdots < i_p < n} r^{B}_{F(a_0)\cdots F(a_n)} \circ \left( F^s_{a_0\cdots a_{i_1}} \otimes F^s_{a_{i_1}\cdots a_{i_2}} \otimes \cdots \otimes F^s_{a_{i_p-1}\cdots a_n} \right) = \sum_{0 \leq i \leq j \leq n} F^s_{a_0\cdots a_{i-j}a_j\cdots a_n} \circ \left( \text{id}^A_{a_0a_1} \otimes \cdots \otimes \text{id}^A_{a_{i-j}a_i} \otimes r^A_{a_{i-j}\cdots a_{i-j+i}} \otimes \cdots \otimes \text{id}^A_{a_{n-1}a_n} \right)$$

Together with $F_{a_0\cdots a_n}$, the map on objects induces a (possibly non-unital) functor $H(F): H(A) \to H(B)$ of graded associative categories. $F$ is called a

quasi-isomorphism if $H(F)$ is an isomorphism. It is called strict if $F_{a_0\cdots a_n} = 0$ unless $n = 1$. In this case (2.1) reduces to

$$r^{B}_{F(a_0)\cdots F(a_n)} \circ \left( F^s_{a_0a_1} \otimes F^s_{a_1a_2} \otimes \cdots \otimes F^s_{a_{n-1}a_n} \right) = F^s_{a_0\cdots a_n} \circ r^A_{a_0\cdots a_n}$$
Consider the commutative associative $k$-algebra $R := R A$ for an $A_\infty$-algebra $A$, generated by $\{E_a\}_{a \in \text{Ob} A}$ subject to $E_a E_b = \delta_{ab} E_a$. Note that $R$ is unital iff $\text{Ob} A$ is finite. Since the $E_a$ are commuting idempotents, we have $R = \bigoplus_{a \in \text{Ob} A} k$ an associative algebra. Consider the $k$-module

$$\mathcal{H} = \mathcal{H}_A = \bigoplus_{a, b \in \text{Ob} A} \text{Hom}_A(a, b)$$

(3.1)

with the grading

$$\mathcal{H}^n = \bigoplus_{a, b \in \text{Ob} A} \text{Hom}_A^n(a, b).$$

(3.2)

We let $\Pi_a : \mathcal{H} \rightarrow \text{Hom}_A(a, b)$ be the projector onto the subspace $\text{Hom}_A(a, b)$. The binary decomposition (3.1) defines an $R$-bimodule structure on $\mathcal{H}$. Namely, $E_a$ acts on the left by the projector $a \Pi$ of $\mathcal{H}$ onto

$$a \mathcal{H} = \bigoplus_{b \in \text{Ob} A} \text{Hom}_A(a, b)$$

(3.3)

and $E_b$ acts on the right by the projector $\Pi_b$

$$\mathcal{H} b = \bigoplus_{a \in \text{Ob} A} \text{Hom}_A(a, b)$$

(3.4)

Lemma. The $k$-module $\mathcal{H} \otimes^R k = \mathcal{H} \otimes_R \cdots \otimes_R \mathcal{H}$ is given by

$$\mathcal{H} \otimes^R k = \bigoplus_{a_0, \ldots, a_n} \text{Hom}_A(a_0, a_1) \otimes \text{Hom}_A(a_1, a_2) \otimes \cdots \otimes \text{Hom}_A(a_{n-1}, a_n)$$

(3.5)

with the obvious $R$-bimodule structure.
Proof. The relation imposed by the tensor product is

$$\mathcal{H} \otimes_R \mathcal{H} = (\mathcal{H} \otimes \mathcal{H}) / (x \otimes_R y - x \otimes_R y)_{x, y \text{ homogeneous}}$$

from which the claim is clear. $\square$

We define the total products $r_n : \mathcal{H}[1]^{\otimes_R n} \rightarrow \mathcal{H}[1]$ via

$$r_n(x^{(1)} \otimes \cdots \otimes x^{(n)}) = \bigoplus_{a_0, \ldots, a_{n-1}} \sum_{a_0, \ldots, a_n} \rho_{a_0, \ldots, a_n}(x^{(i)}_{a_0a_1} \otimes \cdots \otimes x^{(n)}_{a_{n-1}a_n})$$

where $x^{(i)} = \bigoplus_{a, b \in \text{Ob}} x^{(i)}_{ab} \in \mathcal{H}[1]$ with $x^{(i)}_{ab} \in \text{Hom}_R(a, b)[1]$. Since $r_n$ is clearly $R$-bilinear, we can view $r_n$ as an element of

$$r_n \in \text{Hom}_{\text{R-Mod}_R}^{1}(\mathcal{H}[1]^{\otimes_R n}, \mathcal{H}[1])$$

These maps obey the $A_\infty$-relations

$$\sum_{\substack{i+j \leq n \leq i+j} \bigoplus_{0 \leq j \leq 1}} (-1)^{i+j+\cdots+j} r_{n-1,i+1}(x_1 \otimes \cdots \otimes x_i \otimes r_j(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n) = 0$$

Composing with the quotient $\mathcal{H}[1]^{\otimes_R n} \rightarrow \mathcal{H}[1]^{\otimes_R n}$ defines $r_n$ on $\mathcal{H}[1]^{\otimes_R n}$, and $(\mathcal{H}, \{r_n\}_{n \geq 1})$ is thus an $A_\infty$-algebra over $R$ (not over $R$ because $\mathcal{H}$ is an $R$-bimodule with left and right actions do not necessarily agree).
Minimal models (§3.3 of [L])

Let $\mathcal{A}$ be an $A_{\infty}$-category, and $R, \mathcal{H} = \mathcal{H}_{\ast}$ as above. We view $\mathcal{H}$ as defined on $\mathcal{H}$ with the tilde grading.

**Def.** A strict homotopy retraction of $\mathcal{A}$ is a homotopy retract of the $R$-complex $(\mathcal{H}, r_{\ast})$ (notice $r_{n} = m_{n}$), i.e., a pair $(P, G)$ with $P \in \text{Hom}_{R \text{GrMod}}(\mathcal{H}, \mathcal{H})$ and $G \in \text{Hom}_{R \text{GrMod}}(\mathcal{H}, \mathcal{H}[-1])$ such that

1. $P^{2} = P$
2. $\text{id}_{\mathcal{H}} - P = r_{\ast} G + Gr_{\ast}$

Note that by $\text{Hom}_{R \text{GrMod}}(\ast, \ast)$ we mean degree zero maps, and (2) implies $Pr_{n} = r_{n} P$. The $R$-bilinearity means $P, G$ amount to the data of

$$P_{ab} : \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, b)$$

$$G_{ab} : \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{A}}(a, b)[-1]$$

such that $P^{2} = P_{ab}$ and $\text{id} - P_{ab} = (r_{1})_{ab} G_{ab} + G_{ab} (r_{1})_{ab}$. The submodule (graded, $R$-bimodule)

$$B := \text{Im} P \subseteq \mathcal{H}$$

is given by $\bigoplus_{a, b \in \text{Ob} \mathcal{A}} B_{ab}$ where $B_{ab} = \text{Im} P_{ab}$. We let $i : B \rightarrow \mathcal{H}$ be the inclusion and $p : \mathcal{H} \rightarrow B$ the map induced by $P$, so that $i \circ p = P$.

Clearly $(i \circ p)(B) \subseteq B$, so $B$ is a subcomplex of $\mathcal{H}$. 
**Important** For our conventions on trees see (\texttt{ainfcat3}). $J_n$ is defined on p.\texttt{3} there.

**Note on Koszul signs** Recall that for $k$-linear maps $\varphi, \psi$ of degree $a, b$ the map $\varphi \otimes \psi$ is defined by $(\varphi \otimes \psi)(x \otimes y) = (-1)^{a+b} \varphi(x) \otimes \psi(y)$.

This is the Koszul sign convention.

Given a valid plane tree $T$ let $E(T)$ denote the set of all edges, $E_i(T)$ all internal edges, and $E_e(T)$ all external edges. Set $e_i(T) := \text{Card} E_i(T)$ the number of internal edges. For each $T \in J_n$ we define a morphism of graded $R$-bimodules $\rho \in \text{Hom}_{\text{Mod}_R}(B^{\otimes R^n}, B)$ as follows:

(a) associate the inclusion $i$ with every leaf of $T$.

(b) associate the surjection $p$ with the root of $T$.  

(c) associate $v_k$ with each internal vertex of valency $k+1$ (note $k \geq 2$)

(d) associate $A$ with each internal edge of $T$.

Example

\begin{align*}
\text{However notice that due to Koszul signs, with } v_n \text{ of degree } 1, \\
\text{and } A \text{ of degree } -1, \text{ we have } \\
p \circ r_2 \circ (A \otimes A) \circ (v_2 \otimes v_2) \circ (i \otimes i \otimes i \otimes i) = 6.2
\end{align*}

It is therefore important that we fix a convention for which of the two alternatives in (6.2) that we will use. The two choices are called in (\texttt{ainfcat3}) the height and branch denotations. After some effort we are convinced the former is very difficult to make work in the context of the minimal model theorem, so we use branch.
To be more precise about what (6.1) means we introduce the notion of augmented plane tree.

**Def** Given a valid plane tree $T$ the augmentation $\tilde{A}(T)$ of $T$ is the plane tree obtained from $T$ by inserting a new vertex (of valency 2) on each internal edge of $T$. 

![Diagram of augmented plane tree](image)
Given $T_G$ an a valid plane tree, we decorate the augmented tree $A(T)$ according to (6.1), that is we define $D$ to be:

- We assign $L_v := B[i]$, a graded $R$-bimodule, for every leaf $v$ (inc. the root).

- To all edges $e$ of $A(T)$ we assign $M_e := E[i]$. (8.1)

- The maps $B[i] = L_v \rightarrow M_e = E[i]$ for each edge $e$ incident at a non-root leaf $v$ are all $i$, and the map $E[i] - M_e \rightarrow L_v = B[i]$ at the root is $p$.

- To vertices of $A(T)$ coming from an internal edge of $T$ we assign $G$.  

- To internal vertices of $A(T)$ of valency $k+1$ for $k \geq 2$ we assign $Y_k$, of degree $+1$.

**Def.** Given $T_G$ and the homotopy retract-data for $A$ on p.5, we define the homogeneous $R$-bilinear map $\rho_T : B[i] \otimes_R n \rightarrow B[i]$ to be the denotation $\langle D \rangle$ of the decoration (8.1) of the augmented plane tree $A(T)$, multiplied by a sign factor $(-1)^{e_i(T)}$. Here $\langle D \rangle$ is the branch denotation $\langle D \rangle_B$ of $\langle \text{inference} \rangle$.

**Lemma.** $\rho_T$ is homogeneous of degree $2 - n$, and hence degree $+1$ as $B[i] \otimes_R n \rightarrow B[i]$.

**Proof.** The degree of $\rho_T$ is $e_i(T) \cdot (-1) + \sum_{\text{int. vert. } v \text{ of } T} (3 - \text{valency}(v))$, but this is $3 \# \text{int. vert.} - 2 \# \text{int. edges} - \# \text{ext. edges}$. There is an injection, 

\[ \{ \text{int. edges} \} \rightarrow \{ \text{int. vert.} \} \] 

sending an edge to its source, which only misses one vertex (the one adjacent to the root). Hence $|\rho_T| = 3 - \# \text{ext. edges} = 2 - n$. $\square$
Def. For $n \geq 2$ we define the degree +1 map

\[
\rho_n := \sum_{T \in T_n} \rho_T \in \text{Hom}_R\text{Mod}_R(B[[r]]^{\otimes n}, B[[r]]). \tag{9.1}
\]

and we set $\rho_1 := p \circ r \circ i$ (i.e. $r_1 | B$). Note that $\rho_2 = p \circ r_2 \circ i^{\otimes 2} = p \circ r_2 | B \otimes B$.

Example. We have $T_3 = \{ \begin{array}{ccc}
\mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 \\
& & \\
& & \\
& & \\
& & \\
\end{array} \}$ and thus

\[
\rho_{T_1} = (-1)^1 p \circ r_2 \circ (i \otimes \text{Gr}_2 i^{\otimes 2}), \tag{9.2}
\]
\[
\rho_{T_2} = (-1)^2 p \circ r_2 \circ (\text{Gr}_2 i^{\otimes 2} \otimes i),
\]
\[
\rho_{T_3} = (-1)^3 p \circ r_3 \circ i^{\otimes 3}.
\]

and finally $\rho_3 = \rho_{T_1} + \rho_{T_2} + \rho_{T_3}$.

Recall eval from p. 6 (ainfcat2).

Lemma. For any $T \in T_n$ and $a_1, \ldots, a_n \in B[[r]]$ we have

\[
\rho_T(a_1 \otimes \cdots \otimes a_n) = (-1)^{\varepsilon_i(T)} \text{eval}_D(a_1 \otimes \cdots \otimes a_n). \tag{9.3}
\]

Proof. By p. 6 (ainfcat3) the difference is a sign \( \sum_{i=1}^{n} \sum_{v > \text{Gr}_i} \tilde{a}_c \tilde{\phi}_v \), where \( \tilde{a}_c = |a_c| + 1 \), \( v \) runs overall vertices in $A(T)$, \( \text{Gr}_i \) means the $i$th non-root leaf, and \( |\tilde{\phi}_v| \) is the degree of the insertion in $D$ at $v$. But vertices $v > \text{Gr}_i$ in $A(T)$ can be paired up (vertices from internal edges in $T$ with their source vertex) in a way that makes it clear this sign is zero. \( \square \)

Upshot. We can evaluate $\rho_T$ ignoring all Koszul signs!
Theorem The maps \( \{p_n\}_{n \geq 1} \) satisfy the forward suspended \( A_\infty \)-relations \((3.6) \) above or \((4.3) \), i.e. for \( n \geq 1 \)

\[
\sum_{i \geq 0, j \geq 1 \atop 1 \leq i + j \leq n} p_{n-j+1} \circ (\text{id}_{B[i]} \otimes p_j \otimes \text{id}_{B[j]}) = 0. \tag{10.1}
\]

Notes For \( n = 1 \) the relation we need is \( p_1^2 = 0 \), which is immediate since \( r_1^2 = 0 \).

The relation for \( n = 2 \) is \( p_2 + p_1 (p_1 \otimes 1) + p_2 (1 \otimes p_1) = 0 \), which is \( r_1, p_2 + p_1 (r_1 \otimes 1) + p_2 (1 \otimes r_1) = 0 \), which follows by multiplying \( r_1, p_2 + p_1 (r_1 \otimes 1) + p_2 (1 \otimes r_1) = 0 \) on the left by \( p \).

Proof For \( n \geq 1 \) consider the following \( R \)-bilinear map \( \mathfrak{g}[[1]]^n \rightarrow \mathfrak{g}[[1]] \),

\[
(r)_1^n := r_1 \circ r_n + \sum_{i = 0}^{n-1} r_n \circ (\text{id}_{\mathfrak{g}[[1]]^i} \otimes r_2 \otimes \text{id}_{\mathfrak{g}[[1]]^{(n-i)-1}}) \tag{10.2}
\]

and similarly define \( (p)_1^n \) for the products \( p \) on \( B[[1]] \). The \( A_\infty \)-relations are equivalent to \( r_1^2 = 0 \) together with

\[
(r)_{1 \geq 2}^n = - \sum_{i \geq 0, j \geq 2 \atop 1 \leq i + j \leq n, j \leq n-1} r_{n-j+1} \circ (\text{id}_{\mathfrak{g}[[1]]^i} \otimes r_j \otimes \text{id}_{\mathfrak{g}[[1]]^{(n-j)-1}}) \tag{10.3}
\]

Since \( p_1^2 = 0 \) to complete the proof it suffices to show for \( n \geq 2 \) that

\[
(p)_1^n = - \sum_{i \geq 0, j \geq 2 \atop 1 \leq i + j \leq n, j \leq n-1} p_{n-j+1} \circ (\text{id}_{B[[1]]^i} \otimes p_j \otimes \text{id}_{B[[1]]^{(n-j)-1}}). \tag{10.4}
\]

Now by definition \( p_n = \sum_{\tau \in \mathcal{T}_n} (-1)^{e(\tau)} \langle D_\tau \rangle \) where \( D_\tau \) is the canonical decoration of \( A(\tau) \). The strategy is to expand the RHS of \((10.4)\) using this definition, "merge" the trees from \( p_{n-j+1} \) and \( p_j \) then use \( r, v + g v = \text{id}_{\mathfrak{g}} - p \) to recover the LHS.
Given $f \in \text{End}_{\text{Mod}}(\partial D)$ of degree zero, $T \in \mathcal{T}$ and an internal edge $e_i$, let $D_{f,e_i}$ be the decoration of $A(T)$ which puts $f$ rather than $A_i$ at the vertex corresponding to $e_i$. We define

$$\rho^f_{T,e_i} := (-1)^{e_i(T)} \langle D_{f,e_i} \rangle$$  \hspace{1cm} (11.1)

We also set

$$\rho^f_n := \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T), e_i(T) > 1} \rho^f_{T,e_i} \in \text{Hom}_{\text{Mod}}(B[i]^{\otimes n}, B[i]).$$

Given $e \in E_i(T)$ we write $\hat{\rho}_{T,e}$ for $\rho^r_{T,e}$. Given $e \in E_i(T)$ let $v$ be the leaf to which $e_i$ is adjacent (possibly $v = r$ the root). Define $D_v$ to be the decoration replacing $\phi_v$ (which is either $i$ or $p$) by $r_i \circ \phi_v = \phi_v \circ \rho_i$ if $v$ is a non-root leaf, and by $\rho_i \circ \phi_v = \phi_v \circ r_i$ if $v = r$, and set

$$\hat{\rho}_{T,e} := (-1)^{e_i(T)} \langle D_v \rangle$$

$$\hat{\rho}_n = \sum_{T \in \mathcal{T}_n} \sum_{e \in E_i(T)} \hat{\rho}_{T,e}.$$

We begin by proving

Claim A For $0 \leq i \leq n-1$

$$\rho_n \circ (\text{id}_{B[i]} \otimes \rho_{i} \otimes \text{id}_{B^{(n-i-1)}}) = \sum_{T \in \mathcal{T}_n} \hat{\rho}_{T,e}$$

where $e$ is the edge in $T$ adjacent to the $i$th leaf.
Proof of claim. Fix $0 \leq i \leq n-1$, $T \in \mathcal{J}_n$ and let $D_e$ as defined above. Then

$$
\langle D_e \rangle = (-1)^s \langle D_T \rangle \cdot \left( \hat{1}^\otimes (n-i) \otimes \rho \otimes 1^\otimes (n-i) \right)
$$

(D means the standard denotation)

where $s$ is the sum of the degrees $|\phi_q|$ of insertions on $A(T)$ at vertices $q$ which satisfy $q > \tilde{v}$ according to the relation "$>$" of $p$. $\mathcal{J}_n$ where $\tilde{v}$ is the $i$th leaf vertex in $T$, e.g. for $i=1$ the indicated vertex are $> \tilde{v}$

![Diagram](image)

But now observe that the vertices $q$ contributing to this sum come in pairs (like the $r_2, \lambda$ in the above example) of a vertex in $A(T)$ originating from an internal edge in $T$, and its source vertex. Since these pairs cancel, we conclude $s = 0$. □

It is obvious that

$$
\rho^i \rho_n = \sum_T (-1)^{e(T)} \rho^i \langle D_T \rangle = \sum_T \hat{\rho}_{T,e} \quad (e \text{ adjacent to } r).
$$

Hence

$$
\rho^i = \sum_{T \in \mathcal{J}_n} \sum_{e \in E(T)} \hat{\rho}_{T,e} = \hat{\rho}_n - \sum_{T \in \mathcal{J}_n} \sum_{e \in E(T)} \hat{\rho}_{T,e}.
$$

$$
= \hat{\rho}_n - \sum_{T \in \mathcal{J}_n} \sum_{e \in E(T)} \rho^i_{T,e}.
$$
Now using \( r_{1} A + G_{r_{1}} = \text{id} \cdot e - P \) we have

\[
\hat{\rho}_{n} = \sum_{T \in \mathcal{T}_{n}} \sum_{e \in E_{i}(T)} \left[ \rho_{T,e}^{\text{id} \cdot e} - \rho_{T,e}^{P} \right]
\]

\[= \hat{\rho}_{n} - \rho_{n}^{\text{id} \cdot e} + \rho_{n}^{P}.
\]

That is,

\[
\hat{\rho}_{n} = (P)_{n} + \rho_{n}^{\text{id} \cdot e} - \rho_{n}^{P}. \tag{12.1}
\]

Next we calculate \( \hat{\rho}_{n} \) in a different way. Set \( \hat{\rho}_{T} := \sum_{e \in E(T)} \hat{\rho}_{T,e} \) so \( \hat{\rho}_{n} = \sum_{T} \hat{\rho}_{T} \).

For \( n \geq 2 \) and \( T \in \mathcal{T}_{n} \), we can organise the sum \( \hat{\rho}_{T} = \sum_{e \in E(T)} \hat{\rho}_{T,e} \) as a sum over internal vertices of \( T \),

\[
\hat{\rho}_{T} = \sum_{v \text{ internal vertex}} \hat{\rho}_{T,v}
\]

where \( \hat{\rho}_{T,v} \) is the sum of \( \rho_{T,e}^{r_{1}A} \) for every internal edge \( e \) which is incoming to \( v \),
\( \rho_{T,e}^{Gr_{i}} \) for every internal edge outgoing from \( v \), and \( \hat{\rho}_{T,e} \) for every external edge incident at \( v \). That is,

\[
\hat{\rho}_{T,v} = \sum_{e \in E_{i}(T) \text{ e ends at } v} \rho_{T,e}^{r_{1}A} + \sum_{e \in E_{i}(T) \text{ e begins at } v} \rho_{T,e}^{Gr_{i}} + \sum_{e \in E_{o}(T) \text{ e incident at } v} \hat{\rho}_{T,e}. \tag{12.2}
\]
Recall $\rho^+_{T, e}, \rho^-_{T, e}, \rho^\circ_{T, e}$ are signed versions of $\langle D_{r, a, e}, e \rangle_T, \langle D_{a, r, e}, e \rangle_T, \langle D_{e} \rangle_T$ respectively.

Suppose that $v$ has valency $k + 1$ with incident edges (in $T$)

(13.1)

\[
\begin{array}{c}
\vdots \\

(\text{some or all of which may be external edges})
\end{array}
\]

The idea is to recognise $(r)_x^k$ from earlier on the RHS of (13.1), apply (10.3), and then the result on p. 12 of aintcat3.

The contribution of (13.1) to $\langle DT \rangle$ is via the operator

\[
\nu_k \circ (T_1 \otimes \cdots \otimes T_k)
\]

(13.2)

where $T_i$ is the denotation of the $i$th subtree (of $A(T)$). Let us write $\langle - \rangle$ somewhat ambiguously to reflect denotations of decorations of $A(T)$ where we only describe the deviation from $D_T$ near $v$ (and since we are in the branch denotation, not height, it is genuinely safe to do so). Then,

\[
\langle DT \rangle = \langle \nu_k \circ (T_1 \otimes \cdots \otimes T_k) \rangle
\]

(13.3)

\[
\langle D_{r, a, e, i} \rangle = \langle \nu_r \circ (T_i \otimes \cdots \otimes v_i T_i \otimes \cdots \otimes T_k) \rangle
\]

\[
\langle D_{a, r, e, i} \rangle = \langle C_r \circ \nu_k \circ (T_1 \otimes \cdots \otimes T_k) \rangle
\]

For the same reason elaborated above, $|T_j| = 0$ for all $j$, since insertions on edges and internal vertices precisely cancel. Thus
\[ \langle Dr, a, e_i \rangle = \langle r_k \circ (T_1 \otimes \cdots \otimes r_i \circ T_i \otimes \cdots \otimes T_k) \rangle \]
\[ = \langle r_k \circ (I^{(i-1)} \otimes r_i \otimes 1^{(k-i)}) \circ (T_1 \otimes \cdots \otimes T_k) \rangle \]

Hence from (12.2), we conclude by (10.3),
\[ \hat{P}_{T_j} V = (-1)^{e_i(T)} \langle (r)^{\otimes k} \rangle_{T_j V} \text{ meaning } (r)^{\otimes k} \text{ is inserted at } v \text{ in } A(T) \]
\[ = (-1)^{e_i(T)} \left\langle \sum_{i > 0, j \geq 2, i + j \leq k, j \leq k-1} r_{k-j+1} \circ \left( \text{id}_{\mathcal{X}[1]} \otimes r_j \otimes \text{id}_{\mathcal{X}[1]}^{(k-j-i)} \right) \right\rangle_{T_j V} \]
\[ = (-1)^{e_i(T)+1} \sum_{i > 0, j \geq 2, i + j \leq k, j \leq k-1} \left\langle \sum_{i > 0, j \geq 2} r_{k-j+1} \circ \left( \text{id}_{\mathcal{X}[1]} \otimes r_j \otimes \text{id}_{\mathcal{X}[1]}^{(k-j-i)} \right) \right\rangle_{T_j V} \]

Claim B Given \( T \in \mathcal{I}_n \), an internal vertex \( v \), and integers \( i > 0, j > 2 \) with \( i + j \leq k \), \( j \leq k-1 \) where \( v \) has valency \( k+1 \), let \( T' = \text{ins}(T, v, i, j) \) as defined on p.10 of [ainfcrats] and let \( e' \) be the created edge, \( D; \text{id}_\mathcal{X}, e' \) the decoration of \( A(T') \) obtained from the standard one (i.e. p.8) by inserting \( \text{id}_\mathcal{X} \) rather than \( A \) at \( e' \). We claim that
\[ \left\langle r_{k-j+1} \circ \left( \text{id}_{\mathcal{X}[1]} \otimes r_j \otimes \text{id}_{\mathcal{X}[1]}^{(k-j-i)} \right) \right\rangle_{T_j V} = \langle D; \text{id}_\mathcal{X}, e' \rangle. \] (14.1)

Proof of claim with the vicinity of \( v \) as in (13.1) and the notation \( T_j \) as above,
\[ \text{LHS of (14.1)} = \langle r_{k-j+1} \circ (\text{id} \otimes r_j \otimes \text{id}) \circ (T_1 \otimes \cdots \otimes T_k) \rangle \]
\[ = \langle r_{k-j+1} \circ (T_1 \otimes \cdots \otimes r_j T_j \otimes \cdots \otimes T_k) \rangle \]
\[ = \text{RHS of (14.1)}. \]
From Claim B and (14.1) we obtain

$$\hat{\rho}_{T,v} = (-1)^{e_i(T) + 1} \sum_{i > 0, j > 2 \atop i + j \leq k, j \leq k-1} \langle D_{i,i}, e' \rangle_{\text{ins}(T,v,i,j)}$$

We can partition $J_n$ by the number of internal edges, writing

$$J_n = \bigsqcup_{c > 0} J_n^{(c)}$$

where $T \in J_n^{(c)}$ iff $e_i(T) = c$.

By the Lemma on p. 12 of the original text, there is a bijection for $n > 2$ and $c > 10$

$$J_n^{(c+1)} \overset{+}{\longrightarrow} \{ (Q,v,i,j) \mid Q \in J_n^{(c)}, v \text{ an internal vertex}, \ i \geq 0, i + j \leq |v| - 1, 2 \leq j \leq |v| - 2 \}$$

where $(J_n^{(c+1)})^+$ denotes the set of pairs $(T,e)$ where $T \in J_n^{(c+1)}$ and $e \in E_I(T)$.

Hence, assuming $n > 2$

$$J_n^+ = \bigsqcup_{c > 0} (J_n^{(c+1)})^+ = \bigsqcup_{c > 0} \{ (Q,v,i,j) \mid Q \in J_n^{(c)}, \ldots \}$$

which shows

$$\hat{\rho}_n = \sum_{T \in J_n} \sum_v \hat{\rho}_{T,v} = \sum_{T \in J_n} \sum_{v \in J_n} \sum_{i > 0, j > 2 \atop i + j \leq k, j \leq k-1} (-1)^{e_i(T) + 1} \langle D_{i,i}, e' \rangle_{\text{ins}(T,v,i,j)}$$

$$= \sum_{T' \in J_n} \sum_{e \in E_I(T')} (-1)^{e_i(T')} \langle D_{i,i}, e' \rangle_{\text{ins}(T,v,i,j)}$$

$$= \hat{\rho}_n^{\text{id}\times e}$$
Comparing with (12.1) we conclude for \( n > 2 \) that
\[
\left( \rho \right)_n^p = \rho_n. \tag{16.1}
\]

Recall \( P = i \circ p \). To show (10.4) and complete the proof, it is therefore enough to check
\[
\text{Claim: } \rho_n^p = - \sum_{i \geq 0, j \geq 2} \sum_{i + j \leq n, j \leq n - 1} \rho_{n-j+1} \circ (id_{B[i]} \otimes \rho_j \otimes id_{\mathbb{R}[i]}) \tag{16.2}
\]

Proof of claim: By definition
\[
\rho_n^p = \sum_{\tau \in \mathfrak{S}_n} \sum_{e \in E_i(\tau)} (-1)^{e_\tau(\tau)} \left< D_{\pi e} \right> \tag{16.2}
\]

For \( n \) fixed there is a bijection
\[
\sigma : \prod_{2 \leq j \leq n-1} \mathcal{T}_j \times \mathcal{T}_{n-j} \times \{0, \ldots, n-j\} \rightarrow \mathcal{T}_n^+ \tag{16.3}
\]

which is defined on \((Q, Q', i)\) by attaching \( Q' \) to \( Q \) at the \((i+1)\)st leaf, via a new internal edge which is marked, i.e.
\[
\sigma \left( \begin{array}{c}
\circ \quad i \\
\circ \quad i+1 \\
\end{array} \right) = \begin{array}{c}
\circ \\
\circ \\
\end{array} \]

Note that \( \sigma(Q, Q', i) \) has \( e_i(Q) + e_i(Q') + 1 \) internal edges.
Hence (16.2) may be rewritten as

\[
\rho_n^p = \sum_{2 \leq j \leq n-1} \sum_{\alpha \in J_{n-j+1}} \sum_{0 \leq i \leq j} (-1)^{\epsilon_i(\alpha) + \epsilon_\alpha(\alpha) + 1} \langle D_{\alpha} \rangle \circ (\text{id}^\alpha \otimes \langle D_{\alpha'} \rangle \otimes \text{id}^{\delta_{n-j-i}}) \\
= \sum_{i \geq 0, j \geq 2} \{ \sum_{\alpha \in J_{n-j+1}} (-1)^{\epsilon_i(\alpha)} \langle D_{\alpha} \rangle \} \circ (\text{id}^\alpha \otimes \{ \sum_{\alpha' \in J_j} \langle D_{\alpha'} \rangle \} \otimes \text{id}^{\delta_{n-j-i}}) \\
= -\sum_{i \geq 0, j \geq 2} \rho_{n-j+1} \circ (\text{id}_{B[j]} \otimes \rho_j \otimes \text{id}_{F[j]}^{\delta_{n-j-i}}).
\]

as claimed. \(\square\)

which completes the proof of the Theorem. \(\square\)
Appendix (Height vs Branch denotation)

Given $T \in J_n$ we have defined

$$\rho_T = (-1)^{\epsilon_1(T)} \langle D \rangle_H.$$ 

Now, by (7.2) of airact we have

$$\langle D \rangle_H = (-1)^{J(A(T),D)} \langle D \rangle_B$$

where $J(A(T),D) = \sum \omega < \omega', \text{depth}(\omega') < \text{depth}(\omega) | \phi_\omega | \phi_{\omega'} |$ and $\phi_\omega$ is the morphism assigned to $\omega$ by $D$. The only decorations of nonzero degree in $D$ are $v_r$'s (degree +1) on internal vertices and $g_i$'s (degree -1) on vertices created on midpoint of edges of $T$. Note the depth is computed in $A(T)$.

**Lemma** $J(A(T),D) = \sum_{d \geq 1} \binom{Nd}{2}$ where $Nd$ is the number of internal vertices at depth $d$ in $T$.

**Proof** If $r_k$ decorates a vertex adjacent to the root it does not contribute to $J(A(T), D)$. Let $v$ be an internal vertex of $T$, viewed as a vertex in $A(T)$, and suppose $v$ is not adjacent to the root, so in $T$ the edge emanating from $v$ acquires a midpoint vertex $\omega_v$. As $v$ varies over all internal vertices of $T$ the $v, \omega_v$ enumerate all the contributing vertices to $J(A(T), D)$. Moreover all the $v$'s have odd depth in $A(T)$, as $\text{depth}_{A(T)}(v) = 1 + 2(\text{depth}_T(v) - 1)$, and consequently $\text{depth}_{A(T)}(\omega_v) = 2(\text{depth}_T(v) - 1)$ is always even. Let $V$ denote the set of internal vertices of $T$ not adjacent to the root. Then for $v,v'$ in $V$ or $A(T)$ is the same given $v, v' \in V$ with $v < v'$ there are three possible relationships between $(v, \omega_v)$ and $(v', \omega_{v'})$ indicated in the following diagram.
\(A\) \(\text{depth}_T v < \text{depth}_T v'\)

\(B\) \(\text{depth}_T v = \text{depth}_T v'\)

\(C\) \(\text{depth}_T v > \text{depth}_T v'\)

Now a pair in configuration \(A\) does not contribute to \(T(A(T), D)\), a pair in config. \(B\) contributes \(-1\) and a pair in config. \(C\) contributes zero. So we have

\[
T(A(T), D) = \sum_{v, v' \in V \atop v < v', \text{depth}_T(v) = \text{depth}_T(v')} - 1
\]

Among the vertices at a fixed depth \(<\) is a total order, so we conclude that

\[
T|A(T), D) = \sum_{d > 1} \binom{Nd}{2}
\]

as claimed. \(\square\)