Notes on A_∞-categories (checked)

In ainf - ainf4 we looked A_∞-algebras and we are now developing examples in the ainfmf series. In this note we study A_∞-categories and particularly the forwarded suspended multiplications Lazaroiu uses in his paper

[L] "Generating the superpotential on a D-brane category"

Throughout k is a field of characteristic zero, all v. spaces are over k. A small A_∞-category $\mathcal{A}$ is specified by a set of objects $\mathrm{Ob}\,\mathcal{A}$ and by graded vector spaces $\mathrm{Hom}_{\mathcal{A}}(a,b)$ for any $a,b \in \mathrm{Ob}\,\mathcal{A}$ together with linear maps

$$\mu_{a_n,\ldots,a_0} : \mathrm{Hom}_{\mathcal{A}}(a_{n-1},a_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_0,a_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a_0,a_n) \qquad (1.1)$$

of degree $2-n$ subject to A_∞-constraints, to be given below. The graded vector spaces may be either $\mathbb{Z}$ or $\mathbb{Z}_2$-graded, and in each case suspension functors etc. should be interpreted appropriately in the following. The degree of homogeneous $x \in \mathrm{Hom}_{\mathcal{A}}(a,b)$ is denoted $|x|$. In particular for a pair a,b we have

$$\mu_{ba} : \mathrm{Hom}_{\mathcal{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,b) \qquad \deg +1$$

and for $a,b,c \in \mathrm{Ob}\,\mathcal{A}$ a degree zero map

$$\mu_{cba} : \mathrm{Hom}_{\mathcal{A}}(b,c) \otimes \mathrm{Hom}_{\mathcal{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,c).$$

For objects a, b of $\mathcal{A}$ let $s_{ab} : \mathrm{Hom}_{\mathcal{A}}(a,b) \to \mathrm{Hom}_{\mathcal{A}}(a,b)[1]$ be the suspension operator of $\mathrm{Hom}_{\mathcal{A}}(a,b)$. Denoting the degree of $x \in \mathrm{Hom}_{\mathcal{A}}(a,b)[1]$ by $\tilde{x} = |x| - 1$ we have $s(x) = x$ and s is a map of degree -1. We write s_{ab} for $s_{ab}[n]$ and

$$s^n_{ab} : \mathrm{Hom}_{\mathcal{A}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a,b)[n] \qquad n > 0$$

$$s^n_{ab} = s_{ab}[n-1] \circ \cdots \circ s_{ab}[1] \circ s_{ab}$$

and $s^0_{ab} = id$ and for $n < 0$

$$s^n_{ab} = s^{-1}_{ab}[n+1] \circ \cdots \circ s^{-1}_{ab}[-1] \circ s^{-1}_{ab}$$

Next we introduce 'forward' compositions

$$m_{a_0 \ldots a_n} : \mathrm{Hom}_{\mathcal{A}}(a_0, a_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_{n-1}, a_n) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a_0, a_n) \tag{2.1}$$

by the formula

$$m_{a_0 \cdots a_n}(x_1 \otimes \cdots \otimes x_n) = (-1)^{\sum_{1 \le i < j \le n} |x_i||x_j|} M_{a_n \cdots a_0}(x_n \otimes \cdots \otimes x_1). \tag{2.2}$$

Of course $|m_{a_0 \cdots a_n}| = 2 - n$.

The <u>suspended forward compositions</u> are

$$r_{a_0 \cdots a_n} : \mathrm{Hom}_{\mathcal{A}}(a_0, a_1)[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_{n-1}, a_n)[1] \longrightarrow \mathrm{Hom}_{\mathcal{A}}(a_0, a_n)[1] \qquad (3.1)$$

defined by

$$r_{a_0 \cdots a_n} = s_{a_0 a_n} \circ m_{a_0 \cdots a_n} \circ (s^{-1}_{a_0 a_1} \otimes \cdots \otimes s^{-1}_{a_{n-1} a_n}) \qquad (3.2)$$

This is a map of degree $+1$. We have to observe that the s are maps of degree -1 so (3.2) involves Koszul signs when applied to a tensor, e.g.

$$m_{ab}(x) = r_{ab}(x)$$

But

$$\begin{aligned} r_{abc}(x_1 \otimes x_2) &= s \circ m_{abc} \circ (s^{-1}_{ab} \otimes s^{-1}_{bc})(x_1 \otimes x_2) \\ &= s \circ m_{abc}(s^{-1}_{ab}(x_1) \otimes s^{-1}_{bc}(x_2)) \cdot (-1)^{\widetilde{x_1}} \\ &= (-1)^{\widetilde{x_1}} m_{abc}(x_1 \otimes x_2). \end{aligned} \qquad (3.3)$$

$$r_{abcd}(x_1 \otimes x_2 \otimes x_3) = (-1)^{\widetilde{x_2}} m_{abcd}(x_1 \otimes x_2 \otimes x_3).$$

and

$$\mu_{ab}(x) = r_{ba}(x)$$

$$\begin{aligned}
\mu_{abc}(x_1 \otimes x_2) &= (-1)^{|x_1||x_2|} m_{cba}(x_2 \otimes x_1) \\
&= (-1)^{|x_1||x_2| + \tilde{x}_2} r_{cba}(x_2 \otimes x_1) \qquad (4.1) \\
&= (-1)^{(\tilde{x}_1 + 1)(\tilde{x}_2 + 1) + \tilde{x}_2} r_{cba}(x_2 \otimes x_1) \\
&= (-1)^{\tilde{x}_1 \tilde{x}_2 + \tilde{x}_1 + 1} r_{cba}(x_2 \otimes x_1).
\end{aligned}$$

In terms of the forward suspended compositions the A_∞-constraints are given by

$$\sum_{\substack{i \geq 0, j \geq 1 \\ 1 \leq i+j \leq n}} (-1)^{\tilde{x}_1 + \cdots + \tilde{x}_i} r_{a_0 \cdots a_i, a_{i+j} \cdots a_n} (x_1 \otimes \cdots \otimes x_i \otimes r_{a_i \cdots a_{i+j}}(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n) = 0 \qquad \forall n \geq 1. \qquad (4.2)$$

where $x_j \in \mathrm{Hom}_{\mathfrak{A}}(a_{j-1}, a_j)[1]$ is any sequence of 'forward composable' morphisms. Using the Koszul rule this can also be written

$$\sum_{\substack{i \geq 0, j \geq 1 \\ 1 \leq i+j \leq n}} r_{a_0 \cdots a_i, a_{i+j} \cdots a_n} \circ (id_{a_0 a_1} \otimes \cdots \otimes id_{a_{i-1} a_i} \otimes r_{a_i \cdots a_{i+j}} \otimes id_{a_{i+j} a_{i+j+1}} \otimes \cdots \otimes id_{a_{n-1} a_n}) = 0 \qquad \forall n \geq 1 \qquad (4.3)$$

Comparison to ainf

We write (4.2) in terms of the "original" multiplications μ in the one object case, to compare with ainf. In this case all subscripts are $a_0 = \cdots = a_n = a$ and we omit them. For $A := \mathrm{Hom}_{\mathcal{A}}(a,a)$ and $x_1, \ldots, x_n \in A$ (4.2) reads in terms of m's (we write r_j, m_j, μ_j for the obvious things)

$$r_j\,(x_{i+1} \otimes \cdots \otimes x_{i+j}) \qquad \# \text{ of } s^{-1}\text{'s is } j$$

$$= m_j \circ \underbrace{(s^{-1} \otimes \cdots \otimes s^{-1})}\,(x_{i+1} \otimes \cdots \otimes x_{i+j})$$

$$= (-1)^{\substack{\tilde{x}_{i+1} + \cdots + \tilde{x}_{i+j-1} \\ + \tilde{x}_{i+1} + \cdots + \tilde{x}_{i+j-2} \\ + \cdots \\ + \tilde{x}_{i+1}}} m_j\,(x_{i+1} \otimes \cdots \otimes x_{i+j}) \qquad (5.1)$$

$$= (-1)^{(j-1)\tilde{x}_{i+1} + (j-2)\tilde{x}_{i+2} + \cdots 2\tilde{x}_{i+j-2} + \tilde{x}_{i+j-1}}\; m_j\,(x_{i+1} \otimes \cdots \otimes x_{i+j})$$

Writing $\kappa := r_j\,(x_{i+1} \otimes \cdots \otimes x_{i+j})$

$$r(x_1 \otimes \cdots \otimes x_i \otimes \kappa \otimes x_{i+j+1} \otimes \cdots \otimes x_n) \qquad (5.2)$$

$$= m \circ \overbrace{(s^{-1} \otimes \cdots \otimes s^{-1})}^{n-j+1}\,(x_1 \otimes \cdots \otimes x_i \otimes \kappa \otimes x_{i+j+1} \otimes \cdots \otimes x_n)$$

$$= (-1)^{\substack{(n-j)\tilde{x}_1 + \cdots + (n\overset{+1}{-}i-j)\tilde{x}_i + (n-i-j)\tilde{\kappa} + (n-i-j-1)\tilde{x}_{i+j+1} \\ + \cdots + \tilde{x}_{n-1}}}$$

$$m(x_1 \otimes \cdots \otimes x_i \otimes \kappa \otimes x_{i+j+1} \otimes \cdots \otimes x_n)$$

Now $\tilde{\kappa} = \tilde{x}_{i+1} + \cdots + \tilde{x}_{i+j} + 1$

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Hence

$$r(x_1 \otimes \cdots \otimes x_i \otimes r(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n) \qquad (6.1)$$

$$= (-1)^{\phi} m(x_1 \otimes \cdots \otimes x_i \otimes m(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n)$$

where

$$\left.\begin{aligned} \phi = \; & (n-j)\tilde{x}_1 + \cdots + (n-i-j+1)\tilde{x}_i \\ & + (n-i-j)(\tilde{x}_{i+1} + \cdots + \tilde{x}_{i+j} + 1) \\ & + (n-i-j-1)\tilde{x}_{i+j+1} + \cdots + \tilde{x}_{n-1} \end{aligned}\right\} \text{from } (5.2)$$

$$\left. + (j-1)\tilde{x}_{i+1} + (j-2)\tilde{x}_{i+2} + \cdots + 2\tilde{x}_{j+i-2} + \tilde{x}_{i+j-1} \right\} (5.1)$$

$$\begin{aligned} = \; & (n-j)\tilde{x}_1 + \cdots + (n-i-j+1)\tilde{x}_i \\ & + (n-i-1)\tilde{x}_{i+1} + (n-i-2)\tilde{x}_{i+2} + \cdots + (n-i-j+2)\tilde{x}_{j+i-2} \\ & + (n-i-j+1)\tilde{x}_{i+j-1} + (n-i-j)\tilde{x}_{i+j} + [n-i-j] \\ & + (n-i-j-1)\tilde{x}_{i+j+1} + \cdots + \tilde{x}_{n-1} \end{aligned}$$

$$= \sum_{a=1}^{i} (n-j+1-a)\tilde{x}_a + \sum_{a=i+1}^{n-1} (n-a)\tilde{x}_a + n-i-j$$

$$= \sum_{a=1}^{n-1} (n-a)\tilde{x}_a + \sum_{a=1}^{i} (1-j)\tilde{x}_a + n-i-j$$

$$= \sum_{a=1}^{n-1} (n-a)\tilde{x}_a + \underset{\uparrow}{(1-j)\sum_{a=1}^{i}\tilde{x}_a} + n-i-j$$

this cancels with the sign in (4.2)

In terms of the m's the A_∞-constraints read (i.e. (4.21)) ainfcat

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$$\sum_{\substack{i\geq 0, j\geq 1\\ 1\leq i+j\leq n}} (-1)^{\sum_{a=1}^{n-1}(n-a)\tilde{x}_a + j\sum_{a=1}^{i}\tilde{x}_a + i + j} m_{n-j+1}(x_1\otimes\cdots\otimes x_i \otimes m_j(x_{i+1}\otimes\cdots\otimes x_{i+j})\otimes x_{i+j+1}\otimes\cdots\otimes x_n) = 0 \tag{7.1}$$

And finally, in terms of the μ's: we have

(7.2)

$$|\underbrace{m_j(x_{i+1}\otimes\cdots\otimes x_{i+j})}_{\kappa}| = 2-j+|x_{i+1}|+\cdots+|x_{i+j}|$$

Hence

$$m_j(x_{i+1}\otimes\cdots\otimes x_{i+j}) = (-1)^{\overbrace{\sum_{i+1\leq a<b\leq i+j}|x_a||x_b|}^{\phi'}} \mu(x_{i+j}\otimes\cdots\otimes x_{i+1}) \tag{7.3}$$

Hence

$$m_{n-j+1}(x_1\otimes\cdots\otimes x_i\otimes \overbrace{m_j(x_{i+1}\otimes\cdots\otimes x_{i+j})}^{\kappa}\otimes x_{i+j+1}\otimes\cdots\otimes x_n)$$

$$= (-1)^{\text{sign of reversing order of } x_1\otimes\cdots\otimes x_i\otimes x_{i+1}\otimes\cdots\otimes x_n + j(|x_1|+\cdots+|x_i|) + j(|x_{i+j+1}|+\cdots+|x_n|)}$$

call this $|rev|$

$$\mu_{n-j+1}(x_n\otimes\cdots\otimes x_{i+j+1}\otimes \mu_j(x_{i+j}\otimes\cdots\otimes x_{i+1})\otimes x_i\otimes\cdots\otimes x_1) \tag{7.4}$$

Now $j\sum_{a=1}^{i}|x_a| + j\sum_{a=i+j+1}^{n}|x_a| + \sum_{a=1}^{n-1}(n-a)\tilde{x}_a + j\sum_{a=1}^{i}\tilde{x}_a$

$$(\text{mod } 2) \quad \equiv \quad j\sum_{a=i+j+1}^{n}|x_a| + ji \quad + \text{ term indept of } i, j.$$

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The A_∞-constraint becomes in terms of the μ's
(removing the common factor of $|rev|$)

$$\sum_{\substack{i\geq 0,\, j\geq 1\\ 1\leq i+j\leq n}} (-1)^{j\sum_{a=i+j+1}^{n}|x_a| + ji + i + j + n}\, \mu_{n-j+1}\big(x_n\otimes\cdots\otimes x_{i+j+1}\otimes \mu_j(x_{i+j}\otimes\cdots\otimes x_{i+1})\otimes x_i\otimes\cdots\otimes x_1\big) = 0. \qquad \forall n\geq 1 \tag{8.1}$$

which is the same as

$$\sum_{\substack{i\geq 0,\, j\geq 1\\ 1\leq i+j\leq n}} (-1)^{ij+i+j+n}\, \mu_{n-j+1}\big(x_n\otimes\cdots\otimes x_{i+j+1}\otimes \mu_j(x_{i+j}\otimes\cdots\otimes x_{i+1})\otimes x_i\otimes\cdots\otimes x_1\big) = 0 \qquad \forall n\geq 1 \tag{8.2}$$

The signs in ainf are $r+st$ where in (8.2) $r = n-i-j$, $s = j$ and $t = i$, so they are the same. Since what we have said is not specific to the algebra case:

Proposition Given a set of objects $\mathrm{Ob}\,A$ and degree $2-n$ k-linear maps $M_{a_n\ldots a_0}$ as in (1.1), the associated suspended forward compositions (3.1) satisfy their A_∞-constraints (4.3) if and only if the μ's satisfy the constraints (8.2).