In (ainf) - (ainf^4) we looked A_\infty-algebras and we are now developing examples in the (ainf^4) series. In this note we study A_\infty-categories and particularly the forward-suspended multiplications Lazavoiu used in his paper

[1] “Generating the superpotential on a D-brane category”

Throughout k is a field of characteristic zero, all v.s. spaces are over k.
A small A_\infty-category \mathcal{A} is specified by a set of objects \text{Ob} \mathcal{A}
and by graded vector spaces \text{Hom}_\mathcal{A}(a, b) for any \text{a, b} \in \text{Ob} \mathcal{A}
together with linear maps

\[
\begin{align*}
M_{a_0, \ldots, a_n} : \text{Hom}_\mathcal{A}(a_{n-1}, a_n) \otimes \cdots \otimes \text{Hom}_\mathcal{A}(a_0, a_1) & \rightarrow \text{Hom}_\mathcal{A}(a_0, a_n) \\
(1.1)
\end{align*}
\]

of degree 2 - n subject to A_\infty-constraints, to be given below. The graded vector spaces may be either \mathbb{Z} or \mathbb{Z}_2-graded, and in each case suspension function etc. should be interpreted appropriately in the following. The degree of homogeneous \text{x} \in \text{Hom}_\mathcal{A}(a, b) is denoted |x|.

In particular for a pair \text{a, b} we have

\[
M_{b a} : \text{Hom}_\mathcal{A}(a, b) \rightarrow \text{Hom}_\mathcal{A}(a, b) \quad \text{deg} + 1
\]

and for \text{a, b, c} \in \text{Ob} \mathcal{A} a degree zero map

\[
M_{c b a} : \text{Hom}_\mathcal{A}(b, c) \otimes \text{Hom}_\mathcal{A}(a, b) \rightarrow \text{Hom}_\mathcal{A}(a, c)
\]
For objects $a, b$ of $A$, let $s_{ab} : \text{Hom}_A(a, b) \to \text{Hom}_A(a, b)[1]$ be the suspension operator of $\text{Hom}_A(a, b)$. Denoting the degree of $x \in \text{Hom}_A(a, b)[1]$ by $\overline{x} = |x| - 1$ we have $s(x) = x$ and $s$ is a map of degree $-1$. We write $s_{ab}$ for $s_{ab}[n]$ and

$$s^n_{ab} : \text{Hom}_A(a, b) \to \text{Hom}_A(a, b)[n] \quad n > 0$$

$$s^n_{ab} = s_{ab} \circ s_{ab} \circ \cdots \circ s_{ab} \circ s_{ab}$$

and $s^0_{ab} = \text{id}$ and for $n < 0$

$$s^n_{ab} = s_{ab}^{-1} \circ s_{ab}^{n+1} \circ \cdots \circ s_{ab}^{-1} \circ s_{ab}^{-1}$$

Next we introduce "forward" compositions

$$m_{a_0 \ldots a_n} : \text{Hom}_A(a_0, a_1) \otimes \cdots \otimes \text{Hom}_A(a_{n-1}, a_n) \to \text{Hom}_A(a_0, a_n)$$

by the formula

$$m_{a_0 \ldots a_n}(x_1 \otimes \cdots \otimes x_n) = (-1)^{\sum_{1 \leq i < j \leq n} |z_i||z_j|} m_{a_n \ldots a_0}(x_n \otimes \cdots \otimes x_1).$$

(2.2)

Of course $|m_{a_0 \ldots a_n}| = 2^{-n}$. 
The suspended forward compositions are

\[ \text{Ra}_a \cdots \text{Ra}_n : \text{Hom}_{\text{at}}(a_0, a_1)[1] \otimes \cdots \otimes \text{Hom}_{\text{at}}(a_{n-1}, a_n)[1] \]

\[ \downarrow \]

\[ \text{Hom}_{\text{at}}(a_0, a_n)[1] \]

(3.1)

defined by

\[ \text{Ra}_a \cdots \text{Ra}_n = S_{a_0 a_n} \circ \text{Ma}_a \cdots \text{Ma}_n \circ (S_{a_0 a_1}^{-1} \otimes \cdots \otimes S_{a_{n-1} a_n}^{-1}) \]

(3.2)

This is a map of degree +1. We have to observe that the s are maps of degree -1 so (3.2) involves Koszul signs when applied to a tensor, e.g.

\[ m_{ab}(x) = \text{Ra}_{ab}(x) \]

But

\[ \text{Ra}_{abc}(x_1 \otimes x_2) = S \circ m_{abc} \circ (S_{ab}^{-1} \otimes S_{bc}^{-1})(x_1 \otimes x_2) \]

\[ = S \circ m_{abc}(S_{ab}^{-1}(x_1) \otimes S_{bc}^{-1}(x_2)) \cdot (-1)^{x_1} \tilde{x}_1 \]

\[ = (-1)^{\tilde{x}_1} m_{abc}(x_1 \otimes x_2). \]

(3.3)

\[ \text{Ra}_{abcd}(x_1 \otimes x_2 \otimes x_3) = (-1)^{\tilde{x}_2} m_{abcd}(x_1 \otimes x_2 \otimes x_3). \]
and

\[ M_{ab}(x) = \rho_{ba}(x) \]

\[ M_{abc}(x_1 \otimes x_2) = (-1)^{|x_1||x_2|} M_{cba}(x_2 \otimes x_1) \]

\[ = (-1)^{|x_1||x_2|} + \tilde{x}_2 \rho_{cba}(x_2 \otimes x_1) \]

\[ = (-1)^{(x_i+1)(x_{i+1})} + \tilde{x}_2 \rho_{cba}(x_2 \otimes x_1) \]

\[ = (-1)^{x_i \tilde{x}_2 + \tilde{x}_1} + l \rho_{cba}(x_2 \otimes x_1) \]

In terms of the forward suspended compositions the $A_{\infty}$-constraint are given by

\[ \sum_{i \geq 0, j \geq 1} (-1)^{x_i + \cdots + x_i} r_{a_0 \cdots a_i, a_i \cdots a_n} (x_1 \otimes \cdots \otimes x_i \otimes r_{a_i \cdots a_{ij+1}, a_{ij+1} \cdots a_n} (x_{i+1} \otimes \cdots \otimes x_{ij+1} \otimes \cdots \otimes x_{ij} \otimes x_{i+1} \otimes \cdots \otimes x_n)) = 0 \quad \forall n \geq 1. \]

where \( x_j \in \text{Hom}_{\mathcal{A}} [(a_{j-1}, a_j)] \) is any sequence of 'forward composable' morphisms. Using the Kosull rule this can also be written

\[ \sum_{i \geq 0, j \geq 1} r_{a_0 \cdots a_i, a_i \cdots a_n} (\text{id}_{a_0 a_1} \otimes \cdots \otimes \text{id}_{a_{i-1}, a_i} \otimes r_{a_{i} \cdots a_{ij}, a_{ij} \cdots a_n} (\text{id}_{a_{ij+1} a_{ij+2}} \otimes \cdots \otimes \text{id}_{a_{n-1}, a_n})) = 0 \quad \forall n \geq 1. \]
we write (4.2) in terms of the "original" multiplications μ in the one object case, to compare with (4.1). In this case all subscripts are a_0 = ... = a_n = a and we omit them. For A := Hom_A(a, a) and x, ..., x_n ∈ A (4.2) reads in terms of m's (we write r_j, m_j, m_j for the obvious things)

\[ r_j^a (x_{i_1} \otimes \cdots \otimes x_{i_j}) \] 

\[ = m_j \circ (s^{-1} \otimes \cdots \otimes s^{-1}) (x_{i_1} \otimes \cdots \otimes x_{i_j}) \]

\[ = (-1)^j \tilde{x}_{i_1} + \cdots + \tilde{x}_{i_{j-1}} + \tilde{x}_{i_{j-1} + \cdots + \tilde{x}_{i_{j-2}}}

+ \cdots

+ \tilde{x}_{i_1} m_j (x_{i_1} \otimes \cdots \otimes x_{i_j}) \] 

\[ = (-1)^{j-1} \tilde{x}_{i_1} + (j-2) \tilde{x}_{i_2} + \cdots + 2 \tilde{x}_{i_{j-2}} + \tilde{x}_{i_{j-1}} \]

Writing \( \kappa := r_j (x_{i_1} \otimes \cdots \otimes x_{i_j}) \)

\[ r (x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_{i_{j+1}} \otimes \cdots \otimes x_n) \] 

\[ = m \circ (s^{-1} \otimes \cdots \otimes s^{-1}) (x_1 \otimes \cdots \otimes x_i \otimes \kappa \otimes x_{i_{j+1}} \otimes \cdots \otimes x_n) \]

\[ = (-1)^{n-j} \tilde{x}_1 + \cdots + ([n-i-j] \tilde{x}_i + \tilde{x}_{i_{j-1} + \cdots + \tilde{x}_{n-1}})

+ \tilde{x}_{i_{j+1}}

\[ m (x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_{i_{j+1}} \otimes \cdots \otimes x_n) \]

Now \( \tilde{\kappa} = \tilde{x}_{i_1} + \cdots + \tilde{x}_{i_j} \) + 1
Hence

\[ r( \bar{x}_1 \otimes \cdots \otimes \bar{x}_i \otimes r( \bar{x}_{i+1} \otimes \cdots \otimes \bar{x}_{i+j}) \otimes \bar{x}_{i+j+1} \otimes \cdots \otimes \bar{x}_n ) \]  

\[ = (-1)^{\phi} m( \bar{x}_1 \otimes \cdots \otimes \bar{x}_i \otimes m( \bar{x}_{i+1} \otimes \cdots \otimes \bar{x}_{i+j}) \otimes \bar{x}_{i+j+1} \otimes \cdots \otimes \bar{x}_n ) \]

where

\[ \phi = (n-j)\bar{x}_i + \cdots + (n-i-j+1)\bar{x}_i \]

\[ + (n-i-j)(\bar{x}_{i+1} + \cdots + \bar{x}_{i+j} + 1) \]  

\[ + (n-i-j-1)\bar{x}_{i+j+1} + \cdots + \bar{x}_{n-1} \]

\[ + (j-1)\bar{x}_{i+1} + (j-2)\bar{x}_{i+2} + \cdots + 2\bar{x}_{j+i-2} + \bar{x}_{j+i-1} \]  

\[ = (n-j)\bar{x}_1 + \cdots + (n-i-j+1)\bar{x}_i \]

\[ + (n-i-1)\bar{x}_{i+1} + (n-i-2)\bar{x}_{i+2} + \cdots + (n-i-j+2)\bar{x}_{j+i-2} \]

\[ + (n-i-j+1)\bar{x}_{i+j-1} + (n-i-j)\bar{x}_{i+j} + [n-i-j] \]

\[ + (n-i-j-1)\bar{x}_{i+j+1} + \cdots + \bar{x}_{n-1} \]

\[ = \sum_{a=1}^{i} (n-j+1-a)\bar{x}_a + \sum_{a=i+1}^{n-1} (n-a)\bar{x}_a + n-i-j \]

\[ = \sum_{a=1}^{n-1} (n-a)\bar{x}_a + \sum_{a=1}^{i} (1-j)\bar{x}_a + n-i-j \]

\[ = \sum_{a=1}^{n-1} (n-a)\bar{x}_a + (1-j)\sum_{a=1}^{i} \bar{x}_a + n-i-j \]

this cancels with the sign in (4.21)
In terms of the $m$'s the $A_{\infty}$-constraint read (i.e. (4.2))

$$\sum_{i \neq j \geq 1 \leq i+j \leq n} (-1)^{i+j} \sum_{a=1}^{n-1} (n-a) x_a + j \sum_{a=1}^{i} x_a + i+j$$

$$m_{n-j+1}(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n) = 0$$

And finally, in terms of the $m$'s: we have

$$|m_j(x_{i+1} \otimes \cdots \otimes x_{i+j})| = 2-j + |x_{i+1} + \cdots + x_{i+j}|$$

Hence

$$m_j(x_{i+1} \otimes \cdots \otimes x_{i+j}) = (-1)^{i+j} \sum_{i+1 \leq a \leq b \leq i+j} |x_a||x_b|$$

$$m(x_{i+1} \otimes \cdots \otimes x_{i+j}) = (-1)^{i+j} \sum_{i+1 \leq a \leq b \leq i+j} |x_a||x_b|$$

Hence

$$m_{n-j+1}(x_1 \otimes \cdots \otimes x_i \otimes m_j(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_n)$$

$$= (-1)^{i+j} \sum_{\text{sign of reversing order of } x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n}$$

$$+ j(|x_{i+1} + \cdots + x_{i+j}|)$$

$$+ j(|x_{i+j+1} + \cdots + x_n|)$$

$$= m_{n-j+1}(x_n \otimes \cdots \otimes x_{i+j+1} \otimes m_j(x_{i+j} \otimes \cdots \otimes x_{i+1})$$

$$\otimes x_{i+1} \otimes \cdots \otimes x_1)$$

Now

$$j \sum_{a=1}^{i} |x_a| + j \sum_{a=i+1}^{n} |x_a| + \sum_{a=1}^{n-1} (n-a) \tilde{x}_a + j \sum_{a=1}^{i} \tilde{x}_a$$

$$= j \sum_{a=i+1}^{n} |x_a| + ji + \text{term indep of } i,j.$$

$$\text{(mod 2)}$$
The $A_{\infty}$-constraint becomes in terms of the $\mu$'s
(removing the common factor of $\lambda_{\infty}$)

\[
\sum_{i \geq 0, j \geq 1} (-1)^{i+j+i+j+n} a_{i+j+1} \mu_{n-j+1}(x_n \otimes \cdots \otimes x_{i+j+1}) \otimes x_i \otimes \cdots \otimes x_1 = 0.
\]

which is the same as

\[
\sum_{i \geq 0, j \geq 1} (-1)^{i+j+i+j+n} \mu_{n-j+1}(x_n \otimes \cdots \otimes x_{i+j+1} \otimes \mu_j(x_{i+j} \otimes \cdots \otimes x_1) \otimes x_i \otimes \cdots \otimes x_1) = 0.
\]

The signs in $\text{ainf}$ are $r+s+t$ where in (8.2) $r = n-i-j$, $s = j$ and $t = i$, so they are the same. Since what we have said is not specific to the algebra case:

**Proposition.** Given a set of objects $0 \leq t$ and degree $2-n$ $k$-linear maps $\mu_n$ as in (1.1), the associated suspended forward composition (3.1) satisfy their $A_{\infty}$-constraint (4.3) if and only if the $\mu$'s satisfy the constraint (8.2).