Notes on A_∞-algebras IV (checked)

These notes continue (ainf) – (ainf3). Our focus here is on A_∞-modules over Koszul complexes. So throughout k is a commutative ring, and $t_1, \ldots, t_m$ a sequence in k. Let A be the dg-algebra

$$A = \Lambda_k(k\theta_1 \oplus \cdots \oplus k\theta_m) \qquad |\theta_i| = -1$$

$$b_1 = \sum_{i=1}^n t_i \theta_i^* \qquad \text{write } F = \bigoplus_{i=1}^m k\theta_i$$

viewed as an A_∞-algebra, i.e. $b_3 = b_4 = \cdots = 0$ and

$$b_2 : A(1)^{\otimes 2} \longrightarrow A(1)$$

$$b_2(a \otimes b) = (-1)^{|a|} a \wedge b$$

define an A_∞-algebra structure.

#

An A_∞-module over A is a $\mathbb{Z}$-graded k-module M and maps

$$b_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow M(1) \qquad n \geq 1, \ |b_n| = 1$$

Now if we use symbols ω to stand for sequences $\theta_{i_1} \cdots \theta_{i_j}$ with $i_1 < \cdots < i_j$ then the data of the maps b_n is equivalent to specifying, for each sequence $\omega_1, \ldots, \omega_k$ of length $k \geq 0$, a k-linear map

$$b(\omega_1, \ldots, \omega_k) = b(- \otimes \omega_1 \otimes \cdots \otimes \omega_k)$$
$$: M(1) \longrightarrow M(1)$$

For $k=0$ this is a degree 1 map

$$b_1 : M(1) \longrightarrow M(1)$$

These are subject to the constraint (4.1) of ainf2, i.e. for $n \geq 1$

$$0 = \sum_{s+t=n} b_u(b_s \otimes \mathbb{1}^{\otimes t}) + \sum_{i=1}^{2} \sum b_u(\mathbb{1}^{\otimes >0} \otimes b_i \otimes \mathbb{1}^{\otimes t})$$

Evaluated on $M(1) \otimes A(1)^{\otimes n-1}$ on say $x \otimes w_1 \otimes \cdots \otimes w_{n-1}$

$$0 = \sum b_u(b_s(x \otimes w_1 \otimes \cdots \otimes w_{s-1}) \otimes w_s \otimes \cdots \otimes w_{n-1})$$
$$+ (-1)^{1+|x|} \sum_{i=1}^{n-1} (-1)^{|w_1| + \cdots + |w_{i-1}| + i-1} b_n(x \otimes w_1 \otimes \cdots \otimes b_1(w_i) \otimes \cdots \otimes w_{n-1})$$
$$+ (-1)^{1+|x|} \sum_{i=1}^{n-2} (-1)^{|w_1| + \cdots + |w_{i-1}| + i-1} b_{n-1}(x \otimes w_1 \otimes \cdots \otimes b_2(w_i \otimes w_{i+1}) \otimes \cdots)$$

$$= \sum b_u(b_s(x \otimes w_1 \otimes \cdots \otimes w_{s-1}) \otimes w_s \otimes \cdots \otimes w_{n-1})$$
$$+ (-1)^{|x|+1} \sum_{i=1}^{n-1} (-1)^{|w_1| + \cdots + |w_{i-1}| + i+1} b_n(x \otimes w_1 \otimes \cdots \otimes b_1(w_i) \otimes \cdots \otimes w_{n-1})$$
$$+ (-1)^{|x|+1} \sum_{i=1}^{n-2} (-1)^{|w_1| + \cdots + |w_{i-1}| + i+1 + |w_i|} b_{n-1}(x \otimes w_1 \otimes \cdots \otimes w_i w_{i+1} \otimes \cdots) \qquad (2.2)$$

Example $m = 1$ so

$$A = \overset{-1}{k\theta} \xrightarrow{t} \overset{0}{k}$$

and an A_∞-module has maps

$$b_1 : M(1) \longrightarrow M(1)$$
$$b_2(- \otimes \theta),\ b_2(- \otimes 1) : M(1) \longrightarrow M(1)$$
$$b_3(- \otimes \theta \otimes \theta),\ b_3(- \otimes \theta \otimes 1),\ b_3(- \otimes 1 \otimes \theta),$$
$$b_3(- \otimes 1 \otimes 1) : M(1) \longrightarrow M(1)$$

....

which satisfy relations

$\boxed{n=1}$ $\qquad b_1^2 = 0$

$\boxed{n=2}$ $\qquad b_2(b_1 \otimes 1) + b_1 b_2 + b_2(1 \otimes b_1) = 0$

i.e. $M(1)$ is a dg-module $\quad M(1) \otimes A(1) \to M(1)$

$\boxed{n=3}$ $\qquad b_3(b_1 \otimes 1 \otimes 1) + b_2(b_2 \otimes 1) + b_1 b_3$
$$+ b_3(1 \otimes b_1 \otimes 1 + 1 \otimes 1 \otimes b_1) + b_2(1 \otimes b_2) = 0$$

$$M(1) \otimes A(1)^{\otimes 2} \to M(1)$$

The condition $n=2$ evaluated on $x \otimes \theta$ says

$$b_2(b_1(x) \otimes \theta) + b_1 b_2(x \otimes \theta) + (-1)^{|x|+1} t\, b_2(x \otimes 1) = 0$$

i.e. roughly that $b_2(- \otimes \theta)$ is a null-homotopy for the action of t.

Evaluated on $x \otimes 1$ it says

$$b_2(b_1(x)\otimes 1) + b_1 b_2(x \otimes 1) = 0$$

Basically that $b_2(- \otimes 1)$ is a closed operation (most likely 1 for us!).
The condition $n=3$ evaluated on $x \otimes \theta \otimes \theta$ says

$$b_3(b_1(x)\otimes \theta \otimes \theta) + b_2(b_2(x\otimes\theta)\otimes\theta) + b_1 b_3(x\otimes\theta\otimes\theta)$$
$$+ \underset{(-1)^{|x|}}{b_3}(x\otimes t\otimes\theta - x\otimes\theta\otimes t) + \cancel{b_2(1\otimes\theta)} = 0$$

i.e. assuming we can arrange $b_3(x\otimes 1\otimes\theta) = b_3(x\otimes\theta\otimes 1)$
this states that $b_3(-\otimes\theta\otimes\theta)$ gives a null-homotopy for
$b_2(b_2(-\otimes\theta)\otimes\theta)$ i.e. $b_2(-\otimes\theta)^2$.

Example Say (X, D) is a MF of W over a k-algebra R, say
free of finite rank. Then say $R = k[x]$

$$D^2 = W$$
$$\partial_x(D)D + D\partial_x(D) = \partial_x W =: t$$

But how do we argue that $\partial_x(D)^2$ is null-homotopic?

$$D = \begin{pmatrix} 0 & x^2 \\ x^2 & 0 \end{pmatrix} \quad W = x^4 \quad \partial_x(D) = \begin{pmatrix} 0 & 2x \\ 2x & 0 \end{pmatrix}$$
$$\partial_x(D)^2 = \begin{pmatrix} 4x^2 & 0 \\ 0 & 4x^2 \end{pmatrix} = 4x^2$$

Now $\partial_x(D)^2$ is a closed map $X \longrightarrow X$, and
$$\mathrm{End}(X) = X^\vee \otimes k[x]/x^2 = k[x]/x^2 \oplus k[x]/x^2[1]$$

So $\partial_x(D)^2$ is in fact null-homotopic.