Notes on A_∞-algebras III (checked)

31/12/12

1

ainf3

This note continues ainf, ainf2 and we have the same conventions. In particular k is a commutative ring. We begin with morphisms of A_∞-algebras. If $(C,\Delta), (D,\Delta)$ are $\mathbb{Z}$-graded coalgebras a <u>morphism</u> $f:(C,\Delta)\to(D,\Delta)$ is a degree zero map such that

$$\begin{tikzcd} C\otimes C \arrow[r, "f\otimes f"] & D\otimes D \\ C \arrow[u, "\Delta"] \arrow[r, "f"'] & D \arrow[u, "\Delta"'] \end{tikzcd}$$

commutes. Let A, B be $\mathbb{Z}$-graded k-modules.

<u>Lemma</u> There is a bijection between morphisms of coalgebras

$$f: TA(1) \longrightarrow TB(1)$$

and sequences of degree zero maps, $n \geq 1$

$$f_n : A(1)^{\otimes n} \longrightarrow B(1)$$

<u>Proof</u> Given f, define f_n to be $A(1)^{\otimes n} \hookrightarrow TA(1) \xrightarrow{f} TB(1) \twoheadrightarrow B(1)$, and given f_n define f to be the map with component

$$A(1)^{\otimes n} \longrightarrow B(1)^{\otimes u}$$
$$\sum f_{i_1}\otimes\cdots\otimes f_{i_u} \qquad (1.2)$$

with all $i_j \geq 1$ and $\sum i_j = n$.

First we show that f defined in this way is a morphism of coalgebras. We have

$$\Delta f(v_1 \otimes \cdots \otimes v_n) = \sum \Delta(f_{i_1} \otimes \cdots \otimes f_{i_u})(v_1 \otimes \cdots \otimes v_n)$$

and

$$(f \otimes f)\Delta(v_1 \otimes \cdots \otimes v_n) = (f \otimes f)\big((v_1) \otimes (v_2 \otimes \cdots \otimes v_n) + \cdots + (v_1 \otimes \cdots \otimes v_{n-1}) \otimes (v_n) \big)$$

$$\begin{aligned} &= f(v_1) \otimes f(v_2 \otimes \cdots \otimes v_n) \\ &+ f(v_1 \otimes v_2) \otimes f(v_3 \otimes \cdots \otimes v_n) \\ &+ \cdots \\ &+ f(v_1 \otimes \cdots \otimes v_{n-1}) \otimes f(v_n) \end{aligned}$$

Let us apply the projection to $B(1)^{\otimes a} \otimes B(1)^{\otimes b}$ on both sides.

$$\pi \Delta f(v) = \text{rebracketing of } \sum (\underbrace{f_{i_1} \otimes \cdots \otimes f_{i_u}}_{\cap\, B(1)^{\otimes a+b}})(v_1 \otimes \cdots \otimes v_n)$$

$a+b=u$
$\sum i_j = n.$

$$\begin{aligned} \pi(f \otimes f)\Delta(v_1 \otimes \cdots \otimes v_n) &= f(v_1)_a \otimes f(v_2 \otimes \cdots \otimes v_n)_b \\ &+ f(v_1 \otimes v_2)_a \otimes f(v_3 \otimes \cdots \otimes v_n)_b \\ &+ \cdots \\ &+ f(v_1 \otimes \cdots \otimes v_{n-1})_a \otimes f(v_n)_b \end{aligned}$$

But is easily seen that these are the same. (handle $n=1$ separately)

Next we show that f is a morphism of coalgebras it agrees with $\sum f_{i_1} \otimes \cdots \otimes f_{i_n}$. We prove this by induction, proving at the same time that $f(A(1)^{\otimes n}) \subseteq B(1) \oplus B(1)^{\otimes 2} \oplus \cdots \oplus B(1)^{\otimes n}$.

$\boxed{n=1}$

$$\Delta f = (f \otimes f)\Delta$$

$$\Delta f(v_1) = (f \otimes f)\Delta(v_1) = 0. \Rightarrow f(v_1) \in B(1)$$

$$\therefore \; f(v_1) = f_1(v_1).$$

$n \geq 1$

Suppose both claims hold for $v \in A(1)^{\otimes n}$ and let $v_{n+1} \in A(1)$ be given. Let π be the projection to $B(1)^{\otimes a} \otimes B(1)^{\otimes 1}$, $1 \leq a$

$$\pi \Delta f(v \otimes v_{n+1}) = \text{rebracketing of } f(v \otimes v_{n+1})_{a+1}$$

$$\pi (f \otimes f)\Delta(v \otimes v_{n+1})$$

$$= \pi(f \otimes f)\begin{pmatrix} (v_1) \otimes (v_2 \otimes \cdots \otimes v_n \otimes v_{n+1}) \\ + \\ (v_1 \otimes v_2) \otimes (v_3 \otimes \cdots \otimes v_n \otimes v_{n+1}) \\ + \\ \vdots \\ (v_1 \otimes \cdots \otimes v_n) \otimes (v_{n+1}) \end{pmatrix}$$

$$= f(v_1)_a \otimes f(v_2 \otimes \cdots \otimes v_{n+1})_1$$
$$+ \cdots + f(v_1 \otimes \cdots \otimes v_n)_a \otimes f(v_{n+1})_1$$

From this we see $f(v \otimes v_{n+1}) \in B(1)^{\otimes \leq n+1}$ and that f agrees with (1.2). □

Lemma Let A, B be A_∞-algebras, and

$$f_n : A(1)^{\otimes n} \longrightarrow B(1) \qquad n \geq 1$$

a sequence of degree zero maps. The following are equivalent

(a) The associated $f : \overline{T}A(1) \longrightarrow \overline{T}B(1)$ is a morphism of dg coalgebras, i.e. $fb = bf$.

(b) We have, as maps $A(1)^{\otimes n} \longrightarrow B(1)$, for $n \geq 1$ $\qquad$ (4.2)

$$\sum f_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}) = \sum b_u(f_{i_1} \otimes \cdots \otimes f_{i_s})$$

where the first sum is over all decompositions $n = r+s+t$, $s \geq 1$ $r, t \geq 0$ and the second sum is over $s \geq 1$ and $i_j \geq 1$, $\sum i_j = n$.

Proof The maps b are coderivations, so

$$(fb \otimes f + f \otimes fb)\Delta$$
$$\|$$
$$\Delta fb = (f \otimes f)\Delta b = (f \otimes f)(b \otimes 1 + 1 \otimes b)\Delta$$
$$\Delta bf = (b \otimes 1 + 1 \otimes b)\Delta f = (b \otimes 1 + 1 \otimes b)(f \otimes f)\Delta$$
$$= (bf \otimes f + f \otimes bf)\Delta$$

$$\therefore \quad \Delta(fb - bf) = ([fb - bf] \otimes f + f \otimes [fb - bf])\Delta$$

From this formula we deduce that to show $fb = bf$ it is necessary and sufficient that the components $A(1)^{\otimes n} \hookrightarrow \overline{T}A(1) \xrightarrow{fb - bf} \overline{T}B(1) \twoheadrightarrow B(1)$ vanish. But this is (4.2). $\square$

<u>Def^N</u> A <u>morphism</u> $f : A \longrightarrow B$ of A_∞-algebras is a sequence of degree zero maps

$$f_n : A(1)^{\otimes n} \longrightarrow B(1) \qquad n \geq 1$$

satisfying the equivalent conditions of the above lemma.

<u>Note</u> It is obvious that A_∞-algebras form a category, with a fully faithful functor

$$A_\infty\text{-alg} \longrightarrow \text{dg-coalgebras}$$
$$(A, b_n) \longmapsto (\overline{T}A(1), \Delta, b)$$

Given $f : A \longrightarrow B$, $g : B \longrightarrow C$ the composite has

$$(g \circ f)_n = \sum g_u(f_{i_1} \otimes \cdots \otimes f_{i_u})$$

where the sum is over all $i_j \geq 1$ with $\sum i_j = n$, and $u \geq 1$. The identity $1_A : A \longrightarrow A$ has components $(1_A)_1 = 1_{A(1)}$, $(1_A)_n = 0, n > 1$.

<u>Example</u> In (4.2) the condition for $n = 1$ says

$$f_1 b_1 = b_1 f_1$$

i.e. f_1 is a morphism of complexes. For $n = 2$ it says

$$f_1 b_2 + f_2(1 \otimes b_1 + b_1 \otimes 1) = b_2(f_1 \otimes f_1) + b_1 f_2$$

i.e. f_1 preserves the multiplication b_2 <u>up to homotopy</u> given by f_2.