Notes on A_∞-algebras II (checked)

This note continues ainf and we use the same conventions. But we allow k to be any commutative ring, everything in ainf works in this generality.

Recall the definition of an A_∞-algebra: it consists of a $\mathbb{Z}$-graded k-module A and maps (k-linearity is implicit)

$$m_n : A^{\otimes n} \longrightarrow A$$

satisfying some identities. Equivalently, with $V = A(1)$ the tensor coalgebra (reduced) is

$$\overline{T}V = V \oplus V^{\otimes 2} \oplus \cdots$$

and a codifferential $b : \overline{T}V \longrightarrow \overline{T}V$ specifies exactly the same data, with m_n recovered as (9.1) of ainf, i.e. the map $b_n : A(1)^{\otimes n} \longrightarrow A(1)$ with signs. We maintain the above notation.

A strict unit for an A_∞-algebra A is an element $1 \in A^0$ which is a unit for m_2 and such that, for $n \neq 2$, the map b_n takes the value 0 whenever one of its arguments equals 1. Unfortunately, strict unitality is not preserved by A_∞-quasi-isomorphism.

A homological unit for A is a unit for the associative algebra H^*A with the multiplication induced by m_2.

A-infinity modules

Let A be a homologically unital A_∞-algebra. An A_∞-module is a $\mathbb{Z}$-graded k-module M with maps

$$b_n : M(1) \otimes (A(1))^{\otimes n-1} \longrightarrow M(1) \quad n \geq 1 \tag{2.1}$$

of degree 1 such that for $n \geq 1$, we have as maps $M(1) \otimes A(1)^{\otimes n-1} \longrightarrow M(1)$

$$\sum b_u (\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}) = 0 \tag{2.2}$$

where the sum is over all $n = r+s+t$ (so $u = r+1+t$), i.e. the identity in p. 9 of ainf for an A_∞-algebra, plus we insist that the induced action

$$H^* M \otimes H^* A \longrightarrow H^* M \tag{2.3}$$

is unital.

Remark • The map $b_1 : M \longrightarrow M$ has degree one and for $n=1$, (2.2) reads

$$b_1(b_1) = 0 \qquad \text{i.e. } b_1^2 = 0 \text{ is a differential}$$

• The map $b_2 : M(1) \otimes A(1) \longrightarrow M(1)$ defines m_2 by

$$\begin{tikzcd} M(1) \otimes A(1) \arrow[r, "b_2"] & M(1) \\ M \otimes A \arrow[u, "s^{\otimes 2}"] \arrow[r, "m_2"] & M \arrow[u, "s"'] \end{tikzcd}$$

i.e. $\quad m_2(x \otimes a) = (-1)^{|x|} b_2(x \otimes a)$

and for $n=2$, (2.2) reads

$$\overset{0,1}{b_2}(b_1 \otimes \mathbb{1} + \mathbb{1} \otimes b_1) + \overset{1,0}{b_1}(b_2) = 0 \qquad (3.1)$$

Now $b_1 = m_1$ on both A and M so this says that b_2 is <u>closed</u>, or if we evaluate (3.1) on $x \otimes a$ (in $M(1) \otimes A(1)$ so x gets $(-1)^{|x|+1}$!)

$$0 = b_2(m_1(x) \otimes a + (-1)^{|x|+1} x \otimes m_1(a)) + m_1 b_2(x \otimes a)$$

$$= (-1)^{|x|+1} m_2(m_1(x) \otimes a) - m_2(x \otimes m_1(a)) + (-1)^{|x|} m_1 m_2(x \otimes a)$$

i.e. $$m_1 m_2(x \otimes a) = m_2(m_1(x) \otimes a + (-1)^{|x|} x \otimes m_1(a))$$

(this is just immediate from p. 9 of ainf)

$\Rightarrow$ (M, m_1) is a complex, $m_2 : M \otimes A \to M$ is a morphism of complexes. Hence passes to

$$m_2 : H^*M \otimes H^*A \longrightarrow H^*M$$

which is an honest right module structure for the same reason H^*A is an associative algebra.

Obviously if we define m_n via

$$\begin{tikzcd} M(1) \otimes A(1)^{\otimes n-1} \arrow[r, "b_n"] & M(1) \\ M \otimes A^{\otimes n-1} \arrow[u, "s^{\otimes n}"] \arrow[r, "m_n"] & M \arrow[u, "s"'] \end{tikzcd}$$

Then (2.2) holds iff. (2.1) on p. 2 of ainf holds, so this is an alternative defn.

<u>Example</u> Suppose A is a dg-algebra. Then an A_∞-module consists of a graded k-module M and operations b_n subject to constraints (2.2) only for, given $n \geq 1$, pairs (r,t) with either $r=0$ and t arbitrary, or $r>0$ and $1 \leq s \leq 2$, i.e. we have

$$\begin{aligned} 0 &= \sum b_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}) \\ &= \sum_{s+t=n} b_u(b_s \otimes \mathbb{1}^{\otimes t}) + \sum b_u(\mathbb{1}^{\otimes >0} \otimes b_1 \otimes \mathbb{1}) \\ &\quad + \sum b_u(\mathbb{1}^{\otimes >0} \otimes b_2 \otimes \mathbb{1}) \end{aligned} \tag{4.1}$$

So (M, m_1) is a complex, $m_2 : M \otimes A \longrightarrow M$ is a morphism of complexes, and $n=3$ in (2.2) yields on $M(1) \otimes A(1)^{\otimes 2}$

$$\begin{aligned} 0 = b_3(b_1 \otimes \mathbb{1}^{\otimes 2}) &+ b_2(b_2 \otimes \mathbb{1}^{\otimes 1}) + b_1(b_3) \\ &+ b_3(\mathbb{1} \otimes b_1 \otimes \mathbb{1}) + b_3(\mathbb{1}^{\otimes 2} \otimes b_1) \\ &+ b_2(\mathbb{1} \otimes b_2) \end{aligned}$$

i.e. $$b_1 b_3 + b_3(b_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes b_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes b_1)$$

$$+ b_2(b_2 \otimes \mathbb{1} + \mathbb{1} \otimes b_2) = 0$$

So the action of A on M is not associative, but b_3 gives $b_3 : M(1) \otimes A(1)^{\otimes 2} \longrightarrow M(1)$ a homotopy between

$$(m \cdot a) \cdot b \quad \text{and} \quad m \cdot (a \cdot b)$$

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<u>Example</u> Suppose A is an algebra, sitting in degree zero. Then with $m_1 = 0$ and $m_n = 0$ $n \geq 3$ this is an A_∞-algebra (unital). An A_∞-A-module consists of a $\mathbb{Z}$-graded k-module M and operations

$$b_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow M(1) \qquad (|b_n| = 1$$

subject to constraints (4.1), i.e. for $n \geq 1$

$$0 = \sum_{s+t=n} b_u(b_s \otimes \mathbb{1}^{\otimes t}) + \sum b_u(\mathbb{1}^{\otimes \geq 0} \otimes b_2 \otimes \mathbb{1})$$

If the constraints $b_n = 0$ for $n \geq 3$ then the action

$$b_2 : M(1) \otimes A(1) \longrightarrow M(1)$$

satisfies $b_2(b_2 \otimes \mathbb{1} + \mathbb{1} \otimes b_2) = 0$, i.e. $M(1)$ is (up to signs) an dg A-module.

Def^N Let (C,Δ) be a $\mathbb{Z}$-graded coalgebra as on p. 6 of ainf2. A C-comodule is a $\mathbb{Z}$-graded k-module M and map (degree zero)

$$\rho : M \longrightarrow M \otimes C$$

making the diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "1 \otimes \Delta"] \\ M \otimes C \arrow[r, "\rho \otimes 1"'] & M \otimes C \otimes C \end{tikzcd}$$

commute. A morphism of comodules is a degree zero map $M \longrightarrow N$ commuting with ρ.

Def^N A dg-coalgebra over k is a $\mathbb{Z}$-graded coalgebra C equipped with a coderivation d satisfying $d^2 = 0$, $|d| = 1$. (i.e. a coalgebra (w/o unit) in the monoidal cat of $\mathbb{Z}$-graded k-complexes)
A dg-comodule over C is a comodule with $d_M : M \to M$, $d_M^2 = 0$, $|d_M| = 1$ satisfying $\rho d_M = (d_M \otimes 1 + 1 \otimes d)\rho$.

By Lemma on p. 9 of ainf given a $\mathbb{Z}$-graded k-module A, the structure of an A_∞-algebra is precisely given by making the coalgebra $TA(1)$ into a dg-coalgebra, i.e. specifying a codifferential lo. That is,

$$A \text{ an } A_\infty\text{-algebra} \iff TA(1) \text{ a dg-coalgebra}$$

To understand A_∞-modules as dg-comodules we need the full tensor coalgebra. Let V be a $\mathbb{Z}$-graded k-module and

$$TV = k \oplus \bar{T}V = \bigoplus_{i \geq 0} V^{\otimes i}$$

with the coproduct $\tilde{\Delta}$ defined by $\tilde{\Delta}(1) = 1 \otimes 1$ and

$$\begin{aligned}\tilde{\Delta}(v_1 \otimes \cdots \otimes v_m) &= \sum_{i=0}^{m} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_m) \\ &= (1) \otimes (v_1 \otimes \cdots \otimes v_m) + \Delta(v_1 \otimes \cdots \otimes v_m) \\ &\quad + (v_1 \otimes \cdots \otimes v_m) \otimes (1)\end{aligned}$$

where Δ is the coproduct on $\bar{T}V$. This is easily checked to be a $\mathbb{Z}$-graded coalgebra.

$\#$

Let A be an A_∞-algebra with codifferential b on $\bar{T}A(1)$, which we extend to $TA(1)$ by declaring $b = 0$ on k. This makes $TA(1)$ into a dg-coalgebra $(TA(1), \tilde{\Delta}, b)$. The tensor product

$$M(1) \otimes TA(1)$$

is then a $\mathbb{Z}$-graded $TA(1)$-comodule via

$$\rho : M(1) \otimes TA(1) \longrightarrow M(1) \otimes TA(1) \otimes TA(1)$$

$$\rho(x \otimes \omega) = x \otimes \tilde{\Delta}\omega.$$

Here M is any $\mathbb{Z}$-graded k-module

We begin with a useful technical fact. Let M, N be $\mathbb{Z}$-graded k-modules.

<u>Lemma</u> There is a bijection between degree q morphisms of $TA(1)$-comodules

$$f : M(1) \otimes TA(1) \longrightarrow N(1) \otimes TA(1)$$

and sequences of degree q-maps

$$f_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow N(1) \qquad n \geq 1$$

<u>Proof</u> To the map f we associate $(f_n)_{n\geq 1}$ defined by

$$\begin{tikzcd} M(1) \otimes A(1)^{\otimes n-1} \arrow[r, hook] & M(1) \otimes TA(1) \arrow[d, "f"] & \\ & N(1) \otimes TA(1) \arrow[r, two heads] & N(1) \end{tikzcd} \qquad (7.2)$$

and to a sequence $(f_n)_{n\geq 1}$ we associate

$$f = \sum_{\substack{s \geq 1 \\ t \geq 0}} (f_s \otimes 1^{\otimes t}) \qquad (7.3)$$

To prove this f is a morphism of comodules we first check $\rho f(x \otimes 1) = (f \otimes 1)\rho(x \otimes 1)$ which is easy, and then

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(b) $\rho f(x\otimes v) = (f\otimes 1)\rho(x\otimes v)$ $\qquad v = v_1\otimes\cdots\otimes v_m$

$$LHS = \rho\sum(f_s\otimes \mathbb{1}^{\otimes t})(x\otimes v)$$

$$= \rho\sum_{s=1}^{m+1} f_s(x\otimes v_1\otimes\cdots\otimes v_{s-1})\otimes v_s\otimes\cdots\otimes v_m$$

$$= \sum_{s=1}^{m+1} f_s(x\otimes v_1\otimes\cdots\otimes v_{s-1})\otimes\tilde{\Delta}(v_s\otimes\cdots\otimes v_m)$$

$$RHS = (f\otimes 1)(x\otimes\tilde{\Delta}v)$$

$$= (f\otimes 1)\Big(x\otimes 1\otimes v + x\otimes v\otimes 1 + \sum_{i=1}^{m-1} x\otimes(v_1\otimes\cdots\otimes v_i)\otimes(v_{i+1}\otimes\cdots\otimes v_m)\Big)$$

$$= f_1(x)\otimes v + \sum_{s=1}^{m+1} f_s(x\otimes v_1\otimes\cdots\otimes v_{s-1})\otimes(v_s\otimes\cdots\otimes v_m)\otimes(1)$$
$$+ \sum_{i=1}^{m-1} f(x\otimes v_1\otimes\cdots\otimes v_i)\otimes(v_{i+1}\otimes\cdots\otimes v_m)$$

$$= f_1(x)\otimes v + \sum_{s=1}^{m+1} f_s(x\otimes v_1\otimes\cdots\otimes v_{s-1})\otimes(v_s\otimes\cdots\otimes v_m)\otimes(1)$$
$$+ \sum_{i=1}^{m-1}\begin{bmatrix} f_1(x)\otimes(v_1\otimes\cdots\otimes v_i) \\ + f_2(x\otimes v_1)\otimes(v_2\otimes\cdots\otimes v_i) \\ + \\ \vdots \\ + \\ f_{i+1}(x\otimes v_1\otimes\cdots\otimes v_i)\otimes(1)\end{bmatrix}\otimes(v_{i+1}\otimes\cdots\otimes v_m)$$

$$= \sum_{s=1}^{m} f_s(x\otimes v_1\otimes\cdots\otimes v_{s-1})\otimes\Big(\begin{array}{l}(1)\otimes(v_s\otimes\cdots\otimes v_m)\\ +(v_s\otimes\cdots\otimes v_m)\otimes(1)\end{array}\Big)$$
$$+ \sum_{i=1}^{m-1}\begin{bmatrix} f_1(x)\otimes(v_1\otimes\cdots\otimes v_i) \\ + \cdots \\ + f_i(x\otimes v_1\otimes\cdots\otimes v_{i-1})\otimes(v_i)\end{bmatrix}\otimes(v_{i+1}\otimes\cdots\otimes v_m)$$
$$+ f_s(x\otimes v)\otimes(1)\otimes(1) \quad = LHS.$$

It remains to show every morphism of comodules f is determined in this way by its components f_n of (7.2)
We proceed by induction using

$$(f\otimes 1)\rho = \rho f$$

to prove that f never increases exterior degree $\overset{0}{M(1)} \oplus \overset{1}{M(1)\otimes A(1)} \oplus \overset{2}{\cdots}$
that (7.3) holds. We begin with

$\boxed{\text{ext. degree zero}}$

$$(f\otimes 1)\rho(x\otimes 1) = \rho f(x\otimes 1)$$

$$\Rightarrow \underset{\underset{f(x\otimes 1)\otimes 1}{\shortparallel}}{(f\otimes 1)(x\otimes 1\otimes 1)} = \rho f(x\otimes 1)$$

If $f(x\otimes 1) \in N(1) \oplus N(1)\otimes A(1) \oplus \cdots$
is $(p_0, p_1, p_2, \ldots)$

$$\begin{aligned}\rho f(x\otimes 1) &= (\rho p_0, \rho p_1, \rho p_2, \ldots) \\ &\in N(1) \oplus N(1)\otimes k\otimes A(1) \oplus N(1)\otimes A(1)\otimes A(1) \oplus \cdots \\ &\quad \oplus N(1)\otimes A(1)\otimes k \oplus N(1)\otimes A(1)^{\otimes 2}\otimes k \\ &\quad \oplus N(1)\otimes k\otimes A(1)^{\otimes 2}\end{aligned}$$

whereas

$$f(x\otimes 1)\otimes 1 \in N(1)\oplus(N(1)\otimes A(1)\otimes k)\oplus(N(1)\otimes A(1)^{\otimes 2}\otimes k)\oplus\cdots$$

must be zero

But since e.g. if $p_1 = \sum y_i\otimes a_i$ $\rho p_1 = \sum \underbrace{y_i\otimes 1\otimes a_i}_{\text{must be zero}} + y_i\otimes a_i\otimes 1$, i.e. $f(x\otimes 1)\in N(1)$,
we conclude that in fact $p_i = 0,\ i\geq 1$,
proving both claims since then

$$f(x\otimes 1) = f_1(x)\otimes 1$$

$\boxed{\text{Inductive step}}$ Suppose the claims hold for all $x\otimes v \in M(1)\otimes A(1)^{\otimes n-2}$. Then for $v' \in A(1)$

$$(f\otimes 1)\rho(x\otimes(v\otimes v')) = \rho f(x\otimes(v\otimes v')) \qquad (10.1)$$

But

$$\rho(x\otimes(v\otimes v')) = x\otimes \underset{\text{in } A(1)^{\otimes n-1}\subset TA(1)}{\tilde{\Delta}(v\otimes v')}$$

$$= x\otimes(1)\otimes(v\otimes v')$$
$$+\, x\otimes(v\otimes v')\otimes(1)$$
$$+\, x\otimes(v_1)\otimes(v_2\otimes\cdots\otimes v_{n-2}\otimes v')$$
$$+\cdots+ x\otimes(v_1\otimes\cdots\otimes v_{n-2})\otimes(v')$$

$$(f\otimes 1)\rho(x\otimes(v\otimes v')) = f(x\otimes 1)\otimes(v\otimes v') \qquad N(1)\otimes k\otimes A(1)^{\otimes n-1}$$
$$+\, f(x\otimes v\otimes v')\otimes(1) \qquad N(1)\otimes A(1)^{\otimes\leq n-1}\otimes k$$
$$+\, f(x\otimes v_1)\otimes(v_2\otimes\cdots\otimes v') \qquad N(1)\otimes A(1)^{\otimes\leq 1}\otimes -$$
$$+\cdots+ f(x\otimes v_1\otimes\cdots\otimes v_{n-2})\otimes v' \qquad N(1)\otimes A(1)^{\otimes\leq n-2}\otimes -$$

Again we see that, using the inductive hyp.

$$f(x\otimes(v\otimes v'))\in N(1)\oplus N(1)\otimes A(1)\oplus\cdots\oplus N(1)\otimes A(1)^{\otimes n-1}$$

i.e. f never increases exterior degree. To see that (7.3) holds we apply the projection $\pi_{p,q}: N(1)\otimes TA(1)\otimes TA(1)\longrightarrow N(1)\otimes A(1)^{\otimes p}\otimes A(1)^{\otimes q}$, to (10.1). On the LHS this results in ($v' = v_{n-1}$ as needed), $q>0$

$$\pi_p(f(x\otimes v_1\otimes\cdots\otimes v_{n-q-1}))\otimes(v_{n-q}\otimes\cdots\otimes v_{n-2}\otimes v')$$

$$\overset{\text{ind. hyp.}}{=} (f_s\otimes \mathbb{1}^{\otimes t})(x\otimes v_1\otimes\cdots\otimes v_{n-q-1})\otimes(v_{n-q}\otimes\cdots\otimes v_{n-2}\otimes v')$$

for the appropriate s,t to land in $N(1)\otimes A(1)^{\otimes p}$, i.e. $p=t+1$, $s+t=n-q$

$$\therefore s = n-q-(p-1) = n-q-p+1$$

On the RHS we just get $f(x\otimes(v\otimes v'))_{p+q}$ with odd bracketing. Over all p,q we get (7.3). □

Now we return to the subject of p. 6, namely coderivations on $M(1)\otimes TA(1)$, M a $\mathbb{Z}$-graded k-module.

<u>Lemma</u> There is a bijection between degree 1 maps

$$b : M(1)\otimes TA(1) \longrightarrow M(1)\otimes TA(1)$$

satisfying the condition

$$\rho(b + 1\otimes \underset{\uparrow\ \text{from } A}{b}) = (b\otimes 1 + 1\otimes \overset{\text{from } A}{b}\otimes 1 + 1\otimes 1\otimes b)\rho \qquad (11.1)$$

and sequences of degree 1-maps

$$b_n : M(1)\otimes A(1)^{\otimes n-1} \longrightarrow M(1) \qquad n\geq 1$$

<u>Proof</u> To a map b we associate

$$\begin{tikzcd} M(1)\otimes A(1)^{\otimes n-1} \arrow[r, hook] & M(1)\otimes TA(1) \arrow[d, "b"] & \\ & M(1)\otimes TA(1) \arrow[r, two heads] & M(1) \end{tikzcd}$$

and to $(b_n)_{n\geq 1}$ we associate

$$b = \sum_{\substack{s\geq 1 \\ t\geq 0}} b_s \otimes \mathbb{1}^{\otimes t} \qquad (11.2)$$

First we prove that, given $(b_n)_{n\geq 1}$, the b defined in this way satisfies (11.1). We have agreement trivially on $M(1)\otimes k$, and for $v_1\otimes\cdots\otimes v_m \in A(1)^{\otimes m}$, $m\geq 1$

$$LHS = \rho(b+1\otimes b)(x\otimes v) = \rho\sum(\mathbb{1}^{\otimes r}\otimes b_s\otimes \mathbb{1}^{\otimes t})(x\otimes v)$$

$$= \rho\sum(b_s\otimes\mathbb{1}^{\otimes t})(x\otimes v) + \rho\sum(\mathbb{1}^{\otimes r>0}\otimes b_s\otimes\mathbb{1}^{\otimes t})(x\otimes v)$$

using p.7

$$= \sum(b_s\otimes\mathbb{1}^{\otimes t}\otimes\mathbb{1})\rho(x\otimes v) + (-1)^{|x|}\rho\left(x\otimes\sum(\mathbb{1}^{\otimes r}\otimes b_s\otimes\mathbb{1}^{\otimes t})(v)\right)$$

$$= \sum(b_s\otimes\mathbb{1}^{\otimes t}\otimes\mathbb{1})\rho(x\otimes v) + (-1)^{|x|}x\otimes\tilde{\Delta}b(v)$$

b on $TA(1)$

b is a coderivation on $TA(1)$

$$= \sum(b_s\otimes\mathbb{1}^{\otimes t}\otimes\mathbb{1})(x\otimes\tilde{\Delta}v) + (-1)^{|x|}x\otimes(b\otimes 1+1\otimes b)\tilde{\Delta}v$$

$$= \sum(b_s\otimes\mathbb{1}^{\otimes t}\otimes\mathbb{1})(x\otimes\tilde{\Delta}v) + (-1)^{|x|}x\otimes(b\otimes 1+1\otimes b)\tilde{\Delta}v$$

on $TA(1)$

$$= (b\otimes 1)(x\otimes\tilde{\Delta}v) + (1\otimes b)(x\otimes\tilde{\Delta}v)$$

$$= RHS$$

Now it only remains to show that if we begin with b satisfying (11.1) and extract its components b_n, then b can be recovered as (11.2).

We again call "exterior degree" the decomposition

$$M(1)\otimes TA(1) = \overset{0}{M(1)} \oplus \overset{1}{M(1)\otimes A(1)} \oplus \overset{2}{M(1)\otimes A(1)^{\otimes 2}} \oplus \cdots$$

We prove by induction that b only lowers ext-degree (i.e. sends ext-degree n to $\leq n$) and satisfies (11.2).

$\boxed{\text{ext-degree zero}}$

$$\begin{aligned} \rho b(x\otimes 1) &= (b\otimes 1 + 1\otimes b \overset{+1\otimes 1\otimes b}{\cancel{\otimes}})\rho(x\otimes 1) \\ &= (b\otimes 1 \overset{+1\otimes 1\otimes b}{+} 1\otimes b\cancel{\otimes})(x\otimes 1\otimes 1) \\ &= (b\otimes 1)(x\otimes 1\otimes 1) = b(x\otimes 1)\otimes 1 = b_1(x\otimes 1)\otimes 1 \end{aligned}$$

So by the same argument as p. 9, $b(x\otimes 1)\in M(1)$ and (11.2) holds.

$\boxed{\text{Inductive step}}$ Suppose the claim holds for all $x\otimes v\in M(1)\otimes A(1)^{\otimes n-2}$. Then for $v_{n-1}\in A(1)$

$$(1\otimes b + 1\otimes b\otimes 1 + 1\otimes 1\otimes b)\rho(x\otimes(v\otimes v_{n-1})) = \rho(b+1\otimes b)(x\otimes(v\otimes v_{n-1})) \qquad (13.2)$$

But if we project $M(1)\otimes TA(1)\otimes TA(1) \overset{\pi}{\twoheadrightarrow} M(1)\otimes A(1)^{\otimes m}\otimes k$ the RHS is

$$(b+1\otimes b)(x\otimes(v\otimes v_{n-1}))_m \otimes (1)$$

while the LHS is

$$\pi\Big(b\otimes 1 \quad \ldots\Big)\big(x\otimes(1)\otimes(v\otimes v_{n-1})$$
$$+1\otimes b\otimes 1 \qquad + x\otimes(v\otimes v_{n-1})\otimes(1)$$
$$+1\otimes 1\otimes b \qquad + x\otimes(v_1)\otimes(v_2\otimes\cdots\otimes v_{n-2}\otimes v_{n-1})$$
$$+ \vdots$$
$$+ x\otimes(v_1\otimes\cdots\otimes v_{n-2})\otimes(v_{n-1})\big)$$

which vanishes for $m>n$, proving the first claim. Also

$$= \pi\Big[\, b(x\otimes 1)\otimes v\otimes v_{n-1} \qquad + \pi(1\otimes b\otimes 1)\big(x\otimes(v\otimes v_{n-1})\otimes(1)\big)$$
$$+ b(x\otimes v\otimes v_{n-1})\otimes(1)$$
$$+ b(x\otimes v_1)\otimes v_2\otimes\cdots$$
$$+\cdots+ b(x\otimes v_1\otimes\cdots\otimes v_{n-2})\otimes v_{n-1}\Big]$$

Now by the inductive hypothesis this equals $\sum \mathbb{1}^{\otimes r}\otimes b_s\otimes \mathbb{1}^{\otimes t}$ applied to $x\otimes v\otimes v_{n-1}$ as claimed, or at least the $M(1)\otimes A(1)^{\otimes m}$ component of said. Thus the two sides of (11.2) agree, or rather, we have used (13.2) to deduce that

$$b + 1\otimes b = \sum\big(b_s\otimes \mathbb{1}^{\otimes t}\big) + 1\otimes b$$

Hence $b = \sum b_s\otimes \mathbb{1}^{\otimes t}$ as claimed. $\square$

Here is an alternative way to state the Lemma: there is a bijection between degree 1 maps

$$b_M : M(1) \otimes TA(1) \longrightarrow M(1) \otimes TA(1)$$

satisfying

$$\rho b_M = (b_M \otimes 1 + 1 \otimes 1 \otimes b)\rho \qquad (15.1)$$

and sequences of degree 1-maps

$$b_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow M(1)$$

given by sending b_M to the components (15.2)

$$M(1) \otimes A(1)^{\otimes n-1} \hookrightarrow M(1) \otimes TA(1) \xrightarrow{b_M} M(1) \otimes TA(1) \xrightarrow{\pi} M(1)$$

and associating to $(b_n)_{n \geq 1}$ the map

$$b_M = \sum_{\substack{s \geq 1 \\ t \geq 0}} b_s \otimes \mathbb{1}^{\otimes t} + 1 \otimes b$$

↑ from A.

$$= \sum \mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}$$

(The relation being that b on p. 11 is $b_M - 1 \otimes b$). The point is that the projection π in (15.2) kills $1 \otimes b$.

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<u>Lemma</u> With the same notation, $b_M^2 = 0$ if and only if the sequence $(b_n)_{n \geq 1}$ satisfies (2.2), i.e. defines an A_∞-module structure.

<u>Proof</u> The map (15.1) is a coderivation, so

$$\begin{aligned} \rho b_M^2 &= (b_M \otimes 1 + 1 \otimes b)\rho b_M \\ &= (b_M \otimes 1 + 1 \otimes b)(b_M \otimes 1 + 1 \otimes b)\rho \\ &= (b_M^2 \otimes 1 + 1 \otimes b^2)\rho \\ &= (b_M^2 \otimes 1)\rho \end{aligned}$$

i.e. b_M^2 is a morphism of $TA(1)$-comodules. It has degree two, but p. 7 applies to show that $b_M^2 = 0$ iff. the components

$$\begin{array}{ccccc} M(1) \otimes A(1)^{\otimes n-1} & \hookrightarrow M(1) \otimes TA(1) & \xrightarrow{b_M^2} & M(1) \otimes TA(1) \\ & & & \downarrow \\ & & & M(1) \end{array}$$

all vanish. But these components are exactly the things in (2.2). $\square$

<u>Upshot</u> A series of maps $b_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow M(1)$ defining an A_∞-module is equivalent to the specification of a dg-comodule structure on $M(1) \otimes TA(1)$ over $TA(1)$, <u>plus</u> the condition (2.3).

i.e. A_∞-modules $=$ dg-$TA(1)$-comodules

Morphisms of modules

We also have three ways to describe morphisms of A_∞-modules. Let M, N be A_∞-modules. A morphism

$$f : M \longrightarrow N$$

is a sequence of degree zero maps

$$f_n : M(1) \otimes A(1)^{\otimes n-1} \longrightarrow N(1) \qquad n \geq 1$$

satisfying, as maps $M(1) \otimes A(1)^{\otimes n-1} \longrightarrow N(1)$

$$\sum f_u(\mathbb{1}^{\otimes r} \otimes b_s \otimes \mathbb{1}^{\otimes t}) = \sum b_u(f_r \otimes \mathbb{1}^{\otimes s}) \qquad (9.1)$$

where the left sum is taken over all decompositions $n = r+s+t$, $s \geq 1$, $t \geq 0$ and we put $u = r+1+t$ and the right hand sum is taken over all $n = r+s$, $r \geq 1$, $s \geq 0$ and $u = 1+s$.

If we rewrite this in terms of the maps m_n defining the A_∞-structure on M, N and maps g_n defined by the diagram

$$\begin{tikzcd} M(1) \otimes A(1)^{\otimes n-1} \arrow[r, "f_n"] & N(1) \\ M \otimes A^{\otimes n-1} \arrow[u, "s^{\otimes n}"] \arrow[r, "g_n"'] & N \arrow[u, "s"'] \end{tikzcd}$$

Then (9.1) reads (we checked the signs) $\qquad (9.2)$

$$\sum (-1)^{r+st} g_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = \sum (-1)^{(r+1)s} m_u(g_r \otimes \mathbb{1}^{\otimes s}).$$

The first relation $n=1$ says

$$f_1 b_1 = b_1 f_1 \qquad \text{or} \qquad g_1 m_1 = m_1 g_1 \tag{10.1}$$

i.e. f is a morphism of complexes. The second $n=2$ says, as maps $M(1) \otimes A(1) \longrightarrow N(1)$ that

$$f_2(b_1 \otimes 1) + f_1(b_2) = b_2(f_1 \otimes 1) + b_1(f_2) \tag{10.2}$$

Now $b_2 : M(1) \otimes A(1) \longrightarrow M(1)$ gives the action. If we write $b_2(x \otimes a)$ as $x \cdot a$ then (10.2) says

$$f_2(b_1(x) \otimes a) - b_1 f_2(x \otimes a)$$
$$= f_1(x) \cdot a - f_1(x \cdot a)$$

i.e. f_1 does not commute with the action of A, but it does <u>up to homotopy</u> given by f_2.

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According to p.(15) the A_∞-modules M, N determine dg-comodules $M(1)\otimes TA(1)$, $N(1)\otimes TA(1)$ over $TA(1)$ with codifferentials b_M, b_N.

<u>Lemma</u> The bijection of p.(7) between degree zero morphisms

$$f : M(1)\otimes TA(1) \longrightarrow N(1)\otimes TA(1)$$

of $TA(1)$-comodules and sequences

$$f_n : M(1)\otimes A(1)^{\otimes n-1} \longrightarrow N(1)$$

identifies maps f satisfying $b_N f = f b_M$ with sequences (f_n) satisfying (9.1), i.e. morphisms of A_∞-modules.

<u>Proof</u> We have,

$$\rho b_M = (b_M\otimes 1 + 1\otimes b)\rho$$
$$\rho b_N = (b_N\otimes 1 + 1\otimes b)\rho$$

So

$$\begin{aligned}\rho b_N f &= (b_N\otimes 1 + 1\otimes b)\rho f\\ &= (b_N\otimes 1 + 1\otimes b)(f\otimes 1)\rho\\ &= (b_N f\otimes 1 + f\otimes b)\rho\end{aligned}$$

$$\begin{aligned}\rho f b_M &= (f\otimes 1)\rho b_M\\ &= (f\otimes 1)(b_M\otimes 1 + 1\otimes b)\rho\\ &= (f b_M\otimes 1 + f\otimes b)\rho\end{aligned}$$

So

$$\rho(b_N f - f b_M) = \{(b_N f - f b_M)\otimes 1\}\rho$$

i.e. $b_N f - f b_M$ is a morphism of comodules.

Hence $b_N f - f b_M$ vanishes precisely when its components do, by p. 7. These components

$$\begin{array}{ccc} M(1)\otimes A(1)^{\otimes n-1} & \hookrightarrow & M(1)\otimes TA(1) \\ & & \downarrow {\scriptstyle b_N f - f b_M} \\ & & N(1)\otimes TA(1) \twoheadrightarrow N(1) \end{array}$$

are precisely the equations (9.1) $\square$

That is

Morphisms of A_∞-modules = morphisms of dg $TA(1)$-comodules.

Obviously dg $TA(1)$-comodules form a category.

Def^N Given morphisms $f: M \to N$ and $g: N \to L$ of A_∞-modules the <u>composite</u> $g\circ f$ is the morphism corresponding to the composite $M(1)\otimes TA(1) \to N(1)\otimes TA(1) \to L(1)\otimes TA(1)$ of morphisms of dg $TA(1)$-comodules. Thus

$$(g\circ f)_n = \sum g_u (f_s \otimes \mathbb{1}^{\otimes t})$$

where the sum is over $s+t=n$, $s\geq 1, t\geq 0$ and we set $u = 1+t$.

The identity $1_M: M \to M$ corresponds to $1_{M(1)\otimes TA(1)}$, i.e.

$$(1_M)_n : M(1)\otimes A(1)^{\otimes n-1} \to M(1)$$

is $1_{M(1)}$ for $n=1$ and zero otherwise.

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With these definitions the A_∞-modules form a category A-Mod, and there is a fully faithful functor

$$A\text{-Mod} \longrightarrow TA(1)\text{-dg-comod}$$

$$(M, b_n) \longmapsto (M(1) \otimes TA(1), \rho, b_M)$$

<u>Role of the homological unit</u> we have not used the unit, so everything said so far will still hold if we drop this from the definition of a module.