Intro to A_∞-algebras (talk)

introainf

10/11/15

"If I would only understand the beautiful consequences following from the concise proposition $d^2=0$" Henri Cartan.

QUESTION Let (M,d) be a complex. Can we recover (M,d) from H^*M? Clearly the answer is No, if we want to recover (M,d) exactly.

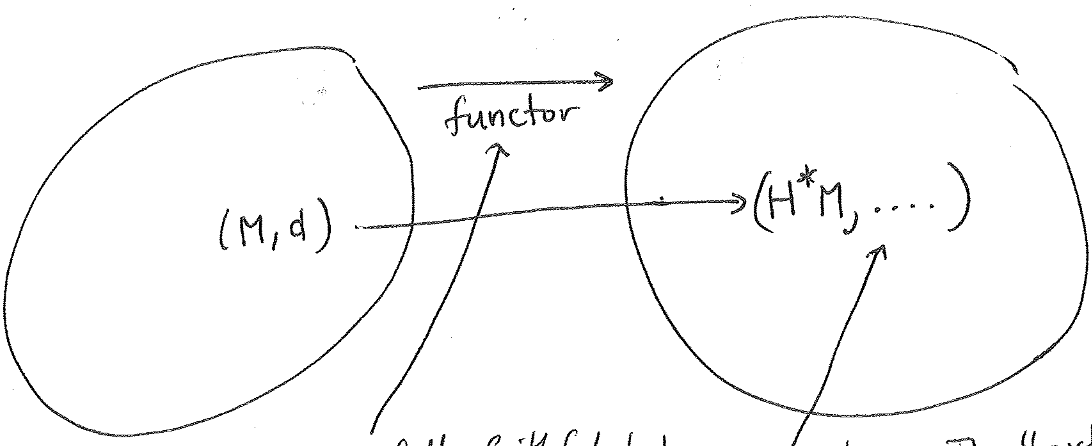
$$\begin{tikzcd} (M,d) \arrow[r] & H^*M \arrow[l, dashed, bend left, "?"] \end{tikzcd}$$

Example. In $\mathbb{D}(k)$, k a field $(M,d) \cong \bigoplus_n H^n(M)[n]$

- Also true in $\mathbb{D}(R)$, gl.dim. $R \leq 1$ (e.g. $\mathbb{Z}$, $k[x]$).

  But not in general.

The problem is that (M,d) has more info than $\bigoplus_n H^n(M)[n]$ (this is the point of the derived category, in fact)



to make this fully faithful take A_∞ here. The "extra info".

Motivation

(∞.dim.) | (f.d.)

(1) $C^*(X;k) \longrightarrow H^*(X;k)$

singular cochain cpx. | + Massey products.

(2) Triangulated cat $\mathcal{T}$ w/ generator G $\longrightarrow$ $\mathrm{End}^*_{\mathcal{T}}(G)$

+ A_∞-higher products.

e.g. $\mathcal{T} = \mathbb{D}^b(\mathrm{coh}\, X)$

$H^*(\text{state space}, \partial_{BRST})$

(3) Topological field theory (open) $\longleftrightarrow$ Calabi-Yau triangulated categories

$\mathbb{D}^b(\mathrm{coh}X)$, $\mathrm{hmf}(W)$, $\mathrm{Fuk}(Y)$

Topological string theory (open) $\longleftrightarrow$ Calabi-Yau A_∞-categories

Gaberdiel–Zwiebach '97
Herbst–Lazaroiu–Lerche '04
Costello '07

(4) Mirror symmetry a la Kontsevich.

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<u>Outline</u>
- A_∞-algebras, A_∞-modules
- Minimal model theorems
- Examples.

k is a field.

<u>Def^N</u> (Stasheff) An A_∞-algebra is a $\mathbb{Z}$-graded v. space

$$A = \bigoplus_{n \in \mathbb{Z}} A^n$$

with <u>operations</u> $m_n : A^{\otimes n} \longrightarrow A$ $\quad n \geq 1$, k-linear, degree $2-n$.

$$m_1 : A \to A \quad \text{degree } +1$$
$$m_2 : A^{\otimes 2} \to A \quad \text{degree } 0 \qquad a \cdot b := m_2(a \otimes b)$$
$$m_3 : A^{\otimes 3} \to A \quad \text{degree } -1.$$
$$\vdots$$

satisfying some equations; one for each $n \geq 1$

(n=1) $\bullet$ $m_1^2 = 0$

(n=2) $\bullet$ $m_1 m_2 = m_2(m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$

$$m_1(a \cdot b) = m_2(m_1(a) \otimes b + (-1)^{|a|} a \otimes m_1(b))$$
$$= m_1(a) \cdot b + (-1)^{|a|} a \cdot m_1(b)$$

$\Rightarrow$ m_1 is a (graded) derivation

(n=3) $\bullet$ $m_2(\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes \mathbb{1}$
$+ \mathbb{1} \otimes \mathbb{1} \otimes m_1)$

of maps $A^{\otimes 3} \longrightarrow A$

a, b, c cycles $\Rightarrow$ $\quad a \cdot (b \cdot c) - (a \cdot b) \cdot c = \underbrace{m_1 m_3(a \otimes b \otimes c)}_{\text{boundary.}}$

...

More generally, for $n \geq 1$

$$\begin{tikzcd} A^{\otimes n} \arrow[rr, dashed] \arrow[d, phantom, "\wr\!\!\wr"] & & A \\ A^{\otimes r} \otimes A^{\otimes s} \otimes A^{\otimes t} \arrow[r, "\mathbb{1} \otimes m_s \otimes \mathbb{1}"'] & A^{\otimes r} \otimes A \otimes A^{\otimes t} \arrow[ur, "m_{r+1+t}"'] & \end{tikzcd}$$

$r, t \geq 0$
$s \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0.$$

<u>Lemma</u> $(H^*(A), m_2)$ is an associative algebra.

<u>Example</u>
- If $m_n = 0$ for $n \geq 3$ (A, m_1, m_2) is a DG-algebra.
- Singular cochain cpx $(C^*(X; k), \partial, \cup)$ is DG-alg.

  $(H^*(X; k), \cup)$ assoc. alg.

- $\mathbb{Z}_2$-graded $A = A_0 \oplus A_1$, same as $\mathbb{Z}$-graded.
- Say $(A, \{m_n\})$ is <u>minimal</u> if $m_1 = 0$.
- $A^{(d)} = k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$ $\quad |\varepsilon| = 1$. <u>Choose $d > 2$</u>

$m_n = 0$ unless $n = 2, d$

$m_d : A^{\otimes d} \longrightarrow A$

$m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = (-1)^{d-1} \cdot 1.$

<u>Lemma</u> $(k[\varepsilon]/\varepsilon^2, m_2, m_d)$ is a $\mathbb{Z}_2$-graded A_∞-algebra.

<u>Proof</u>

$$\xrightarrow{1\otimes m_d \otimes 1} \xrightarrow{m_d} \qquad A^{\otimes 2d-1} \longrightarrow A \qquad \text{(I)}$$

$$\left.\begin{array}{ll} \xrightarrow{1\otimes m_2 \otimes 1} \xrightarrow{m_d} & A^{\otimes d+1} \longrightarrow A \\ \xrightarrow{1\otimes m_d \otimes 1} \xrightarrow{m_2} & A^{\otimes d+1} \longrightarrow A \end{array}\right\} \qquad \text{(II)}$$

$$\xrightarrow{1\otimes m_2 \otimes 1} \xrightarrow{m_2} \qquad A^{\otimes 3} \longrightarrow A \qquad \text{(III)}$$

(III) $\longleftrightarrow$ A is assoc. alg. ✓

(I)

$$\underbrace{\underbrace{a_1 \otimes \cdots \otimes a_d}_{m_d} \otimes \overbrace{a_{d+1} \otimes \cdots \otimes a_{2d-1}}^{d-1}}_{m_d}$$

$$m_d(m_d(a_1\otimes \cdots \otimes a_d) \otimes a_{d+1} \otimes \cdots)$$
$$+ (-1)^{d-1+d|a_1|} m_d(a_1 \otimes m_d(a_2 \otimes \cdots) \otimes \cdots)$$
$$+ \cdots + (-1)^{(d-1)(d-1) + d(|a_1| + \cdots + |a_{d-1}|)} m_d(a_1 \otimes \cdots \otimes a_{d-1} \otimes m_d(\cdots))$$
$$= 0$$

(II)

$$\underbrace{\underbrace{a_1 \otimes a_2}_{m_2} \otimes \cdots \otimes a_d \otimes a_{d+1}}_{m_d}$$

$$(-1)^d m_2(m_d(a_1 \otimes \cdots \otimes a_d) \otimes a_{d+1}) \overset{(-1)^{d|a_1|}}{-} m_2(a_1 \otimes m_d(a_2 \otimes \cdots \otimes a_{d+1}))$$
$$+ m_d(m_2(a_0 \otimes a_1) \otimes \cdots) - m_d(a_1 \otimes m_2(a_2 \otimes a_3) \otimes \cdots)$$
$$+ \cdots + (-1)^{d-1} m_d(a_1 \otimes \cdots \otimes m_2(a_d \otimes a_{d+1})) = 0.$$

On $\varepsilon \otimes \cdots \otimes \varepsilon$

$$(-1)^d m_2\left((-1)^{d-1} 1 \otimes \varepsilon \right) - (-1)^d m_2\left(\varepsilon \otimes (-1)^{d-1} 1 \right) = 0 \quad \checkmark$$

On $1 \otimes \varepsilon \otimes \cdots \otimes \varepsilon$

$$-(-1)^0 m_2\left(1 \otimes (-1)^{d-1} 1\right) + m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = 0 \quad \checkmark$$

on $\varepsilon \otimes 1 \otimes \varepsilon \otimes \cdots \otimes \varepsilon$ ✓

$\vdots \qquad \vdots \qquad$ $\# 1$'s > 1 all zero.

$\varepsilon \otimes \cdots \otimes \varepsilon \otimes 1$ ✓

$\square$.

<u>Def^N</u> An <u>A_∞-module</u> over an A_∞-algebra $(A, \{m_n\}_{n \geq 1})$ is a $\mathbb{Z}$-graded k-module M with operations

$$m_n : M \otimes A^{\otimes(n-1)} \longrightarrow M \qquad \text{degree } 2-n$$

satisfying the same A_∞-constraints, e.g.

- $m_1^2 = 0 \qquad (M, m_1)$ is a cpx
- $m_1 m_2 = m_2(m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$

  $m_1(m \cdot a) = m_1(m) \cdot a + (-1)^{|m|} m \cdot m_1(a)$
- $m_2 : M \otimes A \rightarrow M$ makes M an A-module up to htpy $m_3, \ldots$

<u>Example</u>
- Modules over an algebra
- DG-modules over a DG-algebra
- H^*M is a H^*A-module.

<u>Proposition</u> Each (resp. minimal) homologically unital A_∞-algebra is A_∞-quasi-isomorphic (resp. A_∞-isomorphic) to a strictly unital A_∞-algebra.

<u>Def^N</u> $D_\infty A := \left(\text{category of } A_\infty\text{-modules}\right)[\text{quasi-isos}^{-1}]$

(triangulated)

$$\operatorname{per} A = \underset{\text{shift, extensions, summands}}{\langle A \rangle} \subseteq D_\infty(A)$$

<u>Proposition</u> Let A be an assoc. alg.

$$\mathbb{D}(\operatorname{Mod} A) \xrightarrow{\cong} D_\infty A.$$

$$(M, m_1) \longmapsto (M, m_1)$$

$$\text{ll2 } A_\infty \text{ quasi-iso}$$

$$(H^* M, 0, 0, \underbrace{m_3, m_4, \dots}_{\text{extra info.}})$$

This is the sense in which (M, m_1) can be "recovered" from its cohomology.

Ho(stable Quillen model cat)

<u>Theorem</u> (Keller, Lefevre-Hasegawa) Let $\mathcal{T}$ be a k-linear algebraic triangulated category w/ split idempotents and generator G.

$$A := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(G, G[n])$$

has an A_∞-structure with $m_1 = 0$, s.t.

$$\mathcal{T} \xrightarrow{\cong} \operatorname{per} A$$

$$U \longmapsto \bigoplus_n \operatorname{Hom}_{\mathcal{T}}(G, U[n])$$

<u>Example</u> Recall $(A^{(d)} = k[\varepsilon]/\varepsilon^2, m_2, m_d)$ $d > 2$ from before.

$m_d(\varepsilon \otimes \cdots \otimes \varepsilon) = (-1)^{d-1}$.

$$\mathrm{hmf}(k[x], x^d) \xrightarrow{\simeq} \mathrm{per}\, A^{(d)}$$

pair (A, B) of sq. matrices

$AB = BA = x^d \cdot I$.

(see Dyckerhoff-Kapranov, Nadler)

Q/ what is A_∞-module over $A^{(d)}$?

$\boxed{d=3}$ $\mathbb{Z}_2$-graded vector space M. Say $m_1 = 0$, m_2, m_d nonzero.

$$\left[\begin{array}{l} m_2 : M \otimes A \longrightarrow M \\ \partial := m_2(- \otimes \varepsilon) : M \longrightarrow M \text{ odd} \quad \boxed{\partial^2 = 0} \end{array}\right.$$

$$\left[\begin{array}{l} m_3 : M \otimes A \otimes A \longrightarrow M \quad \text{odd} \\ h := m_3(- \otimes \varepsilon \otimes \varepsilon) : M \longrightarrow M \text{ odd} \end{array}\right.$$

A_∞-constraints $\Rightarrow$

$$m_3(- \otimes 1 \otimes 1) = h^3$$
$$m_3(- \otimes 1 \otimes \varepsilon) = \partial h^3$$
$$m_3(- \otimes \varepsilon \otimes 1) = h^3 \partial$$

$h\partial - \partial h = a$ $\longleftarrow$ $a(m) = (-1)^{|m|} m$

$\therefore (M, \partial)$ is acyclic.

$h^4 = 0$

$\ulcorner (ah)\partial + \partial(ah)$

$(ah)^4 = 0 \lrcorner$