# The Zariski Site

## 1. Motivation

We know that geometry over a commutative ring $k$ is embodied, for a system of equations $\mathfrak{J} \subseteq k[(x_i)_{i\in I}]$, by the varieties $V_{\mathfrak{J}}(k')$ in each $k$-algebra $k'$, and that this category of varieties is opposite to the category **k-alg** (identifying $V_A$ with $A$). We think of the variety $V_A$ as being made manifest by all the varieties $V_A(k')$. In the classical case where $k$ is an algebraically closed field, we can ignore all the $k$-algebras except for $k$ itself (see our Geometry notes), so that in this case the variety *is* the subset $V_A(k)$ of $k^I$. We traditionally put a topology on this variety, called the *Zariski topology*, by defining a subset of $V_A(k) \subseteq k^I$ to be *closed* if it is the locus in $V_A(k)$ of some ideal of $A$. This topology has a basis consisting of the open sets

$$D(f) = \{t \in V_A(k) \mid t(f) \neq 0\} \quad \forall f \in A$$

Recall that in the classical case the Nullstellensatz tells us that the points of $V_A(k)$ are in one-to-one correspondence with the maximal ideals of $A$, that is, the maximal ideals of $k[(x_i)]$ containing $\mathfrak{J}$. Every point $t : A \longrightarrow k$ of $V_A(k)$ is uniquely determined by its kernel $\mathfrak{m}_t = t^{-1}(0)$, so that the topology above has an equivalent definition in terms of maximal ideals, where $D(f)$ is identified with the set of all maximal ideals $\mathfrak{m}$ to which $f$ does not belong.

Can we generalise the Zariski topology to the general case, where our collection of $k$-algebras must be enlarged from the single object $k$ to the entire category? Here, our varieties take their values in many algebras. Can we define the analogue of Zariski topologies on these algebras? Let $k'$ be an arbitrary $k$-algebra. The points of the variety in $k'$ are the morphisms of $k$-algebras $t : A \longrightarrow k'$. Each such point determines an ideal $t^{-1}(0)$ of $A$, which we will denote $I(t)$ (notice that, generally, we may have $I(t) = I(t')$ with $t \neq t'$). A naive generalisation of the Zariski topology would lead us to define, for each ideal $I$ of $A$, subsets

$$V(I) = \{t \in V_A(k') \mid f(t) = 0 \text{ for all } f \in I\} = \{t \mid I \subseteq I(t)\}$$
$$D(f) = \{t \in V_A(k') \mid f(t) \neq 0\} = \{t \mid f \notin I(t)\}$$

But these sets do not form a topology - the problem is that while $V(I) \cup V(J) \subseteq V(I \cap J)$, the reverse inclusion may not hold. In the classical case, where all the $I(t)$ are prime, $I \cap J \subseteq I(t)$ would imply that either $I$ or $J$ were in $I(t)$, as required. Hence, provided that the algebra $k'$ is a domain, we can define the normal Zariski topology on $V_A(k')$. In particular, we can consider all the geometric points (points of $V_A$ in fields) to exist in topologies of this form.

In the classical case, we know that a variety $V_A(k)$ is covered by open sets $D(f_j)$ iff. the $f_j$ generate the ideal (1) of $A$. Generally, $V_A(k') = \cup_j D(f_j)$ iff. $(f_j)$ is not contained in $I(t)$ for any point $t : A \longrightarrow k'$ of $V_A(k')$. This would be equivalent to the $f_j$ generating (1) if we knew that the $I(t)$ included all the maximal ideals (which is true, classically). Now consider $D_A^f$ as a subfunctor of $V_A$ which picks out the points of $V_A(k')$ at which $f$ is nonzero. Generalising the classical case, we should say that the $D_A^{f_i}$ covered $V_A$ if $(f_i)$ generated the unit ideal of $A$, which is iff. $(f_i)$ is contained in no $I(t) = t^{-1}(0)$, for any *geometric* point $t : A \longrightarrow k'$, which is iff. the $D_A^{f_i}$ cover each geometric point (in the sense that if $t : A \longrightarrow k'$ is such a point, then $t \in D_A^{f_j}(k')$ for some $j$). Let us remark upon this idea, noting that a finite number of the $f_j$ will always suffice to generate (1), and hence to cover $V_A$:

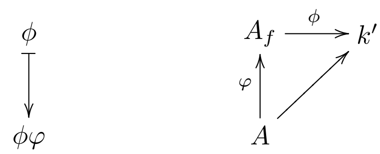
Remark 1 (First definition of a cover). A *cover* of a $k$-algebra $A$ (equivalently, a variety $V_A$) is a finite collection $f_1, \ldots, f_n$ such that $(f_1, \ldots, f_n) = (1)$, or, equivalently, such that the collection of points $D_A^{f_i}(k')$ contains every geometric point of $V_A$.

Before we go any further, let us review the current setup: we have a fixed base ring $k$, the category **k-alg** of $k$-algebras, the category $\mathbf{Var}(k)$ of varieties over $k$, realised as the full subcategory of all representable covariant functors $\mathbf{k}-\mathbf{alg} \longrightarrow \mathbf{Sets}$, which is dual to **k-alg** . This last category can also be identified with the category of all affine schemes over $Spec(k)$, by the following:

$$A = V_A = Spec(A)$$

We saw above that the obvious generalisation of the Zarsiki topology doesn't work for varieties with points in $k$-algebras that are not fields. To realise what the correct generalisation is, consider the following. Let $A$ be a $k$-algebra, with $f \in A$. Consider the morphism $\varphi : A \longrightarrow A_f$ of $k$-algebras. It induces a morphism of varieties

$$V_\varphi : V_{A_f} \longrightarrow V_A$$
$$V_\varphi(k') : Hom_{\mathbf{k}-\mathbf{alg}}(A_f, k') \longrightarrow Hom_{\mathbf{k}-\mathbf{alg}}(A, k')$$



So far the discussion has been free of any particular affine immersion. Let $V_A \longrightarrow E^I$ be such an immersion, so that we identify $A$ with $k[(x_i)]/\mathfrak{J}$, and hence $A_f$ with the $k$-algebra

$$A_f = \frac{A[x]}{(fx-1)} = \frac{\left(\frac{k[(x_i)]}{\mathfrak{J}}\right)[x]}{(fx-1)}$$
$$= \frac{k[(x_i) \cup x]}{(\mathfrak{J}, fx-1)}$$

where the isomorphism is defined by

$$\frac{g+\mathfrak{J}}{(f+\mathfrak{J})^n} \mapsto (g+\mathfrak{J})x^n + (fx-1)$$
$$\mapsto gx^n + (\mathfrak{J}, fx-1)$$

Notice that since $A \longrightarrow A_f$ is an epimorphism, $V_{A_f} \longrightarrow V_A$ is monic. In terms of this immersion, a $k'$-point of $V_{A_f}$ is an $I \cup x$ tuple

$$((t_i)_{i \in I}, t) \qquad t_i, t \in k'$$

satisfying the relations of $\mathfrak{J}$ on the first $I$ positions (together with polynomials in $x$ with these relations as coefficients), together with the relation

$$f\left((t_i)\right) t = 1 \tag{1}$$

The morphism $V_\varphi$ projects this tuple onto its first $I$ coordinates - that is, $(t_i)_{i \in I}$. That this is pointwise monic is obvious, because the tuple $(t_i)$ completely determines $t$ via (1). (of course, it is not necessarily onto since $f((t_i))$ may not always be invertible). Hnece the image of $V_\varphi(k')$ is the set of all tuples $(t_i)_{i\in I}$, $t_i \in k'$, for which $f((t_i))$ is a unit in $k'$. Alternatively, notice that morphisms $A_f \longrightarrow k'$ are in one-to-one correspondence with morphisms $A \longrightarrow k'$ that take $f$ to a unit, which are precisely those tuples $(t_i)$ of $k'^I$ for which $f((t_i))$ is a unit.

This tells us what the correct generalisation of the Zariski topology is. For a polynomial $f \in A$, ideal $I \subseteq A$ and $k$-algebra $k'$, we should define

$$D_A^f(k') = \{t \in V_A(k') \,|\, f(t) \text{ is a unit }\}$$
$$D_A^I(k') = \{t \in V_A(k') \,|\, \text{ some } f \in I, f(t) \text{ is a unit }\}$$

Now notice that for $a, b \in k'$, $ab$ is a unit iff. both $a$ and $b$ are units. Hence $D_A^f(k') \cap D_A^g(k') = D_A^{fg}(k')$ so that the $D_A^f(k')$ form the basis for a topology. We also have $D_A^I(k') = \cup_{f\in I} D_A^f(k')$, etc. If $k'$ is a field, then $f(t) \neq 0$ is exactly the same as requiring $f(t)$ to be a unit, so that in the classical case we just picked the wrong condition (we really picked $f(t) \neq 0$ so that functions $g/f$ would be defined at $t$, so it should be obvious that requiring $f(t)$ to be a *unit* is the correct condition, generally). Hence $D_A^f$ is the subfunctor of $V_A$ with which $V_{A_f}$ is canonically identified.

Notice that a $k'$-point $t : A \longrightarrow k'$ of $V_A$ is contained in $D_A^f$ iff. $t$ factors through $A \longrightarrow A_f$. So that we might say a collection of subobjects $D_A^{f_j}$, $f_j \in A$, "covered" the variety $V_A$ if every $k'$-point $t$, for every $k$-algebra $k'$, factored through some $A \longrightarrow A_{f_j}$. That is, iff.

$$\bigcup_j D_A^{f_j}(k') = V_A(k') \qquad \forall k'$$

However, note once again:

PROPOSITION 1. *A collection $\{f_j\}_{j\in J}$ of elements of $A$ generates the unit ideal if and only if every geometric point factors through some $A \longrightarrow A_{f_j}$*

PROOF. Suppose that $(f_j) = A$, and that $t : A \longrightarrow k'$ is a geometric point. Since $t(1) = 1$, $t$ cannot take all of $A$ to 0, and hence cannot take every $f_j$ to 0. That means that some $t(f_j)$ is nonzero, and consequently is a unit, since $k'$ is a field. Conversely, suppose that geometric points satisfied the condition but that $(f_j)$ were proper. Let $\mathfrak{m}$ be a maximal ideal containing $(f_j)$, so that $t : A \longrightarrow A/\mathfrak{m}$ is a geometric point which cannot factor through any $A \longrightarrow A_{f_j}$. □

That is, the $f_j$ generate the unit ideal iff. the $D_A^{f_j}$ contain every geometric point. Notice that for geometric points the two Zariski topologies (defined classically or with units) coincide, so that the above Proposition is none other than our earlier Remark. For further evidence that this is the correct notion of a "cover", consider the category of affine schemes over $Spec(k)$. We have the following identifications

$$\text{morphism } A \longrightarrow A_f = \text{inclusion of } D_A^f \text{ in } V_A = \text{inclusion of } Spec(A_f) \text{ in } SpecA$$

so that the morphism $A \longrightarrow A_f$ corresponds to the inclusion of the primes of $A_f$ (those primes of $A$ avoiding powers of $f$) in the spectrum of $A$. These primes correspond to the geometric points (modulo the place relation) - the primes of $A_f$ are the geometric points of $V_A$ which factor through $A_f$ (equiv. the geometric points contained in $D_A^f$) and the primes of $A$ correspond to all the geometric points. In $\mathbf{Aff}(k)$, we would say that $Spec(A_{f_j})$ covered $SpecA$ if their union (as sets) was all of $SpecA$. This leads us to our final notion of a cover:

PROPOSITION 2. *The following conditions on a family $f_1, \ldots, f_n$ of elements of a $k$-algebra $A$ are equivalent:*

- *The ideal $(f_1, \ldots, f_n)$ is all of $A$;*
- *Each geometric point of $V_A$ is contained in some $V_{A_{f_i}}$;*
- *The union of the spaces $Spec(A_f)$ is all of $Spec(A)$.*

DEFINITION 1. A *cover* of a variety $V_A$ (resp. algebra $A$, resp. affine scheme $SpecA$) is a finite family $f_1, \ldots, f_n$ of $A$ such that $(f_1, \ldots, f_n) = A$.

## 2. The Zariski Site

Before we can make use of this intuitive development, we need to prove some technical results.

LEMMA 1. *Let $(\mathcal{C}, J)$ be a site. If $\phi : A \longrightarrow B$ is an isomorphism, then the correspondences*

$$\phi : J(A) \longrightarrow J(B)$$
$$\phi^* : J(B) \longrightarrow J(A)$$

*given for a sieve $S$ on $A$ and sieve $T$ on $B$ by $\phi S = \{\phi f \mid f \in S\}$ and $\phi^* T = \{g \mid \phi g \in T\}$, respectively, define a bijection between $J(A)$ and $J(B)$.*

PROOF. By definition, if $T$ is a covering sieve so is $\phi^* T$. Suppose $S \in J(A)$. Then it is easy to see that $\phi S = (\phi^{-1})^* S$, so that $\phi S$ is a cover of $B$. Clearly $\phi^*\phi S = S$ for any $S \in J(A)$, and $\phi\phi^* T \subseteq T$. If $f : D \longrightarrow B \in T$, then $\phi^{-1} f$ is such that $\phi(\phi^{-1} f) \in T$, so $\phi^{-1} f \in \phi^* T$, and so $f \in \phi\phi^* T$. Hence $\phi\phi^* T = T$, also. □

Let $\mathcal{A}$ denote a full representative subcategory of a category $\mathcal{B}$. We wish to show that any Grothendieck topology $J$ on $\mathcal{A}$ has a unique extension to a topology $J'$ on $\mathcal{B}$. First, notice that a sieve $S$ on $A$ in $\mathcal{A}$ need not be a sieve in the category $\mathcal{B}$. Hence we first define a function $\widehat{J}$ on the objects of $\mathcal{A}$

$$\widehat{J}(A) = \{(S) \mid S \in J(A)\}$$

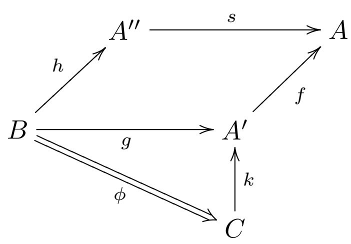
where the closure is taken in $\mathcal{B}$. We now verify that $\widehat{J}$ has the nice properties we expect. Whenever pullbacks are taken in $\mathcal{A}$, we will write $f^*_{\mathcal{A}}S$, and otherwise we assume the pullback is occurring in $\mathcal{B}$. All closures are taken in $\mathcal{B}$ (there is an unfortunate clash of notation here - "closure" has another meaning for sieves, but we will only use it to refer to the smallest sieve containing a given collection of morphisms). Finally, for a $\mathcal{B}$-sieve $R$ on $A \in \mathcal{A}$, let $R \cap \mathcal{A}$ denote all those morphisms of $R$ belonging to $\mathcal{A}$.

Lemma 2. *With the above notation, the following hold:*

(i) *If $(S) = (S')$ for $\mathcal{A}$-sieves $S, S'$ on $A$ in $\mathcal{A}$, then $S = S'$;*
(ii) *For any $\mathcal{B}$-sieve $R$ on an object $A \in \mathcal{A}$, $R = (R \cap \mathcal{A})$;*
(iii) *For $A \in \mathcal{A}$, the maximal $\mathcal{B}$-sieve on $A$ is in $\widehat{J}(A)$;*
(iv) *If $T \in \widehat{J}(A)$ and $f : A' \longrightarrow A$ is a morphism in $\mathcal{A}$, then $f^*T \in \widehat{J}(A')$;*
(v) *If $R$ is a $\mathcal{B}$-sieve on $A \in \mathcal{A}$ and $S \in J(A)$ is such that $f^*R \in \widehat{J}(D)$ for all $f : D \longrightarrow A$ in $S$, then $R \in \widehat{J}(A)$;*
(vi) *The correspondences $S \mapsto \phi S$, $T \mapsto \phi^*T$ for an isomorphism $\phi : A \longrightarrow A'$ in $\mathcal{A}$ define a bijection between $\widehat{J}(A)$ and $\widehat{J}(A')$.*

Proof. (iii) If $g : B \longrightarrow A$, let $\phi : B \longrightarrow C$, $C \in \mathcal{A}$, be an isomorphism. Then $g\phi^{-1}$ is in the maximal sieve on $A$ as defined in $\mathcal{A}$, so that $g \in (t_A)$. Hence the maximal sieve in $\mathcal{B}$ is the closure of the maximal sieve in $\mathcal{A}$.

(iv) Let $T = (S)$, $S \in J(A)$. We show that $f^*(T) = (f^*_{\mathcal{A}}S)$. Let $g : B \longrightarrow A'$ be such that $fg \in (S)$, say $fg = sh$ where $h : B \longrightarrow A''$ is a morphism of $\mathcal{B}$ and $s : A'' \longrightarrow A$ is in $S$. Let $\phi : B \longrightarrow C$ be an isomorphism of $B$ with an object $C$ of $\mathcal{A}$, as in




where everything commutes. Then $g = k\phi$, so to show $g \in (f^*_{\mathcal{A}}S)$ it would suffice to show that $fk \in S$. But $fk = fg\phi^{-1} = sh\phi^{-1} = s(h\phi^{-1}) \in S$, since $S$ is a sieve and $h\phi^{-1} \in \mathcal{A}$. The converse is easy.

(v) Notice we say $S \in J(A)$ because at this stage $\widehat{J}(D)$ for $D$ not in $\mathcal{A}$ doesn't mean anything. Suppose the conditions are satisfied. Let $f^*R = (S_f)$ for a cover $S_f \in J(D)$, and let $R' = R \cap \mathcal{A}$. Then for $f \in S$, by (*ii*) and (*iv*)

$$(S_f) = f^*R = f^*((R \cap \mathcal{A})) = (f^*_{\mathcal{A}}(R \cap \mathcal{A}))$$

Hence by (*i*), $f^*_{\mathcal{A}}(R \cap \mathcal{A}) = S_f \in J(D)$. Hence since $J$ is a topology, $R \cap \mathcal{A} \in J(A)$, which implies that $R = (R \cap \mathcal{A}) \in \widehat{J}(A)$.

□

Proposition 3. *Let $\mathcal{A}$ be a full representative subcategory of $\mathcal{B}$. Then any Grothendieck topology $J$ on $\mathcal{A}$ has a unique extension to a Grothendieck topology $J'$ on $\mathcal{B}$.*

Proof. Let $\widehat{J}$ be the function defined above. Given $B \in \mathcal{B}$, let $A \in \mathcal{A}$ be such that $A \cong B$ via $\phi : B \longrightarrow A$. We define $J'(B)$ by the following condition on sieves $S$ on $B$:

$$S \in J'(B) \quad \text{iff} \quad S = \phi^*T \text{ for some } T \in \widehat{J}(A) \tag{2}$$

We need to show that (2) is independent of the object $A$ and the isomorphism $\phi$ chosen. Suppose $\phi, \psi : B \longrightarrow A$ are both isomorphisms, and that $S$ is a sieve on $B$, with $S = \phi^*T$, $T \in \widehat{J}(A)$. Let

$$\begin{aligned} T' &= (\phi\psi^{-1})^*T \\ &= \{g \mid \phi\psi^{-1}g \in T\} \end{aligned}$$

which is clearly in $\widehat{J}(A)$. Then $\psi^*T' = \phi^*T = S$, since $\phi k \in T$ iff. $\psi k \in T'$. By symmetry $J'$ is independent of the chosen isomorphism. If $B$ is isomorphic to both $A, A'$ in $\mathcal{A}$, then $A \cong A'$, so by the lemma and the fact that $(\phi\psi)^*T = \psi^*\phi^*T$, $J'(B)$ is independent of all our choices. Notice that for $A \in \mathcal{A}$, $J'(A) = \widehat{J}(A)$.

We now check that, thus defined, $J'$ is a Grothendieck topology on $\mathcal{B}$. By Lemma 2, $(iii)$, $t_B \in J'(B)$ for all $B$, and if $S \in J'(B)$ and $f : B' \longrightarrow B$ is a morphism of $\mathcal{B}$, we pick $A, C \in \mathcal{A}$ and isomorphisms $\psi : B' \longrightarrow C, \phi : B \longrightarrow A$ and $T \in \widehat{J}(A)$ such that $S = \phi^*T$. Then let $f' : C \longrightarrow A$ be such that $\phi f = f'\psi$. Then

$$f^*(S) = f^*(\phi^*T) = (\phi f)^*T = (f'\psi)^*T = \psi^*(f'^*T)$$

and hence $f^*S \in J'(C)$. Finally, let $R$ be a sieve on $B$, $T \in J'(B)$ with $A, \phi, S$ as before. Let $\psi_f : D \longrightarrow C_f$ be an isomorphism, $C_f \in \mathcal{A}$, for each $f : D \longrightarrow B$ in $T$. Let $f' : C_f \longrightarrow A$ be such that

$$\begin{array}{ccc} B & \xrightarrow{\phi} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ D & \xrightarrow{\psi_f} & C_f \end{array}$$

commutes. If $f^*R \in J'(D)$ for each $f \in T$, let $S_f = (Q_f) \in \widehat{J}(C_f)$ be such that $f^*R = \psi_f^*S_f$. Since $S \in \widehat{J}(A)$, let $S' \in J(A)$ be such that $S = (S')$. Then since $S = (S \cap \mathcal{A})$, by the Lemma, $S' = S \cap \mathcal{A} \in J(A)$. For $g \in S'$, let $f \in T$ be such that $g = f'\psi_f$. Then

$$\begin{aligned} g^*(\phi R) &= (f'\psi_f)^*(\phi R) \\ &= (\phi^{-1}f'\psi_f)^*R \\ &= f^*R \\ &= \psi_f^*S_f \in \widehat{J}(D) \end{aligned}$$

Hence by Lemma 2$(v)$, $\phi R \in \widehat{J}(A)$, and so $R = \phi^*\phi R \in J'(B)$, as required. Hence $J'$ defines a Grothendieck topology (one checks the above works equally well for objects of $\mathcal{A}$ and $\mathcal{B}$ mixed together), and uniqueness is clear. □

Lemma 3. *In the situation of the previous proposition, a presheaf $P : \mathcal{B}^{op} \longrightarrow$ **Sets** is a $J'$-sheaf if and only if the restriction $P|_{\mathcal{A}}$ of $P$ to $\mathcal{A}$ is a $J$-sheaf.*

Proof. Suppose $P$ is a $J'$-sheaf on $\mathcal{B}$, and let $S \in J(A)$ be an $\mathcal{A}$-cover of $A \in \mathcal{A}$. If $(x_f)_{f \in S}$ is a matching family for $P$ on $S$, then we can extend it to a matching family on the $\mathcal{B}$-cover $(S) \in J'(A)$ by defining, for $g = fh \in (S)$,

$$x_g = x_f \cdot h$$

This is well defined since if $g = fh = f'h'$, for $f, f' \in S$ and $h, h' : D \longrightarrow A$, let $\psi : D \longrightarrow A'$ be an isomorphism of $D$ with an object of $\mathcal{A}$, and let $k = h\psi^{-1}$, $k' = h'\psi^{-1}$. Then $fk = f'k'$, and since $(x_f)$ is matching for $S$,

$$\begin{aligned} x_f \cdot h &= x_f \cdot (k\psi) \\ &= (x_f \cdot k) \cdot \psi \\ &= x_{fk} \cdot \psi \\ &= x_{f'k'} \cdot \psi \\ &= x_f \cdot (k'\psi) \\ &= x_f \cdot h' \end{aligned}$$

It is clear that this extended family is matching for $P$ on $(S)$ in $\mathcal{B}$. Hence let $x \in P(A)$ be unique such that $x \cdot g = x_g$ for all $g \in (S)$, so in particular for $f \in S$, $x \cdot f = x_f$. Clearly if $x' \in P(A)$ also gave $x' \cdot f = x_f$, then for $g = fh$ in $(S)$, $x' \cdot g = x_f \cdot h = x_g$, so $x = x'$.

Conversely, suppose that $P : \mathcal{B}^{\text{op}} \longrightarrow$ **Sets** is a presheaf which restricts to a $J$-sheaf on $\mathcal{A}$. Let $B$ be an object of $B$, $\psi : B \longrightarrow A$ an isomorphism of $B$ with an object of $\mathcal{A}$, $T \in J'(B)$ with $S \in \widehat{J}(A)$ such that $T = \psi^*S$, and let $(x_g)_{g \in T}$ be a matching famliy for $P$ on $T$. Let $S' \in J(A)$ be $S \cap \mathcal{A}$. For $f : A' \longrightarrow A \in S'$, $\psi^{-1}f \in \psi^*S = T$, so let $y_f = x_{\psi^{-1}f} \in P(A')$. Then $(y_f)_{f \in S'}$ are a matching family for the restriction of $P$ to $\mathcal{A}$ on the cover $S'$, since for $k : A'' \longrightarrow A'$ in $\mathcal{A}$,

$$\begin{aligned} y_f \cdot k &= x_{\psi^{-1}f} \cdot k \\ &= x_{\psi^{-1}fk} \\ &= y_{fk} \end{aligned}$$

hence there is a unique $y \in P(A)$ with the property that $y \cdot f = y_f = x_{\psi^{-1}f}$ for all $f \in S'$. We claim $x = y \cdot \psi \in P(B)$ is the unique amalgamation of $(x_g)$. For $g \in T$,

$$\begin{aligned} x \cdot g &= (y \cdot \psi) \cdot g \\ &= y \cdot (\psi g) \\ &= y_{\psi g} \\ &= x_{\psi^{-1}\psi g} \\ &= x_g \end{aligned}$$

The amalgamation is unique, since if $x' \in P(B)$ satisfies $x' \cdot g = x_g$, then $x' \cdot \psi^{-1} \in P(A)$ satisfies

$$(x' \cdot \psi^{-1}) \cdot f = x' \cdot (\psi^{-1} f) = x_{\psi^{-1}f} = y_f$$

Hence $x' \cdot \psi^{-1} = y$, and so $x = y \cdot \psi = x'$, as required. □

We know that in **k-alg** , the coproduct of two algebras $A$ and $B$ is the tensor product $A \otimes_k B$, with injections $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. Consider the diagram

$$\begin{array}{ccc} k & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & A \otimes_k B \end{array}$$

where the maps from $k$ are the structure maps. Recall that

LEMMA 4. *Let $k$ be a commutative ring. Then the category of $k$-algebras* **k-alg** *is isomorphic to the category $k/$***Rng** *of rings under $k$.*

Hence, since coproducts in $k/$**Rng** must correspond to pushouts in **Rng**, we have the following result

LEMMA 5. *In the category* **Rng***, the pushout of a pair of ring morphisms $A \longrightarrow B$, $A \longrightarrow C$ is the diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & B \otimes_A C \end{array}$$

*that is, the pushout is the tensor product of $B$ and $C$ considered as $A$-algebras via the two maps $A \longrightarrow B$, $A \longrightarrow C$.*

Now let $\phi : A \longrightarrow B$ be a morphism of rings, $f \in A$. Then there is an induced morphism $\varphi : A_f \longrightarrow B_{\phi(f)}$ making the following diagram commute

$$\begin{array}{ccc} A & \longrightarrow & A_f \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ B & \longrightarrow & B_{\phi(f)} \end{array}$$

It is not difficult to check that this diagram is a pushout (of rings). Also, if $A$ and $B$ are $k$-algebras, so too are $A_f$ and $B_{\phi(f)}$, and if $B \longrightarrow C$ is a $k$-algebra morphism taking $\phi(f)$ to a unit, the induced morphism of rings $B_{\phi(f)} \longrightarrow C$ is also a morphism of $k$-algebras. Hence, the above diagram is also a pushout of algebras. Notice that the above implies that there is a canonical isomorphism of rings $B \otimes_A A_f \cong B_{\phi(f)}$.

We now return to the subject of interest: the Zariski site. Let $Var(k)$ denote the full, replete subcategory of $\mathbf{Sets}^{\mathbf{k-alg}}$ containing the functors $V_A$ for $k$-algebras $A$, and all isomorphic functors. Let $Var'(k)$ temporarily denote the subcategory consisting of *just* these functors $V_A$ and the morphisms between them (so that $Var'(k)$ is the opposite category of **k-alg** ). A *geometric point* of $V_A$ is morphism $V_q \longrightarrow V_A$, $q$ a field. We say a $Var'(k)$-sieve $S$ on $V_A \in Var'(k)$ is *closed under geometric points* if every geometric point of $V_A$ belongs to $S$. Since representing objects are unique up to isomorphism, it makes sense to say a functor $W \in Var(k)$ *represents a field.* We call morphisms $f : W \longrightarrow V$, where $W$ represents a field, *geometric points* of $V$, and define closure under geometric points for $Var(k)$-sieves in the obvious way.

Suppose for a moment we are doing classical geometry, so that the only points of interest are the morphisms $A \longrightarrow k$, $k$ the algebraically closed base field, and we identify the variety $V_A$ with its values $V_A(k)$ in $k^I$ (after choosing a particular affine immersion $V_A \longrightarrow E^I$). Then a sieve on $V_A$ consists of the inclusion of various subvarieties of $V_A$, and a sieve is closed under geometric points iff. every point of $k$ belongs to one of the subvarieties. But that isn't what we normally mean by a "cover". A cover of $V_A$ should be formed by a union of sets like $V_{A_{f_i}} = D(f_i) \subseteq V_A \subseteq k^I$ with the property that every $V_A$-point of $k$ belongs to this union. Clearly a cover in this sense is closed under geometric morphisms. We are now ready to define a basis for a topology, called the *Zariski topology*, on $Var'(k)$.

PROPOSITION 4. *For an object $V_A$ of $Var'(k)$, the function $K$ defined by*

$$M \in K(V_A) \quad \textit{iff} \quad M = \{\phi_i : V_{A_{f_i}} \longrightarrow V_A \mid 1 \le i \le n, \phi_i = V_{A \longrightarrow A_{f_i}} \text{ and } (f_i)_i = A\}$$

*is a basis for a Grothendieck topology on $Var'(k)$.*

PROOF. That is, we take all finite collections of elements $f_1, \ldots, f_n \in A$ which generate the unit ideal in $A$, and take the duals of the canonical morphisms $A \longrightarrow A_{f_i}$. Recall from Proposition 1 that $f_1, \ldots, f_n$ generate the unit ideal iff. every $k$-algebra morphism from $A$ to a field $q$ factors through one of the morphisms $A \longrightarrow A_{f_i}$ (with the canonical $k$-algebra structure).

If $\phi : V_B \longrightarrow V_A$ is an isomorphism, then it is the dual of an isomorphism $f : A \longrightarrow B$. Let $f_1 = 1 \in A$. Then $f$ can be considered as the canonical morphism of $k$-algebras $f : A \longrightarrow A_{f_1} = B$. Hence $\{\phi\} \in K(V_A)$. Now suppose that $\{\phi_i\}_i$ are a cover of $V_A$, and let $\varphi : V_B \longrightarrow V_A$ be any morphism, say $\varphi = V_g$. Consider the two diagrams

$$\begin{array}{ccc} V_A & \xleftarrow{\phi_i} & V_{A_{f_i}} \\ \uparrow{\scriptstyle\varphi} & & \uparrow \\ V_B & \longleftarrow & V_{B_{g(f_i)}} \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & A_{f_i} \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow & B_{g(f_i)} \end{array}$$

Let $t : B \longrightarrow q$ be any geometric point of $B$. Then $tg$ is a geometric point of $A$, and hence factors through some $A_{f_j}$, since $\{\phi_i\}_i$ is a cover of $V_A$. But then using the second pullback, $t$ factors through $B_{g(f_j)}$. This verifies the stability axiom.

Suppose a family $A \longrightarrow A_{f_i}$ covers $A$, and that for each $i$, $A_{f_i} \longrightarrow (A_{f_i})_{c_{ij}}$ covers $A_{f_i}$. Let $A \longrightarrow k'$ be a geometric point of $V_A$. Then $A \longrightarrow k'$ factors through some $A_{f_j}$, so that we have a geometric point $A_{f_j} \longrightarrow k'$. Hence this morphism factors through some $(A_{f_j})_{c_{jk}}$, so that the original morphism $A \longrightarrow k'$ factors as $A \longrightarrow (A_{f_j})_{c_{jk}} \longrightarrow k'$, as required. □

We call the Grothendieck topology on $Var(k)$ induced by this basis the *Zariski topology.* Notice that if $\mathcal{O} : Var(k)^{\text{op}} \longrightarrow \mathbf{Sets}$ is a presheaf, to show that $\mathcal{O}$ is a sheaf it suffices to check that $\mathcal{O}|_{Var'(k)}$ is a sheaf, so that we can just show it is a sheaf with respect to the above basis.

## 3. Sheaves subsume Schemes

Notice that a presheaf on $Var'(k)$ is just a covariant functor $\mathbf{k-alg} \longrightarrow \mathbf{Sets}$, so that the representable functors $Hom(A, -)$ for $k$-algebras $A$, are presheaves. In particular the presheaf $\mathcal{O} = Hom(k[x], -) = Hom(-, V_{k[x]})$, which takes a variety $V_A$ to the set $A$, is called the *structure sheaf*:

LEMMA 6. *The presheaf $\mathcal{O}$ is a sheaf for the Zariski site.*

PROOF. Let $V_A$ be a variety, and $\{A \longrightarrow A_{f_i}\}_{i \in I}$ a cover of $V_A$. It is not difficult to check that the canonical commutative square

$$\begin{array}{ccc} A & \longrightarrow & A_{f_i} \\ \downarrow & & \downarrow \\ A_{f_j} & \longrightarrow & A_{f_i f_j} \end{array}$$

is a pushout of $k$-algebras for $i, j \in I$. Hence given a matching family $x_i \in A_{f_i}$, set $x_i = y_i / f_i^m$ for some $y_i \in A$ and all $i$, for $m$ sufficiently large. Then $x_i = x_j$ in $A_{f_i f_j}$ means that

$$y_i f_j^m (f_i f_j)^k = y_j f_i^m (f_i f_j)^k$$

for some sufficiently large $k > 0$. Now $1 \in (f_1, \ldots, f_n)$ in $A$; by raising the implied equation to the power $n(m+k)$ we may write $1 = \sum t_i f_i^{m+k}$ for suitable $t_i$. Let $x = \sum t_i y_i f_i^k$. Then

$$\begin{aligned} f_j^{m+k} &= \sum_i t_i y_i f_i^k f_j^{m+k} \\ &= \sum_i t_i f_i^{m+k} f_j^k y_j \\ &= f_j^k y_j \end{aligned}$$

Therefore in $A_{f_i}$, we have $x = f_j^k y_j / f_j^{m+k} = y_j / f_j^m = x_j$. This shows that $x$ is an amalgamation of the $x_i$'s. To show uniqueness, suppose $x = 0$ in $A_{f_i}$ for each $i$. Then for sufficiently large $m$, we have $f_i^m x = 0$ in $A$ for each $i$. But since $1 \in (f_1, \ldots, f_n)$, one can write $1 = \sum s_i f_i^m$, and hence $x = \sum s_i f_i^m x = 0$. □

Corollary 1. *Every representable presheaf $\mathcal{Q} = Hom(B, -) = Hom(-, V_B)$ for a $k$-algebra $B$, is a sheaf for the Zariski site. That is, the Zariski site is subcanonical.*

Proof. Let $A$ be a $k$-algebra with a cover $\varphi_i : A \longrightarrow A_{f_i}$, $i \in I$. A matching family for $\mathcal{Q}$ is a family of $I$-indexed morphisms $\phi_i : B \longrightarrow A_{f_i}$ such that for each $i, j \in I$, the square

$$\begin{array}{ccc} B & \xrightarrow{\phi_i} & A_{f_i} \\ {\scriptstyle \phi_j}\downarrow & & \downarrow \\ A_{f_j} & \longrightarrow & A_{f_i f_j} \end{array}$$

commutes. We define $\phi : B \longrightarrow A$ as follows: let $x \in B$, then $\phi_i(x)$ form a matching family for $\mathcal{O}$, and hence have a unique amalgamation $\phi(x) \in A$. This defines a function $\phi : B \longrightarrow A$ defined by the property that for each $x \in B$, $\phi(x)$ uniquely satisfies $\varphi_i \phi(x) = \phi_i(x)$. Hence, since

$$\varphi_i\left(\phi(x) + \phi(x')\right) = \varphi_i \phi(x) + \varphi_i \phi(x') = \phi_i(x + x')$$

we see that $\phi(x+x') = \phi(x)+\phi(x')$. Checking similarly the other conditions, we see that $\phi$ defines a morphism of $k$-algebras, and it is clearly a unique amalgamation of the $\phi_i$ for $\mathcal{Q}$. □

Recall that the representable presheaves on $Var(k)$ are the representable functors on **k-alg** , which are precisely the elements of $Var(k)$! For example, $\mathcal{O} = Hom(k[x], -)$ is the variety $V_{k[x]}$. But while this means that we can think of the varieties as sheaves, not all such sheaves correspond to varieties. In fact, just as we identify the affine schemes over $Spec(k)$ with the representable functors on **k-alg** , we can identify the schemes over $Spec(k)$ with certain functors $\mathbf{k-alg} \longrightarrow \mathbf{Sets}$, and unsurprisingly, such schemes also determine sheaves on the Zariski site. However, not even all the schemes over $Spec(k)$ can exhaust the category of sheaves!

It is this Grothendieck topos of sheaves over the Zariski site that will allow us to generalise algebraic geometry to noncommutative rings. While the internal language of a topos makes an appearance in commutative algebraic geometry, it only becomes essential when we pass to noncommutative geometry.

Proposition 5. *The function $J$ which assigns to $V_A \in Var'(k)$ the collection of all sieves on $V_A$ which are closed under geometric points, is a Grothendieck topology on $Var'(k)$.*

Proof. Clearly the maximal sieves are in the topology. Suppose that $S \in J(V_A)$ and that $f : V_B \longrightarrow V_A$ is a morphism of $Var'(k)$. Let $\phi : V_q \longrightarrow V_B$ be a geometric point of $V_B$. Then $f\phi$ is a geometric point of $V_A$, and hence belongs to $S$. This implies that $f^*S$ is closed under geometric points, and so $f^*S \in J(V_B)$. Finally, let $R$ be a sieve on $V_A$, and $S \in J(V_A)$ a cover. Suppose that $f^*R \in J(V_C)$ for every $f : V_C \longrightarrow V_A$ in $S$. Suppose that $\phi : V_q \longrightarrow V_A$ is a geometric point of $V_A$. Then since $S$ is a cover, $\phi$ belongs to $S$. Notice that $J(V_q) = \{t_{V_q}\}$, and hence $\phi^*R \in J(V_q)$ implies $\phi^*R = t_{V_q}$, and so $\phi \in R$, as required. □