Triangulated Categories Part III

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The aim of this note is to prove the Brown Representability theorem. This was originally proved by Neeman [Nee01] but our presentation follows recent simplifications due to Krause [Kra02]. For most of this note the only required background is our Triangulated Categories Part I notes.

Contents

1 Brown Representability

First we introduce a portly abelian category $A(\mathcal{S})$ for any preadditive category $\mathcal{S}$. This will be a certain full subcategory of the category of all contravariant additive functors $\mathcal{S} \longrightarrow \mathbf{Ab}$. Modulo the fact that $\mathcal{S}$ is not assumed to be small, this is precisely what we call a right module over a ringoid in our Rings With Several Objects (RSO) notes. For background on portly abelian categories, the reader is referred to (AC,Section 2.4).

Definition 1. Given a preadditive category $\mathcal{A}$, the objects of the portly abelian category $\mathbf{Mod}\mathcal{A} = (\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ are called *right $\mathcal{A}$-modules*. A sequence of right modules

$$M' \longrightarrow M \longrightarrow M''$$

is exact in $\mathbf{Mod}\mathcal{A}$ if and only if the following sequence is exact in $\mathbf{Ab}$ for every $A \in \mathcal{A}$

$$M'(A) \longrightarrow M(A) \longrightarrow M''(A)$$

Similarly kernels, cokernels and images in $\mathbf{Mod}\mathcal{A}$ are computed pointwise. See (AC,Corollary 59) and the proof of (AC,Proposition 44) for details. A morphism $\phi : M \longrightarrow N$ in $\mathbf{Mod}\mathcal{A}$ is a monomorphism or epimorphism if and only if $\phi_A : M(A) \longrightarrow N(A)$ has this property for every $A \in \mathcal{A}$. For any object $A \in \mathcal{A}$ we have the right module $H_A = Hom(-, A) : \mathcal{A} \longrightarrow \mathbf{Ab}$ defined in the obvious way.

Proposition 1 (Yoneda). *If $\mathcal{A}$ is a preadditive category, then*

(i) For any object $A \in \mathcal{A}$ and right $\mathcal{A}$-module T there is a canonical isomorphism of abelian groups $Hom_{\mathcal{A}}(H_A, T) \longrightarrow T(A)$ defined by $\gamma \mapsto \gamma_A(1)$.

(ii) The functor $A \mapsto H_A$ defines a full additive embedding $\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{A}$.

(iii) The objects $\{H_A\}_{A \in \mathcal{A}}$ form a (large) generating family of projectives for $\mathbf{Mod}\mathcal{A}$.

Proof. All three results are proved in the usual way. See (RSO,Lemma 1), (RSO,Lemma 2) and (RSO,Proposition 3). Observe that since the object class of $\mathcal{A}$ is not assumed to be small, the family $\{H_A\}_{A\in\mathcal{A}}$ is *not* indexed by a set, so is *not* a generating family in the usual sense. In particular we *cannot* take coproducts and conclude that $\mathbf{Mod}\mathcal{A}$ has a generator. □

Remark 1. Given the Yoneda embedding $\mathcal{A} \longrightarrow \mathbf{Mod}\mathcal{A}$ it is natural to identify $\mathcal{A}$ with its image in $\mathbf{Mod}\mathcal{A}$. So at least in our intuition, we will tend to confuse A with H_A and identify a morphism $\alpha : A \longrightarrow B$ with its corresponding natural transformation $H_A \longrightarrow H_B$.

Definition 2. Let $\mathcal{S}$ be a preadditive category. We say that a right $\mathcal{S}$-module F is *coherent* if there exists an exact sequence in $\mathbf{Mod}\mathcal{S}$ of the following form

$$H_A \longrightarrow H_B \longrightarrow F \longrightarrow 0 \tag{1}$$

That is, F is the cokernel of some morphism of $\mathcal{S}$ considered as a morphism of $\mathcal{S}$-modules. Clearly any representable functor is coherent. We denote by $A(\mathcal{S})$ the full replete subcategory of $\mathbf{Mod}\mathcal{S}$ consisting of the coherent modules. At the moment we only know that this is a preadditive portly category.

Remark 2. Let $\mathcal{S}$ be a preadditive category. Given a morphism $\varphi : M \longrightarrow N$ in $\mathbf{Mod}\mathcal{S}$ and presentations

$$\begin{array}{ccccccc} H_A & \longrightarrow & H_B & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \\ H_{A'} & \longrightarrow & H_{B'} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

there exist by projectivity vertical morphisms making the above diagram commute.

Lemma 2. *Let $\mathcal{S}$ be a preadditive category. If a right $\mathcal{S}$-module F is coherent, then as a covariant functor $\mathcal{S}^{op} \longrightarrow \mathbf{Ab}$ it preserves products.*

Proof. Suppose that F has a presentation of the form (1). Given a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{S}$ and a coproduct $\bigoplus_i X_i$ we have a commutative diagram in which the rows are exact

$$\begin{array}{ccccccc} Hom(\bigoplus_i X_i, A) & \longrightarrow & Hom(\bigoplus_i X_i, B) & \longrightarrow & F(\bigoplus_i X_i) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_i Hom(X_i, A) & \longrightarrow & \prod_i Hom(X_i, B) & \longrightarrow & \prod_i F(X_i) & \longrightarrow & 0 \end{array}$$

The first two vertical morphisms are clearly isomorphisms, and therefore so is the third. This shows that F sends coproducts to products, as required. □

Remark 3. Suppose we have a commutative diagram of abelian groups

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow[b]{} & B' \end{array} \tag{2}$$

Taking cokernels of the rows, we get a morphism $B/A \longrightarrow B'/A'$. The cokernel of this morphism is the quotient of B'/A' by the subobject $(Im(d) + A')/A'$, so it is isomorphic to the quotient $B'/(Im(d) + A')$. In other words, it is the cokernel of the morphism $B \oplus A' \longrightarrow B'$ induced by the morphisms d, b.

Now let $\mathcal{S}$ be a preadditive category, and suppose we are given a commutative diagram (2) in $\mathcal{S}$. Mapping this diagram to $\mathbf{Mod}\mathcal{S}$ and taking cokernels of the rows yields a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_A & \longrightarrow & H_B & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \\ H_{A'} & \longrightarrow & H_{B'} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

There is an induced morphism $\varphi : M \longrightarrow N$. Let $\kappa : H_{B'} \longrightarrow H$ be the cokernel of the morphism $H_B \oplus H_{A'} \longrightarrow H_{B'}$ induced by the morphisms d, b. Then it is not difficult to check that the induced morphism $N \longrightarrow H$ is the cokernel of φ in $\mathbf{Mod}\mathcal{S}$. This observation motivates the proof of the next result.

Lemma 3. *Let $\mathcal{S}$ be an additive category. If $\varphi : M \longrightarrow N$ is a morphism in $A(\mathcal{S})$ then any cokernel of φ in $\mathbf{Mod}\mathcal{S}$ also belongs to $A(\mathcal{S})$.*

Proof. In light of Remark 2 and Remark 3 we need only observe that the Yoneda embedding is additive, so $H_B \oplus H_{A'} \cong H_{B \oplus A'}$ and so H belongs to $A(\mathcal{S})$. ☐

Definition 3. Let $\mathcal{S}$ be a preadditive category. A morphism $u : X \longrightarrow Y$ is a *weak kernel* of $v : Y \longrightarrow Z$ if $vu = 0$ and if any other morphism $f : T \longrightarrow Y$ with $vf = 0$ factors through u (not necessarily uniquely). In other words, the induced sequence

$$H_X \longrightarrow H_Y \longrightarrow H_Z$$

is exact in $\mathbf{Mod}\mathcal{S}$. If every morphism of $\mathcal{S}$ has a weak kernel, then we say that $\mathcal{S}$ *has weak kernels.*

Lemma 4. *Let $\mathcal{S}$ be an additive category with weak kernels. If $\varphi : M \longrightarrow N$ is a morphism in $A(\mathcal{S})$ then any kernel of φ in $\mathbf{Mod}\mathcal{S}$ also belongs to $A(\mathcal{S})$.*

Proof. Choose presentations of M, N and construct a commutative diagram inducing φ as follows

$$\begin{array}{ccccccc} H_A & \xrightarrow{H_a} & H_B & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle H_c}\downarrow & & \downarrow{\scriptstyle H_d} & & \downarrow{\scriptstyle \varphi} & & \\ H_{A'} & \xrightarrow[H_b]{} & H_{B'} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

We have a commutative diagram in $\mathcal{S}$

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow[b]{} & B' \end{array}$$

Taking our cue from Remark 3 we try to construct a square sitting on top of this one, which induces a kernel of φ when we apply the Yoneda embedding and take cokernels. Take a weak kernel $B'' \longrightarrow B \oplus A'$ of the morphism $B \oplus A' \longrightarrow B'$, and a weak kernel $A'' \longrightarrow B'' \oplus A$ of the morphism $B'' \oplus A \longrightarrow B$. In other words, we have two weak kernels

$$B'' \xrightarrow{\begin{pmatrix} r \\ s \end{pmatrix}} B \oplus A' \xrightarrow{\begin{pmatrix} d & b \end{pmatrix}} B'$$

$$A'' \xrightarrow{\begin{pmatrix} m \\ n \end{pmatrix}} B'' \oplus A \xrightarrow{\begin{pmatrix} r & a \end{pmatrix}} B$$

and a commutative diagram

$$\begin{array}{ccc} A'' & \xrightarrow{m} & B'' \\ {\scriptstyle -n}\downarrow & & \downarrow{\scriptstyle r} \\ A & \xrightarrow[a]{} & B \end{array}$$

Apply the Yoneda embedding to this diagram, and take a cokernel $H_{B''} \longrightarrow K$ of $H_m : H_{A''} \longrightarrow H_{B''}$. We have an induced morphism $\psi : K \longrightarrow M$, which one checks is a pointwise kernel of $\varphi : M \longrightarrow N$. Since K is certainly coherent, the proof is complete. ☐

Proposition 5. *Let $\mathcal{S}$ be an additive category with weak kernels. Then $A(\mathcal{S})$ is a portly abelian category. If $\mathcal{S}$ has coproducts then $A(\mathcal{S})$ is cocomplete and the induced Yoneda functor*

$$H_{(-)} : \mathcal{S} \longrightarrow A(\mathcal{S})$$

preserves coproducts.

Proof. We know from Lemma 3 and Lemma 4 that $A(\mathcal{S})$ has kernels and cokernels, which can be calculated in $\mathbf{Mod}\mathcal{S}$. That is, the inclusion $A(\mathcal{S}) \longrightarrow \mathbf{Mod}\mathcal{S}$ preserves kernels and cokernels. Since the Yoneda functor $H_{(-)} : \mathcal{S} \longrightarrow \mathbf{Mod}\mathcal{S}$ preserves finite coproducts, it is clear that $A(\mathcal{S})$ is closed under finite products and coproducts in $\mathbf{Mod}\mathcal{S}$, and is therefore portly abelian. Now suppose that $\mathcal{S}$ has coproducts, and that we are given a nonempty family $\{F_i\}_{i \in I}$ of objects of $A(\mathcal{S})$. Choose presentations

$$H_{A_i} \xrightarrow{H_{\alpha_i}} H_{B_i} \longrightarrow F_i \longrightarrow 0$$

and let $H_{\oplus_i B_i} \longrightarrow F$ be the cokernel in $\mathbf{Mod}\mathcal{S}$ of $H_{\oplus_i \alpha_i}$. We have a commutative diagram for each $i \in I$

$$\begin{array}{ccccccc} H_{A_i} & \longrightarrow & H_{B_i} & \longrightarrow & F_i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots & & \\ H_{\oplus_i A_i} & \longrightarrow & H_{\oplus_i B_i} & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

and therefore an induced morphism $u_i : F_i \longrightarrow F$. Given $Q \in A(\mathcal{S})$ and morphisms $\phi_i : F_i \longrightarrow Q$ it is easy to see that there is a morphism $\phi : F \longrightarrow Q$ with $\phi u_i = \phi_i$. To prove uniqueness, suppose we have a morphism $\phi : F \longrightarrow Q$ with $Q \in A(\mathcal{S})$ and $\phi u_i = 0$ for every $i \in I$. Lift the morphism ϕ to a morphism of presentations

$$\begin{array}{ccccccc} H_{\oplus_i A_i} & \longrightarrow & H_{\oplus_i B_i} & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ H_C & \longrightarrow & H_D & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Since $\phi u_i = 0$ for each i, we deduce that for each $i \in I$ the composite $H_{B_i} \longrightarrow H_{\oplus_i B_i} \longrightarrow H_D$ vanishes on $H_D \longrightarrow Q$, and therefore factors through $H_C \longrightarrow H_D$ (using projectivity of H_B). It is now not difficult to see that $H_{\oplus_i B_i} \longrightarrow H_D$ must factor through $H_C \longrightarrow H_D$, which implies immediately that the composite $H_{\oplus_i B_i} \longrightarrow F \longrightarrow Q$ is zero. Since the first morphism is an epimorphism, we deduce that $\phi = 0$ as required. This proves that $A(\mathcal{S})$ is cocomplete, and as a special case of the above construction we find that given a coproduct $\{u_i : A_i \longrightarrow \bigoplus_i A_i\}_{i \in I}$ in $\mathcal{S}$, the morphisms $H_{u_i} : H_{A_i} \longrightarrow H_{\oplus_i A_i}$ are a coproduct in $A(\mathcal{S})$. That is, the Yoneda functor into $A(\mathcal{S})$ preserves coproducts (in particular this shows that coproducts in $A(\mathcal{S})$ are *not* computed pointwise). $\square$

Remark 4. Let $\mathcal{T}$ be a triangulated category. Then homotopy kernels in $\mathcal{T}$ are weak kernels, so Proposition 5 implies that $A(\mathcal{T})$ is a portly abelian category, which is cocomplete if $\mathcal{T}$ has coproducts. The canonical functor $\mathcal{T} \longrightarrow A(\mathcal{T})$ is clearly homological.

Definition 4. Let $\mathcal{T}$ be a triangulated category with coproducts. A nonempty set of objects $S \subseteq \mathcal{T}$ is a *perfect generating set for* $\mathcal{T}$ (or *perfectly generates* $\mathcal{T}$) if the following conditions hold:

(G1) Given $X \in \mathcal{T}$ if we have $Hom(k, X) = 0$ for every $k \in S$ then $X = 0$.

(G2) Given a nonempty countable family of morphisms $\{X_i \longrightarrow Y_i\}_{i \in I}$ in $\mathcal{T}$ such that the map $Hom(k, X_i) \longrightarrow Hom(k, Y_i)$ is surjective for every $i \in I, k \in S$, the induced map

$$Hom(k, \bigoplus_{i \in I} X_i) \longrightarrow Hom(k, \bigoplus_{i \in I} Y_i)$$

is also surjective for any $k \in S$.

If a perfect generating set exists for $\mathcal{T}$ then we say that $\mathcal{T}$ is *perfectly generated*.

Remark 5. With the notation of Definition 4, if S is a perfect generating set then so is the set $\{\Sigma^n k \,|\, n \in \mathbb{Z}, k \in S\}$. So a triangulated category $\mathcal{T}$ with coproducts has a perfect generating set if and only if it has a perfect generating set closed under suspension.

Example 1. Let $\mathcal{T}$ be a triangulated category with coproducts, $S \subseteq \mathcal{T}$ a nonempty set of objects. If every object of S is small (AC,Definition 18) then S satisfies condition (G2).

Definition 5. Let $\mathcal{T}$ be an additive category, $S \subseteq \mathcal{T}$ a nonempty class of objects. We denote by $Add(S)$ the smallest full, replete subcategory of $\mathcal{T}$ containing S and closed under coproducts and direct summands in $\mathcal{T}$. That is, $Add(S)$ is the intersection of all subcategories of $\mathcal{T}$ with these properties. Clearly $Add(S)$ is an additive category.

The next lemma explains the importance of the condition (G2).

Lemma 6. *Let $\mathcal{T}$ be an additive category with coproducts and weak kernels, $S \subseteq \mathcal{T}$ a nonempty set of objects of $\mathcal{T}$, and define $\mathcal{S} = Add(S)$. Then*

(i) The additive category $\mathcal{S}$ has weak kernels, and $A(\mathcal{S})$ is a cocomplete portly abelian category.

(ii) The map $F \mapsto F|_{\mathcal{S}}$ gives an exact functor $A(\mathcal{T}) \longrightarrow A(\mathcal{S})$.

Proof. Observe that for every $X \in \mathcal{T}$, there exists an *approximation* $\nu : X' \longrightarrow X$ such that $X' \in \mathcal{S}$ and $Hom(w, X') \longrightarrow Hom(w, X)$ is surjective for every $w \in \mathcal{S}$. To see this, define X' to be $\bigoplus_{k \in S} X_k$ where $X_k = \bigoplus_{f \in Hom(k,X)} k$ and define ν to be the morphism $\nu u_{k,f} = f$. Let $\mathcal{S}'$ be the full subcategory of $\mathcal{T}$ consisting of the objects $w \in \mathcal{T}$ which make $Hom(w, X') \longrightarrow Hom(w, X)$ surjective. One checks that $\mathcal{S}'$ is replete, closed under coproducts and direct summands, and contains S. It therefore contains $\mathcal{S}$, which is what we were trying to show.

(i) It suffices by Proposition 5 to show that $\mathcal{S}$ has weak kernels. Given a morphism $Y \longrightarrow Z$ in $\mathcal{S}$, one obtains a weak kernel by composing a weak kernel $X \longrightarrow Y$ in $\mathcal{T}$ with an approximation $X' \longrightarrow X$.

(ii) Restriction defines an exact functor $\mathbf{Mod}\mathcal{T} \longrightarrow \mathbf{Mod}\mathcal{S}$, and we claim that this restricts to an exact functor $A(\mathcal{T}) \longrightarrow A(\mathcal{S})$. Suppose we are given $F \in A(\mathcal{T})$ and choose a presentation $H_A \longrightarrow H_B \longrightarrow F \longrightarrow 0$. We have an exact sequence in $\mathbf{Mod}\mathcal{S}$

$$H_A|_{\mathcal{S}} \longrightarrow H_B|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0$$

so it suffices by Lemma 3 to show that $H_A|_{\mathcal{S}} \in A(\mathcal{S})$ for any $A \in \mathcal{T}$. Given the object $A \in \mathcal{T}$, let $A' \longrightarrow A$ be an approximation. By definition of an approximation, the morphism $H_{A'} \longrightarrow H_A|_{\mathcal{S}}$ is an epimorphism in $\mathbf{Mod}\mathcal{S}$. Let $X \longrightarrow A'$ be a weak kernel in $\mathcal{T}$, and $X' \longrightarrow X$ another approximation. We have an exact sequence in $\mathbf{Mod}\mathcal{S}$

$$H_{X'} \longrightarrow H_{A'} \longrightarrow H_A|_{\mathcal{S}} \longrightarrow 0$$

which proves that $H_A|_{\mathcal{S}}$ is coherent, as required. The functor $A(\mathcal{T}) \longrightarrow A(\mathcal{S})$ is obviously exact. □

Lemma 7. *Let $\mathcal{T}$ be a triangulated category with coproducts, $S \subseteq \mathcal{T}$ a nonempty set of objects of $\mathcal{T}$, and define $\mathcal{S} = Add(S)$. Then the functor*

$$\mathcal{T} \longrightarrow A(\mathcal{S}), \qquad X \mapsto H_X|_{\mathcal{S}}$$

is homological. It preserves countable coproducts if and only if (G2) *holds for S.*

Proof. The functor $\mathcal{T} \longrightarrow A(\mathcal{S})$ is the composite $\mathcal{T} \longrightarrow A(\mathcal{T}) \longrightarrow A(\mathcal{S})$, so it is clearly homological. We observe that $\mathcal{T} \longrightarrow A(\mathcal{S})$ preserves (countable) coproducts if and only if $A(\mathcal{T}) \longrightarrow A(\mathcal{S})$ does. One implication is clear, since $\mathcal{T} \longrightarrow A(\mathcal{T})$ preserves coproducts. Suppose that $\mathcal{T} \longrightarrow A(\mathcal{S})$

preserves (countable) coproducts. Given objects $F_i \in A(\mathcal{T})$ construct a coproduct as in Proposition 5. Restricting to $\mathcal{S}$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_{A_i}|_{\mathcal{S}} & \longrightarrow & H_{B_i}|_{\mathcal{S}} & \longrightarrow & F_i|_{\mathcal{S}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_{\oplus_i A_i}|_{\mathcal{S}} & \longrightarrow & H_{\oplus_i B_i}|_{\mathcal{S}} & \longrightarrow & F|_{\mathcal{S}} & \longrightarrow & 0 \end{array}$$

By assumption the first two families of vertical morphisms form coproducts, and since coproducts preserve cokernels we deduce that the morphisms $F_i|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}$ are a coproduct in $A(\mathcal{S})$ as well, which is what we wanted to show.

Now we show that $\mathcal{T} \longrightarrow A(\mathcal{S})$ preserves countable coproducts if and only if (G2) holds for S. Suppose that (G2) holds for S and that we are given a countable coproduct $A_i \longrightarrow \bigoplus_i A_i$ in $\mathcal{T}$. As in the proof of Lemma 6 we construct a presentation of $H_{A_i}|_{\mathcal{S}}$ in $\mathbf{Mod}\mathcal{S}$ as follows: take an approximation $A_i' \longrightarrow A_i$, a homotopy kernel $X_i \longrightarrow A_i'$ of this approximation, and another approximation $X_i' \longrightarrow X_i$. By (G2) countable coproducts of approximations are approximations, and by (TRC,Remark 9) coproducts preserve homotopy kernels. Therefore we have a commutative diagram with exact rows in $A(\mathcal{S})$

$$\begin{array}{ccccccc} H_{X_i'} & \longrightarrow & H_{A_i'} & \longrightarrow & H_{A_i}|_{\mathcal{S}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_{\oplus_i X_i'} & \longrightarrow & H_{\oplus A_i'} & \longrightarrow & H_{\oplus_i A_i}|_{\mathcal{S}} & \longrightarrow & 0 \end{array}$$

We know that the first two families of vertical morphisms form coproducts in $A(\mathcal{S})$, and since coproducts preserve cokernels we deduce that the morphisms $H_{A_i}|_{\mathcal{S}} \longrightarrow H_{\oplus_i A_i}|_{\mathcal{S}}$ are a coproduct in $A(\mathcal{S})$, as required.

Conversely, we suppose that $\mathcal{T} \longrightarrow A(\mathcal{S})$ preserves countable coproducts, and prove that (G2) holds for S. Given a nonempty countable family of morphisms $\{X_i \longrightarrow Y_i\}_{i \in I}$ as in the statement of (G2), by assumption the morphisms $H_{X_i}|_{\mathcal{S}} \longrightarrow H_{Y_i}|_{\mathcal{S}}$ are epimorphisms in $A(\mathcal{S})$. Therefore their coproduct $H_{\oplus_i X_I}|_{\mathcal{S}} \longrightarrow H_{\oplus_i Y_i}|_{\mathcal{S}}$ is an epimorphism, which is what we needed to show. □

Theorem 8 (Brown Representability). *Let $\mathcal{T}$ be a triangulated category with coproducts and a perfect generating set. Then an additive functor $F : \mathcal{T}^{op} \longrightarrow$ **Ab** is representable if and only if it is homological and product preserving.*

Proof. Equivalently, we are claiming that a contravariant additive functor $F : \mathcal{T} \longrightarrow \mathbf{Ab}$ is naturally equivalent to H_X for some $X \in \mathcal{T}$ if and only if it is cohomological (TRC,Definition 5) and sends coproducts in $\mathcal{T}$ to products in $\mathbf{Ab}$. By Remark 5 we can assume that $\mathcal{T}$ has a perfect generating set S closed under suspension, and we set $\mathcal{S} = Add(S)$.

Let $F : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ be a homological functor which preserves products. We construct inductively a sequence of objects and morphisms in $\mathcal{T}$

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \longrightarrow \cdots \tag{3}$$

together with a morphism $\pi_i : H_{X_i} \longrightarrow F$ in $\mathbf{Mod}\mathcal{T}$ for each $i \geq 0$. Given $k \in S$ and $x \in F(k)$ we write k_x for the object k and set $X_0 = \bigoplus_{k \in S, x \in F(k)} k_x$ (all coproducts are taken in $\mathcal{T}$). By assumption we have a canonical isomorphism of abelian groups

$$F(X_0) \cong \prod_{k \in S, x \in F(k)} F(k_x)$$

so the sequence $(x)_{k \in S, x \in F(k)}$ in the right-hand product corresponds to an element π_0 of $F(X_0)$, and therefore to a morphism $\pi_0 : H_{X_0} \longrightarrow F$ in $\mathbf{Mod}\mathcal{T}$ with

$$(\pi_0 H_{u_{k,x}})_T(\varphi) = F(\varphi)(x)$$

for any $T \in \mathcal{T}$ and morphism $\varphi : T \longrightarrow u_{k,x}$. Suppose we have already constructed objects $X_0, \ldots, X_i$ morphisms $\phi_0, \ldots, \phi_{i-1}$ and $\pi_0, \ldots, \pi_i$ for some $i \geq 0$. Set $K_i = Ker\pi_i$ and define $T_i = \bigoplus_{k\in S, x\in K_i(k)} k_x$. It is easy to check that K_i sends coproducts in $\mathcal{T}$ to products in **Ab**, so there is a canonical morphism $H_{T_i} \longrightarrow K_i$. Composing with the kernel morphism $K_i \longrightarrow H_{X_i}$ we have a morphism $v_i : T_i \longrightarrow X_i$ in $\mathcal{T}$. Extending this to a triangle

$$T_i \xrightarrow{v_i} X_i \xrightarrow{\phi_i} X_{i+1} \xrightarrow{\xi_i} \Sigma T_i$$

defines the object X_{i+1} and morphism ϕ_i. Since F is homological we have an exact sequence

$$F(\Sigma T_i) \longrightarrow F(X_{i+1}) \longrightarrow F(X_i) \longrightarrow F(T_i)$$

By construction $F(v_i)(\pi_i) = 0$ so there is an element $\pi_{i+1} \in F(X_{i+1})$ such that $F(\phi_i)(\pi_{i+1}) = \pi_i$. In other words, we can write $\pi_i : H_{X_i} \longrightarrow F$ as the composite of $\pi_{i+1} : H_{X_{i+1}} \longrightarrow F$ and $H_{\phi_i} : H_{X_i} \longrightarrow H_{X_{i+1}}$. This completes the construction of the sequence (3) and morphisms π_i.

For each $i \geq 0$ let $\kappa_i : H_{T_i} \longrightarrow K_i$ be the morphism in **Mod**$\mathcal{T}$ constructed above. By construction $(\kappa_i)_k$ is surjective for every $k \in S$, and it follows that $H_{T_i}|_{\mathcal{S}} \longrightarrow K_i|_{\mathcal{S}}$ is an epimorphism in **Mod**$\mathcal{S}$ (the category of all $w \in \mathcal{T}$ such that $(\kappa_i)_w$ is surjective is replete, closed under coproducts and direct summands, therefore contains $\mathcal{S}$). For the same reason, $\pi_i|_{\mathcal{S}} : H_{X_i}|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}$ is an epimorphism. We therefore have an exact sequences in the portly abelian category **Mod**$\mathcal{S}$

$$0 \longrightarrow K_i|_{\mathcal{S}} \longrightarrow H_{X_i}|_{\mathcal{S}} \xrightarrow{\pi_i|_{\mathcal{S}}} F|_{\mathcal{S}} \longrightarrow 0 \qquad (4)$$

$$H_{T_i}|_{\mathcal{S}} \xrightarrow{H_{v_i}|_{\mathcal{S}}} H_{X_i}|_{\mathcal{S}} \xrightarrow{\pi_i|_{\mathcal{S}}} F|_{\mathcal{S}} \longrightarrow 0 \qquad (5)$$

From which we deduce that $F|_{\mathcal{S}}$ and $K_i|_{\mathcal{S}}$ are coherent. For each $i \geq 0$ we have a commutative diagram in $A(\mathcal{S})$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i|_{\mathcal{S}} & \longrightarrow & H_{X_i}|_{\mathcal{S}} & \xrightarrow{\pi_i|_{\mathcal{S}}} & F|_{\mathcal{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & K_{i+1}|_{\mathcal{S}} & \longrightarrow & H_{X_{i+1}}|_{\mathcal{S}} & \xrightarrow[\pi_{i+1}|_{\mathcal{S}}]{} & F|_{\mathcal{S}} & \longrightarrow & 0 \end{array}$$

where we set $\psi_i = H_{\phi_i}|_{\mathcal{S}}$. The composite $K_i|_{\mathcal{S}} \longrightarrow H_{X_i}|_{\mathcal{S}} \longrightarrow H_{X_{i+1}}|_{\mathcal{S}}$ is zero since $\phi_i \circ v_i = 0$, so there is a factorisation $\ell_i : F|_{\mathcal{S}} \longrightarrow H_{X_{i+1}}|_{\mathcal{S}}$. It is clear that $\pi_{i+1}|_{\mathcal{S}} \circ \ell_i = 1$, so the exact sequence (4) splits for $i \geq 1$ and there is an isomorphism $H_{X_i}|_{\mathcal{S}} \cong F|_{\mathcal{S}} \oplus K_i|_{\mathcal{S}}$. Consider the following commutative diagram in $A(\mathcal{S})$

$$\begin{array}{ccccccc} H_{X_1}|_{\mathcal{S}} & \xrightarrow{\psi_1} & H_{X_2}|_{\mathcal{S}} & \xrightarrow{\psi_2} & H_{X_3}|_{\mathcal{S}} & \xrightarrow{\psi_3} & \cdots \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ F|_{\mathcal{S}} \oplus K_1|_{\mathcal{S}} & \xrightarrow[1\oplus 0]{} & F|_{\mathcal{S}} \oplus K_2|_{\mathcal{S}} & \xrightarrow[1\oplus 0]{} & F|_{\mathcal{S}} \oplus K_3|_{\mathcal{S}} & \xrightarrow[1\oplus 0]{} & \cdots \end{array}$$

Taking colimits of the rows we deduce that the morphisms $\{\pi_i|_{\mathcal{S}} : H_{X_i}|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}\}_{i\geq 1}$ are a colimit in $A(\mathcal{S})$ of the direct system in the first row. In the usual way (DTC,Remark 23) we deduce an exact sequence in $A(\mathcal{S})$

$$0 \longrightarrow \bigoplus_{i\geq 1} H_{X_i}|_{\mathcal{S}} \xrightarrow{1-\nu} \bigoplus_{i\geq 1} H_{X_i}|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0 \qquad (6)$$

where the coproducts are taken in $A(\mathcal{S})$. We should observe that $1-\nu$ is a monomorphism by virtue of being a coretraction. To see this, note that $1-\nu$ can be written as a direct sum of the corresponding morphisms for the following two sequences

$$K_1|_{\mathcal{S}} \xrightarrow{0} K_2|_{\mathcal{S}} \xrightarrow{0} K_3|_{\mathcal{S}} \xrightarrow{0} \cdots$$

$$F|_{\mathcal{S}} \xrightarrow{1} F|_{\mathcal{S}} \xrightarrow{1} F|_{\mathcal{S}} \xrightarrow{1} \cdots$$

In the first case $1-\nu = 1$ and in the second case $1-\nu$ is easily checked to be a coretraction, so the direct sum of these two morphisms is a coretraction, which justifies exactness of (6). Now take a homotopy colimit (TRC,Definition 34) of the sequence (3) in $\mathcal{T}$ (with the first term deleted). That is, we have a triangle

$$\bigoplus_{i\geq 1} X_i \xrightarrow{1-\mu} \bigoplus_{i\geq 1} X_i \xrightarrow{q} X \longrightarrow \Sigma \bigoplus_{i\geq 1} X_i \tag{7}$$

and since F is homological, an exact sequence

$$F(X) \xrightarrow{F(q)} F(\bigoplus_{i\geq 1} X_i) \xrightarrow{F(1-\mu)} F(\bigoplus_{i\geq 1} X_i)$$

Under the isomorphism $\prod_{i\geq 1} F(X_i) \cong F(\bigoplus_{i\geq 1} X_i)$ the sequence $(\pi_i)_{i\geq 1}$ corresponds to an element $j \in F(\bigoplus_{i\geq 1} X_i)$. Since $\pi_i = \pi_{i+1} H_{\phi_i}$ for every $i \geq 1$, it is clear that $F(1-\mu)(j) = 0$, so there is $\pi \in F(X)$ with $F(q)(\pi) = j$. That is, we have a morphism $\pi : H_X \longrightarrow F$ with $\pi H_{q_i} = \pi_i$ for every $i \geq 1$, where we let q_i be the ith component of the morphism q defined above.

Since S is a perfect generating set, it follows from Lemma 7 that the functor $\mathcal{T} \longrightarrow A(\mathcal{S})$ is homological and preserves countable coproducts. Applying this functor to (7) yields an exact sequence in $A(\mathcal{S})$

$$\bigoplus_{i\geq 1} H_{X_i}|_{\mathcal{S}} \xrightarrow{1-\nu} \bigoplus_{i\geq 1} H_{X_i}|_{\mathcal{S}} \longrightarrow H_X|_{\mathcal{S}} \longrightarrow H_{\Sigma\oplus_{i\geq 1} X_i}|_{\mathcal{S}} \xrightarrow{H_{\Sigma(1-\mu)}|_{\mathcal{S}}} H_{\Sigma\oplus_{i\geq 1} X_i}|_{\mathcal{S}}$$

We claim that $\lambda = H_{\Sigma(1-\mu)}|_{\mathcal{S}}$ is a monomorphism. It suffices to prove this pointwise, so we take the category of all $w \in \mathcal{S}$ such that λ_w is injective. This is replete, closed under coproducts and direct summands, and contains S since we know the morphism $1-\nu$ of (6) is a monomorphism, and by assumption S is closed under suspension. It follows that our subcategory is all of $\mathcal{S}$, and λ is a monomorphism. Comparing with (6) we infer that $\pi|_{\mathcal{S}} : H_X|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}$ is an isomorphism. Moreover the full subcategory of all $Y \in \mathcal{T}$ such that π_Y is an isomorphism is replete, closed under coproducts and mapping cones, and contains S.

Let $\mathcal{Q}$ be *any* full subcategory of $\mathcal{T}$ with all these properties. We claim that $\mathcal{Q} = \mathcal{T}$. To see this let $Y \in \mathcal{T}$ be given and apply the construction in the first part of the proof to the functor $F = H_Y : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. In the construction of the sequence (3) for $F = H_Y$ we take coproducts of objects in S and mapping cones of morphisms between such objects, so it is clear that every X_i belongs to $\mathcal{Q}$. From (7) we conclude that $X \in \mathcal{Q}$, so we have a morphism $\pi : H_X \longrightarrow H_Y$ with $X \in \mathcal{Q}$ which restricts to an isomorphism on $\mathcal{S}$. Extend the corresponding morphism $\pi : X \longrightarrow Y$ to a triangle

$$W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$$

Given $k \in S$ we apply $Hom(k,-)$ to this triangle and obtain a long exact sequence of abelian groups. Using the fact that S is closed under suspension and π restricts to an isomorphism on $\mathcal{S}$, we deduce that $Hom(k, W) = Hom(k, \Sigma W) = 0$. By (G1) we have $W = 0$ and $\Sigma W = 0$, which implies that $X \longrightarrow Y$ is an isomorphism, from which we deduce $Y \in \mathcal{Q}$ as claimed. Applying this conclusion to the first part of the proof, we see that $\pi : H_X \longrightarrow F$ is an isomorphism, and therefore F is representable. □

Corollary 9. *Let $\mathcal{T}$ be a triangulated category with coproducts and S a perfect generating set. Then $\langle S \rangle = \mathcal{T}$.*

Proof. That is, the smallest localising subcategory of $\mathcal{T}$ containing the objects of S is the whole category. This follows from the observation made in the last part of the proof of Theorem 8. □

Definition 6. Let $\mathcal{T}$ be a triangulated category with coproducts. We say that the *representability theorem holds for* $\mathcal{T}$ if an additive functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and product preserving.

Corollary 10. *Let $\mathcal{T}$ be a triangulated category with coproducts for which the representability theorem holds. Then $\mathcal{T}$ also has products.*

Proof. Given a nonempty family of objects $\{X_i\}_{i\in I}$ of $\mathcal{T}$, the additive functor

$$\prod_{i\in I} Hom_{\mathcal{T}}(-,X_i) : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

is homological and preserves products. It is therefore representable, and any representing object is clearly a product of the family $\{X_i\}_{i\in I}$. □

Lemma 11. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Then F has a right adjoint if and only if for every $D \in \mathcal{D}$ the contravariant functor $H_D F : \mathcal{C} \longrightarrow$ **Sets** is representable.*

Proof. If F has a right adjoint G then $H_D F \cong H_{G(D)}$ so one implication is obvious. For the other, suppose we are given for each $D \in \mathcal{D}$ an object $G(D) \in \mathcal{C}$ representing $H_D F$. That is, there is a bijection natural in X

$$Hom_{\mathcal{C}}(X, G(D)) \longrightarrow Hom_{\mathcal{D}}(F(X), D)$$

A morphism $\alpha : D \longrightarrow D'$ in $\mathcal{D}$ induces a natural transformation $H_\alpha F : H_D F \longrightarrow H_{D'} F$ and therefore a unique morphism $G(\alpha) : G(D) \longrightarrow G(D')$ making the following diagram commute for each $X \in \mathcal{C}$

$$\begin{array}{ccc} Hom_{\mathcal{D}}(F(X), D) & \longrightarrow & Hom_{\mathcal{D}}(F(X), D') \\ \Downarrow & & \Downarrow \\ Hom_{\mathcal{C}}(X, G(D)) & \longrightarrow & Hom_{\mathcal{C}}(X, G(D')) \end{array}$$

This makes G into a functor, which is clearly right adjoint to F. □

Corollary 12. *Let $\mathcal{T}$ be a triangulated category with coproducts for which the representability theorem holds. Then a triangulated functor $\mathcal{T} \longrightarrow \mathcal{S}$ is coproduct preserving if and only if it has a right adjoint.*

Proof. Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a coproduct preserving triangulated functor. Given $D \in \mathcal{C}$ the composite $H_D F : \mathcal{T} \longrightarrow \mathbf{Ab}$ is homological and product preserving, therefore representable. It follows from Lemma 11 that F has a right adjoint. □

Lemma 13. *Let $\mathcal{T}$ be a triangulated category with coproducts, $\mathcal{S} \subseteq \mathcal{T}$ a thick localising subcategory satisfying the representability theorem. Then $\mathcal{S}$ is a bousfield subcategory of $\mathcal{T}$.*

Proof. The inclusion $\mathcal{S} \longrightarrow \mathcal{T}$ is a triangulated functor preserving all coproducts, which by Corollary 12 must have a right adjoint. □

1.1 Dual Notions

Definition 7. Let $\mathcal{T}$ be a triangulated category with products. A nonempty set of objects $S \subseteq \mathcal{T}$ is a *perfect cogenerating set for* $\mathcal{T}$ (or *perfectly cogenerates* $\mathcal{T}$) if it perfectly generates $\mathcal{T}^{\mathrm{op}}$. That is, the following conditions hold:

(H1) Given $X \in \mathcal{T}$ if we have $Hom(X,k) = 0$ for every $k \in S$ then $X = 0$.

(H2) Given a nonempty countable family of morphisms $\{X_i \longrightarrow Y_i\}_{i\in I}$ in $\mathcal{T}$ such that the map $Hom(Y_i, k) \longrightarrow Hom(X_i, k)$ is surjective for every $i \in I, k \in S$, the induced map

$$Hom(\prod_{i\in I} Y_i, k) \longrightarrow Hom(\prod_{i\in I} X_i, k)$$

is also surjective for any $k \in S$.

If S is a perfect cogenerating set then so is the set $\{\Sigma^n k \mid n \in \mathbb{Z}\}$, so $\mathcal{T}$ has a perfect cogenerating set if and only if it has a perfect cogenerating set closed under suspension. If a perfect cogenerating set exists for $\mathcal{T}$ then we say that $\mathcal{T}$ is *perfectly cogenerated* (equivalently, $\mathcal{T}^{\mathrm{op}}$ is perfectly generated).

Theorem 14 (Representability for the Dual). *Let $\mathcal{T}$ be a triangulated category with products and a perfect cogenerating set. Then an additive functor $F : \mathcal{T} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and product preserving.*

Definition 8. Let $\mathcal{T}$ be a triangulated category with products. We say that the *dual representability theorem holds for $\mathcal{T}$* if an additive functor $\mathcal{T} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and preserves products. That is, the representability theorem holds for $\mathcal{T}^{\mathrm{op}}$.

Corollary 15. *Let $\mathcal{T}$ be a triangulated category with products for which the dual representability theorem holds. Then $\mathcal{T}$ also has coproducts.*

Corollary 16. *Let $\mathcal{T}$ be a triangulated category with products for which the dual representability theorem holds. Then a triangulated functor $\mathcal{T} \longrightarrow \mathcal{S}$ is product preserving if and only if it has a left adjoint.*

2 Compactly Generated Triangulated Categories

Definition 9. Let $\mathcal{T}$ be a triangulated category with coproducts. A nonempty set of objects $S \subseteq \mathcal{T}$ is a *compact generating set* for $\mathcal{T}$ (or *compactly generates* $\mathcal{T}$) if it satisfies (G1) and every object $k \in S$ is compact (AC,Definition 18).

If $\mathcal{T}$ admits a compact generating set, then we say that $\mathcal{T}$ is *compactly generated*. It follows from Remark 1 that a compact generating set is a perfect generating set, so any compactly generated triangulated category $\mathcal{T}$ satisfies Brown representability.

In the case of compactly generated triangulated categories, the next result is very useful in identifying the compact objects.

Lemma 17. *Let $\mathcal{T}$ be a compactly generated triangulated category and suppose S is a family of compact generators. Then $\mathcal{T}^c$ is the smallest thick triangulated subcategory of $\mathcal{T}$ containing the objects of S.*

Proof. By Corollary 9 we have $\mathcal{T} = \langle S \rangle$ so this is an immediate consequence of (TRC2,Lemma 49). □

Definition 10. Let $\mathcal{T}$ be a triangulated category. A nonempty set of objects $S \subseteq \mathcal{T}$ is a *symmetric generating set* for $\mathcal{T}$ if it is satisfies (G1) and if there exists a nonempty set of objects $T \subseteq \mathcal{T}$ with the following property:

(G3) For any morphism $X \longrightarrow Y$ the induced map $Hom_{\mathcal{T}}(k, X) \longrightarrow Hom_{\mathcal{T}}(k, Y)$ is surjective for every $k \in S$ if and only if $Hom_{\mathcal{T}}(Y, m) \longrightarrow Hom_{\mathcal{T}}(X, m)$ is injective for every $m \in T$.

If $\mathcal{T}$ has coproducts then it is clear that (G3) implies (G2), so any symmetric generating set for a triangulated category with coproducts is a perfect generating set.

Lemma 18. *If $\mathcal{T}$ is a triangulated category then $\mathcal{T}$ has a symmetric generating set if and only if $\mathcal{T}^{op}$ does.*

Proof. It suffices to show that if $\mathcal{T}$ has a symmetric generating set then so does $\mathcal{T}^{\mathrm{op}}$. Let S be a symmetric generating set for $\mathcal{T}$ with T as in the definition. We claim that T is a symmetric generating set for $\mathcal{T}^{\mathrm{op}}$. To prove (G1), suppose that $X \in \mathcal{T}$ is such that $Hom_{\mathcal{T}}(X, m) = 0$ for every $m \in \mathcal{T}$. Then $Hom_{\mathcal{T}}(X, m) \longrightarrow Hom_{\mathcal{T}}(0, m)$ is injective for every $m \in T$ and so $Hom_{\mathcal{T}}(k, 0) \longrightarrow Hom_{\mathcal{T}}(k, X)$ is surjective for every $k \in S$. Therefore $X = 0$ by (G1) for S.

For (G3), suppose we are given a morphism $Y \longrightarrow X$ in $\mathcal{T}$ which we extend to a triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow \Sigma Y$$

Writing out the corresponding long exact sequences, we deduce the following chain of equivalences

$$\begin{gathered} Hom_{\mathcal{T}}(X,m) \longrightarrow Hom_{\mathcal{T}}(Y,m) \text{ surjective for every } m \in T \\ \Updownarrow \\ Hom_{\mathcal{T}}(\Sigma^{-1}Z,m) \longrightarrow Hom_{\mathcal{T}}(\Sigma^{-1}X,m) \text{ injective for every } m \in T \\ \Updownarrow \\ Hom_{\mathcal{T}}(k,\Sigma^{-1}X) \longrightarrow Hom_{\mathcal{T}}(k,\Sigma^{-1}Z) \text{ surjective for every } k \in S \\ \Updownarrow \\ Hom_{\mathcal{T}}(k,Y) \longrightarrow Hom_{\mathcal{T}}(k,X) \text{ injective for every } k \in S \end{gathered}$$

which proves that T is a symmetric generating set for $\mathcal{T}^{\mathrm{op}}$. □

Proposition 19. *Let $\mathcal{T}$ be a triangulated category with coproducts. Then any compact generating set for $\mathcal{T}$ is also a symmetric generating set.*

Proof. Let S be a compact generating set for $\mathcal{T}$. Consider the abelian group $\mathbb{Q}/\mathbb{Z}$, which is an injective cogenerator for the abelian category **Ab**. For each $k \in S$ we have a homological product preserving functor

$$\begin{gathered} Q^k : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab} \\ Q^k(X) = Hom_{\mathbf{Ab}}(Hom_{\mathcal{T}}(k,X), \mathbb{Q}/\mathbb{Z}) \end{gathered}$$

Since $\mathcal{T}$ is perfectly generated it satisfies the representability theorem, and we can find objects $T_k \in \mathcal{T}$ representing these functors and define $T = \{T_k\}_{k \in S}$. Using the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, it is now easy to check that S satisfies (G3) and is therefore a symmetric generating set. □

So given a triangulated category $\mathcal{T}$ with coproducts the different types of generating sets fit into the following implication: compact $\Longrightarrow$ symmetric $\Longrightarrow$ perfect.

Corollary 20. *If a triangulated category $\mathcal{T}$ is compactly generated, then $\mathcal{T}$ is perfectly cogenerated. In particular $\mathcal{T}$ has products.*

In particular if $\mathcal{T}$ is a compactly generated triangulated category, then the representability theorem *and* the dual representability theorem hold for $\mathcal{T}$. That is, if F is an additive functor $\mathcal{T} \longrightarrow \mathbf{Ab}$ or $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ then F is representable if and only if it is homological and preserves products.

Corollary 21. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangulated functor with $\mathcal{T}$ compactly generated. Then*

(i) F has a right adjoint if and only if it preserves coproducts.

(ii) F has a left adjoint if and only if it preserves products.

Lemma 22. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}, G : \mathcal{S} \longrightarrow \mathcal{T}$ be triangulated functors with $\mathcal{T}$ compactly generated, and suppose that F is left adjoint to G. Then F preserves compactness if and only if G preserves coproducts.*

Proof. Suppose that G preserves coproducts, and let $k \in \mathcal{T}$ be compact. For any coproduct $\bigoplus_i Y_i$ in $\mathcal{S}$ we have an isomorphism

$$\begin{aligned} Hom_{\mathcal{S}}(F(k), \oplus_i Y_i) &\cong Hom_{\mathcal{T}}(k, G(\oplus_i Y_i)) \\ &\cong Hom_{\mathcal{T}}(k, \oplus_i G(Y_i)) \\ &\cong \oplus_i Hom_{\mathcal{T}}(k, G(Y_i)) \\ &\cong \oplus_i Hom_{\mathcal{S}}(F(k), Y_i) \end{aligned}$$

from which it follows that $F(k)$ is compact in $\mathcal{S}$. Note that this direction does not need $\mathcal{T}$ to be compactly generated. Now suppose that F preserves compactness and let $\oplus_i Y_i$ be a coproduct in $\mathcal{S}$. The canonical morphism $\oplus_i G(Y_i) \longrightarrow G(\oplus_i Y_i)$ induces a morphism

$$Hom_{\mathcal{T}}(k, \oplus_i G(Y_i)) \longrightarrow Hom_{\mathcal{T}}(k, G(\oplus_i Y_i))$$

for every object $k \in \mathcal{T}$, which one checks as above is an isomorphism provided k is compact (since we know $F(k)$ is also compact). But then in the triangle

$$\oplus_i G(Y_i) \longrightarrow G(\oplus_i Y_i) \longrightarrow Z \longrightarrow \Sigma \oplus_i G(Y_i)$$

we must have $Hom_{\mathcal{T}}(k, Z) = 0$ for every compact object k. Since $\mathcal{T}$ is compactly generated this implies $Z = 0$, from which we deduce that G preserves coproducts. □

Lemma 23. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangulated functor with $\mathcal{T}$ compactly generated, and let S be a compact generating set for $\mathcal{T}$. Then F preserves compactness if and only if $F(k)$ is compact for every $k \in S$.*

Proof. The compact objects of $\mathcal{S}$ form a thick triangulated subcategory, and therefore so does the subcategory of objects in $\mathcal{T}$ mapping into compacts of $\mathcal{S}$. The claim now follows from Lemma 17. □

3 Portly Considerations

Let $\mathcal{T}$ be a portly triangulated category with coproducts. It is clear what we mean by a *perfect generating set*, a *perfect cogenerating set*, a *compact generating set* and a *symmetric generating set* for $\mathcal{T}$, and therefore what we mean when we say that $\mathcal{T}$ is *perfectly generated*, *perfectly cogenerated* or *compactly generated*. A compact generating set is a perfect generating set.

Definition 11. Let $\mathcal{C}$ be a portly category. We say that $\mathcal{C}$ is *mildly portly* if the object conglomerate of $\mathcal{C}$ is actually a class, and every morphism conglomerate of $\mathcal{C}$ is small. Replacing each morphism conglomerate by a bijective set, we can define a (noncanonical) category $\mathcal{D}$ together with an isomorphism of categories $\mathcal{C} \longrightarrow \mathcal{D}$ which is the identity on objects. In particular if a portly triangulated category $\mathcal{T}$ is mildly portly, then it is triisomorphic to a triangulated category.

The only difference between a mildly portly triangulated category and a triangulated category is some pedantic distinction between small conglomerates and sets (which many authors simply ignore). So one would expect Brown representability to hold under appropriate hypotheses. However, since the morphism conglomerates may not be sets, we have to modify what we mean by a *representable functor*.

Definition 12. Let $\mathcal{C}$ be a preadditive mildly portly category and $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ an additive functor. We say that F is *representable* if there is an additive isomorphism of portly categories $T : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{D}$ a preadditive category (not just a portly category) such that the functor FT is representable. Equivalently, FT is representable for *every* additive isomorphism $\mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{D}$ a preadditive category. If $\mathcal{C}$ happens to be a category, this agrees with the usual definition.

If $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ is representable and $Q : \mathcal{C}' \longrightarrow \mathcal{C}$ an additive equivalence of mildy portly preadditive categories, then QF is also representable. Representability is also stable under natural equivalence of functors $\mathcal{C} \longrightarrow \mathbf{Ab}$.

Lemma 24. *Let $\mathcal{C}$ be a mildly portly preadditive category and $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ an additive functor. Then F is representable if and only if there exists $X \in \mathcal{C}$ together with an isomorphism of (large) abelian groups natural in Y*

$$Hom_{\mathcal{C}}(X, Y) \longrightarrow F(Y)$$

We say that the object X represents *F, and this representing object is unique up to isomorphism.*

Definition 13. Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts. We say that the *representability theorem holds for* $\mathcal{T}$ if an additive functor $\mathcal{T}^{\text{op}} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and product preserving. This property is stable under triequivalence of mildly portly triangulated categories.

If $\mathcal{T}$ is a mildly portly triangulated category with products we say that the *dual representability theorem holds for* $\mathcal{T}$ if the representability theorem holds for $\mathcal{T}^{\text{op}}$, that is, an additive functor $F : \mathcal{T} \longrightarrow \mathbf{Ab}$ is representable if and only if it is homological and product preserving. This property is also stable under triequivalence.

Theorem 25. *Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts and a perfect generating set. Then the representability theorem holds for $\mathcal{T}$.*

Proof. With the definition of a representable functor given in Definition 12 this follows at once from Theorem 8. □

Corollary 26. *Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts for which the representability theorem holds. Then $\mathcal{T}$ also has products.*

Corollary 27. *Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts for which the representability theorem holds. Then a triangulated functor $\mathcal{T} \longrightarrow \mathcal{S}$ into another mildly portly triangulated category is coproduct preserving if and only if it has a right adjoint.*

Dually

Corollary 28. *Let $\mathcal{T}$ be a mildly portly triangulated category with products for which the dual representability theorem holds. Then a triangulated functor $\mathcal{T} \longrightarrow \mathcal{S}$ into another mildly portly triangulated category is product preserving if and only if it has a left adjoint.*

Lemma 29. *Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts, $\mathcal{S} \subseteq \mathcal{T}$ a thick localising portly subcategory satisfying the representability theorem. Then $\mathcal{S}$ is a bousfield subcategory of $\mathcal{T}$.*

Lemma 30. *If $\mathcal{T}$ is a mildly portly triangulated category then $\mathcal{T}$ has symmetric generating set if and only if $\mathcal{T}^{op}$ does.*

Proposition 31. *Let $\mathcal{T}$ be a mildly portly triangulated category with coproducts. Then any compact generating set for $\mathcal{T}$ is also a symmetric generating set.*

Corollary 32. *If a mildly portly triangulated category $\mathcal{T}$ is compactly generated, then $\mathcal{T}$ is perfectly cogenerated.*

In particular if $\mathcal{T}$ is a compactly generated mildly portly triangulated category, then the representability theorem *and* the dual representability theorem hold for $\mathcal{T}$. That is, if F is an additive functor $\mathcal{T} \longrightarrow \mathbf{Ab}$ or $\mathcal{T}^{\text{op}} \longrightarrow \mathbf{Ab}$ then F is representable if and only if it is homological and preserves products.

Corollary 9 is still true with $\mathcal{T}$ a mildly portly triangulated category. Lemma 17 is still true with $\mathcal{T}$ a mildly portly triangulated category. Clearly Lemma 22 and Lemma 23 are still true with $\mathcal{S}, \mathcal{T}$ both mildly portly.

4 Representability for Linear Categories

Throughout this section let k be a commutative ring. Recall the definition of a *k-linear category* from (AC,Definition 35) and a k-linear triangulated category (TRC,Definition 32). Given a k-linear category $\mathcal{A}$ an additive functor $T : \mathcal{A}^{\text{op}} \longrightarrow k\mathbf{Mod}$ is said to be *k-linear* if for every pair $A, B \in \mathcal{A}$ the map

$$Hom_{\mathcal{A}}(A, B) \longrightarrow Hom_k(TB, TA)$$

is a morphism of k-modules. The k-linear functors form a portly abelian subcategory $\mathbf{Mod}_k\mathcal{A}$ of the portly abelian category $(\mathcal{A}^{\text{op}}, k\mathbf{Mod})$ of all additive functors $\mathcal{A}^{\text{op}} \longrightarrow k\mathbf{Mod}$. A sequence in $\mathbf{Mod}_k\mathcal{A}$ of the form

$$M' \longrightarrow M \longrightarrow M''$$

is exact if and only if for the following sequence in $k\mathbf{Mod}$ is exact for every $A \in \mathcal{A}$

$$M'(A) \longrightarrow M(A) \longrightarrow M''(A)$$

Similarly kernels, cokernels and images in $\mathbf{Mod}_k\mathcal{A}$ are computed pointwise. A morphism $\phi : M \longrightarrow N$ in $\mathbf{Mod}_k\mathcal{A}$ is a monomorphism or epimorphism if and ony if $\phi_A : M(A) \longrightarrow N(A)$ has this property for every $A \in \mathcal{A}$. For any object $A \in \mathcal{A}$ the functor $H_A = Hom(-, A)$ defines an object of $\mathbf{Mod}_k\mathcal{A}$. In order to avoid a clash of notation with earlier results, we denote the morphism sets in $\mathbf{Mod}_k\mathcal{A}$ by $Hom_{k\mathcal{A}}(M, N)$. It is clear that $\mathbf{Mod}_k\mathcal{A}$ is a k-linear portly category.

Proposition 33. *If $\mathcal{A}$ is a k-linear category, then*

(i) For any object $A \in \mathcal{A}$ and T in $\mathbf{Mod}_k\mathcal{A}$ there is a canonical isomorphism of k-modules $Hom_{k\mathcal{A}}(H_A, T) \longrightarrow T(A)$ defined by $\gamma \mapsto \gamma_A(1)$, which is natural in A and T.

(ii) The functor $A \mapsto H_A$ defines a full k-linear embedding $\mathcal{A} \longrightarrow \mathbf{Mod}_k\mathcal{A}$.

(iii) The objects $\{H_A\}_{A\in\mathcal{A}}$ form a (large) generating family of projectives for $\mathbf{Mod}_k A$.

Definition 14. Let $\mathcal{S}$ be a k-linear category. We say that an object F of $\mathbf{Mod}_k\mathcal{A}$ is *coherent* if there exists an exact sequence in $\mathbf{Mod}_k\mathcal{A}$ of the following form

$$H_A \longrightarrow H_B \longrightarrow F \longrightarrow 0 \tag{8}$$

Clearly any representable functor is coherent. We denote by $A_k(\mathcal{S})$ the full replete subcategory of $\mathbf{Mod}_k\mathcal{A}$ consisting of the coherent functors. At the moment we only know that this is a k-linear portly category.

If $\mathcal{S}$ is a k-linear category and $\varphi : M \longrightarrow N$ a morphism of $\mathbf{Mod}_k\mathcal{S}$ then given two presentations of the form (8) we can lift φ to a morphism of the presentations. As in Lemma 2 a coherent functor $F \in \mathbf{Mod}_k\mathcal{S}$ preserves products as a functor $\mathcal{S}^{\text{op}} \longrightarrow k\mathbf{Mod}$. The observation of Remark 3 is also still valid.

Lemma 34. *Let $\mathcal{S}$ be a k-linear additive category. If $\varphi : M \longrightarrow N$ is a morphism in $A_k(\mathcal{S})$ then any cokernel of φ in $\mathbf{Mod}_k S$ also belongs to $A_k(\mathcal{S})$.*

Lemma 35. *Let $\mathcal{S}$ be a k-linear additive category with weak kernels. If $\varphi : M \longrightarrow N$ is a morphism in $A_k(\mathcal{S})$ then any kernel of φ in $\mathbf{Mod}_k S$ also belongs to $A_k(\mathcal{S})$.*

Proposition 36. *Let $\mathcal{S}$ be a k-linear additive category with weak kernels. Then $A_k(\mathcal{S})$ is a portly abelian category. If $\mathcal{S}$ has coproducts then $A_k(\mathcal{S})$ is cocomplete and the induced Yoneda functor*

$$H_{(-)} : \mathcal{S} \longrightarrow A_k(\mathcal{S})$$

preserves coproducts.

Lemma 37. *Let $\mathcal{T}$ be a k-linear additive category with coproducts and weak kernels, $S \subseteq \mathcal{T}$ a nonempty set of objects of $\mathcal{T}$, and define $\mathcal{S} = Add(S)$. Then* *(i) The additive category $\mathcal{S}$ has weak kernels, and $A_k(\mathcal{S})$ is a cocomplete portly abelian category.*

(ii) The map $F \mapsto F|_{\mathcal{S}}$ gives an exact functor $A_k(\mathcal{T}) \longrightarrow A_k(\mathcal{S})$.

Lemma 38. *Let $\mathcal{T}$ be a k-linear triangulated category with coproducts, $S \subseteq \mathcal{T}$ a nonempty set of objects, and define $\mathcal{S} = Add(S)$. Then the functor*

$$\mathcal{T} \longrightarrow A_k(\mathcal{S}), \quad X \mapsto H_X|_{\mathcal{S}}$$

is homological. It preserves countable coproducts if and only if $(G2)$ holds for S.

Theorem 39 (Linear Brown Representability). *Let $\mathcal{T}$ be a k-linear triangulated category with coproducts and a perfect generating set. Then a k-linear functor $F : \mathcal{T}^{op} \longrightarrow k\mathbf{Mod}$ is representable if and only if it is homological and product preserving.*

Clearly if $\mathcal{T}$ is a k-linear mildly portly triangulated category with coproducts then the analogue of Theorem 39 holds for $\mathcal{T}$, once we define a k-linear functor $F : \mathcal{T}^{\mathrm{op}} \longrightarrow k\mathbf{Mod}$ to be *representable* if there is $X \in \mathcal{T}$ together with an isomorphism of (large) k-modules natural in Y

$$Hom_{\mathcal{T}}(Y, X) \longrightarrow F(Y)$$

References

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