Triangulated Categories Part II

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Contents

The results in this note are all from [Nee01] or [BN93]. For necessary background on cardinals (particularly *regular* and *singular* cardinals) see our notes on Basic Set Theory (BST). In our BST notes there is no mention of grothendieck universes, whereas all our notes on category theory (including this one) are implicitly working inside a fixed grothendieck universe $\mathfrak{U}$ using the conglomerate convention for $\mathfrak{U}$ (FCT,Definition 5). So some explanation of how these two situations interact is in order.

Definition 1. The expression *ordinal* has the meaning given in (BST,Definition 2), and a cardinal is a special type of ordinal (BST,Definition 7). Under the conglomerate convention, ordinals and cardinals are *a priori* conglomerates, not necessarily sets. We say an ordinal α is *small* if it is a small conglomerate. We make the following observations

- If $\alpha \prec \beta$ are ordinals with β small, then α is also small.
- If an ordinal α is small, so is $\alpha^+ = \alpha \cup \{\alpha\}$.
- The first infinite cardinal $\omega = \aleph_0$ is always small, by our convention that grothendieck universes are always infinite (FCT,Definition 4). Therefore every finite cardinal is small.

Remark 1. Although in our BST notes we consider the finite cardinals $0, 1$ to be regular, it is our convention throughout this note that all regular cardinals are infinite. This just saves us from writing expressions like "small infinite regular cardinal" repeatedly.

Remark 2. Recall that when we say a category $\mathcal{C}$ is *complete*, or even just *has coproducts*, we mean that all *set-indexed* colimits (resp. coproducts) exist in $\mathcal{C}$ (we are working under the conglomerate convention, so a set is an element of our universe $\mathfrak{U}$). If β is a small cardinal, we say that $\mathcal{C}$ *has β-coproducts* if any family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ indexed by a set I of cardinality $< \beta$ has a coproduct. For example, $\mathcal{C}$ has finite coproducts iff. it has $\aleph_0$-coproducts and has countable coproducts iff. it has $\aleph_1$-coproducts.

Lemma 1. *If a cardinal κ is small, then so is the successor cardinal of κ. In particular for any ordinal α if the cardinal $\aleph_\alpha$ is small, so is $\aleph_{\alpha^+}$.*

Proof. See (BST,Definition 15) for the definition of the cardinal successor (not to be confused with the ordinal successor). Let κ be a small cardinal, so that κ is in bijection with some $x \in \mathfrak{U}$ and therefore satisfies $\kappa < c(\mathfrak{U})$ by (BST,Lemma 43). Since the cardinal $c(\mathfrak{U})$ is strongly inaccessible (BST,Proposition 44) it follows that $\kappa < 2^\kappa < c(\mathfrak{U})$. If δ is the cardinal successor of κ then we must have $\delta \le 2^\kappa$ and therefore $\delta < c(\mathfrak{U})$. From (BST,Lemma 43) we conclude that δ is also small. □

1 Finer Localising Subcategories

Definition 2. Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and S a nonempty class of objects of $\mathcal{T}$. Then $\langle S\rangle^\beta$ denotes the smallest triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ satisfying the following conditions

(i) The objects of S belong to $\mathcal{S}$.

(ii) Any coproduct in $\mathcal{T}$ of fewer than β objects of $\mathcal{S}$ belongs to $\mathcal{S}$. That is, if $\{X_i\}_{i\in I}$ is a family of objects of $\mathcal{S}$ indexed by a nonempty set I of cardinality $< \beta$ then any coproduct of this family in $\mathcal{T}$ belongs to $\mathcal{S}$.

(iii) The subcategory $\mathcal{S} \subseteq \mathcal{T}$ is thick.

Remark 3. The triangulated subcategory $\langle S\rangle^\beta$ exists, because we can take the intersection of all the triangulated subcategories of $\mathcal{T}$ satisfying conditions $(i), (ii), (iii)$ (observe that this collection is going to be a conglomerate, not a set).

In this section we want to show that provided S is a set, the subcategory $\langle S\rangle^\beta$ is essentially small. The first observation is

Lemma 2. *Let $\mathcal{T}$ be a triangulated category with coproducts, $\beta \le \gamma$ small infinite cardinals and S a nonempty class of objects of $\mathcal{T}$. Then $\langle S\rangle^\beta \subseteq \langle S\rangle^\gamma$.*

Definition 3. Let $\mathcal{T}$ be a triangulated category and S a nonempty class of objects of $\mathcal{T}$. We let $\overline{T}(S)$ denote the smallest triangulated subcategory of $\mathcal{T}$ containing the objects of S (that is, the intersection of all such subcategories).

Lemma 3. *Let S be a nonempty set of objects in a triangulated category $\mathcal{T}$. Then the category $\overline{T}(S)$ is essentially small.*

Proof. Define $T_1(S)$ to be the full subcategory of $\mathcal{T}$ whose objects are the objects of S. The category $T_1(S)$ is certainly small. We define a small full subcategory $T_{n+1}(S)$ for $n > 0$ inductively as follows: suppose $T_n(S)$ has been defined and is small. Choose for every morphism $f : X \longrightarrow Y$ in $T_n(S)$ a particular triangle in $\mathcal{T}$

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \qquad (1)$$

Let $T_{n+1}(S)$ be the full subcategory of $\mathcal{T}$ whose objects are the objects of $T_n(S)$ together with the following objects:

(i) The object C_f for every morphism $f : X \longrightarrow Y$ in $T_n(S)$.

(ii) The object $\Sigma^{-1}X$ for every $X \in T_n(S)$.

This is a small full subcategory of $\mathcal{T}$.

Now put $T(S) = \bigcup_{n=1}^\infty T_n(S)$, which is a small full subcategory of $\mathcal{T}$. Given any morphism $f : X \longrightarrow Y$ in $T(S)$, it lies in some $T_n(S)$. But then $T_{n+1}(S)$ contains an object C_f fitting into a triangle (1). It is clear that $T(S)$ is closed under Σ^{-1}, so by (TRC,Lemma 33) the replete closure $\overline{T}(S)$ of $T(S)$ is a triangulated subcategory of $\mathcal{T}$ (and is therefore the smallest triangulated subcategory containing S). Since the inclusion $T(S) \longrightarrow \overline{T}(S)$ is an equivalence, we see that $\overline{T}(S)$ is essentially small. □

For use in the proof of the next result, we make some technical definitions.

Definition 4. Let $\mathcal{T}$ be a triangulated category with coproducts and β a small infinite cardinal. Let Q be the conglomerate of all families of objects of $\mathcal{T}$ indexed by cardinals $< \beta$ (to be precise, elements of Q are functions $f : \alpha \longrightarrow \mathcal{T}$ where $\alpha < \beta$ is a cardinal). Simultaneously choose a coproduct in $\mathcal{T}$ for every family in Q, and denote this assignment by $\mathcal{Q}$. We call $\mathcal{Q}$ an *assignment of canonical coproducts of size* $< \beta$.

Let S be a nonempty set of objects of $\mathcal{T}$. Let $CP_{\beta,\mathcal{Q}}(S)$ denote the set of objects of $\mathcal{T}$ consisting of the objects of S together with the canonical coproduct (i.e. the one chosen by $\mathcal{Q}$) for every family of objects of S indexed by a cardinal $< \beta$. The intuitive meaning of this construction is clear: we start with S, and add a coproduct for every family of objects of S smaller than β.

Definition 5. Let $\mathcal{T}$ be a triangulated category. An *assignment of mapping cones* is a function $\mathcal{C}$ which assigns to every morphism $f : X \longrightarrow Y$ in $\mathcal{T}$ an object C_f which occurs in a triangle of the form

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X$$

Such assignments certainly exist.

Remark 4. For the duration of this remark we drop the conglomerate convention (FCT,Definition 5). In the proof of Proposition 4 we are going to use a construction by transfinite recursion. To do this carefully we have to define two "constructions" $\tau(x), \mu(x)$ on all sets x (BST,Remark 3). We define the construction τ as follows

(i) If $x = (\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \mathcal{C}, S)$ is a tuple consisting of a universe $\mathfrak{U}$, a triangulated category $\mathcal{T}$ with coproducts, a small infinite cardinal β, an assignment $\mathcal{Q}$ of canonical coproducts of size $< \beta$, an assignment $\mathcal{C}$ of mapping cones, and a nonempty set of objects $S \subseteq \mathcal{T}$ (here the meaning of "category" and "set" are relative to $\mathfrak{U}$) then $\tau(x)$ is defined in the following way: first form the set $CP_{\beta,\mathcal{Q}}(S)$. Then, using the mapping cones selected by $\mathcal{C}$, we can canonically form the set $\varphi(x) = T_{\mathcal{C}}(CP_{\beta,\mathcal{Q}}(S))$ defined in the proof of Lemma 3. Then $\tau(x) = (\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \mathcal{C}, \varphi(x))$.

(ii) If x is not of this form, then $\tau(x)$ is the empty set.

We define the construction μ as follows

(a) If x is nonempty and consists of tuples of the form given in (i) above, all of which are equal to each other in the first five places $\{(\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \mathcal{S}, S_k)\}_{k\in K}$, then $\mu(x)$ is constructed from the union over all the tuples of the sets in the last position. That is, $\mu(x) = (\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \bigcup_k S_k)$.

(b) If x does not have the form described in (a), then $\mu(x)$ is the empty set.

We remark that the "output" in case (a) need not be a tuple of the form given in (i), since the union set may not be small if K is too large. But in our application, this will not concern us.

Proposition 4. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and S a nonempty set of objects of $\mathcal{T}$. Then the category $\langle S\rangle^\beta$ is essentially small.*

Proof. Let our infinite cardinal β be given, and let γ be the successor cardinal (BST,Definition 15). Then by Lemma 2 we have $\langle S\rangle^\beta \subseteq \langle S\rangle^\gamma$ and to show the former category is essentially small it suffices to show that the latter category is essentially small. So we may reduce to the case where $\beta > \aleph_0$ is an infinite successor cardinal, and therefore regular (BST,Proposition 38).

First observe that any triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ satisfying condition (ii) of Definition 2 for β is automatically thick, since $\beta > \aleph_0$ and (TRC,Corollary 85) implies that a triangulated subcategory closed under countable coproducts is thick. Therefore $\langle S\rangle^\beta$ is the smallest triangulated subcategory satisfying conditions $(i), (ii)$ of Definition 2.

Choose an assignment $\mathcal{Q}$ of canonical coproducts of size $< \beta$ to the objects of $\mathcal{T}$, and also an assignment $\mathcal{C}$ of mapping cones. Using the constructions τ, μ defined in Remark 4 and (BST,Theorem

17) with initial conglomerate $z = (\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \mathcal{C}, T_{\mathcal{C}}(S))$ we define a function f_α on α^+ for every ordinal α. For any ordinal $\alpha \preceq \beta$ one checks that

$$f_\alpha(\alpha) = (\mathfrak{U}, \mathcal{T}, \beta, \mathcal{Q}, \mathcal{C}, S_\alpha)$$

for some set S_α of objects of $\mathcal{T}$. These sets S_α for $\alpha \preceq \beta$ have the following properties

$$S_0 = T_{\mathcal{C}}(S)$$
$$S_{\alpha^+} = T_{\mathcal{C}}(CP_{\beta,\mathcal{Q}}(S_\alpha))$$
$$S_\alpha = \bigcup_{\gamma \prec \alpha} S_\gamma \text{ for any limit ordinal } \alpha$$

By construction we have $S_\gamma \subseteq S_\alpha$ for any ordinals $\gamma \prec \alpha \preceq \beta$, and in particular S is contained in every S_α. Let $\overline{S}_\alpha$ denote full replete subcategory of $\mathcal{T}$ whose objects are those isomorphic to an object of S_α. It is straightforward using (TRC,Lemma 33) to check that $\overline{S}_\alpha$ is a triangulated subcategory of $\mathcal{T}$ which contains S, for every ordinal $\alpha \preceq \beta$. In particular this is true of $\overline{S}_\beta$.

Next we claim that the triangulated subcategory $\overline{S}_\beta$, which certainly satisfies condition (i) of Definition 2, also satisfies the condition (ii). Suppose $\{X_i\}_{i \in I}$ is a nonempty family of objects of $\overline{S}_\beta$ indexed by a nonempty set I with $|I| < \beta$. We can assume that $I = |I|$ is a cardinal $< \beta$ and also that every X_i belongs to S_β.

Any infinite cardinal is a limit ordinal, so $S_\beta = \bigcup_{\gamma \prec \beta} S_\gamma$. For each $i \in I$ choose an ordinal $\gamma_i \prec \beta$ with $X_i \in S_{\gamma_i}$. Let γ be the ordinal $\gamma = \bigcup_i \gamma_i$. Since it is a union of $I < \beta$ cardinals all of size $< \beta$ it follows from (BST,Proposition 42) and regularity of β that $\gamma \prec \beta$. Therefore S_γ contains every X_i, and by construction $S_{\gamma^+} \subseteq S_\beta$ contains some coproduct in $\mathcal{T}$ of the family $\{X_i\}_{i \in I}$. This shows that $\overline{S}_\beta$ satisfies the conditions of Definition 2, from which we deduce $\langle S \rangle^\beta \subseteq \overline{S}_\beta$. But $\overline{S}_\beta$ is an essentially small category, so the smaller category $\langle S \rangle^\beta$ is also essentially small, which is what we wanted to show. □

Definition 6. Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a triangulated subcategory and β a small infinite cardinal. We say that $\mathcal{S}$ is *β-localising* if it is closed under coproducts in $\mathcal{T}$ indexed by sets of cardinality $< \beta$. Dually we say that $\mathcal{S}$ is *β-colocalising* if it is closed under products indexed by sets of cardinality $< \beta$.

Remark 5. Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and S a nonempty class of objects of $\mathcal{T}$. Then by definition the triangulated subcategory $\langle S \rangle^\beta$ is β-localising.

The following result may seem obvious, but a careful proof requires some subtle set theory.

Lemma 5. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a triangulated subcategory. Then $\mathcal{S}$ is localising if and only if it is β-localising for every small infinite cardinal β.*

Proof. See (TRC,Definition 37) for the definition of a localising subcategory. The condition is clearly necessary, so suppose that $\mathcal{S}$ is a triangulated subcategory which is β-localising for every small infinite cardinal β. If $\{S_i\}_{i \in I}$ is any nonempty family of objects of $\mathcal{S}$, by which we mean that I is a set and each $S_i \in \mathcal{S}$, then the cardinal $\kappa = |I|$ is small. If this cardinal is finite, then since $\mathcal{S}$ is additive it trivially contains any coproduct of the S_i. So we may assume κ is a small infinite cardinal. By Lemma 1 the successor cardinal β of κ is also small, and $\kappa < \beta$ so since $\mathcal{S}$ is assumed to be β-localising any coproduct $\bigoplus_{i \in I} S_i$ in $\mathcal{T}$ must belong to $\mathcal{S}$. □

Remark 6. If $\mathcal{T}$ is a triangulated category with coproducts, and $\mathcal{S}$ a β-localising subcategory for any small infinite cardinal $\beta > \aleph_0$, then $\mathcal{S}$ contains countable coproducts of its objects and is therefore thick (TRC,Corollary 85).

Lemma 6. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a triangulated subcategory and β a small infinite cardinal. If $\mathcal{S}$ is β-localising then so is its thick closure $\widehat{\mathcal{S}}$.*

Proof. Suppose we are given a family of objects $\{X_i\}_{i\in I}$ of $\widehat{\mathcal{S}}$ indexed by a set I of cardinality $< \beta$. By definition for each $i \in I$ there exists an object Y_i with $X_i \oplus Y_i \in \mathcal{S}$. Then

$$(\bigoplus_i X_i) \oplus (\bigoplus_i Y_i) = \bigoplus_i (X_i \oplus Y_i)$$

which by assumption belongs to $\mathcal{S}$. Therefore $\widehat{\mathcal{S}}$ is β-localising. □

Proposition 7. *Let $\mathcal{T}$ be a triangulated category with coproducts and S a nonempty class of objects of $\mathcal{T}$. Then the triangulated subcategory $\langle S\rangle = \bigcup_\beta \langle S\rangle^\beta$ is localising, and is the smallest localising subcategory containing S.*

Proof. Observe that since every small infinite cardinal is in bijection with an element of our universe $\mathfrak{U}$, we can form the conglomerate of all small infinite cardinals. So the union $\bigcup_\beta \langle S\rangle^\beta$ of the class $\langle S\rangle^\beta$, as β ranges over all small infinite cardinals, makes sense. Using (TRC,Lemma 33) one checks easily that $\langle S\rangle$ is a triangulated subcategory of $\mathcal{T}$. To see that it is localising, let $\{X_i\}_{i\in I}$ be any nonempty family of objects of $\langle S\rangle$, which we may as well assume is infinite. The cardinal $\beta = |I|$ is a small infinite cardinal, and for each $i \in I$ we choose a small infinite cardinal β_i with $X_i \in \langle S\rangle^{\beta_i}$.

By (BST,Proposition 44) the cardinal $c(\mathfrak{U})$ is regular, so the infinite cardinal $\bigcup_i \beta_i$ is still small. Combining this with Lemma 1 we can find a small infinite cardinal γ with $\gamma > \beta$ and $\gamma > \beta_i$ for each $i \in I$. Then all the X_i belong to $\langle S\rangle^\gamma$ and since $\beta < \gamma$ we can use the fact that $\langle S\rangle^\gamma$ is γ-localising to conclude that any coproduct of the X_i in $\mathcal{T}$ belongs to $\langle S\rangle^\gamma$ and therefore to $\langle S\rangle$. This shows that $\langle S\rangle$ is localising. For the final statement, observe that any localising subcategory containing S must contain $\langle S\rangle^\beta$ for every small infinite cardinal β, and therefore contains $\langle S\rangle$. □

Definition 7. Let $\mathcal{T}$ be a triangulated category with coproducts and S a nonempty class of objects of $\mathcal{T}$. We call the localising subcategory $\langle S\rangle$ the *localising subcategory generated by* S.

Now that we have refined the notion of a localising subcategory to a β-localising subcategory, we can also refine some results from (TRC,Section 4).

Lemma 8. *Let β be a small infinite cardinal and $\mathcal{T}$ a triangulated category with β-coproducts. Let $\mathcal{S}$ be a β-localising subcategory of $\mathcal{T}$. Then the portly triangulated category $\mathcal{T}/\mathcal{S}$ has β-coproducts and the canonical functor $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ preserves β-coproducts.*

Proof. See Remark 2 for the definition of a category with β-coproducts. It suffices to show that F preserves β-coproducts, and the proof is exactly the one given in (TRC,Lemma 91). □

Example 1. Let $\mathcal{T}$ be a triangulated category with coproducts, and $\mathcal{S}$ a β-localising subcategory for some small infinite cardinal $\beta > \aleph_0$. It is not true in general that $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ preserves *all* coproducts. If it did then $\mathcal{S}$ would be localising, for given any family of objects $\{X_i\}_{i\in I}$ in $\mathcal{S}$ we would have

$$F(\bigoplus_i X_i) = \bigoplus_i F(X_i) = 0$$

By Remark 6 the subcategory $\mathcal{S}$ is thick, so from $F(\bigoplus_i X_i) = 0$ we deduce that $\bigoplus_i X_i \in \mathcal{S}$. Therefore $\mathcal{S}$ is localising.

2 Perfect Classes

Definition 8. Let $\mathcal{T}$ be a triangulated category with coproducts and β a small infinite cardinal. A class of objects $S \subseteq \mathcal{T}$ is called *β-perfect* if it satisfies the following conditions

(P0) S contains some zero object of $\mathcal{T}$.

(P1) Given a nonempty family of objects $\{X_i\}_{i\in I}$ of $\mathcal{T}$ indexed by a set I of cardinality $< \beta$ and a morphism $\varphi : k \longrightarrow \bigoplus_{i\in I} X_i$ with $k \in S$, there are morphisms $f_i : k_i \longrightarrow X_i$ with $k_i \in S$ and a commutative diagram of the following form

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow^{\varphi} \\ \bigoplus_i k_i & \xrightarrow[\bigoplus_i f_i]{} & \bigoplus_i X_i \end{array}$$

(P2) Given a nonempty family of morphisms $\{f_i : k_i \longrightarrow X_i\}_{i\in I}$ of $\mathcal{T}$ indexed by a set I of cardinality $< \beta$ with each $k_i \in S$, if there is a morphism $k \longrightarrow \bigoplus_i k_i$ with $k \in S$ making the following composite zero

$$k \longrightarrow \bigoplus_i k_i \xrightarrow{\bigoplus_i f_i} \bigoplus_i X_i$$

then each f_i can be factored through an object $l_i \in S$ as follows

$$k_i \xrightarrow{g_i} l_i \xrightarrow{h_i} X_i$$

such that the following composite also vanishes

$$k \longrightarrow \bigoplus_i k_i \xrightarrow{\bigoplus_i g_i} \bigoplus_i l_i$$

Example 2. Let $\mathcal{T}$ be a triangulated category with coproducts and β a small infinite cardinal. If 0 is any zero object of $\mathcal{T}$ then the class $\{0\}$ is β-perfect. The whole category $\mathcal{T}$ is also β-perfect. If $S \subseteq \mathcal{T}$ is a β-perfect class then it is also γ-perfect for any small infinite cardinal $\gamma < \beta$.

In the next two results, $\mathcal{T}$ is a triangulated category with coproducts and β is a small infinite cardinal.

Lemma 9. *Let $T \subseteq \mathcal{T}$ be a β-perfect class for $\mathcal{T}$. Suppose that $S \subseteq T$ is an equivalent class; that is, any object of T is isomorphic to some object of S. Then S is also β-perfect.*

Lemma 10. *Let S be a β-perfect class for $\mathcal{T}$ and let T be the class of all objects in $\mathcal{T}$ which are direct summands of objects of S. Then T is also β-perfect.*

Proof. Observe that the definition of T is fragile, in the following sense: an object $X \in \mathcal{T}$ belongs to T if and only if there exists an object $Y \in \mathcal{T}$ and a coproduct $X \oplus Y$ in $\mathcal{T}$ with $X \oplus Y \in S$. Since S is not replete, it does not follow that *every* such coproduct must belong to S.

Any object $X \in S$ is the coproduct of itself with any zero object of $\mathcal{T}$, from which we deduce that T contains S and every zero object of $\mathcal{T}$. In particular T satisfies P0. To establish P1, let a nonempty family $\{X_i\}_{i\in I}$ with $|I| < \beta$ and a morphism $\varphi : k \longrightarrow \bigoplus_i X_i$ with $k \in T$ be given. There exists some $k' \in \mathcal{T}$ and a coproduct $k \oplus k' \in S$. Consider the composite

$$k \oplus k' \xrightarrow{(1\ \ 0)} k \xrightarrow{\varphi} \bigoplus_i X_i$$

Since S is β-perfect this must factor in the following way

$$k \oplus k' \longrightarrow \bigoplus_i k_i \xrightarrow{\bigoplus f_i} \bigoplus_i X_i$$

with every $k_i \in S \subseteq T$. Composing $k \oplus k' \longrightarrow \bigoplus_i k_i$ with the injection $k \longrightarrow k \oplus k'$ we have a factorisation of the original morphism φ, as required.

It remains to check P2. Suppose we are given a vanishing composite

$$k \longrightarrow \bigoplus_i k_i \xrightarrow{\bigoplus_i f_i} \bigoplus_i X_i$$

with $k, k_i \in T$. Choose coproducts $k \oplus k' \in S$ and $k_i \oplus k_i' \in S$ for each $i \in I$. Then the following composite is zero

$$k \oplus k' \longrightarrow k \longrightarrow \bigoplus_i k_i \xrightarrow{\oplus_i \binom{1}{0}} \bigoplus_i (k_i \oplus k_i') \xrightarrow{\oplus_i (f_i \ 0)} \bigoplus_i X_i$$

Since S is β-perfect, for each $i \in I$ the morphism $(f_i \ \ 0) : k_i \oplus k_i' \longrightarrow X_i$ must factor through an object $l_i \in S$ as $k_i \oplus k_i' \longrightarrow l_i \longrightarrow X_i$ with the following composite equal to zero

$$k \oplus k' \longrightarrow k \longrightarrow \bigoplus_i k_i \xrightarrow{\oplus_i \binom{1}{0}} \bigoplus_i (k_i \oplus k_i') \longrightarrow \bigoplus_i l_i$$

But then composite of this morphism with the injection $k \longrightarrow k \oplus k'$ is also zero, so the proof is complete. $\square$

Definition 9. Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory. An object $k \in \mathcal{T}$ is called *β-good for $\mathcal{S}$* if whenever we have a nonempty family of objects $\{X_i\}_{i \in I}$ of $\mathcal{T}$ indexed by a set I of cardinality $< \beta$ and a morphism $\varphi : k \longrightarrow \bigoplus_{i \in I} X_i$ there are morphisms $f_i : k_i \longrightarrow X_i$ with $k_i \in \mathcal{S}$ and a commutative diagram of the following form

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow^{\varphi} \\ \bigoplus_i k_i & \xrightarrow[\oplus_i f_i]{} & \bigoplus_i X_i \end{array}$$

Example 3. Let $\mathcal{T}, \beta, \mathcal{S}$ be as in Definition 9. Any zero object $0 \in \mathcal{T}$ is β-good for $\mathcal{S}$. If k is β-good for $\mathcal{S}$ then so are $\Sigma k, \Sigma^{-1} k$ and any object isomorphic to k in $\mathcal{T}$. Any direct summand in $\mathcal{T}$ of a β-good object for $\mathcal{S}$ is also β-good for $\mathcal{S}$. If k is β-good for $\mathcal{S}$ then it is also β-good for any larger triangulated subcategory $\mathcal{S}' \supseteq \mathcal{S}$.

Lemma 11. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory. If k is an object of $\mathcal{T}$ which is β-good for $\mathcal{S}$ then*

- *Given a nonempty family of morphisms $\{f_i : k_i \longrightarrow X_i\}_{i \in I}$ of $\mathcal{T}$ indexed by a set I of cardinality $< \beta$ with each $k_i \in \mathcal{S}$, if there is a morphism $k \longrightarrow \bigoplus_i k_i$ making the following composite zero*

  $$k \longrightarrow \bigoplus_i k_i \xrightarrow{\oplus_i f_i} \bigoplus_i X_i$$

  then each f_i can be factored through an object $l_i \in \mathcal{S}$ as $k_i \longrightarrow l_i \longrightarrow X_i$ with the following composite equal to zero

  $$k \longrightarrow \bigoplus_i k_i \longrightarrow \bigoplus_i l_i$$

Proof. Suppose we are in the situation described above. For each $i \in I$ let $w_i : Y_i \longrightarrow k_i$ be a homotopy kernel of f_i. It follows from (TRC,Remark 9) that the morphism $\bigoplus_i w_i$ is a homotopy kernel of $\bigoplus_i f_i$. As the composite of $\bigoplus f_i$ with our morphism $\alpha : k \to \bigoplus_i k_i$ is zero, we deduce that α factors as $k \longrightarrow \bigoplus Y_i \longrightarrow \bigoplus_i k_i$. But then since k is β-good for $\mathcal{S}$, the morphism $k \longrightarrow \bigoplus_i Y_i$ must factor as $k \longrightarrow \bigoplus_i j_i \longrightarrow \bigoplus_i Y_i$ with $j_i \in \mathcal{S}$. The composite

$$j_i \longrightarrow Y_i \longrightarrow k_i \xrightarrow{f_i} X_i$$

is zero. So if we choose for each $i \in I$ a homotopy cokernel $g_i : k_i \longrightarrow l_i$ of $j_i \longrightarrow k_i$, we obtain a morphism $h_i : l_i \longrightarrow X_i$ with $f_i = h_i \circ g_i$. Since j_i, k_i both belong to the triangulated subcategory $\mathcal{S}$, so does the homotopy cokernel l_i. It is clear by construction that the composite $k \longrightarrow \bigoplus_i k_i \longrightarrow \bigoplus_i l_i$ vanishes, so the proof is complete. $\square$

Lemma 12. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory whose objects are all β-good for $\mathcal{S}$. Then $\mathcal{S}$ is a β-perfect class.*

Proof. This follows immediately from Lemma 11. ☐

In fact we will see in the next few results that to show $\mathcal{S}$ is β-perfect, it suffices to show that it has a large enough generating class of β-good objects.

Proposition 13. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory. Then the full subcategory of $\mathcal{T}$ defined by*

$$\mathcal{Q} = \{k \in \mathcal{S} \,|\, k \text{ is } \beta\text{-good for } \mathcal{S}\}$$

is a triangulated subcategory of $\mathcal{T}$. If $\mathcal{S}$ is thick then so is $\mathcal{Q}$.

Proof. It is clear from Example 3 that $\mathcal{Q}$ is replete and closed under Σ, Σ^{-1}, so by (TRC,Lemma 33) it suffices to show that $\mathcal{Q}$ is closed under mapping cones (if $\mathcal{S}$ is thick then Example 3 also shows that $\mathcal{Q}$ is thick). So suppose we have a triangle in $\mathcal{T}$

$$k \xrightarrow{f} l \longrightarrow m \longrightarrow \Sigma k \tag{2}$$

with $k, l \in \mathcal{Q}$. Let a nonempty family $\{X_i\}_{i \in I}$ of objects of $\mathcal{T}$ be given, where $|I| < \beta$, and let $\bigoplus_i X_i$ be any coproduct in $\mathcal{T}$. Suppose that we are given a morphism $h : m \longrightarrow \bigoplus_i X_i$. Composing with $l \longrightarrow m$ yields a morphism $l \longrightarrow \bigoplus_i X_i$ making the following composite vanish

$$k \longrightarrow l \longrightarrow \bigoplus_i X_i$$

Since l is β-good for $\mathcal{S}$ there exists a factorisation of $l \longrightarrow \bigoplus_i X_i$ through some morphism $\bigoplus_i f_i : \bigoplus_i l_i \longrightarrow \bigoplus_i X_i$ with $l_i \in \mathcal{S}$. The composite with $k \longrightarrow l$ vanishes, so from Lemma 11 we deduce that every f_i factors as

$$l_i \xrightarrow{g_i} m_i \xrightarrow{h_i} X_i$$

with $m_i \in \mathcal{S}$ and the following composite equal to zero

$$k \longrightarrow l \longrightarrow \bigoplus_i l_i \xrightarrow{\bigoplus_i g_i} \bigoplus_i m_i$$

Since (2) is a triangle, we infer that $l \longrightarrow \bigoplus_i m_i$ factors through $l \longrightarrow m$, with factorisation $g : m \longrightarrow \bigoplus_i m_i$ say. The composite $\bigoplus_i h_i \circ g$ need not agree with the given morphism h, but their composites with $l \longrightarrow m$ do agree. Their difference $h - \bigoplus_i h_i \circ g$ therefore factors through Σk. Since $\Sigma k \in \mathcal{Q}$ this factorisation $\Sigma k \longrightarrow \bigoplus_i X_i$ must factor as

$$\Sigma k \longrightarrow \bigoplus_i m_i' \xrightarrow{\bigoplus_i h_i'} \bigoplus_i X_i$$

with $m_i' \in \mathcal{S}$. The direct sum $(\bigoplus_i m_i) \oplus (\bigoplus_i m_i')$ is also a coproduct $\bigoplus_i (m_i \oplus m_i')$, and we denote by q the sum of the following two composites

$$m \xrightarrow{g} \bigoplus_i m_i \longrightarrow \bigoplus_i (m_i \oplus m_i')$$

$$m \longrightarrow \Sigma k \longrightarrow \bigoplus_i m_i' \longrightarrow \bigoplus_i (m_i \oplus m_i')$$

By construction h is equal to the composite of $\bigoplus_i (h_i \quad h_i') : \bigoplus_i (m_i \oplus m_i') \longrightarrow \bigoplus_i X_i$ with q, which shows that m is β-good for $\mathcal{S}$ and consequently that $m \in \mathcal{Q}$. Therefore $\mathcal{Q}$ is a triangulated subcategory of $\mathcal{T}$, and the proof is complete. ☐

Lemma 14. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and S a nonempty class of objects of $\mathcal{T}$. Suppose that every object of S is β-good for the triangulated subcategory $\overline{T}(S) \subseteq \mathcal{T}$. Then $\overline{T}(S)$ is a β-perfect class.*

Proof. We define a full subcategory $\mathcal{R}$ of $\mathcal{T}$ as follows

$$\mathcal{R} = \{k \in \overline{T}(S) \,|\, k \text{ is } \beta\text{-good for } \overline{T}(S)\}$$

By Lemma 12 it suffices to show that $\mathcal{R} = \overline{T}(S)$, so we need only show that $\mathcal{R}$ is a triangulated subcategory of $\mathcal{T}$. But this is a special case of Proposition 13, so we are done. □

Lemma 15. *Let $\mathcal{T}$ be a triangulated category with coproducts, α, β small infinite cardinals and S a nonempty class of objects of $\mathcal{T}$. Suppose that every object of S is β-good for the triangulated subcategory $\langle S\rangle^\alpha \subseteq \mathcal{T}$. Then $\langle S\rangle^\alpha$ is a β-perfect class.*

Proof. We define a full subcategory $\mathcal{S}$ of $\mathcal{T}$ as follows

$$\mathcal{Q} = \{k \in \langle S\rangle^\alpha \,|\, k \text{ is } \beta\text{-good for } \langle S\rangle^\alpha\}$$

To show that $\langle S\rangle^\alpha$ is β-perfect, it suffices by Lemma 12 to show that $\mathcal{Q} = \langle S\rangle^\alpha$. It is therefore enough to show that $\mathcal{Q}$ is a triangulated subcategory of $\mathcal{T}$ satisfying the conditions $(i), (ii), (iii)$ of Definition 2. It is a thick triangulated subcategory of $\mathcal{T}$ by Proposition 13 and the condition (i) is true by hypothesis, so to complete the proof, we have only to show that any coproduct in $\mathcal{T}$ of fewer than α objects of $\mathcal{Q}$ belongs to $\mathcal{Q}$.

Let $\{k_\mu\}_{\mu\in\Lambda}$ be a nonempty family of objects of $\mathcal{Q}$ indexed by a set Λ of cardinality $< \alpha$ and let $\bigoplus_\mu k_\mu$ be any coproduct in $\mathcal{T}$. We know that this coproduct belongs to $\langle S\rangle^\alpha$, so we need only show that it is β-good for $\langle S\rangle^\alpha$. Suppose we are given a nonempty family $\{X_i\}_{i\in I}$ of objects of $\mathcal{T}$ with $|I| < \beta$ and a morphism $\bigoplus_\mu k_\mu \longrightarrow \bigoplus_i X_i$. In other words, we have for each $\mu \in \Lambda$ a morphism $k_\mu \longrightarrow \bigoplus_i X_i$, and this must factor as

$$k_\mu \longrightarrow \bigoplus_{i\in I} k_{\mu,i} \xrightarrow{\bigoplus_i f_{\mu,i}} \bigoplus_{i\in I} X_i$$

for objects $k_{\mu,i} \in \langle S\rangle^\alpha$. It follows that our morphism $\bigoplus_\mu k_\mu \longrightarrow \bigoplus_i X_i$ must factor as

$$\bigoplus_\mu k_{\mu\in\Lambda} \longrightarrow \bigoplus_{i\in I} \bigoplus_{\mu\in\Lambda} k_{\mu,i} \xrightarrow{\quad\bigoplus_i G_i\quad} \bigoplus_{i\in I} X_i$$

where $G_i : \bigoplus_\mu k_{\mu,i} \longrightarrow X_i$ has μth coordinate $f_{\mu,i}$. This shows that $\bigoplus_\mu k_\mu$ is β-good for $\mathcal{S}$, and therefore belongs to $\mathcal{Q}$, which completes the proof. □

Theorem 16. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\{S_i\}_{i\in I}$ a nonempty family of β-perfect classes of $\mathcal{T}$. Then*

(i) The class of objects $\overline{T}(\bigcup_i S_i)$ is β-perfect.

(ii) For any small infinite cardinal α the class of objects $\langle \bigcup_i S_i\rangle^\alpha$ is β-perfect.

Proof. To be clear, whenever we say "family" we generally mean "conglomerate" and in this case we only require the indices I to be a nonempty conglomerate (not necessarily a set, or even a class). In any case, we can certainly form the class $\bigcup_i S_i$ of objects of $\mathcal{T}$. To prove (ii) it suffices by Lemma 15 to show that any $k \in \bigcup_i S_i$ is β-good for $\langle \bigcup_i S_i\rangle^\alpha$, but when one recalls what it means for each S_i to be β-perfect, this is trivial. Using Lemma 14 the proof of (i) is equally straightforward. □

Corollary 17. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory. The conglomerate of all β-perfect classes $S \subseteq \mathcal{T}$ which are contained in $\mathcal{S}$ has a largest member.*

Proof. We can form the family $\{S_i\}_{i\in I}$ of all β-perfect classes of objects $S_i \subseteq \mathcal{T}$ whose objects happen to all lie in $\mathcal{S}$. This is nonempty, since any zero object forms a perfect class. By Theorem 16(i) the class of objects $\overline{T}(\bigcup_i S_i)$ is β-perfect (here $\overline{T}$ refers to the smallest triangulated subcategory *of* $\mathcal{T}$ containing the union). Since $\mathcal{S}$ is triangulated and contains the union $\bigcup_i S_i$, it also contains $\overline{T}(\bigcup_i S_i)$, so $\overline{T}(\bigcup_i S_i)$ is one our of β-perfect classes contained in $\mathcal{S}$. It clearly contains every other such class so it is the unique largest member, as required. □

Definition 10. Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a triangulated subcategory. We denote by $\mathcal{S}_\beta$ the triangulated subcategory of $\mathcal{T}$ whose class of objects is the largest member of the conglomerate of all β-perfect classes of $\mathcal{T}$ contained in $\mathcal{S}$. Clearly $\mathcal{S}_\beta$ is also a triangulated subcategory of $\mathcal{S}$.

Corollary 18. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a thick triangulated subcategory. Then $\mathcal{S}_\beta$ is also a thick subcategory of $\mathcal{T}$.*

Proof. Let T be the thick closure in $\mathcal{T}$ of the triangulated subcategory $\mathcal{S}_\beta$. Since $\mathcal{S}$ is thick we have $\mathcal{S}_\beta \subseteq T \subseteq \mathcal{S}$. By Lemma 10 the class T is β-perfect, so by maximality of $\mathcal{S}_\beta$ we have $T = \mathcal{S}_\beta$, from which we deduce that $\mathcal{S}_\beta$ is thick. □

Corollary 19. *Let $\mathcal{T}$ be a triangulated category with coproducts, α, β small infinite cardinals and $\mathcal{S}$ an α-localising subcategory. Then $\mathcal{S}_\beta$ is also an α-localising subcategory of $\mathcal{T}$.*

Proof. Every triangulated subcategory is $\aleph_0$-localising, so we can assume $\alpha > \aleph_0$, in which case $\mathcal{S}$ must also be thick. The category $\mathcal{S}$ is thick and α-localising and contains $\mathcal{S}_\beta$, so it must contain $\langle \mathcal{S}_\beta \rangle^\alpha$, the smallest thick α-localising subcategory containing $\mathcal{S}_\beta$. But by Theorem 16 the class $\langle \mathcal{S}_\beta \rangle^\alpha$ is β-perfect, so maximality of $\mathcal{S}_\beta$ implies $\mathcal{S}_\beta = \langle \mathcal{S}_\beta \rangle^\alpha$. This shows that $\mathcal{S}_\beta$ is an α-localising subcategory of $\mathcal{T}$, as required. □

Remark 7. Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{S}$ a triangulated subcategory. If $\gamma < \beta$ are small infinite cardinals then it is clear that $\mathcal{S}_\beta \subseteq \mathcal{S}_\gamma$. On the other hand if we fix a small infinite cardinal β and consider triangulated subcategories $\mathcal{R}, \mathcal{S}$ of $\mathcal{T}$ with $\mathcal{R} \subseteq \mathcal{S}$ then it is clear that $\mathcal{R}_\beta \subseteq \mathcal{S}_\beta$.

Lemma 20. *Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{S}$ a triangulated subcategory. Then $\mathcal{S}_{\aleph_0} = \mathcal{S}$.*

Proof. It suffices to show that $\mathcal{S}$ is an $\aleph_0$-perfect class in $\mathcal{T}$. Given any finite set $\{X_1, \ldots, X_n\}$ of objects of $\mathcal{T}$ and a morphism $k \longrightarrow \bigoplus_{i=1}^n X_i$ with $k \in \mathcal{S}$, let $f_i : k \longrightarrow X_i$ be the components. Then we can factor this morphism as

$$k \xrightarrow{\Delta} \bigoplus_{i=1}^n k \xrightarrow{\bigoplus_i f_i} \bigoplus_{i=1}^n X_i$$

where Δ is the diagonal morphism. Now suppose we are given morphisms $f_i : k_i \longrightarrow X_i$ of $\mathcal{T}$ for $1 \le i \le n$ with $k_i \in \mathcal{S}$, such that there is a morphism $\gamma : k \longrightarrow \bigoplus_{i=1}^n k_i$ with $k \in \mathcal{S}$ and $\bigoplus_{i=1}^n f_i \circ \gamma = 0$. Denote the ith component of γ by γ_i. Then we have $f_i \gamma_i = 0$ for $1 \le i \le n$. Choose for each i a homotopy cokernel $c_i : k_i \longrightarrow w_i$. Then since $\mathcal{S}$ is a triangulated subcategory we have $w_i \in \mathcal{S}$ and f_i factorises as

$$k_i \xrightarrow{c_i} w_i \longrightarrow X_i$$

By construction $\bigoplus_i c_i \circ \gamma = 0$, so the proof is complete. □

3 Small Objects

Definition 11. Let $\mathcal{C}$ be a category, A an object of $\mathcal{C}$ and α a small infinite cardinal. We say that A is α-*small* if whenever we have a morphism $u : A \longrightarrow \bigoplus_{i \in I} A_i$ into a nonempty coproduct, there is a nonempty subset $J \subseteq I$ of cardinality $< \alpha$ and a factorisation of u of the following form

$$A \longrightarrow \bigoplus_{j \in J} A_j \longrightarrow \bigoplus_{i \in I} A_i$$

where the second morphism is canonical. This property is stable under isomorphism of objects and automorphisms of $\mathcal{C}$. If A is α-small then A is β-small for any small infinite cardinal $\beta > \alpha$.

Remark 8. An object A is $\aleph_0$-small if and only if it is small, in the sense of (AC,Definition 18). In particular, a small object is α-small for any small infinite cardinal α. Any zero object is small.

Definition 12. Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal. Then the full subcategory whose objects are all the α-small objects of $\mathcal{T}$ will be denoted $\mathcal{T}^{(\alpha)}$. If $\alpha < \beta$ are small infinite cardinals then clearly $\mathcal{T}^{(\alpha)} \subseteq \mathcal{T}^{(\beta)}$.

Lemma 21. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal. Then $\mathcal{T}^{(\alpha)}$ is a thick triangulated subcategory of $\mathcal{T}$.*

Proof. By (TRC,Lemma 33) it suffices to show that $\mathcal{T}^{(\alpha)}$ is closed under Σ, Σ^{-1} and mapping cones (thickness is easily checked). The former is trivial, so it remains to show the latter. Suppose we have a triangle in $\mathcal{T}$

$$k \longrightarrow l \longrightarrow m \xrightarrow{w} \Sigma k \tag{3}$$

with $k, l \in \mathcal{T}^{(\alpha)}$. We have to show that m is α-small. Suppose we are given a nonempty family of objects $\{X_i\}_{i \in I}$ and a coproduct $\bigoplus_i X_i$. Suppose now that we are given a morphism $h : m \longrightarrow \bigoplus_i X_i$. Composing with $l \longrightarrow m$ we have a morphism $l \longrightarrow \bigoplus_i X_i$ whose composite with $k \longrightarrow l$ is zero. But l is α-small, so there is a nonempty subset $J \subseteq I$ with $|J| < \alpha$ and a factorisation of $l \longrightarrow \bigoplus_i X_i$ of the form

$$l \xrightarrow{z} \bigoplus_{j \in J} X_j \xrightarrow{u_J} \bigoplus_{i \in I} X_i$$

The composite

$$k \longrightarrow l \longrightarrow \bigoplus_{j \in J} X_j \longrightarrow \bigoplus_{i \in I} X_i$$

vanishes, but in this diagram the last morphism is a monomorphism, so the composite of the first two morphisms must already vanish. Since (3) is a triangle, we deduce that $l \longrightarrow \bigoplus_{j \in J} X_j$ must factor through m, say via the morphism $g : m \longrightarrow \bigoplus_{j \in J} X_j$. Then the difference $h - u_J g$ vanishes on $l \longrightarrow m$ and therefore factors through some morphism $q : \Sigma k \longrightarrow \bigoplus_{i \in I} X_i$. But Σk is α-small, so q factors as

$$\Sigma k \xrightarrow{z} \bigoplus_{v \in K} X_v \longrightarrow \bigoplus_{i \in I} X_i$$

for some nonempty subset with $|K| < \alpha$. Then our original morphism h factors as

$$m \xrightarrow{\binom{g}{zw}} \bigoplus_{j \in J} X_j \oplus \bigoplus_{v \in K} X_v \longrightarrow \bigoplus_{j \in J \cup K} X_j \longrightarrow \bigoplus_{i \in I} X_i$$

where $|J \cup K| \leq |J| + |K| < \alpha$ since α is infinite. Therefore m is α-small, and the proof is complete. ☐

Lemma 22. *Let $\mathcal{T}$ be a triangulated category with coproducts and α a small infinite cardinal. If α is regular, then $\mathcal{T}^{(\alpha)}$ is an α-localising subcategory of $\mathcal{T}$.*

Proof. Let $\{k_\mu\}_{\mu \in M}$ be a nonempty family of objects of $\mathcal{T}^{(\alpha)}$, where M is a set of cardinality $< \alpha$, and suppose we have a morphism $\bigoplus_\mu k_\mu \longrightarrow \bigoplus_i X_i$ for some nonempty family $\{X_i\}_{i \in I}$ of objects of $\mathcal{T}$. That is, for every $\mu \in M$ we have a morphism $k_\mu \longrightarrow \bigoplus_i X_i$. Because k_μ is α-small, there exists a nonempty subset $I_\mu \subseteq I$ with $|I_\mu| < \alpha$, and a factorisation of $k_\mu \longrightarrow \bigoplus_i X_i$ as follows

$$k_\mu \longrightarrow \bigoplus_{j \in I_\mu} X_j \longrightarrow \bigoplus_{i \in I} X_i$$

The morphism $\bigoplus_\mu k_\mu \longrightarrow \bigoplus_i X_i$ then factorises as

$$\bigoplus_\mu k_\mu \longrightarrow \bigoplus_{j \in \bigcup_\mu I_\mu} X_j \longrightarrow \bigoplus_{i \in I} X_i$$

Here the cardinality of $\bigcup_{\mu \in M} I_\mu$ is bounded by the sum of $|M| < \alpha$ cardinals all of which are $< \alpha$. Since α is regular, this cardinal must also be $< \alpha$, so $\bigoplus_\mu k_\mu$ is α-small and the proof is complete. ☐

4 Compact Objects

Let $\mathcal{T}$ be a triangulated category with coproducts. In Section 3 we learned how to construct for every small infinite cardinal α a thick triangulated subcategory $\mathcal{T}^{(\alpha)}$ of α-small objects in $\mathcal{T}$. In Section 2 we learned that given any triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ and a small infinite cardinal β, there is a way to construct a triangulated subcategory $\mathcal{S}_\beta \subseteq \mathcal{S}$. In this section, the idea will be to combine these constructions and study $\left\{\mathcal{T}^{(\alpha)}\right\}_\beta$.

Lemma 23. *Let $\mathcal{T}$ be a triangulated category with coproducts and α a small infinite cardinal. Let S be an α-perfect class of α-small objects. Then S is β-perfect for* every *small infinite cardinal β.*

Proof. Let k be an object of S, and suppose $\{X_i\}_{i\in I}$ is a family of objects of $\mathcal{T}$ indexed by a set I of cardinality $< \beta$. Suppose we are given a morphism $k \longrightarrow \bigoplus_i X_i$. Since k is α-small, this factors as

$$k \longrightarrow \bigoplus_{j\in J} X_j \longrightarrow \bigoplus_{i\in I} X_i$$

where $|J| < \alpha$. Since S is α-perfect, the first morphism in this diagram factors as

$$k \longrightarrow \bigoplus_{j\in J} k_j \xrightarrow{\bigoplus_j f_j} \bigoplus_{j\in J} X_j$$

for some objects $k_j \in S$. For $i \in I \setminus J$ we define $k_i = 0$ (some zero object in S) and $f_i = 0$. We deduce a factorisation of our original morphism $k \longrightarrow \bigoplus_i X_i$ as

$$k \longrightarrow \bigoplus_{i\in I} k_i \xrightarrow{\bigoplus_i f_i} \bigoplus_{i\in I} X_i$$

which shows that S satisfies the condition P1 of a β-perfect class. Suppose now that we are given a nonempty family of morphisms $\{f_i : k_i \longrightarrow X_i\}_{i\in I}$ of $\mathcal{T}$ indexed by some set I of cardinality $< \beta$ with each $k_i \in S$, and a morphism $w : k \longrightarrow \bigoplus_i k_i$ with $k \in S$ and the composite $(\bigoplus_i f_i) \circ w$ equal to zero. Because k is α-small, the morphism $k \longrightarrow \bigoplus_i k_i$ must factor as

$$k \longrightarrow \bigoplus_{j\in J} k_j \longrightarrow \bigoplus_{i\in I} k_i$$

for some $|J| < \alpha$. The composite

$$k \longrightarrow \bigoplus_{j\in J} k_j \xrightarrow{\bigoplus_j f_j} \bigoplus_{j\in J} X_j$$

vanishes, and since S is α-perfect, we deduce that for each $j \in J$ the morphism $f_j : k_j \longrightarrow X_j$ factors as

$$k_j \xrightarrow{g_j} l_j \xrightarrow{h_j} X_j$$

with $l_j \in S$ and the composite of $k \longrightarrow \bigoplus_j k_j$ with $\bigoplus_j g_j$ equal to zero. For $i \in I \setminus J$ define $g_i : k_i \longrightarrow l_i$ to be the identity. Then the composite of $k \longrightarrow \bigoplus_i k_i$ with $\bigoplus_i g_i$ still vanishes, so the proof is complete. □

Definition 13. Let $\mathcal{T}$ be a triangulated category with coproducts and α a small infinite cardinal. We define the triangulated subcategory $\mathcal{T}^\alpha$ of $\mathcal{T}$ to be given by $\{\mathcal{T}^{(\alpha)}\}_\alpha$, which by Corollary 18 is thick. By construction $\mathcal{T}^\alpha \subseteq \mathcal{T}^{(\alpha)}$ is an α-perfect class in $\mathcal{T}$. It is the *largest α-perfect class consisting of α-small objects.*

Lemma 24. *Let $\mathcal{T}$ be a triangulated category with coproducts and $\alpha < \beta$ small infinite cardinals. Then $\mathcal{T}^\alpha \subseteq \mathcal{T}^\beta$.*

Proof. We have $\mathcal{T}^{(\alpha)} \subseteq \mathcal{T}^{(\beta)}$ and therefore $\mathcal{T}^\alpha \subseteq \mathcal{T}^{(\beta)}$. On the other hand, the objects of $\mathcal{T}^\alpha$ form an α-perfect class of α-small objects, so $\mathcal{T}^\alpha$ is also β-perfect by Lemma 23. Using maximality we have $\mathcal{T}^\alpha \subseteq \mathcal{T}^\beta$, as required. □

Lemma 25. *Let $\mathcal{T}$ be a triangulated category with coproducts and α a small infinite cardinal. If α is regular then $\mathcal{T}^\alpha$ is an α-localising subcategory of $\mathcal{T}$.*

Proof. We know from Lemma 22 that $\mathcal{T}^{(\alpha)}$ is α-localising. It then follows from Corollary 19 that $\mathcal{T}^\alpha = \{\mathcal{T}^{(\alpha)}\}_\alpha$ is α-localising. □

Remark 9. Let $\mathcal{T}$ be a triangulated category with coproducts. In the special case $\alpha = \aleph_0$, every triangulated subcategory of $\mathcal{T}$ forms an α-perfect class. In particular we have $\{\mathcal{T}^{(\aleph_0)}\}_{\aleph_0} = \mathcal{T}^{(\aleph_0)}$ and therefore $\mathcal{T}^{\aleph_0} = \mathcal{T}^{(\aleph_0)}$.

Definition 14. Let $\mathcal{T}$ be a triangulated category with coproducts and α a small infinite cardinal. We call the objects of $\mathcal{T}^\alpha$ the *α-compact* objects. The property of being α-compact is certainly stable under isomorphism of objects. We say that X is *compact* if it is $\aleph_0$-compact, which by Remark 9 is equivalent to X being a small object. We sometimes write $\mathcal{T}^c$ for the subcategory $\mathcal{T}^{\aleph_0}$ of compact objects. If X is α-compact it is β-compact for any small infinite cardinal $\beta > \alpha$.

5 Portly Considerations

As in (TRC,Section 7) we collect in this section all the "portly" versions of results that we will need.

Definition 15. Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and S a nonempty conglomerate of objects of $\mathcal{T}$. Then $\langle S\rangle^\beta$ denotes the smallest portly triangulated category $\mathcal{S}$ of $\mathcal{T}$ satisfying the following conditions

(i) The objects of S belong to $\mathcal{S}$.

(ii) Any coproduct in $\mathcal{T}$ of fewer than β objects of $\mathcal{S}$ belongs to $\mathcal{S}$. That is, if $\{X_i\}_{i\in I}$ is a family of objects of $\mathcal{S}$ indexed by a nonempty set I of cardinality $< \beta$ then any coproduct of this family in $\mathcal{T}$ belongs to $\mathcal{S}$.

(iii) The subcategory $\mathcal{S} \subseteq \mathcal{T}$ is thick.

It is clear that Lemma 2 holds for portly triangulated categories. It also makes sense to define $\overline{T}(S)$ for any portly triangulated category $\mathcal{T}$ and nonempty conglomerate S of objects of $\mathcal{T}$ to be the smallest *portly* triangulated subcategory containing the objects of S. It is clear what we mean if we say that a portly triangulated subcategory is *β-localising* or *β-colocalising* for a small infinite cardinal β. By definition $\langle S\rangle^\beta$ is β-localising.

Lemma 26. *Let $\mathcal{T}$ be a portly triangulated category, $\mathcal{S}$ a portly triangulated subcategory. Then $\mathcal{S}$ is localising if and only if it is β-localising for every small infinite cardinal β.*

Remark 10. If $\mathcal{T}$ is a portly triangulated category with coproducts, and $\mathcal{S}$ a β-localising portly subcategory for any small infinite cardinal $\beta > \aleph_0$, then $\mathcal{S}$ contains countable coproducts of its objects and is therefore thick.

Lemma 27. *Let $\mathcal{T}$ be a portly triangulated category, $\mathcal{S}$ a portly triangulated subcategory and β a small infinite cardinal. If $\mathcal{S}$ is β-localising then so is its thick closure $\widehat{\mathcal{S}}$.*

Proposition 28. *Let $\mathcal{T}$ be a portly triangulated category with coproducts and S a nonempty conglomerate of objects of $\mathcal{T}$. Then the portly triangulated subcategory $\langle S\rangle = \bigcup_\beta \langle S\rangle^\beta$ is localising, and is the smallest localising portly subcategory containing S.*

Let $\mathcal{T}$ be a portly triangulated category with coproducts and β a small infinite cardinal. Given a conglomerate S of objects of $\mathcal{T}$, it is clear what we mean if we say that S is *β-perfect*. Example 2 still stands for portly triangulated categories, as do Lemma 9 and Lemma 10. Given a portly triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ it is clear what we mean if we say an object $k \in \mathcal{T}$ is *β-good for $\mathcal{S}$*. One checks that Example 3, Lemma 11 are true with "triangulated category" and "triangulated subcategory" replaced by their portly equivalents.

Lemma 29. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a portly triangulated subcategory whose objects are all β-good for $\mathcal{S}$. Then $\mathcal{S}$ is a β-perfect conglomerate.*

Proposition 30. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a portly triangulated subcategory. Then the full portly subcategory of $\mathcal{T}$ defined by*

$$\mathcal{Q} = \{k \in \mathcal{S} \,|\, k \text{ is } \beta\text{-good for } \mathcal{S}\}$$

is a portly triangulated subcategory of $\mathcal{T}$. If $\mathcal{S}$ is thick then so is $\mathcal{Q}$.

Lemma 31. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and S a nonempty conglomerate of objects of $\mathcal{T}$. Suppose that every object of S is β-good for the portly triangulated subcategory $\overline{T}(S) \subseteq \mathcal{T}$. Then $\overline{T}(S)$ is a β-perfect conglomerate.*

Lemma 32. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, α, β small infinite cardinals and S a nonempty conglomerate of objects of $\mathcal{T}$. Suppose that every object of S is β-good for the portly triangulated subcategory $\langle S \rangle^\alpha \subseteq \mathcal{T}$. Then $\langle S \rangle^\alpha$ is a β-perfect conglomerate.*

Theorem 33. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and $\{S_i\}_{i\in I}$ a nonempty family of β-perfect conglomerates of $\mathcal{T}$. Then*

(i) The conglomerate of objects $\overline{T}(\bigcup_i S_i)$ is β-perfect.

(ii) For any small infinite cardinal α the conglomerate of objects $\langle \bigcup_i S_i \rangle^\alpha$ is β-perfect.

Corollary 34. *Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a portly triangulated subcategory. The conglomerate of all β-perfect conglomerates $S \subseteq \mathcal{T}$ which are contained in $\mathcal{S}$ has a largest member.*

Definition 16. Let $\mathcal{T}$ be a portly triangulated category with coproducts, β a small infinite cardinal and $\mathcal{S}$ a portly triangulated subcategory. We denote by $\mathcal{S}_\beta$ the portly triangulated subcategory of $\mathcal{T}$ whose class of objects is the largest member of the conglomerate of all β-perfect conglomerates of $\mathcal{T}$ contained in $\mathcal{S}$. Clearly $\mathcal{S}_\beta$ is also a portly triangulated subcategory of $\mathcal{S}$.

As before it is clear that if $\mathcal{S}$ is thick or α-localising, then so is $\mathcal{S}_\beta$. The portly versions of Remark 7 and Lemma 20 are also true. If $\mathcal{C}$ is a portly category and α a small infinite cardinal, it is clear what we mean if we say an object of $\mathcal{C}$ is α-*small*, and Remark 8 still holds.

Definition 17. Let $\mathcal{T}$ be a portly triangulated category with coproducts, α a small infinite cardinal. Then the full portly subcategory whose objects are all the α-small objects of $\mathcal{T}$ will be denoted $\mathcal{T}^{(\alpha)}$. This is a thick portly triangulated subcategory of $\mathcal{T}$, and if α is regular it is also α-localising. If $\alpha < \beta$ are small infinite cardinals then clearly $\mathcal{T}^{(\alpha)} \subseteq \mathcal{T}^{(\beta)}$.

Lemma 35. *Let $\mathcal{T}$ be a portly triangulated category with coproducts and α a small infinite cardinal. Let S be an α-perfect conglomerate of α-small objects. Then S is β-perfect for* every *small infinite cardinal β.*

Definition 18. Let $\mathcal{T}$ be a portly triangulated category with coproducts and α a small infinite cardinal. We define the thick portly triangulated subcategory $\mathcal{T}^\alpha$ of $\mathcal{T}$ to be given by $\{\mathcal{T}^{(\alpha)}\}_\alpha$. This is the largest α-perfect conglomerate consisting of α-small objects. If α is regular then $\mathcal{T}^\alpha$ is α-localising, and if $\alpha < \beta$ are small infinite cardinals we have $\mathcal{T}^\alpha \subseteq \mathcal{T}^\beta$. It is also still true that $\mathcal{T}^{\aleph_0} = \mathcal{T}^{(\aleph_0)}$. We call the objects of $\mathcal{T}^\alpha$ the α-*compact* objects.

6 Thomason Localisation

Remark 11. Let $\mathcal{T}$ be a triangulated category with coproducts and S a nonempty class of objects of $\mathcal{T}$. Recall that $\langle S \rangle$ denotes the smallest localising subcategory containing the objects of S (see Definition 7), and is equal to the union

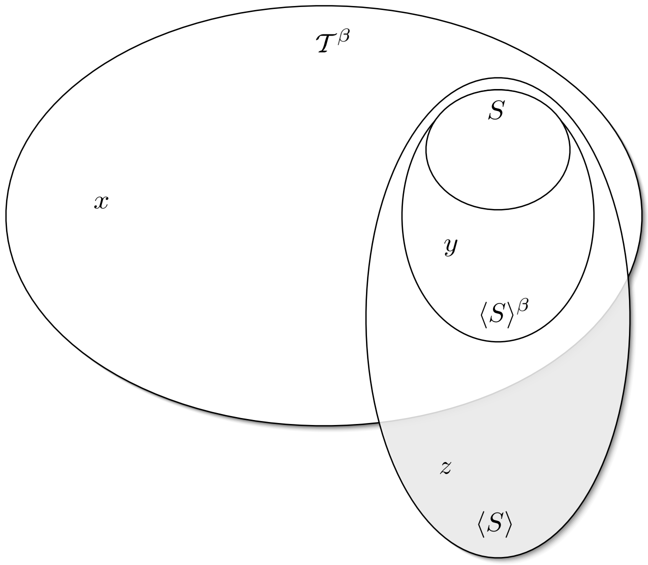
$$\langle S \rangle = \bigcup_\beta \langle S \rangle^\beta$$

where $\langle S\rangle^\beta$ is the smallest thick β-localising subcategory containing S (see Definition 2).

Lemma 36. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small regular cardinal and $S \subseteq \mathcal{T}^\beta$ a nonempty class of objects. Then $\langle S\rangle^\beta \subseteq \mathcal{T}^\beta$.*

Proof. We already know that $\mathcal{T}^\beta$ is thick and contains S, and it is β-localising by Lemma 25, so the inclusion $\langle S\rangle^\beta \subseteq \mathcal{T}^\beta$ is immediate. □

Theorem 37. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small regular cardinal and $S \subseteq \mathcal{T}^\beta$ a nonempty class of objects. Given objects $x \in \mathcal{T}^\beta$ and $z \in \langle S\rangle$, any morphism $x \longrightarrow z$ factors through some object of $\langle S\rangle^\beta$.*



Proof. We know that the localising subcategory $\langle S\rangle$ is the union of the ascending chain of subcategories $\langle S\rangle^{\aleph_0} \subseteq \langle S\rangle^{\aleph_1} \subseteq \cdots$ and the idea is that $\langle S\rangle^\beta$ is the "last" such subcategory that we know for certain is contained in $\mathcal{T}^\beta$ (by Lemma 36). We claim that any morphism from an object of $\mathcal{T}^\beta$ to an object of $\langle S\rangle$ factors through this "last stage" of the construction still inside $\mathcal{T}^\beta$.

Let $\mathcal{S}$ be the full subcategory of $\mathcal{T}$ consisting of all those objects z which satisfy the theorem, for every $x \in \mathcal{T}^\beta$. That is, an object $z \in \mathcal{T}$ belongs to $\mathcal{S}$ if and only if every morphism $x \longrightarrow z$ with $x \in \mathcal{T}^\beta$ factors through an object of $\langle S\rangle^\beta$. We complete the proof by showing that $\langle S\rangle \subseteq \mathcal{S}$, which we do by proving that $\mathcal{S}$ is a localising subcategory of $\mathcal{T}$ which contains S. It is clear that any object of $\langle S\rangle^\beta$ belongs to $\mathcal{S}$. In particular, $S \subseteq \mathcal{S}$. So we need to show that $\mathcal{S}$ is a triangulated subcategory, closed under all coproducts in $\mathcal{T}$.

Proof that $\mathcal{S}$ is triangulated. It is clear that $\mathcal{S}$ is replete and closed under Σ, Σ^{-1} so it only remains to show it is closed under mapping cones (TRC,Lemma 33). Suppose that we are given a triangle in $\mathcal{T}$

$$z \longrightarrow z' \longrightarrow z'' \longrightarrow \Sigma z$$

with $z, z' \in \mathcal{S}$. Let a morphism $f : x \longrightarrow z''$ be given, with $x \in \mathcal{T}^\beta$. The composite of f with $z'' \longrightarrow \Sigma z$ must factor through some object $y \in \langle S\rangle^\beta$, and the composite

$$x \longrightarrow y \longrightarrow \Sigma z \longrightarrow \Sigma z' \tag{4}$$

clearly vanishes. If we complete $x \longrightarrow y$ to a triangle $x \longrightarrow y \longrightarrow C \longrightarrow \Sigma x$ then the morphism $y \longrightarrow \Sigma z'$ of (4) must factor through $y \longrightarrow C$. That is, we have a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & C \\ \downarrow & & \downarrow \\ \Sigma z & \longrightarrow & \Sigma z' \end{array} \tag{5}$$

Since both x, y belong to the triangulated subcategory $\mathcal{T}^\beta$, we have $C \in \mathcal{T}^\beta$ also. Since $\Sigma z' \in \mathcal{S}$, the morphism $C \longrightarrow \Sigma z'$ must factor through some $y' \in \langle S \rangle^\beta$. So we can replace (5) with a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & y' \\ \downarrow & & \downarrow \\ \Sigma z & \longrightarrow & \Sigma z' \end{array} \tag{6}$$

in which the top row only involves objects of $\langle S \rangle^\beta$. Also observe that the composite of $x \longrightarrow y$ with the top row still vanishes. Now complete (6) to a morphism of triangles

$$\begin{array}{ccccccc} \Sigma^{-1}y'' & \longrightarrow & y & \longrightarrow & y' & \longrightarrow & y'' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ z'' & \longrightarrow & \Sigma z & \longrightarrow & \Sigma z' & \longrightarrow & \Sigma z'' \end{array}$$

Clearly $y'' \in \langle S \rangle^\beta$ and since the composite $x \longrightarrow y \longrightarrow y'$ vanishes the morphism $x \longrightarrow y$ factors through $\Sigma^{-1}y''$. The morphism $g : x \longrightarrow \Sigma^{-1}y'' \longrightarrow z''$ is not necessarily f, but the difference $f - g$ gives zero when composed with $z'' \longrightarrow \Sigma z$, and must therefore factor through $z' \longrightarrow z''$. We know that $z' \in \mathcal{S}$ so the factorisation $q : x \longrightarrow z'$ must itself factor through some $\overline{y} \in \langle S \rangle^\beta$. But then $f : x \longrightarrow z''$ factors as

$$x \longrightarrow \overline{y} \oplus \Sigma^{-1}y'' \longrightarrow z''$$

and $\overline{y} \oplus \Sigma^{-1}y''$ belongs to $\langle S \rangle^\beta$. Therefore $z'' \in \mathcal{S}$, as required.

Proof that $\mathcal{S}$ is localising. Let $\{z_i\}_{i \in I}$ be an arbitrary family of objects of $\mathcal{S}$ and $\bigoplus_i z_i$ a coproduct in $\mathcal{T}$. We want to show that this coproduct belongs to $\mathcal{S}$. Suppose we are given a morphism $f : x \longrightarrow \bigoplus_i z_i$ with $x \in \mathcal{T}^\beta$. Since $\mathcal{T}^\beta \subseteq \mathcal{T}^{(\beta)}$ the object x is β-small, so there is a nonempty subset $J \subseteq I$ of cardinality $< \beta$ and a factorisation

$$x \xrightarrow{g} \bigoplus_{j \in J} z_j \longrightarrow \bigoplus_{i \in I} z_i$$

But the class $\mathcal{T}^\beta$ is β-perfect, so the morphism g factors as

$$x \longrightarrow \bigoplus_{j \in J} x_j \xrightarrow{\bigoplus_j h_j} \bigoplus_{j \in J} z_j$$

for some collection of morphisms $h_j : x_j \longrightarrow z_j$ with $x_j \in \mathcal{T}^\beta$. For each $j \in J$ the fact that $z_j \in \mathcal{S}$ means that h_j factors as $x_j \longrightarrow y_j \longrightarrow z_j$ with $y_j \in \langle S \rangle^\beta$. But then f factors as

$$x \longrightarrow \bigoplus_{j \in J} y_j \longrightarrow \bigoplus_{i \in I} z_i$$

and $\bigoplus_{j \in J} y_j$, being a coproduct of fewer than β objects of $\langle S \rangle^\beta$, belongs to $\langle S \rangle^\beta$. This is the necessary factorisation of f, which completes the proof. □

Remark 12. Both Lemma 36 and Theorem 37 are still in the case where $\mathcal{T}$ is a portly triangulated category.

As we observed in (TRC,Section 2), one of the problems with Verdier's construction of the quotient is that the resulting categories are *portly categories*. That is, their morphism "sets" are not necessarily sets: they are in general only conglomerates. In the remainder of this section we will apply what we have learned to study verdier quotients, and in particular we will give a criterion for the verdier quotient to have morphism conglomerates which are at least small.

Proposition 38. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small regular cardinal and $S \subseteq \mathcal{T}^\beta$ a nonempty class of objects. There is a canonical triangulated functor*

$$\varrho : \mathcal{T}/\langle S \rangle^\beta \longrightarrow \mathcal{T}/\langle S \rangle$$

and we claim that for $x \in \mathcal{T}^\beta$ and arbitrary $y \in \mathcal{T}$ the morphism

$$\varrho_{x,y} : Hom_{\mathcal{T}/\langle S\rangle^\beta}(x, y) \longrightarrow Hom_{\mathcal{T}/\langle S\rangle}(x, y)$$

is an isomorphism.

Proof. Since $\langle S\rangle^\beta \subseteq \langle S\rangle$ there is by (TRC,Remark 50) an induced triangulated functor ϱ defined on objects to be the identity and on morphisms by $\varrho([f, g]) = [f, g]$. Throughout fix objects $x \in \mathcal{T}^\beta, y \in \mathcal{T}$. First we show that $\varrho_{x,y}$ is surjective. Suppose we are given a morphism $[f, \alpha] : x \longrightarrow y$ in $\mathcal{T}/\langle S\rangle$, alternatively written as the following diagram

$$\begin{array}{ccccc} & & p & & \\ & \stackrel{f}{\swarrow} & & \stackrel{\alpha}{\searrow} & \\ x & & & & y \end{array} \qquad (7)$$

with $f \in Mor_{\langle S\rangle}$. Extend $f : p \longrightarrow x$ to a triangle $p \longrightarrow x \longrightarrow z \longrightarrow \Sigma p$. Then $z \in \langle S\rangle$ and by Theorem 37 the morphism $x \longrightarrow z$ must factor through some $z' \in \langle S\rangle^\beta$. We can extend the factorisation $x \longrightarrow z'$ to a triangle as in the first row of the following diagram, and then extend the middle commutative square to a morphism of triangles

$$\begin{array}{ccccccc} p' & \longrightarrow & x & \longrightarrow & z' & \longrightarrow & \Sigma p' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\ p & \xrightarrow{f} & x & \longrightarrow & z & \longrightarrow & \Sigma p \end{array}$$

The morphism $fg : p' \longrightarrow x$ belongs to $Mor_{\langle S\rangle^\beta}$ since $z' \in \langle S\rangle^\beta$. The diagram

$$\begin{array}{ccccc} & & p' & & \\ & \stackrel{fg}{\swarrow} & & \stackrel{\alpha g}{\searrow} & \\ x & & & & y \end{array}$$

is therefore a morphism in $\mathcal{T}/\langle S\rangle^\beta$, whose image under $\varrho_{x,y}$ is clearly $[f, g]$, as required.

To show that $\varrho_{x,y}$ is injective, suppose we are given a morphism $[f, \alpha] : x \longrightarrow y$ in $\mathcal{T}/\langle S\rangle^\beta$ represented by a diagram of the form (7) whose image under $\varrho_{x,y}$ is zero. Extending f to a triangle as before, we observe that $z \in \langle S\rangle^\beta$ which is contained in $\mathcal{T}^\beta$. Since this is a triangulated subcategory, we deduce that $p \in \mathcal{T}^\beta$ also. From $\varrho_{x,y}[f, \alpha] = 0$ and (TRC,Lemma 55) we deduce that $\alpha : p \longrightarrow y$ must factor through some object $z \in \langle S\rangle$. But then Theorem 37 implies that the factorisation $p \longrightarrow z$ must factor through some $z' \in \langle S\rangle^\beta$. Since we have now factored α through an object of $\langle S\rangle^\beta$ it follows again by (TRC,Lemma 55) that $[f, \alpha] = 0$ already in $\mathcal{T}/\langle S\rangle^\beta$. □

Corollary 39. *Let $\mathcal{T}$ be a triangulated category with coproducts, β a small regular cardinal and $S \subseteq \mathcal{T}^\beta$ a nonempty class of objects. Then the canonical triangulated functor*

$$\vartheta : \mathcal{T}^\beta/\langle S\rangle^\beta \longrightarrow \mathcal{T}/\langle S\rangle$$

is a full embedding.

Proof. We know that $\langle S\rangle^\beta$ is contained in $\mathcal{T}^\beta$, so the verdier quotients make sense. The functor $\mathcal{T}^\beta \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\langle S\rangle$ sends objects of $\langle S\rangle^\beta$ to zero, so there is an induced triangulated functor $\vartheta : \mathcal{T}^\beta/\langle S\rangle^\beta \longrightarrow \mathcal{T}/\langle S\rangle$. The proof of Proposition 38 shows that this functor is fully faithful. □

All of this becomes useful when $\mathcal{T} = \bigcup_\beta \mathcal{T}^\beta$. Then we know

Corollary 40. *Let $\mathcal{T}$ be a triangulated category with coproducts, and suppose that*

$$\mathcal{T} = \bigcup_\beta \mathcal{T}^\beta$$

That is, every object of $\mathcal{T}$ is β-compact for some small infinite cardinal β. Suppose that $S \subseteq \mathcal{T}^\alpha$ is a small nonempty class of objects for some small infinite cardinal α. Then the portly category $\mathcal{T}/\langle S\rangle$ has small morphism conglomerates.

Proof. To be clear, recall that in a portly category the morphisms $Hom_{\mathcal{T}/\langle S\rangle}(x,y)$ are arbitrary conglomerates. We are claiming that under some hypothesis, these conglomerates are small (that is, in bijection with sets). We are *not* claiming that they *are* sets, so $\mathcal{T}/\langle S\rangle$ is still not a "genuine" category. For the proof, we fix objects $x,y\in\mathcal{T}$ and let β be a small infinite cardinal with $x\in\mathcal{T}^\beta$. We may as well assume that β is regular and $\beta>\alpha$ (since successor cardinals are regular, and successors of small cardinals are small). Then we have $S\subseteq\mathcal{T}^\beta$ so by Proposition 38 there is a bijection

$$Hom_{\mathcal{T}/\langle S\rangle^\beta}(x,y)\longrightarrow Hom_{\mathcal{T}/\langle S\rangle}(x,y) \tag{8}$$

On the other hand S is small, so by Proposition 4 the category $\langle S\rangle^\beta$ is essentially small. From (TRC,Proposition 69) we deduce that the conglomerate on the left in (8) is small. Therefore $Hom_{\mathcal{T}/\langle S\rangle}(x,y)$ is also small, which is what we wanted to show. □

Lemma 41. *Let $F:\mathcal{D}\longrightarrow\mathcal{E}$ be a fully faithful triangulated functor between portly triangulated categories. Then the essential image of F is a portly triangulated subcategory of $\mathcal{E}$.*

Proof. By the *essential image* we mean the full portly subcategory of $\mathcal{E}$ consisting of all those objects of $\mathcal{E}$ isomorphic to $F(X)$ for some $X\in\mathcal{D}$. Denote the essential image by $Im(F)$. There is an equivalence $\mathcal{D}\cong Im(F)$ of ordinary categories, so $Im(F)$ is certainly replete and additive. Using (TRC,Lemma 33) it is easy to check that in fact $Im(F)$ is a portly triangulated subcategory. □

Lemma 42. *Let $\mathcal{T}$ be a triangulated category with coproducts satisfying $\mathcal{T}=\bigcup_\beta\mathcal{T}^\beta$, α a small infinite cardinal and $S\subseteq\mathcal{T}^\alpha$ a nonempty class. Then for any small regular cardinal $\beta\geq\alpha$ the objects of $\mathcal{T}^\beta$ form a β-perfect class of β-small objects in $\mathcal{T}/\langle S\rangle$. In particular*

$$\mathcal{T}^\beta\subseteq\big(\mathcal{T}/\langle S\rangle\big)^\beta$$

Proof. We first we show that any object of $\mathcal{T}^\beta$ is β-small in $\mathcal{T}/\langle S\rangle$. Suppose we are given a morphism $k\longrightarrow\bigoplus_i X_i$ into a nonempty coproduct in $\mathcal{T}/\langle S\rangle$, where $k\in\mathcal{T}^\beta$. By (TRC,Lemma 91) we may as well assume that this coproduct is the image in $\mathcal{T}/\langle S\rangle$ of a coproduct $u_i:X_i\longrightarrow\bigoplus_i X_i$ in $\mathcal{T}$. Since $S\subseteq\mathcal{T}^\alpha\subseteq\mathcal{T}^\beta$ we are in the situation of Proposition 38, and our morphism $k\longrightarrow\bigoplus_i X_i$ is the image under the functor $\mathcal{T}/\langle S\rangle^\beta\longrightarrow\mathcal{T}/\langle S\rangle$ of some morphism, represented say by a diagram

$$\begin{array}{ccc} & p & \\ {\scriptstyle f}\swarrow & & \searrow \\ k & & \bigoplus_i X_i \end{array}$$

with $f\in Mor_{\langle S\rangle^\beta}$. This means that there is a triangle $p\longrightarrow k\longrightarrow z\longrightarrow\Sigma p$ in $\mathcal{T}$ with $z\in\langle S\rangle^\beta$, which by Lemma 36 is contained in $\mathcal{T}^\beta$. Therefore also $p\in\mathcal{T}^\beta$. The β-smallness of p implies that the morphism $p\longrightarrow\bigoplus_i X_i$ factors in $\mathcal{T}$ as follows

$$p\longrightarrow\bigoplus_{j\in J}X_j\longrightarrow\bigoplus_{i\in I}X_i$$

for some subset $J\subseteq I$ of cardinality $<\beta$. The β-compactness of p in $\mathcal{T}$ says that the morphism $p\longrightarrow\bigoplus_{j\in J}X_j$ factors as

$$p\longrightarrow\bigoplus_{j\in J}k_j\longrightarrow\bigoplus_{j\in J}X_j$$

with every $k_j\in\mathcal{T}^\beta$. So we have factored the morphism $k\longrightarrow\bigoplus_i X_i$ in $\mathcal{T}/\langle S\rangle$ into a composite

$$k\longrightarrow\bigoplus_{j\in J}k_j\longrightarrow\bigoplus_{j\in J}X_j\longrightarrow\bigoplus_{i\in I}X_i$$

which shows that k is β-small, and also that it is β-good for any portly triangulated subcategory of $\mathcal{T}/\langle S\rangle$ containing the objects of $\mathcal{T}^\beta$. By Corollary 39 the canonical functor $\mathcal{T}^\beta/\langle S\rangle^\beta\longrightarrow\mathcal{T}/\langle S\rangle$ is fully faithful, which together with Lemma 41 shows that the full subcategory of $\mathcal{T}/\langle S\rangle$ consisting of objects isomorphic to some object of $\mathcal{T}^\beta$ (in $\mathcal{T}/\langle S\rangle$) is a portly triangulated subcategory of $\mathcal{T}/\langle S\rangle$. Call this portly triangulated subcategory $\mathcal{E}$.

From what we have already shown, every object of $\mathcal{E}$ is β-good for $\mathcal{E}$. Then from Lemma 29 we infer that $\mathcal{E}$ is a β-perfect class. Lemma 9 for portly categories implies that $\mathcal{T}^\beta$ is β-perfect, which completes the proof. □

Lemma 43. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal and $S \subseteq \mathcal{T}^\alpha$ a nonempty class of objects with $\mathcal{T} = \langle S\rangle$. Then $\mathcal{T} = \bigcup_\beta \mathcal{T}^\beta$.*

Proof. We have to show that any object is β-compact for some small infinite cardinal β. Let an object $x \in \mathcal{T}$ be given. Since $\mathcal{T} = \langle S\rangle = \bigcup_\beta \langle S\rangle^\beta$ there exists a small infinite cardinal β with $x \in \langle S\rangle^\beta$. We can assume that $\beta \geq \alpha$ and replacing β by its successor cardinal if necessary, we can assume β is regular. Then $S \subseteq \mathcal{T}^\alpha \subseteq \mathcal{T}^\beta$ and by Lemma 36 we know that $\langle S\rangle^\beta \subseteq \mathcal{T}^\beta$. Therefore $x \in \mathcal{T}^\beta$, as required. □

Lemma 44. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal and $S \subseteq \mathcal{T}^\alpha$ a nonempty class of objects with $\mathcal{T} = \langle S\rangle$. Given any small regular cardinal $\beta \geq \alpha$ we have $\langle S\rangle^\beta = \mathcal{T}^\beta$.*

Proof. We already know that $\langle S\rangle^\beta \subseteq \mathcal{T}^\beta$, so it suffices to show the reverse inclusion. Given an object $x \in \mathcal{T}^\beta$ the identity $1 : x \longrightarrow x$ is a morphism from an object of $\mathcal{T}^\beta$ to an object of $\langle S\rangle$. Therefore by Theorem 37 it must factor through some object $y \in \langle S\rangle^\beta$. This morphism $x \longrightarrow y$ is a coretraction, so x must be a direct summand of y. Since $\langle S\rangle^\beta$ is thick we conclude that $x \in \langle S\rangle^\beta$, which is what we wanted to show. □

Remark 13. Put another way, Lemma 44 says that $(\langle S\rangle)^\beta = \langle S\rangle^\beta$. In general this is not true, so we usually try to avoid expressions like the one on the left.

Remark 14. Since Theorem 37 works for portly triangulated categories, it is immediate that Lemma 44 is still true if we replace $\mathcal{T}$ by a portly triangulated category (in which case S is only required to be a nonempty subconglomerate).

Remark 15. Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{S}$ a triangulated subcategory. Suppose we are given a portly triangulated subcategory $\mathcal{Q} \subseteq \mathcal{T}/\mathcal{S}$. Then the full subcategory of $\mathcal{T}$ whose objects are the objects of $\mathcal{Q}$ is a triangulated subcategory of $\mathcal{T}$. If $\mathcal{S}$ is localising in $\mathcal{T}$ and $\mathcal{Q}$ is localising in $\mathcal{T}/\mathcal{S}$ then its preimage is localising in $\mathcal{T}$.

Proposition 45. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal and $S, T \subseteq \mathcal{T}^\alpha$ nonempty classes of objects with $\mathcal{T} = \langle T\rangle$. Suppose $\beta \geq \alpha$ is a small regular cardinal. Then we have*

$$\mathcal{T}^\beta \subseteq (\mathcal{T}/\langle S\rangle)^\beta$$

and moreover everything on the right hand side is a direct summand of something on the left.

Proof. We know from Lemma 43 that the hypothesis of Lemma 42 are satisfied, so we at least have the inclusion $\mathcal{T}^\beta \subseteq (\mathcal{T}/\langle S\rangle)^\beta$. As in the proof of Lemma 42 we let $\mathcal{E}$ denote the portly triangulated subcategory of $\mathcal{T}/\langle S\rangle$ given by closing the objects of $\mathcal{T}^\beta$ with respect to isomorphisms. Since $(\mathcal{T}/\langle S\rangle)^\beta$ is thick it contains the thick closure $\widehat{\mathcal{E}}$ of $\mathcal{E}$. We have to prove the reverse inclusion.

Since β is regular the subcategory $\mathcal{T}^\beta$ is β-localising in $\mathcal{T}$. As coproducts in $\mathcal{T}/\langle S\rangle$ can be calculated in $\mathcal{T}$, it follows that $\mathcal{E}$ is β-localising, and therefore by Lemma 27 so is $\widehat{\mathcal{E}}$. Therefore $\widehat{\mathcal{E}}$ is a thick, β-localising portly triangulated subcategory, which contains the objects of T. We therefore have an inclusion $\langle T\rangle^\beta \subseteq \widehat{\mathcal{E}}$ (where the former subcategory is calculated in $\mathcal{T}/\langle S\rangle$).

On the other hand $\mathcal{T} = \langle T\rangle$, so it follows that $\mathcal{T}/\langle S\rangle$ is also the smallest localising portly subcategory of itself containing the objects of T (use Remark 15). The portly triangulated category $\mathcal{T}/\langle S\rangle$ has coproducts (TRC,Lemma 91) so by Lemma 44 applied to the class T in the portly triangulated category $\mathcal{T}/\langle S\rangle$ we get

$$(\mathcal{T}/\langle S\rangle)^\beta = \langle T\rangle^\beta \subseteq \widehat{\mathcal{E}}$$

which is what we wanted to show. □

Corollary 46. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal and $S, T \subseteq \mathcal{T}^\alpha$ nonempty classes of objects with $\mathcal{T} = \langle T \rangle$. Suppose $\beta \geq \alpha$ is a small regular cardinal with $\beta > \aleph_0$. Then the essential image of the canonical triangulated functor*

$$\vartheta : \mathcal{T}^\beta / \langle S \rangle^\beta \longrightarrow \mathcal{T}/\langle S \rangle$$

is precisely $(\mathcal{T}/\langle S \rangle)^\beta$. In particular there is a canonical equivalence of triangulated categories

$$\mathcal{T}^\beta / \langle S \rangle^\beta \longrightarrow (\mathcal{T}/\langle S \rangle)^\beta$$

Proof. From Corollary 39 we know that this functor is fully faithful and distinct on objects. In the proof of Proposition 45 we observed that the essential image $\mathcal{E}$ is β-localising. Since $\beta > \aleph_0$ it follows from (TRC,Lemma 91) and Remark 10 that $\mathcal{E}$ is already thick, so $\mathcal{E} = \widehat{\mathcal{E}} = (\mathcal{T}/\langle S \rangle)^\beta$. □

Lemma 47. *Let $\mathcal{T}$ be a triangulated category with coproducts, α a small infinite cardinal and $S \subseteq \mathcal{T}^\alpha$ a nonempty class. Let $\mathcal{S} = \langle S \rangle$ be the localising subcategory generated by S and suppose that $\beta \geq \alpha$ is a small regular cardinal. Then*

$$\mathcal{S} \cap \mathcal{T}^\beta \subseteq \mathcal{S}^\beta$$

Proof. We will show that $\mathcal{S} \cap \mathcal{T}^\beta$ is a β-perfect class of objects in $\mathcal{S}$ which is contained in $\mathcal{S}^{(\beta)}$. Maximality of $\mathcal{S}^\beta$ will then give the desired result.

First we show that every element of this intersection is β-small in $\mathcal{S}$. Let $k \in \mathcal{S} \cap \mathcal{T}^\beta$ be given, and suppose we have a morphism $k \longrightarrow \bigoplus_{i \in I} X_i$ into a nonempty coproduct in $\mathcal{S}$. Since $\mathcal{S}$ is localising this is also a coproduct in $\mathcal{T}$, and since $k \in \mathcal{T}^\beta$ is β-small there must be a subset $J \subseteq I$ of cardinality $< \beta$ and a factorisation

$$k \longrightarrow \bigoplus_{j \in J} X_j \longrightarrow \bigoplus_{i \in I} X_i$$

This middle coproduct also belongs to $\mathcal{S}$, which proves that k is β-small in $\mathcal{S}$ and yields the inclusion $\mathcal{S} \cap \mathcal{T}^\beta \subseteq \mathcal{S}^{(\beta)}$.

Now we want to show that $\mathcal{S} \cap \mathcal{T}^\beta$ is a β-perfect class in $\mathcal{S}$. It is a triangulated subcategory, so by Lemma 12 it suffices to show that every object of $\mathcal{S} \cap \mathcal{T}^\beta$ is β-good for $\mathcal{S} \cap \mathcal{T}^\beta$ in $\mathcal{S}$. Suppose we are given a family of objects $\{X_i\}_{i \in I}$ of $\mathcal{S}$ indexed by a set I of cardinality $< \beta$ and a morphism $k \longrightarrow \bigoplus_i X_i$ with $k \in \mathcal{S} \cap \mathcal{T}^\beta$. Since $k \in \mathcal{T}^\beta$ this morphism factors as

$$k \longrightarrow \bigoplus_{i \in I} k_i \xrightarrow{\bigoplus_i f_i} \bigoplus_{i \in I} X_i$$

with $k_i \in \mathcal{T}^\beta$. But then each $f_i : k_i \longrightarrow X_i$ is a morphism from an object of $\mathcal{T}^\beta$ to an object of $\mathcal{S} = \langle S \rangle$. So by Theorem 37 it must factor as

$$k_i \longrightarrow k_i' \longrightarrow X_i$$

for some $k_i' \in \langle S \rangle^\beta \subseteq \mathcal{S} \cap \mathcal{T}^\beta$. Thus we can factor our original morphism $k \longrightarrow \bigoplus_i X_i$ as

$$k \longrightarrow \bigoplus_{i \in I} k_i' \longrightarrow \bigoplus_{i \in I} X_i$$

which proves that k is β-good for $\mathcal{S} \cap \mathcal{T}^\beta$ and completes the proof. □

Putting the results of this section together, we have the following

Theorem 48. *Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{S}$ a localising subcategory. Suppose that we have a small infinite cardinal α and nonempty classes of objects $T \subseteq \mathcal{T}^\alpha, S \subseteq \mathcal{S} \cap \mathcal{T}^\alpha$ such that*

$$\mathcal{T} = \langle T \rangle \text{ and } \mathcal{S} = \langle S \rangle$$

Then for any small regular cardinal $\beta \geq \alpha$ we have

$$\langle S\rangle^\beta = \mathcal{S}^\beta = \mathcal{S} \cap \mathcal{T}^\beta$$
$$\langle T\rangle^\beta = \mathcal{T}^\beta$$

The canonical triangulated functor $\mathcal{T}^\beta/\mathcal{S}^\beta \longrightarrow \mathcal{T}/\mathcal{S}$ is fully faithful and induces a triangulated functor $\theta : \mathcal{T}^\beta/\mathcal{S}^\beta \longrightarrow (\mathcal{T}/\mathcal{S})^\beta$ with the following properties

- *If $\beta = \aleph_0$ then every object of $(\mathcal{T}/\mathcal{S})^\beta$ is a direct summand of some object of $\mathcal{T}^\beta/\mathcal{S}^\beta$.*
- *If $\beta > \aleph_0$ then θ is an equivalence.*

Proof. Firstly we observe that $\langle S\rangle^\beta$, the smallest thick β-localising subcategory of $\mathcal{T}$ containing S, is also the smallest thick β-localising subcategory of $\mathcal{S}$ containing S, so there is no ambiguity in the notation. From Lemma 47 we know that $\mathcal{S} \cap \mathcal{T}^\beta \subseteq \mathcal{S}^\beta$. So we have inclusions

$$\langle S\rangle^\beta \subseteq \mathcal{S} \cap \mathcal{T}^\beta \subseteq \mathcal{S}^\beta$$

By Lemma 44 applied to the triangulated category $\mathcal{S}$ we have $\langle S\rangle^\beta = \mathcal{S}^\beta$, so all three inclusions are equalities. The equality $\langle T\rangle^\beta = \mathcal{T}^\beta$ follows immediately from Lemma 44. By Corollary 39 the canonical triangulated functor

$$\mathcal{T}^\beta/\mathcal{S}^\beta \longrightarrow \mathcal{T}/\mathcal{S}$$

is a full embedding. By Lemma 42 the image of this functor is contained in $(\mathcal{T}/\mathcal{S})^\beta$ so there is an induced triangulated functor $\theta : \mathcal{T}^\beta/\mathcal{S}^\beta \longrightarrow (\mathcal{T}/\mathcal{S})^\beta$. We know from Proposition 46 that every object of the portly triangulated category $(\mathcal{T}/\mathcal{S})^\beta$ is a direct summand (in $(\mathcal{T}/\mathcal{S})^\beta$ or $\mathcal{T}/\mathcal{S}$, the condition being the same) of some object of $\mathcal{T}^\beta$. In the case where $\beta > \aleph_0$ we showed in Corollary 46 that θ is an equivalence of categories. □

6.1 In the countable case

The results of the previous section were proved in great generality, for arbitrary small infinite cardinals. In applications the simplest case $\alpha = \aleph_0$ is the most common, so it is worthwhile recording the results in this special case explicitly.

Remark 16. Let $\mathcal{T}$ be a triangulated category with coproducts and let S be a nonempty class of objects of $\mathcal{T}$. Then the triangulated subcategory $\langle S\rangle^{\aleph_0}$ is simply the smallest thick triangulated subcategory of $\mathcal{T}$ containing the objects of S.

Lemma 49. *Let $\mathcal{T}$ be a triangulated category with coproducts, and suppose there is a nonempty class $S \subseteq \mathcal{T}^c$ of compact objects such that $\mathcal{T} = \langle S\rangle$. Then $\mathcal{T}^c$ is the smallest thick triangulated subcategory of $\mathcal{T}$ containing the objects of S.*

Proof. This is Lemma 2.2 of [Nee92], and it is a special case of Theorem 48 with $\alpha = \aleph_0$ and $\mathcal{S} = \mathcal{T}$. □

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