Tor

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1 Tor on the Right

Throughout this note R is a ring (not necessarily commutative). The abelian category $\mathbf{Mod}R$ has enough projectives and enough injectives.

Definition 1. Let B be a left R-module, so that $-\otimes_R B : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$ is a right exact functor. For $i \geq 0$ we define the abelian groups

$$Tor_i^R(A, B) = L_i(- \otimes_R B)(A)$$

We usually drop the ring from the notation. The functor $Tor_i(-, B)$ is additive and covariant for $i \geq 0$. Since $- \otimes_R B$ is right exact the functors $Tor_0(-, B)$ and $- \otimes_R B$ are naturally equivalent. We simply write $Tor(-, B)$ for $Tor_1(-, B)$.

The group $Tor_i(A, B)$ is only determined up to isomorphism, and to calculate it we find a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and calculate the homology of the sequence

$$\cdots \longrightarrow P_2 \otimes_R B \longrightarrow P_1 \otimes_R B \longrightarrow P_0 \otimes_R B \longrightarrow 0$$

We think of Tor_i as assigning to any pair A, B consisting of a right R-module A and a left R-module B an isomorphism class of abelian groups, which has the following properties:

- For any projective module P we have $Tor_i(P, B) = 0$ for $i \neq 0$, since this is a property of any left derived functor.

- For any flat module F we have $Tor_i(A, F) = 0$ for $i \neq 0$, since the higher left derived functors of the exact functor $-\otimes_R F$ are zero. In particular this is true for free and projective modules.

For any exact sequence of right R-modules

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

there are morphisms $\omega_1 : Tor(A'', B) \longrightarrow A' \otimes_R B$ and $\omega_n : Tor_n(A'', B) \longrightarrow Tor_{n-1}(A', B)$ for $n \geq 2$ which are canonical and make the following sequence exact

$$\begin{aligned} \cdots \longrightarrow Tor_2(A', B) \longrightarrow Tor_2(A, B) \longrightarrow Tor_2(A'', B) \longrightarrow \\ \longrightarrow Tor(A', B) \longrightarrow Tor(A, B) \longrightarrow Tor(A'', B) \longrightarrow \\ \longrightarrow A' \otimes_R B \longrightarrow A \otimes_R B \longrightarrow A'' \otimes_R B \longrightarrow 0 \end{aligned}$$

This sequence is called the *long exact Tor sequence in the first variable.* It is natural in the exact sequence, in the sense that if we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

Then the following diagrams commute for $n \geq 2$

$$\begin{tikzcd} Tor(A'', B) \arrow[r] \arrow[d] & A' \otimes_R B \arrow[d] \\ Tor(C'', B) \arrow[r] & C' \otimes_R B \end{tikzcd} \qquad \begin{tikzcd} Tor_n(A'', B) \arrow[r, "\omega_n"] \arrow[d] & Tor_{n-1}(A', B) \arrow[d] \\ Tor_n(C'', B) \arrow[r, "\omega_n"'] & Tor_{n-1}(C', B) \end{tikzcd}$$

The long exact sequence is also natural in the module B. Let $\beta : B \longrightarrow B'$ be a morphism of modules, and let $- \otimes \beta$ denote the natural transformation $- \otimes_R B \longrightarrow - \otimes_R B'$ defined by $(- \otimes \beta)_A = A \otimes \beta$. Let $\mathcal{P}$ be a fixed assignment of projective resolutions. Then there is a natural transformation $L_n(- \otimes \beta) : L_n(- \otimes_R B) \longrightarrow L_n(- \otimes_R B')$ and we denote by $Tor_n(A, \beta)$ the morphism $L_n(- \otimes \beta)_A : Tor_n(A, B) \longrightarrow Tor_n(A, B')$. Note that for another morphism $\gamma : B' \longrightarrow B''$, $L_n(- \otimes_R \gamma)L_n(- \otimes_R \beta) = L_n(- \otimes_R \gamma\beta)$ so for any object A

$$Tor_n(A, \gamma)Tor_n(A, \beta) = Tor_n(A, \gamma\beta)$$

This defines a covariant additive functor $Tor_n(A, -) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$. For any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ we obtain the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & Tor_n(A', B) \arrow[r] \arrow[d] & Tor_n(A, B) \arrow[r] \arrow[d] & Tor_n(A'', B) \arrow[r, "\omega_n"] \arrow[d] & Tor_{n-1}(A', B) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Tor_n(A', B') \arrow[r] & Tor_n(A, B') \arrow[r] & Tor_n(A'', B') \arrow[r, "\omega_n"'] & Tor_{n-1}(A', B') \arrow[r] & \cdots \end{tikzcd}$$

Proposition 1. *For $n \geq 0$ and a morphism $\alpha : A \longrightarrow A'$ of right R-modules and a morphism $\beta : B \longrightarrow B'$ of left R-modules*

$$Tor_n(A', \beta)Tor_n(\alpha, B) = Tor_n(\alpha, B')Tor_n(A, \beta) \tag{1}$$

It follows that Tor_n defines a functor $\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ for $n \geq 0$, with $Tor_n(\alpha, \beta) : Tor_n(A, B) \longrightarrow Tor_n(A', B')$ given by the equivalent expressions in (1). The partial functors are the functors $Tor_n(A, -)$ and $Tor_n(-, B)$ defined above.

Proof. This follows for arbitrary β and monomorphisms (or epimorphisms) α by commutativity of (1). Since $\mathbf{Mod}R$ has epi-mono factorisations it then follows for arbitrary α. If we use a different assignment of projective resolutions to calculate the bifunctor Tor_n the results will be canonically naturally equivalent. $\square$

For a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ of left R-modules the corresponding sequence of natural transformations $- \otimes B' \longrightarrow - \otimes B \longrightarrow - \otimes B''$ is exact on projectives. Then for every object A there are canonical connecting morphisms $\omega_n : Tor_n(A, B'') \longrightarrow Tor_{n-1}(A, B')$ for $n \geq 1$ which fit into an exact sequence

$$\cdots \longrightarrow Tor_n(A, B') \longrightarrow Tor_n(A, B) \longrightarrow Tor_n(A, B'') \longrightarrow Tor_{n-1}(A, B') \longrightarrow \cdots$$

This sequence is called the *long exact* Tor *sequence in the second variable*. It is natural in both A and the exact sequence. For a morphism $\alpha : A \longrightarrow A'$ of right R-modules the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & Tor_n(A, B') \arrow[r] \arrow[d] & Tor_n(A, B) \arrow[r] \arrow[d] & Tor_n(A, B'') \arrow[r, "\omega_n"] \arrow[d] & Tor_{n-1}(A, B') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Tor_n(A', B') \arrow[r] & Tor_n(A', B) \arrow[r] & Tor_n(A', B'') \arrow[r, "\omega_n"'] & Tor_{n-1}(A', B') \arrow[r] & \cdots
\end{tikzcd}$$

And for a commutative diagram of left R-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0
\end{tikzcd}$$

The following diagram commutes for any right R-module A

$$\begin{tikzcd}
\cdots \arrow[r] & Tor_n(A, B') \arrow[r] \arrow[d] & Tor_n(A, B) \arrow[r] \arrow[d] & Tor_n(A, B'') \arrow[r, "\omega_n"] \arrow[d] & Tor_{n-1}(A, B') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Tor_n(A, C') \arrow[r] & Tor_n(A, C) \arrow[r] & Tor_n(A, C'') \arrow[r, "\omega_n"'] & Tor_{n-1}(A, C') \arrow[r] & \cdots
\end{tikzcd}$$

We have shown that for every assignment of projective resolutions $\mathcal{P}$ to $\mathbf{Mod}R$ we obtain a bifunctor $Tor_n^{\mathcal{P}}(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ for $n \geq 0$ with the property that short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. The connecting morphisms for these sequences depend only on $\mathcal{P}$.

If $\mathcal{Q}$ is another assignment of resolutions to $\mathbf{Mod}R$ then we obtain another bifunctor $Tor_n^{\mathcal{Q}}(-,-)$ for $n \geq 0$ which is canonically naturally equivalent to $Tor_n^{\mathcal{P}}(-,-)$. The connecting morphisms for the two assignments $\mathcal{P}, \mathcal{Q}$ agree in the following sense: for an object B and an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & Tor_n^{\mathcal{P}}(A', B) \arrow[r] \arrow[d, Rightarrow] & Tor_n^{\mathcal{P}}(A, B) \arrow[r] \arrow[d, Rightarrow] & Tor_n^{\mathcal{P}}(A'', B) \arrow[r, "\omega_n^{\mathcal{P}}"] \arrow[d, Rightarrow] & Tor_{n-1}^{\mathcal{P}}(A', B) \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\cdots \arrow[r] & Tor_n^{\mathcal{Q}}(A', B) \arrow[r] & Tor_n^{\mathcal{Q}}(A, B) \arrow[r] & Tor_n^{\mathcal{Q}}(A'', B) \arrow[r, "\omega_n^{\mathcal{Q}}"'] & Tor_{n-1}^{\mathcal{Q}}(A', B) \arrow[r] & \cdots
\end{tikzcd}$$

Similarly for an object A and an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & Tor_n^{\mathcal{P}}(A, B') \arrow[r] \arrow[d, Rightarrow] & Tor_n^{\mathcal{P}}(A, B) \arrow[r] \arrow[d, Rightarrow] & Tor_n^{\mathcal{P}}(A, B'') \arrow[r, "\omega_n^{\mathcal{P}}"] \arrow[d, Rightarrow] & Tor_{n-1}^{\mathcal{P}}(A, B') \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\cdots \arrow[r] & Tor_n^{\mathcal{Q}}(A, B') \arrow[r] & Tor_n^{\mathcal{Q}}(A, B) \arrow[r] & Tor_n^{\mathcal{Q}}(A, B'') \arrow[r, "\omega_n^{\mathcal{Q}}"'] & Tor_{n-1}^{\mathcal{Q}}(A, B') \arrow[r] & \cdots
\end{tikzcd}$$

Both these claims follow direct from our Derived Functor notes.

2 Tor on the Left

Definition 2. Let A be a right R-module, so that $A \otimes_R - : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a right exact functor. For $i \geq 0$ we define the abelian groups

$$\underline{Tor}_i(A,B) = L_i(A \otimes_R -)(B)$$

We usually drop the ring from the notation. The functor $\underline{Tor}_i(A,-)$ is additive and covariant for $i \geq 0$. Since $A \otimes_R -$ is right exact the functors $\underline{Tor}_0(A,-)$ and $A \otimes_R -$ are naturally equivalent. We simply write $\underline{Tor}(A,-)$ for $\underline{Tor}_1(A,-)$

The group $\underline{Tor}_i(A,B)$ is only determined up to isomorphism, and to calculate it we find a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ and calculate the homology of the sequence

$$\cdots \longrightarrow A \otimes_R P_2 \longrightarrow A \otimes_R P_1 \longrightarrow A \otimes_R P_0 \longrightarrow 0$$

We think of $\underline{Tor}_i$ as assigning to any pair A, B consisting of a right R-module A and a left R-module B an isomorphism class of abelian groups, which has the following properties:

- For any projective module P we have $\underline{Tor}_i(A,P) = 0$ for $i \neq 0$, since this is a property of any left derived functor.
- For any flat module F we have $\underline{Tor}_i(F,B) = 0$ for $i \neq 0$, since the higher left derived functors of the exact functor $F \otimes_R -$ are zero. In particular this is true for free and projective modules.

For any exact sequence of left R-modules

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

there are morphisms $\omega_1 : \underline{Tor}(A,B'') \longrightarrow A \otimes_R B'$ and $\omega_n : \underline{Tor}_n(A,B'') \longrightarrow \underline{Tor}_n(A,B')$ for $n \geq 2$ which are canonical and make the following sequence exact

$$\begin{aligned}
\cdots \longrightarrow \underline{Tor}_2(A,B') \longrightarrow \underline{Tor}_2(A,B) \longrightarrow \underline{Tor}_2(A,B'') \longrightarrow \\
\longrightarrow \underline{Tor}(A,B') \longrightarrow \underline{Tor}(A,B) \longrightarrow \underline{Tor}(A,B'') \longrightarrow \\
\longrightarrow A \otimes_R B' \longrightarrow A \otimes_R B \longrightarrow A \otimes_R B'' \longrightarrow 0
\end{aligned}$$

This sequence is called the *long exact Tor sequence in the second variable.* It is natural in the exact sequence, in the sense that if we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0
\end{tikzcd}$$

Then the following diagrams commute for $n \geq 2$

$$\begin{tikzcd}
\underline{Tor}(A,B'') \arrow[r] \arrow[d] & A \otimes_R B' \arrow[d] \\
\underline{Tor}(A,C'') \arrow[r] & A \otimes_R C'
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{Tor}_n(A,B'') \arrow[r, "\omega_n"] \arrow[d] & \underline{Tor}_{n-1}(A,B') \arrow[d] \\
\underline{Tor}_n(A,C'') \arrow[r, "\omega_n"'] & \underline{Tor}_{n-1}(A,C')
\end{tikzcd}$$

This long exact sequence is also natural in the module A. Let $\alpha : A \longrightarrow A'$ be a morphism of modules, and let $\alpha \otimes -$ denote the natural transformation $A \otimes_R - \longrightarrow A' \otimes_R -$ defined by $(\alpha \otimes -)_B = \alpha \otimes B$. Let $\mathcal{P}$ be a fixed assignment of projective resolutions. Then there is a natural transformation $L_n(\alpha \otimes -) : L_n(A \otimes_R -) \longrightarrow L_n(A' \otimes_R -)$ and we denote by $\underline{Tor}_n(\alpha, B)$ the morphism $L_n(\alpha \otimes -)_B : \underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_n(A',B)$. Note that for another morphism $\gamma : A' \longrightarrow A''$, $L_n(\gamma \otimes_R -)L_n(\alpha \otimes_R -) = L_n(\gamma\alpha \otimes_R -)$ for for any object B

$$\underline{Tor}_n(\gamma, B)\underline{Tor}_n(\alpha, B) = \underline{Tor}_n(\gamma\alpha, B)$$

This defines a covariant additive functor $\underline{Tor}_n(-,B) : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$. For any exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ we obtain the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{Tor}_n(A,B') \arrow[r] \arrow[d] & \underline{Tor}_n(A,B) \arrow[r] \arrow[d] & \underline{Tor}_n(A,B'') \arrow[r, "\omega^n"] \arrow[d] & \underline{Tor}_{n-1}(A,B') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{Tor}_n(A',B') \arrow[r] & \underline{Tor}_n(A',B) \arrow[r] & \underline{Tor}_n(A',B'') \arrow[r, "\omega^n"'] & \underline{Tor}_{n-1}(A',B') \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 2. *For $n \geq 0$ and a morphism $\alpha : A \longrightarrow A'$ of right R-modules and a morphism $\beta : B \longrightarrow B'$ of left R-modules*

$$\underline{Tor}_n(A',\beta)\underline{Tor}_n(\alpha,B) = \underline{Tor}_n(\alpha,B')\underline{Tor}_n(A,\beta) \tag{2}$$

It follows that $\underline{Tor}_n$ defines a functor $\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ for $n \geq 0$, with $\underline{Tor}_n(\alpha,\beta) : \underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_n(A',B')$ given by the equivalent expressions in (2). The partial functors are the functors $\underline{Tor}_n(A,-)$ and $\underline{Tor}_n(-,B)$ defined above.

Proof. The proof is straightforward. Once again if we use a different assignment of projective resolutions to calculate the bifunctor $\underline{Tor}_n$ the results are canonically naturally equivalent. $\square$

For a short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of right R-modules the corresponding sequence of natural transformations $A' \otimes - \longrightarrow A \otimes - \longrightarrow A'' \otimes -$ is exact on projectives. Then for every object B there are canonical connecting morphisms $\omega_n : \underline{Tor}_n(A'',B) \longrightarrow \underline{Tor}_{n-1}(A',B)$ for $n \geq 1$ with the property that the following sequence is exact

$$\cdots \longrightarrow \underline{Tor}_n(A',B) \longrightarrow \underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_n(A'',B) \longrightarrow \underline{Tor}_{n-1}(A',B) \longrightarrow \cdots$$

This sequence is called the *long exact Tor sequence in the first variable*. It is natural in both B and the exact sequence. For a morphism $\beta : B \longrightarrow B'$ of left R-modules the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{Tor}_n(A',B) \arrow[r] \arrow[d] & \underline{Tor}_n(A,B) \arrow[r] \arrow[d] & \underline{Tor}_n(A'',B) \arrow[r, "\omega^n"] \arrow[d] & \underline{Tor}_{n-1}(A',B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{Tor}_n(A',B') \arrow[r] & \underline{Tor}_n(A,B') \arrow[r] & \underline{Tor}_n(A'',B') \arrow[r, "\omega^n"'] & \underline{Tor}_{n-1}(A',B') \arrow[r] & \cdots
\end{tikzcd}$$

And for a commutative diagram of right R-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0
\end{tikzcd}$$

The following diagram commutes for any left R-module B

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{Tor}_n(A',B) \arrow[r] \arrow[d] & \underline{Tor}_n(A,B) \arrow[r] \arrow[d] & \underline{Tor}_n(A'',B) \arrow[r, "\omega^n"] \arrow[d] & \underline{Tor}_{n-1}(A',B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{Tor}_n(C',B) \arrow[r] & \underline{Tor}_n(C,B) \arrow[r] & \underline{Tor}_n(C'',B) \arrow[r, "\omega^n"'] & \underline{Tor}_{n-1}(C',B) \arrow[r] & \cdots
\end{tikzcd}$$

We have shown that for every assignment of projective resolutions $\mathcal{P}$ to $R\mathbf{Mod}$ we obtain a bifunctor $\underline{Tor}_n^{\mathcal{P}}(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ for $n \geq 0$ with the property that short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. The connecting morphisms for these sequences depend only on $\mathcal{P}$.

If $\mathcal{Q}$ is another assignment of resolutions to $R\mathbf{Mod}$ then we obtain another bifunctor $\underline{Tor}_n^{\mathcal{Q}}(-,-)$ for $n \geq 0$ which is canonically naturally equivalent to $\underline{Tor}_n^{\mathcal{P}}(-,-)$. The connecting morphisms for the two assignments $\mathcal{P}, \mathcal{Q}$ agree in the following sense: for an object A and an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{Tor}_n^{\mathcal{P}}(A,B') \arrow[r] \arrow[d, Rightarrow] & \underline{Tor}_n^{\mathcal{P}}(A,B) \arrow[r] \arrow[d, Rightarrow] & \underline{Tor}_n^{\mathcal{P}}(A,B'') \arrow[r, "\omega_n^{\mathcal{P}}"] \arrow[d, Rightarrow] & \underline{Tor}_{n-1}^{\mathcal{P}}(A,B') \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\cdots \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A,B') \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A,B) \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A,B'') \arrow[r, "\omega_n^{\mathcal{Q}}"'] & \underline{Tor}_{n-1}^{\mathcal{Q}}(A,B') \arrow[r] & \cdots
\end{tikzcd}$$

Similarly for an object B and an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{Tor}_n^{\mathcal{P}}(A',B) \arrow[r] \arrow[d, Rightarrow] & \underline{Tor}_n^{\mathcal{P}}(A,B) \arrow[r] \arrow[d, Rightarrow] & \underline{Tor}_n^{\mathcal{P}}(A'',B) \arrow[r, "\omega_n^{\mathcal{P}}"] \arrow[d, Rightarrow] & \underline{Tor}_{n-1}^{\mathcal{P}}(A',B) \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\cdots \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A',B) \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A,B) \arrow[r] & \underline{Tor}_n^{\mathcal{Q}}(A'',B) \arrow[r, "\omega_n^{\mathcal{Q}}"'] & \underline{Tor}_{n-1}^{\mathcal{Q}}(A',B) \arrow[r] & \cdots
\end{tikzcd}$$

Both these claims follow directly from our Derived Functor notes.

3 Balancing Tor

We choose once and for all assignments of projective resolutions $\mathcal{P}$ for $\mathbf{Mod}R$ and $\mathcal{Q}$ for $R\mathbf{Mod}$, with respect to which all derived functors are calculated. We have defined two bifunctors $Tor_n(-,-)$ and $\underline{Tor}_n(-,-)$ for $n \geq 0$. The first is calculated by taking the left derived functors of the functors $-\otimes_R B$ and the second by taking the left derived functors of the functors $A \otimes_R -$. We claim that these two bifunctors are naturally equivalent. We begin with the case $n = 0$.

Lemma 3. *There are canonical natural equivalences of bifunctors $\underline{Tor}_0(-,-) \cong -\otimes_R -$ and $-\otimes_R - \cong Tor_0(-,-)$.*

Proof. It is clear what we mean by the bifunctor $-\otimes_R - : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$. We already know there are canonical natural equivalences $Tor_0(-,B) \cong -\otimes_R B$ and $\underline{Tor}_0(A,-) \cong A \otimes_R -$ and it is not hard to check that these isomorphisms are natural in the other variable. $\square$

Proposition 4. *For $n \geq 0$ there is a canonical natural equivalence of bifunctors $\Phi^n : Tor_n(-,-) \cong \underline{Tor}_n(-,-)$.*

Proof. We proceed by induction on n, having already proved the result for $n = 0$. Assume that there is a canonical natural equivalence Φ^n and let a right R-module A and a left R-module B be given. We have to define a canonical isomorphism $\Phi^{n+1}_{A,B}$ which is natural in A and B. Choose a projective presentation of B

$$0 \longrightarrow K \longrightarrow P \longrightarrow B \longrightarrow 0$$

We know that $Tor_i(A,P) = 0 = \underline{Tor}_i(A,P)$ for $i \neq 0$. Now we show how to define the isomorphism $\Phi^{n+1}_{A,B} : Tor_{n+1}(A,B) \longrightarrow \underline{Tor}_n(A,B)$. There are two cases: if $n = 1$ then the long exact sequences for Tor and $\underline{Tor}$ in the second variable give a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Tor(A,B) \arrow[r, "\omega_1"] \arrow[d, dotted, "\Phi^1_{A,B}"'] & Tor_0(A,K) \arrow[r] \arrow[d, Rightarrow, "\Phi^0_{A,K}"'] & Tor_0(A,P) \arrow[r] \arrow[d, Rightarrow] & Tor_0(A,B) \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & \underline{Tor}(A,B) \arrow[r, "\omega_1"'] & \underline{Tor}_0(A,K) \arrow[r] & \underline{Tor}_0(A,P) \arrow[r] & \underline{Tor}_0(A,B) \arrow[r] & 0
\end{tikzcd}$$

This induces a unique isomorphism $\Phi^1_{A,B}$ making the diagram commute. For $n > 1$ the connecting morphisms $Tor_n(A,B) \longrightarrow Tor_{n-1}(A,K)$ and $\underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_{n-1}(A,K)$ in the two sequences are isomorphisms, and we define $\Phi^{n+1}_{A,B}$ to be the unique morphism fitting into the following commutative diagram

$$\begin{tikzcd}
Tor_{n+1}(A,B) \arrow[r, Rightarrow] \arrow[d, dotted, "\Phi^{n+1}_{A,B}"'] & Tor_n(A,K) \arrow[d, Rightarrow, "\Phi^n_{A,K}"'] \\
\underline{Tor}_{n+1}(A,B) \arrow[r, Rightarrow] & \underline{Tor}_n(A,K)
\end{tikzcd}$$

Next we have to show that the isomorphism $\Phi^{n+1}_{A,B}$ does not depend on the chosen presentation. Suppose we have a commutative diagram with exact rows and middle objects projective:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & Q \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$

Consider the following cube for $n \geq 1$

$$\begin{tikzcd}
& Tor_{n+1}(A,B') \arrow[rr] \arrow[dd, Rightarrow] & & Tor_n(A,M) \arrow[dd, Rightarrow, "\Phi^n_{A,M}"'] \\
Tor_{n+1}(A,B) \arrow[ur] \arrow[rr] \arrow[dd, Rightarrow] & & Tor_n(A,K) \arrow[ur] \arrow[dd, Rightarrow, "\Phi^n_{A,K}"' near end] & \\
& \underline{Tor}_{n+1}(A,B') \arrow[rr] & & \underline{Tor}_n(A,M) \\
\underline{Tor}_{n+1}(A,B) \arrow[ur] \arrow[rr] & & \underline{Tor}_n(A,K) \arrow[ur] &
\end{tikzcd}$$

If we use the above technique to produce isomorphisms $Tor_{n+1}(A,B) \longrightarrow \underline{Tor}_{n+1}(A,B)$ and $Tor_{n+1}(A,B') \longrightarrow \underline{Tor}_{n+1}(A,B')$ using the given presentations then in either case ($n = 1$ or otherwise) these morphisms make the front and back squares commute. The top and bottom squares commute by naturality of the connecting morphisms, and the right square commutes by the inductive hypothesis. Since $\underline{Tor}_{n+1}(A,B') \longrightarrow \underline{Tor}_n(A,M)$ is a monomorphism, it follows that the left square also commutes. This implies that $\Phi^{n+1}_{A,B}$ is independent of the chosen resolution and natural in B.

To prove naturality in A we construct a similar diagram and use naturality with respect to morphisms of the long exact Tor and $\underline{Tor}$ sequences in the second variable. Since by the inductive hypothesis Φ^n depends only on the assignments of resolutions $\mathcal{P}, \mathcal{Q}$, it follows that this is true of Φ^{n+1} as well. $\square$

For any ring R and assignments $\mathcal{P}, \mathcal{Q}$ of projective resolutions to $\mathbf{Mod}R$ and $R\mathbf{Mod}$ respectively, there is a natural equivalence of the bifunctors $Tor^{\mathcal{P}}_n(-,-)$ and $\underline{Tor}^{\mathcal{Q}}_n(-,-)$ for $n \geq 0$. So every pair A, B consisting of a right R-module A and a left R-module B together with an integer $n \geq 0$ determines an isomorphism class of abelian groups. We can calculate a representative of this class in the following ways

- Choose a projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ of A and calculate the homology of the following chain complex of abelian groups

$$\cdots \longrightarrow P_2 \otimes_R B \longrightarrow P_1 \otimes_R B \longrightarrow P_0 \otimes_R B \longrightarrow 0$$

- Choose a projective resolution $\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$ of B and calculate the homology of the following chain complex of abelian groups

$$\cdots \longrightarrow A \otimes_R Q_2 \longrightarrow A \otimes_R Q_1 \longrightarrow A \otimes_R Q_0 \longrightarrow 0$$

If there is no chance of confusion we simply refer to any of these groups by $Tor_n(A,B)$ and drop $\underline{Tor}$ from the notation. Occasionally, however, we will insist on the distinction.

4 Properties of Tor

4.1 Dimension Shifting

Let B be a left R-module. Then any flat right R-module acyclic for the additive functor $- \otimes_R B : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$. Similarly for any left R-module A, any flat left R-module is acyclic for the functor $A \otimes_R - : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$. Hence the following results are immediate consequences of our notes on dimension shifting.

Proposition 5. *Let B be a left R-module, and suppose we have an exact sequence of right R-modules with all F_i flat and $m \geq 0$*

$$0 \longrightarrow M \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

Then there are canonical isomorphisms $\rho_n : Tor_n(A,B) \longrightarrow Tor_{n-m-1}(M,B)$ for $n \geq m+2$, and an exact sequence

$$0 \longrightarrow Tor_{m+1}(A,B) \longrightarrow M \otimes_R B \longrightarrow F_m \otimes_R B$$

These are both natural in B, in the sense that for a morphism $B \longrightarrow B'$ the following two diagrams commute for $n \geq m+2$ and $m \geq 0$

$$\begin{tikzcd}
Tor_n(A,B) \arrow[r] \arrow[d] & Tor_{n-m-1}(M,B) \arrow[d] \\
Tor_n(A,B') \arrow[r] & Tor_{n-m-1}(M,B')
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Tor_{m+1}(A,B) \arrow[r] \arrow[d] & M \otimes_R B \arrow[r] \arrow[d] & F_m \otimes_R B \arrow[d] \\
0 \arrow[r] & Tor_{m+1}(A,B') \arrow[r] & M \otimes_R B' \arrow[r] & F_m \otimes_R B'
\end{tikzcd}$$

Proposition 6. *Let A be a right R-module, and suppose we have an exact sequence of left R-modules with all F_i flat and $m \geq 0$*

$$0 \longrightarrow M \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

Then there are canonical isomorphisms $\rho_n : \underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_{n-m-1}(A,M)$ for $n \geq m+2$ and an exact sequence

$$0 \longrightarrow \underline{Tor}_{m+1}(A,B) \longrightarrow A \otimes_R M \longrightarrow A \otimes_R F_m$$

These are both natural in A, in the sense that for a morphism $A \longrightarrow A'$ the following two diagrams commute for $n \geq m+2$ and $m \geq 0$

$$\begin{tikzcd}
\underline{Tor}_n(A,B) \arrow[r] \arrow[d] & \underline{Tor}_{n-m-1}(A,M) \arrow[d] \\
\underline{Tor}_n(A',B) \arrow[r] & \underline{Tor}_{n-m-1}(A',M)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \underline{Tor}_{m+1}(A,B) \arrow[r] \arrow[d] & A \otimes_R M \arrow[r] \arrow[d] & A \otimes_R F_m \arrow[d] \\
0 \arrow[r] & \underline{Tor}_{m+1}(A',B) \arrow[r] & A' \otimes_R M \arrow[r] & A' \otimes_R F_m
\end{tikzcd}$$

4.2 Tor and Colimits

Proposition 7. *Let R be a ring. For a left R-module B and a right R-module A, the functors $Tor_n(A,-) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ and $Tor_n(-,B) : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$ preserve coproducts and direct limits.*

Proof. The functors $A \otimes_R -$ and $- \otimes_R B$ preserve all colimits, so this follows immediately from our Derived Functor notes. $\square$

Corollary 8. *Let R be a ring. For a left R-module B and a right R-module A, we have*

$$Tor_n(A,B) = \varinjlim Tor_n(A,B_\alpha) = \varinjlim Tor_n(A_\alpha,B)$$

where the direct limits are over all the finitely generated submodules A_α, B_α of A and B respectively.

5 Tor for Commutative Rings

Let R be a commutative ring and $U : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ the forgetful functor, which is faithful and exact. This functor maps the canonical kernels, cokernels, images, zero and biproducts of $R\mathbf{Mod}$ to the corresponding canonical structure on $\mathbf{Ab}$. So if X is a (co)chain complex in $R\mathbf{Mod}$ then the (co)homology modules have as underlying groups the (co)homology groups of the sequence considered as a complex of groups.

Given a left R-module B tensoring with B defines a right exact functor $S : -\otimes_R B : \mathbf{Mod}R \longrightarrow R\mathbf{Mod}$. Denote by T the functor $- \otimes_R B : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$, so that $T = US$. Given $n \geq 0$ and an assignment of projective resolutions $\mathcal{P}$ to $\mathbf{Mod}R$ the functors L_nT and $U \circ L_nS$ are *equal*. So for a right R-module A the Tor group $Tor_n(A,B)$ becomes an R-module in a canonical way, and for $\alpha : A \longrightarrow A'$ the morphism of groups $Tor_n(\alpha,B) : Tor_n(A,B) \longrightarrow Tor_n(A',B)$ is a morphism of these modules. Similarly if $\beta : B \longrightarrow B'$ is a morphism of modules then the morphism of groups $Tor_n(A,B) \longrightarrow Tor_n(A,B')$ is a morphism of modules, so $Tor_n(A,-)$ lifts to a covariant additive functor $\mathbf{Mod}R \longrightarrow R\mathbf{Mod}$. Also $Tor_0(-,B) : \mathbf{Mod}R \longrightarrow R\mathbf{Mod}$ is canonically naturally equivalent to S.

For a fixed assignment of projective resolutions $\mathcal{P}$ the bifunctor $Tor_n(-,-)$ becomes a bifunctor $Tor_n(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$. If $\mathcal{Q}$ is another assignment of projective resolutions to $\mathbf{Mod}R$ then the resulting bifunctors (with values in $R\mathbf{Mod}$) are canonically naturally equivalent. Given an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of right R-modules the connecting morphisms $Tor_n(A'',B) \longrightarrow Tor_{n-1}(A',B)$ are all module morphisms, so the long exact sequence of Tor in the first variable

$$\cdots \longrightarrow Tor_n(A',B) \longrightarrow Tor_n(A,B) \longrightarrow Tor_n(A'',B) \longrightarrow Tor_{n-1}(A',B) \longrightarrow \cdots$$

is a long exact sequence of modules. Similarly if $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is an exact sequence then the connecting morphisms $Tor_n(A,B'') \longrightarrow Tor_{n-1}(A,B')$ are module morphisms and the long exact sequence of Tor in the second variable

$$\cdots \longrightarrow Tor_n(A,B') \longrightarrow Tor_n(A,B) \longrightarrow Tor_n(A,B'') \longrightarrow Tor_{n-1}(A,B') \longrightarrow \cdots$$

is a long exact sequence of modules. If $r \in R$ let $\alpha : A \longrightarrow A$ and $\beta : B \longrightarrow B$ the module morphisms obtained from multiplication by r. Then $Tor_n(\alpha,B) = Tor_n(A,\beta) : Tor_n(A,B) \longrightarrow Tor_n(A,B)$ give the action of r on the abelian group $Tor_n(A,B)$.

Similarly, if we replace S by the right exact functor $A \otimes_R - : R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ and T by $A \otimes_R - : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ for a right R-module A, then all the above remains true with Tor replaced by $\underline{Tor}$. That is, $\underline{Tor}_n(-,-)$ lifts to a functor $\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ which (up to a canonical natural equivalence) is independent of the assignment of projective resolutions to $R\mathbf{Mod}$, and the two long exact sequences are long exact sequences of modules. With r, α, β as above, the group morphisms $\underline{Tor}_n(\alpha,B) = \underline{Tor}_n(A,\beta) : \underline{Tor}_n(A,B) \longrightarrow \underline{Tor}_n(A,B)$ give the action of r on the abelian group $Tor_n(A,B)$.

The canonical natural equivalences $\underline{Tor}_0(-,-) \cong - \otimes_R - \cong Tor_0(-,-)$ give natural equivalences of the module-valued bifunctors. Then our earlier proof shows that for $n \geq 0$ there is a canonical natural equivalence $Tor_n(-,-) \cong \underline{Tor}_n(-,-)$ of bifunctors $\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$.

So associated to any pair of R-modules M, N is an isomorphism class of R-modules $Tor_n^R(M,N)$. We can find a representative of this class by choosing a projective resolution P of N and calculating the homology modules of $\cdots \longrightarrow M \otimes P_1 \longrightarrow M \otimes P_0 \longrightarrow 0$ or choosing a projective resolution Q of M and calculating the homology modules of $\cdots \longrightarrow Q_1 \otimes N \longrightarrow Q_0 \otimes N \longrightarrow 0$.

Lemma 9. *For a commutative ring R and module A the additive functors $\{Tor_n(A,-)\}_{n\geq 0}$ form a universal homological δ-functor between $R\mathbf{Mod}$ and $R\mathbf{Mod}$.*

Proof. We have just shown that $\{Tor_n(A,-)\}_{n\geq 0}$ is a homological δ-functor. Since every module is the quotient of a free module, the functors $Tor_n(A,-)$ are coeffaceable for $n > 0$ and so by (DF,Theorem 74) our δ-functor is universal. $\square$

Lemma 10. *For a commutative ring R and module A the functors $Tor_n(A,-) : R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ and $Tor_n(-,A) : \mathbf{Mod}R \longrightarrow R\mathbf{Mod}$ preserve all coproducts and direct limits.*

Lemma 11. *For a commutative ring R and modules A, B we have an isomorphism of R-modules $Tor_n(A,B) \cong Tor_n(B,A)$ for $n \geq 0$.*

Proof. Let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ be a projective resolution of B (considered as a left R-module). Then the chain complex $\cdots \longrightarrow A \otimes P_1 \longrightarrow A \otimes P_0 \longrightarrow 0$ is isomorphic to the chain complex $\cdots \longrightarrow P_1 \otimes A \longrightarrow P_0 \otimes A \longrightarrow 0$, so both sequences have the homology. Hence $Tor_n(B,A) \cong Tor(A,B)$. $\square$

Example 1. Suppose that $x \in R$ is regular and that M is an R-module. We compute the groups $Tor_i^R(R/(x), M)$. The short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

is a projective resolution of $R/(x)$. So the group $Tor_i^R(R/(x), M)$ is the ith homology group of the complex

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

Hence $Tor_i^R(R/(x), M) = 0$ for $i > 1$, while $Tor_1^R(R/(x), M) = (0 : x) = \{m \,|\, xm = 0\}$ and $Tor_0^R(R/(x), M) = M/xM$.

Proposition 12. *Let R be a principal ideal domain. Then*

(a) $Tor_1^R(A,B)$ is a torsion module.

(b) $Tor_n^R(A,B) = 0$ for $n \geq 2$.

Proof. First consider the case where A is finitely generated. Then either $A \cong R^m$ for some $m \geq 0$ or $A \cong R^m \oplus R/(x_1) \oplus \cdots \oplus R/(x_k)$ for $m \geq 0$ and nonzero x_i. In the first case $Tor_n^R(A,B) = Tor_n^R(R,B)^m$ and in the second case

$$Tor_n^R(A,B) \cong Tor_n^R(R,B)^m \oplus Tor_n^R(R/(x_1),B) \oplus \cdots \oplus Tor_n^R(R/(x_2),B)$$

The module A is the direct limit of its finitely generated submodules A_α, so $Tor_n^R(A,B)$ is the direct limit of the modules $Tor_n(A_\alpha, B)$. Since the direct limit of torsion modules is torsion, (a) and (b) follow easily from the previous Example. $\square$

Let $m, d \neq 0, 1$ be elements of $\mathbb{Z}$ and assume that $d|m$. Then $A = \mathbb{Z}/d$ is a $R = \mathbb{Z}/m$-module, and the following is a free resolution of A as an R-module

$$\cdots \longrightarrow \mathbb{Z}/m \xrightarrow{d} \mathbb{Z}/m \xrightarrow{m/d} \mathbb{Z}/m \xrightarrow{d} \mathbb{Z}/m \longrightarrow \mathbb{Z}/d \longrightarrow 0$$

Hence for any $\mathbb{Z}/m$-module B we have

$$Tor_n^{\mathbb{Z}/m}(\mathbb{Z}/d, B) = \begin{cases} B/dB & \text{if } n = 0 \\ \{b \in B \,|\, db = 0\}/(m/d)B & \text{if } n \text{ is odd }, n > 0 \\ \{b \in B \,|\, (m/d)b = 0\}/dB & \text{if } n \text{ is even }, n > 0 \end{cases}$$

Proposition 13. *Let R be a commutative noetherian ring and suppose A, B are finitely generated R-modules. Then $Tor_i^R(A, B)$ is a finitely generated R-module.*

Proof. Since R is noetherian and A is finitely generated, we can find a projective resolution $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow)$ with all the F_i finite free modules. Then in the following sequence every module is finitely generated

$$\cdots \longrightarrow F_1 \otimes B \longrightarrow F_0 \otimes B \longrightarrow 0$$

so the homology modules $Tor_i^R(A, B)$ will also be finitely generated. $\square$

5.1 Tor for Bimodules

Let R, S be rings and B an R-S-bimodule. Then tensoring with B gives right exact functors $- \otimes_R B : \mathbf{Mod}R \longrightarrow \mathbf{Mod}S$ and $B \otimes_S - : S\mathbf{Mod} \longrightarrow R\mathbf{Mod}$, so that tensor products with B inherit a module structure. We would like to extend this structure to the Tor groups. Let $R\mathbf{Mod}S$ denote the abelian category of R-S-bimodules.

Let $\mathcal{P}$ be an assignment of projective resolutions to the category $\mathbf{Mod}R$. Let B be an R-S-bimodule and let A be a right R-module with assigned resolution P. There are two ways to define a right S-module structure on the abelian group $Tor_i^R(A, B)$:

- Taking the ith left derived functor of $- \otimes_R B : \mathbf{Mod}R \longrightarrow \mathbf{Mod}S$ defines a functor $Tor_i^R(-, B) : \mathbf{Mod}R \longrightarrow \mathbf{Mod}S$, and by the same argument used for commutative rings the functor $Tor_i^R(-, B) : \mathbf{Mod}R \longrightarrow \mathbf{Ab}$ is equal to $Tor_i^R(-, B) : \mathbf{Mod}R \longrightarrow \mathbf{Mod}S$ followed by the forgetful functor $\mathbf{Mod}S \longrightarrow \mathbf{Ab}$. In this way $Tor_i^R(A, B)$ acquires a right S-module structure, which we calculate by taking the homology of the following sequence of right S-modules
$$\cdots \longrightarrow P_2 \otimes_R B \longrightarrow P_1 \otimes_R B \longrightarrow P_0 \otimes_R B \longrightarrow 0$$

- For $s \in S$ right multiplication by s defines a morphism of left R-modules $\beta : B \longrightarrow B$, and this gives rise to a morphism of abelian groups $Tor_i^R(A, \beta) : Tor_i^R(A, B) \longrightarrow Tor_i^R(A, B)$ which is the effect on homology of the following morphism of chain complexes of groups
$$\begin{tikzcd} \cdots \arrow[r] & P_2 \otimes_R B \arrow[r] \arrow[d, "1\otimes\beta"'] & P_1 \otimes_R B \arrow[r] \arrow[d, "1\otimes\beta"'] & P_0 \otimes_R B \arrow[r] \arrow[d, "1\otimes\beta"'] & 0 \\ \cdots \arrow[r] & P_2 \otimes_R B \arrow[r] & P_1 \otimes_R B \arrow[r] & P_0 \otimes_R B \arrow[r] & 0 \end{tikzcd}$$
In this way $Tor_i^R(A, B)$ acquires a second right S-module structure, which is clearly the same as the first.

So given resolutions for $\mathbf{Mod}R$, a right R-module A and an R-S-bimodule B there is a canonical right S-module structure on $Tor_i^R(A, B)$. It is not hard to check that the morphisms induced on the Tor groups by right R-module morphisms $A \longrightarrow A'$ and bimodule morphisms $B \longrightarrow B'$ are also morphisms of right S-modules. So the bifunctor $Tor_i^R(-, -) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ lifts canonically to a bifunctor $Tor_i^R(-, -) : \mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$. The bifunctor $Tor_0^R(-, -)$ is canonically naturally equivalent to $- \otimes_R - : \mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$. If you choose two different assignments of resolutions to $\mathbf{Mod}R$ then you get canonically naturally bifunctors $\mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$.

Using our Derived Functor notes on change of base, the two types of long exact sequence in $Tor^R(-, -)$ are sequences of right S-modules. That is, if we have an exact sequence of right

R-modules $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ then for an R-S-bimodule B the connecting morphisms $Tor_i^R(A'', B) \longrightarrow Tor_{i-1}^R(A', B)$ are morphisms of right S-modules and we have a long exact sequence of S-modules

$$\cdots \longrightarrow Tor_i^R(A', B) \longrightarrow Tor_i^R(A, B) \longrightarrow Tor_i^R(A'', B) \longrightarrow Tor_{i-1}^R(A', B) \longrightarrow \cdots$$

And if we have an exact sequence of R-S-bimodules $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ then for a right R-module A the connecting morphisms $Tor_i^R(A, B'') \longrightarrow Tor_{i-1}^R(A, B')$ are morphisms of right S-modules and we have a long exact sequence of S-modules

$$\cdots \longrightarrow Tor_i^R(A, B') \longrightarrow Tor_i^R(A, B) \longrightarrow Tor_i^R(A, B'') \longrightarrow Tor_{i-1}^R(A, B') \longrightarrow \cdots$$

Now suppose we are given an assignment of projective resolutions $\mathcal{Q}$ to the category $R\mathbf{Mod}$, let B be an R-S-bimodule whose assigned resolution as a left R-module is Q and let A be a right R-module. We can define a right S-module structure on the abelian group $\underline{Tor}_i^R(A, B)$ as follows:

- For $s \in S$ right multiplication by s defines a morphism of left R-modules $\beta : B \longrightarrow B$, and this gives rise to a morphism of abelian groups $\underline{Tor}_i^R(A, \beta) : \underline{Tor}_i^R(A, B) \longrightarrow \underline{Tor}_i^R(A, B)$ which is calculated in the following way: lift β to a morphism of chain complexes $\varphi : Q \longrightarrow Q$ and consider the effect on homology of the following morphism of chain complexes of groups

$$\begin{tikzcd} \cdots \arrow[r] & A \otimes_R Q_2 \arrow[r] \arrow[d, "1\otimes\varphi_2"'] & A \otimes_R Q_1 \arrow[r] \arrow[d, "1\otimes\varphi_1"'] & A \otimes_R Q_0 \arrow[r] \arrow[d, "1\otimes\varphi_0"'] & 0 \\ \cdots \arrow[r] & A \otimes_R Q_2 \arrow[r] & A \otimes_R Q_1 \arrow[r] & A \otimes_R Q_0 \arrow[r] & 0 \end{tikzcd}$$

  This defines a right S-module structure on the group $\underline{Tor}_i^R(A, B)$.

So given resolutions for $R\mathbf{Mod}$, a right R-module A and an R-S-bimodule B there is a canonical right S-module structure on $\underline{Tor}_i^R(A, B)$. It is a consequence of the fact that Tor is balanced that the canonical isomorphism of groups $\underline{Tor}_i^R(A, B) \cong Tor_i^R(A, B)$ is actually an isomorphism of S-modules, with the structures just defined. It follows that bifunctor $\underline{Tor}_i^R(-, -) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ lifts canonically to a bifunctor $\underline{Tor}_i^R(-, -) : \mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$ which is naturally equivalent to $Tor_i^R(-, -) : \mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$ defined earlier. Also $\underline{Tor}_0^R$ is naturally equivalent to $- \otimes_R - : \mathbf{Mod}R \times R\mathbf{Mod}S \longrightarrow \mathbf{Mod}S$.

Remark 1. We would like to say that the two types of long exact sequences in $\underline{Tor}$ are long exact sequences of right S-modules. This would follow if we knew that the isomorphism of Tor and $\underline{Tor}$ was natural with respect to connecting morphisms.

Next we study the case where the bimodule is in the first variable. Let $\mathcal{P}$ be an assignment of projective resolutions to the category $R\mathbf{Mod}$. Let A be an S-R-bimodule and let B be a left R-module with assigned resolution P. There are two ways to define a left S-module structure on the abelian group $\underline{Tor}_i^R(A, B)$:

- Taking the ith left derived functor of $A \otimes_R - : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ defines a functor $\underline{Tor}_i^R(A, -) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$, and the functor $\underline{Tor}_i^R(A, -) : R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is equal to $\underline{Tor}_i^R(A, -) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ followed by the forgetful functor $S\mathbf{Mod} \longrightarrow \mathbf{Ab}$. In this way $\underline{Tor}_i^R(A, B)$ acquires a left S-module structure, which we calculate by taking homology of the following sequence of left S-modules

$$\cdots \longrightarrow A \otimes_R P_2 \longrightarrow A \otimes_R P_1 \longrightarrow A \otimes_R P_0 \longrightarrow 0$$

- For $s \in S$ left multiplication by s defines a morphism of right R-modules $\alpha : A \longrightarrow A$, and this gives rise to a morphism of abelian groups $\underline{Tor}_i^R(\alpha, B) : \underline{Tor}_i^R(A, B) \longrightarrow \underline{Tor}_i^R(A, B)$ which is the effect on homology of the following morphism of chain complexes of groups

$$\begin{tikzcd} \cdots \arrow[r] & A \otimes_R P_2 \arrow[r] \arrow[d, "\alpha\otimes 1"'] & A \otimes_R P_1 \arrow[r] \arrow[d, "\alpha\otimes 1"'] & A \otimes_R P_0 \arrow[r] \arrow[d, "\alpha\otimes 1"'] & 0 \\ \cdots \arrow[r] & A \otimes_R P_2 \arrow[r] & A \otimes_R P_1 \arrow[r] & A \otimes_R P_0 \arrow[r] & 0 \end{tikzcd}$$

In this way $\underline{Tor}_i^R(A,B)$ acquires a second left S-module structure, which is clearly the same as the first.

So given resolutions for $R\mathbf{Mod}$, a left R-module B and an S-R-bimodule A there is a canonical left S-module structure on $\underline{Tor}_i^R(A,B)$. It is not hard to check that the morphisms induced on the $\underline{Tor}$ groups by left R-module morphisms $B \longrightarrow B'$ and bimodule morphisms $A \longrightarrow A'$ are also morphisms of left S-modules. So the bifunctor $\underline{Tor}_i^R(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ lifts canonically to a bifunctor $\underline{Tor}_i^R(-,-) : S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$. The bifunctor $\underline{Tor}_0^R(-,-)$ is canonically equivalent to $- \otimes_R - : S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$. If you choose two different assignments of resolutions to $R\mathbf{Mod}$ then you get canonically naturally equivalent bifunctors $S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$.

Using our Derived Functor notes on change of base, the two types of long exact sequence in $\underline{Tor}^R(-,-)$ are sequences of left S-modules. That is, if we have an exact sequence of left R-modules $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ then for an S-R-bimodule A the connecting morphisms $\underline{Tor}_i^R(A,B'') \longrightarrow \underline{Tor}_{i-1}^R(A,B')$ are morphisms of left S-modules and we have a long exact sequence of S-modules

$$\cdots \longrightarrow \underline{Tor}_i^R(A,B') \longrightarrow \underline{Tor}_i^R(A,B) \longrightarrow \underline{Tor}_i^R(A,B'') \longrightarrow \underline{Tor}_{i-1}^R(A,B') \longrightarrow \cdots$$

And if we have an exact sequence of S-R-bimodules $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ then for a left R-module B the connecting morphisms $\underline{Tor}_i^R(A'',B) \longrightarrow \underline{Tor}_{i-1}^R(A',B)$ are morphisms of left S-modules and we have a long exact sequence of S-modules

$$\cdots \longrightarrow \underline{Tor}_i^R(A',B) \longrightarrow \underline{Tor}_i^R(A,B) \longrightarrow \underline{Tor}_i^R(A'',B) \longrightarrow \underline{Tor}_{i-1}^R(A',B) \longrightarrow \cdots$$

Now suppose we are given an assignment of projective resolutions $\mathcal{Q}$ to the category $\mathbf{Mod}R$, let A be an S-R-bimodule whose assigned resolution as a right R-module is Q and let B be a left R-module. We can define a left S-module structure on the abelian group $Tor_i^R(A,B)$ as follows:

- For $s \in S$ left multiplication by s defines a morphism of right R-modules $\alpha : A \longrightarrow A$, and this gives rise to a morphism of abelian groups $Tor_i^R(\alpha,B) : Tor_i^R(A,B) \longrightarrow Tor_i^R(A,B)$ which is calculated in the following way: lift α to a morphism of chain complexes $\varphi : Q \longrightarrow Q$ and consider the effect on homology of the following morphism of chain complexes of groups

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \otimes_R B \arrow[r] \arrow[d, "\varphi_2 \otimes 1"'] & Q_1 \otimes_R B \arrow[r] \arrow[d, "\varphi_1 \otimes 1"'] & Q_0 \otimes_R B \arrow[r] \arrow[d, "\varphi_0 \otimes 1"'] & 0 \\
\cdots \arrow[r] & Q_2 \otimes_R B \arrow[r] & Q_1 \otimes_R B \arrow[r] & Q_0 \otimes_R B \arrow[r] & 0
\end{tikzcd}$$

  This defines a left S-module structure on the group $Tor_i^R(A,B)$.

So given resolutions for $\mathbf{Mod}R$, a left R-module B and an S-R-bimodule A there is a canonical left S-module structure on $Tor_i^R(A,B)$. It is a consequence of the fact that Tor is balanced that the canonical isomorphism of groups $Tor_i^R(A,B) \cong \underline{Tor}_i^R(A,B)$ is actually an isomorphism of S-modules, with the structures just defined. It follows that the bifunctor $Tor_i^R(-,-) : \mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ lifts canonically to a bifunctor $Tor_i^R(-,-) : S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ which is naturally equivalent to $\underline{Tor}_i^R(-,-) : S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ defined earlier. Also Tor_0^R is naturally equivalent to $- \otimes_R - : S\mathbf{Mod}R \times R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$.

Remark 2. Once again, we would like to say that the connecting morphisms for the left S-module valued bifunctor $Tor^R(-,-)$ were morphisms of left S-modules.

Since the contents of this section have the potential to confuse, we make some final comments. Given a ring R, an assignment of projective resolutions to $\mathbf{Mod}R$ gives you the bifunctors $Tor_n^R(-,-)$ and an assignment of projective resolutions to $R\mathbf{Mod}$ gives you the bifunctors $\underline{Tor}_n^R(-,-)$.

- If B is an R-S-bimodule then you can define a right S-module structure on the groups $Tor_n^R(A,B)$ and $\underline{Tor}_n^R(A,B)$. These two structures agree via the canonical isomorphism $Tor_n^R(A,B) \cong \underline{Tor}_n^R(A,B)$ of groups. However, it is only for the bifunctor Tor that we could show all the long exact sequences were sequences of right S-modules. So for bimodules in the second variable, Tor plays the major role, with $\underline{Tor}$ used only as an alternative method to calculate the S-module structure.

- If A is an S-R-bimodule then you can define a left S-module structure on the groups $\underline{Tor}_n^R(A,B)$ and $Tor_n^R(A,B)$. These two structures agree via the canonical isomorphism $Tor_n^R(A,B) \cong \underline{Tor}_n^R(A,B)$ of groups. However, it is only for the bifunctor $\underline{Tor}$ that we could show all the long exact sequences were sequences of left S-modules. So for bimodules in the first variable, $\underline{Tor}$ plays the major role, with Tor used only as an alternative method to calculate the S-module structure.

Lemma 14. *Let R, S be rings and A an S-R-bimodule. Then for every $n \geq 0$ we have an additive functor $\underline{Tor}_n^R(A,-) : R\mathbf{Mod} \longrightarrow S\mathbf{Mod}$ and the family $\{\underline{Tor}_n^R(A,-)\}_{n\geq 0}$ is a universal homological δ-functor between $R\mathbf{Mod}$ and $S\mathbf{Mod}$.*

Proof. The results of this section show that $\{\underline{Tor}_n^R(A,-)\}_{n\geq 0}$ is a homological δ-functor. For the rest, use the argument given in the proof of Lemma 9. $\square$

5.2 Criteria for Flatness

In this section all rings are commutative and all modules are left modules. Then

Proposition 15. *Let R be a ring and M an R-module. If I is an ideal of R then the multiplication map $I \otimes_R M \longrightarrow M$ is an injection if and only if $Tor_1^R(R/I, M) = 0$. The module M is flat if and only if this condition is satisifed for every finitely generated ideal I.*

Proof. From the short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ we obtain a long exact Tor sequence containing

$$Tor_1^R(R,M) \longrightarrow Tor_1^R(R/I,M) \longrightarrow I \otimes M \longrightarrow R \otimes M$$

Since R is projective, $Tor_1^R(R,M) = 0$ and the right hand term is M, so the first assertion is proved. We have proved the second assertion in our Stenstrom notes. $\square$

Since $Tor_1^R(R,M) = Tor_1^R(0,M) = 0$ we need only check finitely generated, nonzero, proper ideals I in the Proposition. If R is a ring and M an R-module, then for $x \in R$ we let xM denote the submodule $(x)M = \{xm \,|\, m \in M\}$.

Corollary 16. *Let k be a field. If $R = k[t]/(t^2)$ and M is an R-module, then M is flat if and only if multiplication by t from M to tM induces an isomorphism $M/tM \longrightarrow tM$.*

Proof. Since $t^2 = 0$ it is clear that tM is contained in the kernel of this map. We claim that M is flat iff. tM *is* the kernel - so $tm = 0$ if and only if $m = tn$ for some $n \in M$. The only nonzero proper ideal in R is (t), which is isomorphism as an R-module to $R/(t)$ by the map $R/(t) \longrightarrow (t)$ sending 1 to t. Applying the criterion of Proposition 15 we see that M is flat iff the map $(t) \otimes M \longrightarrow M$ is injective, which is iff the composite

$$M/tM \cong R/(t) \otimes M \cong (t) \otimes M \longrightarrow M$$

is injective. But this is precisely the map $M/tM \longrightarrow tM$ we are interested in. $\square$

If M is an R-module then an element $a \in R$ is *regular on M* if it is nonzero and $a \cdot m \neq 0$ for all nonzero $m \in M$.

Corollary 17. *If $a \in R$ is regular and M is a flat R-module, then a is regular on M. If R is a principal ideal domain, then the converse is also true: M is flat as an R-module if and only if M is torsion free.*

Proof. By assumption the map $R \longrightarrow (a)$ defined by $1 \mapsto a$ is an isomorphism of R-modules. So we obtain a morphism

$$M \cong R \otimes M \cong (a) \otimes M \longrightarrow R \otimes M \cong M$$

This composite is simply multiplication by a, and if M is flat it is a monomorphism, which proves that a is regular on M.

If R is a PID, then every nonzero ideal is generated by a regular element, and we have seen that for x regular, $Tor^R(R/(x), M) = \{m \,|\, xm = 0\}$. So M is flat if and only if every nonzero element of R is regular on M, which is what we mean when we say M is torsion free. □

Next we collect some results about flatness from our Stenstrom notes.

- A module isomorphic to a flat module is flat.
- If $R \longrightarrow S$ is a ring morphism and M is a flat R-module then $M \otimes_R S$ is a flat S-module.
- If F_i are R-modules then any coproduct $\bigoplus_{i \in I} F_i$ over a nonempty index set is flat iff. each F_i is flat.
- Every projective module is flat.
- Direct limits of flat modules are flat.
- A module F is flat iff. $Hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is injective.
- A module F is flat iff. whenever $\sum_{i=1}^n b_i x_i = 0$ for $b_i \in R, x_i \in F$ then there exist $u_1, \ldots, u_m$ in F and $a_{ij} \in R$ ($1 \leq i \leq n, 1 \leq j \leq m$) such that

$$0 = \sum_i b_i a_{ij} \qquad \forall j$$

$$x_i = \sum_j a_{ij} u_j \qquad \forall i$$

- Every finitely presented flat module is projective.
- Let A be a subring of B. If M is a finitely generated flat A-module, and $M \otimes_A B$ is a projective B-module, then M is projective over A. For example if A is an integral domain with field of fractions K then every finitely generated flat module A-module is projective, since $M \otimes_A K$ is always projective.
- If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact with N flat then L is flat iff. M is flat.

Theorem 18 (Local Criterion for Flatness). *Suppose that $(R, \mathfrak{m})$ is a local noetherian ring, and let $(S, \mathfrak{n})$ be a local noetherian R-algebra such that $\mathfrak{m}S \subseteq \mathfrak{n}$. If M is a finitely generated S-module, then M is flat as an R-module if and only if $Tor_1^R(R/\mathfrak{m}, M) = 0$.*

Proof. Another way of expressing the setup is that we have a local morphism $R \longrightarrow S$ of local noetherian rings. If M is flat then $Tor_1^R(R/\mathfrak{m}, M) = 0$ by Proposition 15.

Now suppose S and M are as stated and that $Tor_1^R(R/\mathfrak{m}, M) = 0$. As a preliminary step we show that if N is an R-module of finite length then $Tor_1^R(N, M) = 0$. We may prove this by induction on the length n. If N has length 1 then it is isomorphic to $R/\mathfrak{m}$, so this case follows from the hypothesis. If N' is any proper nonzero submodule of N, then the exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0$ gives rise to an exact sequence of Tor containing the terms

$$Tor_1^R(N', M) \longrightarrow Tor_1^R(N, M) \longrightarrow Tor_1^R(N/N', M)$$

By induction on the length, $Tor_1^R(N', M) = 0 = Tor_1^R(N/N', M)$ so $Tor_1^R(N, M) = 0$ as required.

Now let I be an arbitrary proper nonzero ideal, and suppose that $u \in I \otimes_R M$ is in the kernel of the multiplication map $I \otimes_R M \longrightarrow M$. We shall prove that $u = 0$. The S-module structure on M gives $I \otimes_R M$ the structure of an S-module, and we have $\mathfrak{m}^n(I \otimes_R M) \subseteq \mathfrak{n}^n(I \otimes_R M)$. It is finitely generated as an S-module , so by the Krull intersection theorem $\bigcap_n \mathfrak{n}^n(I \otimes_R M) = 0$, and so $\bigcap_n \mathfrak{m}^n(I \otimes_R M) = 0$. Thus it suffices to show that $u \in \mathfrak{m}^n(I \otimes_R M)$ for every $n \geq 1$.

With $n \geq 1$ fixed, the module $\mathfrak{m}^n(I \otimes_R M)$ is the image in $I \otimes_R M$ of $(\mathfrak{m}^n I) \otimes_R M$. By the Artin-Rees lemma, $\mathfrak{m}^t \cap I \subseteq \mathfrak{m}^n I$ for sufficiently large t, so it suffices to show that u is in the image of $(\mathfrak{m}^t \cap I) \otimes_R M$ for all t. Tensoring the short exact sequence

$$0 \longrightarrow \mathfrak{m}^t \cap I \longrightarrow I \longrightarrow I/(\mathfrak{m}^t \cap I) \longrightarrow 0$$

with M produces the exact sequence

$$(\mathfrak{m}^t \cap I) \otimes_R M \longrightarrow I \otimes_R M \longrightarrow I/(\mathfrak{m}^t \cap I) \otimes_R M \longrightarrow 0$$

It thus suffices to show that u goes to 0 in $I/(\mathfrak{m}^t \cap I) \otimes_R M$. Consider the following commutative diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d] & I/(\mathfrak{m}^t \cap I) \arrow[d, "\varphi"] \\ R \arrow[r] & R/\mathfrak{m}^t \end{tikzcd}$$

Tensoring with M gives

$$\begin{tikzcd} I \otimes_R M \arrow[r] \arrow[d] & I/(\mathfrak{m}^t \cap I) \otimes_R M \arrow[d, "\varphi\otimes 1"] \\ M \arrow[r] & R/\mathfrak{m}^t \otimes_R M \end{tikzcd}$$

Since u goes to zero under the left hand vertical map, we see that it suffices to show that the right-hand vertical map $\varphi \otimes 1$ is injective. Using the isomorphism $I/(\mathfrak{m}^t \cap I) \cong (I + \mathfrak{m}^t)/\mathfrak{m}^t$ it suffices to show that the left hand map in the following short exact sequence gives a monomorphism when tensored with M

$$0 \longrightarrow (I + \mathfrak{m}^t)/\mathfrak{m}^t \longrightarrow R/\mathfrak{m}^t \longrightarrow R/(I + \mathfrak{m}^t) \longrightarrow 0$$

Applying Tor, we get a long exact sequence of which a part is

$$Tor_1^R(R/(I + \mathfrak{m}^t), M) \longrightarrow (I + \mathfrak{m}^t)/\mathfrak{m}^t \otimes_R M \longrightarrow R/\mathfrak{m}^t \otimes_R M$$

so it is enough to show that $Tor_1^R(R/(I+\mathfrak{m}^t), M) = 0$. Since $R/(I+\mathfrak{m}^t)$ is annihilated by $\mathfrak{m}^t$, it is a module of finite length (see Corollary 2.17 in our written Eisenbud notes), and we are done. $\square$

Corollary 19. *Suppose that $(R, \mathfrak{m})$ is a local noetherian ring and M a finitely generated R-module. Then M is flat if and only if $Tor_1^R(R/\mathfrak{m}, M) = 0$.*

Let $R \longrightarrow S$ be a ring morphism, M an R-module and N an S-module. What is the relationship between the groups $Tor_i^S(N, M \otimes_R S)$ and $Tor_i^R(N, M)$? At least in the case where the morphism is $R \longrightarrow R/(x)$ for a regular element x regular on M they are equal.

Lemma 20. *If R is a ring, M an R-module and $x \in R$ a regular element which is regular on M, then for any $R/(x)$-module N we have $Tor_i^{R/(x)}(N, M/xM) = Tor_i^R(N, M)$ for any $i \geq 0$.*

Proof. The functor $- \otimes_R R/(x) : \mathbf{Mod}R \longrightarrow \mathbf{Mod}R/(x)$ has a right adjoint and therefore preserves free modules. So if F is free over R then F/xF is free over $R/(x)$. In particular take a free resolution P of M

$$P : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

Then the complex $R/(x) \otimes_R F : \cdots \longrightarrow R/(x) \otimes_R F_0 \longrightarrow 0$ at least consists of free objects. We claim that $R/(x) \otimes_R F$ is a free resolution of $R/(x) \otimes_R M$, in which case we may compute

$Tor_i^{R/(x)}(N, M/xM)$ as the homology of $N \otimes_R P$ which coincides with $Tor_i^R(N, M)$ as claimed. So we have to prove that the following sequence is exact

$$\cdots \longrightarrow R/(x) \otimes_R F_2 \longrightarrow R/(x) \otimes_R F_1 \longrightarrow R/(x) \otimes_R F_0 \longrightarrow R/(x) \otimes_R M \longrightarrow 0$$

Since $R/(x) \otimes_R -$ is right exact, it is exact in the last two nonzero places. The homology at the higher places are the groups $Tor_i^R(R/(x), M)$. Since x is regular these are zero for $i > 1$ and $Tor_1^R(R/(x), M) = 0$ since x is regular on M. Since the homology groups are all zero, the sequence is exact and the proof is complete. □

Extending scalars preserves flatness, and the next result proves the converse in a special case.

Corollary 21. *Suppose that $(R, \mathfrak{m})$ is a local noetherian ring, and let $(S, \mathfrak{n})$ be a local noetherian R-algebra such that $\mathfrak{m}S \subseteq \mathfrak{n}$. If M is a finitely generated S-module and $x \in \mathfrak{m}$ is a regular element of R which is regular on M, then M is flat over R if and only if M/xM is flat over $R/(x)$.*

Proof. If M is flat then $M/xM \cong R/(x) \otimes_R M$ is flat over $R/(x)$ without any hypothesis, so suppose that M/xM is flat over $R/(x)$. Let $k = R/\mathfrak{m}$ and use the previous Lemma to see that

$$Tor_1^R(k, M) = Tor_1^{R/(x)}(k, M/xM) = 0$$

It follows from Theorem 18 that M is a flat R-module. □

We know that for a flat module M we have $Tor_1^R(M, N) = Tor_1^R(N, M) = 0$ for all modules N. This actually characterises flat modules.

Corollary 22. *Let R be a ring and M an R-module. Then the following are equivalent:*

(i) *M is flat.*

(ii) *$Tor_1^R(R/I, M) = 0$ for any finitely generated ideal I.*

(iii) *$Tor_1^R(N, M) = 0$ for any finitely generated module N.*

(iv) *The map $I \otimes_R M \longrightarrow M$ is injective for all finitely generated ideals I.*