Topological Rings

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July 9, 2005

Consider the category **Top**, which is complete and cocomplete and has a terminal object **1** consisting of the singleton set $\{*\}$ with the discrete topology. Hence we can apply the ideas of this Chapter to topological spaces. This example is particularly important since the axioms for a Gabriel topology (and hence for Grothendieck topologies) arise naturally in the study of topological rings.

1 Topological Groups

To begin with, we define a *topological group* to be a group object in **Top**. Since any morphism $\mathbf{1} \longrightarrow A$ is continuous, this reduces to the following definition:

Definition 1. A *topological group* is an abelian group A together with a topology on A such that the maps

$$a : A \times A \longrightarrow A, \quad (a, b) \mapsto a + b$$
$$v : A \longrightarrow A, \quad a \mapsto -a$$

are continuous. For any subsets $U, V \subseteq A$ we define $U + V = \{u + v \,|\, u \in U, v \in V\}$, and $-U = \{-u \,|\, u \in U\}$. The map v is clearly a homeomorphism, so if U is open then $-U$ is also open.

Lemma 1. *Let A be a topological group. If $c \in A$ then the map $A \longrightarrow A$ defined by $x \mapsto c + x$ is a homeomorphism.*

Proof. The subspace $\{c\} \times A$ of $A \times A$ is clearly homeomorphic to A via $(c, b) \mapsto b$, and the restriction of a to $\{c\} \times A$ is continuous, and obviously bijective. Thus there is a continuous bijection $A \longrightarrow A$ defined by $x \mapsto c + x$. Clearly the morphism $x \mapsto -c + x$ is an inverse, and so we have a homeomorphism $x \mapsto c + x$ for each $c \in A$. $\square$

Remark 1. Let A be a topological group. If $U \subseteq A$ is open and $c \in A$ then the set $U + c = \{u + c \,|\, u \in U\}$ is also open. Taking unions, we see that the sum $U + V$ of any two open sets U, V is open. If $c \in A$ then U is an open neighborhood of c if and only if $U - c$ is an open neighborhood of 0, so the topology of A is completely determined by the open neighborhoods of 0.

Definition 2. Let X be a topological space. If $x \in X$ then a *fundamental system of neighborhoods of x* is a nonempty set $\mathcal{M}$ of open neighborhoods of x with the property that if U is open and $x \in U$, then there is $V \in \mathcal{M}$ with $V \subseteq U$.

Proposition 2. *Let A be a topological group. Then the set $\mathcal{N}$ of open neighborhoods of* 0 *satisfies*

N0. For $U \in \mathcal{N}$ and $c \in U$ there exists $V \in \mathcal{N}$ such that $c + V \subseteq U$.

N1. For each $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $V + V \subseteq U$.

N2. If $U \in \mathcal{N}$ then $-U \in \mathcal{N}$.

If A is any abelian group and $\mathcal{N}$ a nonempty set of subsets of A which satisfies $N0, N1, N2$ and has the property that (a) every element of $\mathcal{N}$ contains 0 and (b) if $U, V \in \mathcal{N}$ then there is $W \in \mathcal{N}$ with $W \subseteq U \cap V$ then there is a unique topology on A making A into a topological group in such a way that $\mathcal{N}$ is a fundamental system of neighborhoods of 0.

Proof. Let A be a topological group with $\mathcal{N}$ as described. Condition $N0$ follows from the fact that the map $x \mapsto x - c$ is a homeomorphism. For $N1$, let $U \in \mathcal{N}$ be given. Since a is continuous and $(0,0) \in a^{-1}U$, there are open sets V_1, V_2 with $(0,0) \in V_1 \times V_2 \subseteq a^{-1}U$. Set $V = V_1 \cap V_2$. The condition $N2$ follows from the fact that v is continuous.

For the converse, let A be an abelian group and $\mathcal{N}$ a nonempty set of subsets of A with the given properties. We define a subset $U \subseteq A$ to be open if for every $x \in U$ there is $W \in \mathcal{N}$ with $x + W \subseteq U$. It is easy to check that this is a topology. Condition $N0$ implies that the elements of $\mathcal{N}$ are open sets.

Next, we claim that if U is open then $c+U$ is open for any $c \in A$. This is clear since if $b \in c+U$ then $b - c \in U$ and so there is $V \in \mathcal{N}$ with $b - c + V \subseteq U$. Thus $b + V \subseteq c + U$, and $c + U$ is open.

To show that A is a topological group we have to show that the maps $a : A \times A \longrightarrow A$ and $v : A \longrightarrow A$ are continuous. Let U be an open set, and suppose $(c, d) \in a^{-1}U$. Then $c + d \in U$, so there is $W \in \mathcal{N}$ such that $c + d + W \subseteq U$. Using $N1$, let $Q \in \mathcal{N}$ be such that $Q + Q \subseteq W$. Then

$$(c, d) \in (c + Q) \times (d + Q) \subseteq a^{-1}U$$

which shows that $a^{-1}U$ is open. Therefore a is continuous. To see that v is continuous, we have to show that if U is open then so is $-U$. But if $c \in -U$ there is $V \in \mathcal{N}$ such that $-c + V \subseteq U$. Hence $c + -V \subseteq -U$, and since $-V \in \mathcal{N}$ by $N2$, we are done. This shows that A is a topological group. Suppose that $\mathcal{J}$ is another topology on A with respect to which A is a topological group, and suppose further that the set $\mathcal{N}$ is a fundamental system of neighborhoods of 0 in this topology. It is not hard to see that this topology must agree with the one we have just defined, which is therefore unique with these properties. □

2 Topological Rings

Definition 3. A *topological ring* is a ring A with a topology making A into an additive topological group, such that the multiplication $m : A \times A \longrightarrow A$, $(b, c) \mapsto bc$ is a continuous map. For any subsets $V, W \subseteq A$ we define $V \cdot W = \{vw \mid v \in V, w \in W\}$.

Lemma 3. *Let A be a topological ring. If $c \in A$ then the maps $A \longrightarrow A$ defined by $x \mapsto cx$ and $x \mapsto xc$ are continuous.*

Proof. The subspace $\{c\} \times A$ of $A \times A$ is clearly homeomorphic to A via $(c, b) \mapsto b$, and the restriction of m to $\{c\} \times A$ is continuous. The same argument works on the right. □

Proposition 4. *Let A be a topological ring. Then the set $\mathcal{N}$ of open neighborhoods of 0 satisfies $N0, N1, N2$ and also*

N3. For $c \in A$ and $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $cV \subseteq U$ and $Vc \subseteq U$.

N4. For each $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $V \cdot V \subseteq U$.

Conversely, if A is any ring and $\mathcal{N}$ a nonempty set of subsets of A which satisfies $N0, N1, N2, N3$ and $N4$ and has the property that (a) every element of $\mathcal{N}$ contains 0 and (b) if $U, V \in \mathcal{N}$ then there is $W \in \mathcal{N}$ with $W \subseteq U \cap V$ then there is a unique topology on A making A into a topological ring in such a way that $\mathcal{N}$ is a fundamental system of neighborhoods of 0.

Proof. Suppose that A is a topological ring. Then by Proposition 2 the set $\mathcal{N}$ satisfies $N0, N1, N2$. Condition $N3$ follows easily from continuity of the maps $x \mapsto cx$ and $x \mapsto xc$. For $N4$, let $U \in \mathcal{N}$ be given and use the fact that m is continuous to find $V_1, V_2 \in \mathcal{N}$ such that $(0,0) \in V_1 \times V_2 \subseteq m^{-1}U$. Then set $V = V_1 \cap V_2$.

Conversely, suppose we are given a ring A and a nonempty set of subsets of A with the given properties. With the topology defined in Proposition 2, A becomes a topological group. We have to show that with this topology, A is a topological ring.

First we show that for $c \in A$ the map $\theta : A \longrightarrow A$ defined by $x \mapsto cx$ is continuous. Let $U \subseteq A$ be open and suppose that $x \in \theta^{-1}U$, that is, $cx \in U$. By definition there is $W \in \mathcal{N}$ with $cx + W \subseteq U$. Let $V \in \mathcal{N}$ be such that $cV \subseteq W$. Then $x + V \subseteq \theta^{-1}U$. Therfore $\theta^{-1}U$ is open and θ is continuous. Similarly we show that the map $x \mapsto xc$ is continuous.

We are now ready to show that the product $m : A \times A \longrightarrow A$ is continuous. Let $U \subseteq A$ be open and suppose $(c, d) \in m^{-1}U$. Let $\theta : A \longrightarrow A$ be left multiplication by c and $\psi : A \longrightarrow A$ right multiplication by d. Then $cd \in U$, so there is $W \in \mathcal{N}$ such that $cd + W \subseteq U$. Let $Q \in \mathcal{N}$ satisfy $Q + Q \subseteq W$, and using $N4$, let $V \in \mathcal{N}$ be such that $V \cdot V \subseteq Q$. Let $P \in \mathcal{N}$ be such that $P + P \subseteq Q$, and set $P_d = \psi^{-1}(P) \cap Q \cap V$ and $P_c = \theta^{-1}(P) \cap Q \cap V$. Then

$$(c, d) \in (c + P_d) \times (d + P_c) \subseteq m^{-1}U$$

since for $v \in P_d, v' \in P_c$, $(c + v)(d + v') = cd + cv' + vd + vv' \in cd + W \subseteq U$. It follows that m is continuous, as required. It follows from Proposition 2 that this topology is the unique topology making A into a topological group with $\mathcal{N}$ a fundamental system of neighborhoods of 0. $\square$

Definition 4. Let A be a ring. A nonempty set $\mathcal{N}$ of subsets of A is *fundamental* if it satisfies the following conditions

(a) Every element of $\mathcal{N}$ contains 0.

(b) If $U, V \in \mathcal{N}$ then there is $W \in \mathcal{N}$ with $W \subseteq U \cap V$.

N0. For $U \in \mathcal{N}$ and $c \in U$ there exists $V \in \mathcal{N}$ such that $c + V \subseteq U$.

N1. For each $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $V + V \subseteq U$.

N2. If $U \in \mathcal{N}$ then $-U \in \mathcal{N}$.

N3. For $c \in A$ and $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $cV \subseteq U$ and $Vc \subseteq U$.

N4. For each $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $V \cdot V \subseteq U$.

By Proposition 4 if A is a topological ring, then the set $\mathcal{N}$ of open neighborhoods of 0 is fundamental. Conversely, if A is a ring and $\mathcal{N}$ a fundamental set of subsets of A, then there is a unique topology on A making A into a topological ring in such a way that $\mathcal{N}$ is a fundamental system of neighborhoods of 0. We call this the topology *generated by* $\mathcal{N}$.

Proposition 5. *Let A be a ring and $\mathcal{G}$ a nonempty set of right ideals. Suppose that the following conditions are satisfied*

T1. If $\mathfrak{a} \in \mathcal{G}$ and $\mathfrak{a} \subseteq \mathfrak{b}$ for a right ideal $\mathfrak{b}$, then $\mathfrak{b} \in \mathcal{G}$.

T2. If $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathcal{G}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{G}$.

T3. If $\mathfrak{a} \in \mathcal{G}$ and $a \in A$, then $(\mathfrak{a} : a) \in \mathcal{G}$.

Then the set $\mathcal{G}$ is fundamental, and $\mathcal{G}$ is precisely the set of open right ideals in the generated topology on A.

Proof. The axioms $(a), N0, N1, N2, N4$ are trivially verified and (b) follows from $T2$. To check $N3$, let $a \in A$ and $\mathfrak{a} \in \mathcal{N}$ be given. By assumption the right ideal $(\mathfrak{a} : a) = \{x \in A \,|\, ax \in \mathfrak{a}\}$ belongs to $\mathcal{G}$. Set $\mathfrak{d} = (\mathfrak{a} : a) \cap \mathfrak{a}$, which is in $\mathcal{G}$ by $T2$. Clearly $a\mathfrak{d} \subseteq \mathfrak{a}$ and $\mathfrak{d}a \subseteq \mathfrak{a}$, which shows that $\mathcal{G}$ is fundamental. Give A the topology generated by $\mathcal{G}$. Every ideal in $\mathcal{G}$ is open in this topology. If $\mathfrak{b}$ is an open right ideal of A then $\mathfrak{b} \supseteq \mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{G}$, and therefore $\mathfrak{b} \in \mathcal{G}$ by $T1$. $\square$

Lemma 6. *Let A be a topological ring. Then the set $\mathcal{G}$ of all open right ideals of A satisfies the conditions $T1, T2, T3$.*

Proof. If $\mathfrak{a}$ is open and $\mathfrak{a} \subseteq \mathfrak{b}$, then for $x \in \mathfrak{b}$ we have $x + \mathfrak{a} \subseteq \mathfrak{b}$. As the union of such open sets, $\mathfrak{b}$ is open. The condition $T2$ is trivial. If $\mathfrak{a}$ is open, $a \in A$ and $x \in (\mathfrak{a} : a)$, then $ax \in \mathfrak{a}$, and since the whole collection of open neighborhoods of 0 satisfies the condition $N3$, we may find a neighborhood V of 0 such that $aV \subseteq \mathfrak{a}$. Hence $x + V \subseteq (\mathfrak{a} : a)$, which is thus open. □

Definition 5. A *right linear topological ring* is a topological ring A which admits a fundamental system of neighborhoods of 0 consisting of right ideals (necessarily open).

Lemma 7. *Let A be a topological ring. Then A is right linear topological if and only if the topology on A is the one generated by the set of all open right ideals.*

Proof. Let A be a topological ring, and let $\mathcal{G}$ be the set of all open right ideals. If the topology on A is the one generated by $\mathcal{G}$, then of course A is a right linear topological ring. For the converse, suppose that A is right linear topological. Then $\mathcal{G}$ must be a fundamental system of open neighborhoods of 0, and the uniqueness part of Proposition 4 implies that the topology on A must be the one generated by $\mathcal{G}$. □