The Relative Proj Construction

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Earlier we defined the Proj of a graded ring. In these notes we introduce a relative version of this construction, which is the **Proj** of a sheaf of graded algebras $\mathscr{S}$ over a scheme X. This construction is useful in particular because it allows us to construct the projective space bundle associated to a locally free sheaf $\mathscr{E}$, and it allows us to give a definition of blowing up with respect to an arbitrary sheaf of ideals.

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1 Relative Proj

See our Sheaves of Algebras notes (SOA) for the definition of sheaves of algebras, sheaves of graded algebras and their basic properties. In particular note that a sheaf of algebras (resp. graded algebras) is not necessarily commutative. Although in SOA we deal with noncommutative algebras over a ring, here "A-algebra" will refer to a commutative algebra over a commutative ring A.

Example 1. Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. In our Special Sheaves of Algebras (SSA) notes we defined the following structures:

- The relative tensor algebra $\mathbb{T}(\mathscr{F})$, which is a sheaf of graded $\mathcal{O}_X$-algebras with the property that $\mathbb{T}^0(\mathscr{F}) = \mathcal{O}_X$. If $\mathscr{F}$ is quasi-coherent then so is $\mathbb{T}(\mathscr{F})$ (SSA,Corollary 7).

- The relative symmetric algebra $\mathbb{S}(\mathscr{F})$ which is a sheaf of commutative graded $\mathcal{O}_X$-algebras with the property that $\mathbb{S}^0(\mathscr{F}) = \mathcal{O}_X$. If $\mathscr{F}$ is quasi-coherent then $\mathbb{S}(\mathscr{F})$ is quasi-coherent and locally generated by $\mathbb{S}^1(\mathscr{F})$ as an $\mathbb{S}^0(\mathscr{F})$-algebra (SSA,Corollary 16), (SSA,Corollary 18).

- The relative polynomial sheaf $\mathscr{F}[x_1, \dots, x_n]$ for $n \geq 1$. This is a sheaf of graded modules, which is quasi-coherent if $\mathscr{F}$ is (SSA,Corollary 44). If $\mathscr{S}$ is a sheaf of algebras on X then $\mathscr{S}[x_1, \dots, x_n]$ becomes a sheaf of graded algebras, which is commutative if $\mathscr{S}$ is. If $\mathscr{S}$ is a quasi-coherent sheaf of commutative algebras, then $\mathscr{S}[x_1, \dots, x_n]$ is locally finitely generated by $\mathscr{S}[x_1, \dots, x_n]_1$ as a $\mathscr{S}[x_1, \dots, x_n]_0$-algebra (SSA,Corollary 46).

Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. Then for any affine open subset $U \subseteq X$ we have a graded $\mathcal{O}_X(U)$-algebra $\mathscr{F}(U)$, and therefore a scheme $Proj\mathscr{F}(U)$ together with a morphism of schemes $\pi_U : Proj\mathscr{F}(U) \longrightarrow Spec\mathcal{O}_X(U) \cong U$ (SOA,Proposition 40). Suppose $U \subseteq V$ are affine open subsets of X. It follows immediately from (RAS,Lemma 9) that the following diagram is a pushout of rings (or equivalently that $\mathscr{F}(U)$ is a coproduct of $\mathcal{O}_X(V)$-algebras)

$$\begin{tikzcd}
\mathcal{O}_X(V) \arrow[r] \arrow[d] & \mathcal{O}_X(U) \arrow[d] \\
\mathscr{F}(V) \arrow[r] & \mathscr{F}(U)
\end{tikzcd}$$

Note that the isomorphism $\mathscr{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathscr{F}(U)$ is an isomorphism of graded $\mathcal{O}_X(U)$-algebras, where the tensor product has the canonical grading, and $\mathscr{F}(V) \longrightarrow \mathscr{F}(U)$ is a morphism of graded rings. If we write $T = \mathscr{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)$ then T_+ is generated as a $\mathcal{O}_X(U)$-module by $\varphi(\mathscr{F}(V)_+)$ so we get morphisms of schemes $ProjT \longrightarrow Proj\mathscr{F}(V)$ and $Proj\mathscr{F}(U) \longrightarrow Proj\mathscr{F}(V)$. It follows from our Proj Construction notes (Section 2 on *Proj* under pullback) that we have a commutative diagram, where the inner square is a pullback

$$\begin{tikzcd}
Proj\mathscr{F}(U) \arrow[dr, Rightarrow] \arrow[drr, bend left] \arrow[ddr] & & \\
 & ProjT \arrow[r] \arrow[d] & Proj\mathscr{F}(V) \arrow[d] \\
 & Spec\mathcal{O}_X(U) \arrow[r] & Spec\mathcal{O}_X(V)
\end{tikzcd}$$

Therefore the outside diagram is also a pullback, and hence so is the following diagram

$$\begin{tikzcd}
Proj\mathscr{F}(U) \arrow[r] \arrow[d, "\pi_U"'] & Proj\mathscr{F}(V) \arrow[d, "\pi_V"] \\
U \arrow[r] & V
\end{tikzcd}$$

Hence $\rho_{U,V} : Proj\mathscr{F}(U) \longrightarrow Proj\mathscr{F}(V)$ is an open immersion with open image $\pi_V^{-1}U$. Since $\rho_{U,V}$ is the morphism of schemes induced by the morphism of graded rings $\mathscr{F}(V) \longrightarrow \mathscr{F}(U)$ it is clear that if $U \subseteq V \subseteq W$ are affine, $\rho_{V,W}\rho_{U,V} = \rho_{U,W}$. Also $\rho_{U,U} = 1$ for any affine open subset $U \subseteq X$.

For open affines $U, V \subseteq X$ let $X_{U,V}$ denote the open subset $\pi_U^{-1}V$ of $Proj\mathscr{F}(U)$ with the induced scheme structure. We want to define an isomorphism of X-schemes $\mathcal{O}_{U,V} : X_{U,V} \cong X_{V,U}$. We do this in several steps:

Step 1 Let W be an affine open subset of $U \cap V$ and let τ_W be the following morphism of schemes over X

$$\tau_W : \pi_U^{-1}W \Longrightarrow Proj\mathscr{F}(W) \longrightarrow Proj\mathscr{F}(V)$$

So τ_W is an open immersion with open image $\pi_V^{-1}W$. Let $\{W_\alpha\}_{\alpha \in \Lambda}$ be the set of all affine open subsets of $U \cap V$, so that the open sets $\pi_U^{-1}W_\alpha$ form an open cover of $X_{U,V}$. We want to show that we can glue the morphisms τ_W over this cover. First we check a special case.

Step 2 Let $W' \subseteq W$ be open affine subsets of $U \cap V$. We claim that $\tau_{W'} = \pi_U^{-1}W' \longrightarrow \pi_U^{-1}W \longrightarrow Proj\mathscr{F}(V)$. It suffices to show that the following diagram commutes

$$\begin{tikzcd}
\pi_U^{-1}W' \arrow[r] \arrow[d, Rightarrow] & \pi_U^{-1}W \arrow[d, Rightarrow] \\
Proj\mathscr{F}(W') \arrow[r] & Proj\mathscr{F}(W)
\end{tikzcd}$$

This is straightforward to check, using the pullback $\pi_U^{-1}W = Proj\mathscr{F}(U) \times_U W$ and properties of the morphisms ρ given above.

Step 3 Now let W, W' be arbitrary open affine subsets of $U \cap V$. We have to show the following diagram commutes

$$\begin{tikzcd}
\pi_U^{-1}(W \cap W') \arrow[r] \arrow[d] & \pi_U^{-1}W \arrow[d] \\
\pi_U^{-1}W' \arrow[r] & Proj\mathscr{F}(V)
\end{tikzcd}$$

But we can cover $W \cap W'$ with affine open subsets, and it suffices to check that both legs of this diagram agree when composed with the inclusions of these open subsets. Therefore we can reduce to the case already established in Step 2. This shows that we can glue the morphisms τ_W to get a morphism of X-schemes $\tau : X_{U,V} \longrightarrow Proj\mathscr{F}(V)$. Let $\theta_{U,V} : X_{U,V} \cong X_{V,U}$ be the factorisation through the open subset $X_{V,U}$. It is not hard to check that $\theta_{U,V}^{-1}(\pi_V^{-1}W) = \pi_U^{-1}W$ for any open affine $W \subseteq U \cap V$. Hence $\theta_{U,V}$ is an isomorphism, since it is locally the isomorphism $\pi_U^{-1}W \longrightarrow \pi_V^{-1}W$. So we have produced an isomorphism $\theta_{U,V}$ of schemes over X with the property that for any open affine subset $W \subseteq U \cap V$ the following diagram commutes

$$\begin{tikzcd}
X_{U,V} \arrow[r, Rightarrow, "\theta_{U,V}"] & X_{V,U} \\
\pi_U^{-1}W \arrow[r, Rightarrow] \arrow[u] & \pi_V^{-1}W \arrow[u]
\end{tikzcd} \tag{1}$$

where the bottom morphism is $\pi_U^{-1}W \longrightarrow Proj\mathscr{F}(W) \longrightarrow \pi_V^{-1}W$. In particular if $U \subseteq V$ then $\theta_{U,V}$ is the canonical isomorphism $Proj\mathscr{F}(U) \cong X_{V,U}$ induced by $\rho_{U,V}$.

Take the family of schemes $\{Proj\mathscr{F}(U)\}_{U \subseteq X}$ indexed by the affine open subsets U of X. For every pair of affine open subsets U, V we have the isomorphism $\theta_{U,V}$ between the open subsets $X_{U,V}$ and $X_{V,U}$ of $Proj\mathscr{F}(U)$ and $Proj\mathscr{F}(V)$ respectively. To glue all the family of schemes $\{Proj\mathscr{F}(U)\}_{U \subseteq X}$ along these open subsets, we first have to check that $\theta_{U,V} = \theta_{V,U}^{-1}$ (see Ex 2.12 for the details). But one can check this locally, in which case it follows directly from the definition using (1). Note also that $\theta_{U,U} = 1$.

Next let U, V, W be open affine subsets of X. Since $\theta_{U,V}$ is a morphism of schemes over X, it is clear that $\theta_{U,V}(X_{U,V} \cap X_{U,W}) = X_{V,U} \cap V_{V,W}$. Next we have to check that $\theta_{U,W} = \theta_{V,W}\theta_{U,V}$ on $X_{U,V} \cap X_{U,W}$, but it suffices to check this on $\pi_U^{-1}Q$ for an affine open subset $Q \subseteq U \cap V \cap W$. But then the morphism $\pi_U^{-1}Q \longrightarrow \pi_W^{-1}Q$ obtained from $\theta_{V,W}\theta_{U,V}$ is just the composite

$$\pi_U^{-1}Q \longrightarrow Proj\mathscr{F}(Q) \longrightarrow \pi_V^{-1}Q \longrightarrow Proj\mathscr{F}(Q) \longrightarrow \pi_W^{-1}Q$$

The middle two morphisms are inverse to each other, which shows that this composite is the same as the morphism determined by $\theta_{U,W}$. Thus our family of schemes and patches satisfies the conditions of the Glueing Lemma, and we have a scheme $\mathbf{Proj}\mathscr{F}$ together with open immersions $\psi_U : Proj\mathscr{F}(U) \longrightarrow \mathbf{Proj}\mathscr{F}$ for each affine open subset $U \subseteq X$. These morphisms have the following properties:

(a) The open sets $Im\psi_U$ cover $\mathbf{Proj}\mathscr{F}$.

(b) For affine open subsets $U, V \subseteq X$ we have $\psi_U(X_{U,V}) = Im\psi_U \cap Im\psi_V$ and $\psi_V|_{X_{V,U}}\theta_{U,V} = \psi_U|_{X_{U,V}}$.

In particular for affine open subsets $U \subseteq V$ we have $X_{U,V} = Proj\mathscr{F}(U)$ so the following diagram commutes

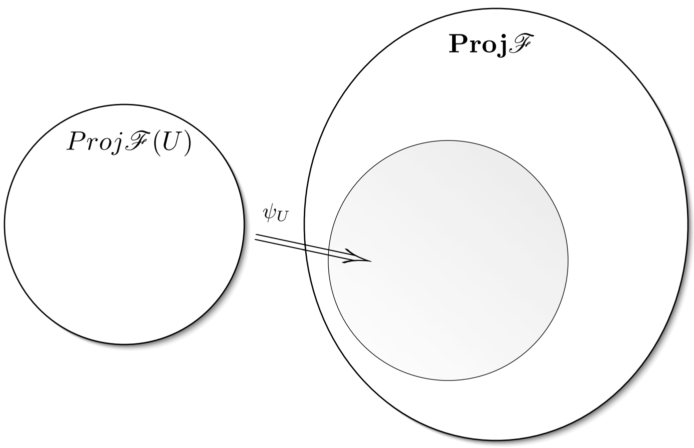
$$\begin{tikzcd} Proj\mathscr{F}(V) \arrow[r, "\psi_V"] & \mathbf{Proj}\mathscr{F} \\ Proj\mathscr{F}(U) \arrow[u, "\rho_{U,V}"] \arrow[ur, "\psi_U"'] & \end{tikzcd} \tag{2}$$

The open sets $Im\psi_U$ are a nonempty open cover of $\mathbf{Proj}\mathscr{F}$, and it is a consequence of (b) above and the fact that $\theta_{U,V}$ is a morphism of schemes over X that the morphisms $Im\psi_U \cong Proj\mathscr{F}(U) \longrightarrow U \longrightarrow X$ can be glued (that is, for open affines U, V the corresponding morphisms agree on $Im\psi_U \cap Im\psi_V$). Therefore there is a unique morphism of schemes $\pi : \mathbf{Proj}\mathscr{F} \longrightarrow X$ with the property that for every open affine subset $U \subseteq X$ the following diagram commutes

$$\begin{tikzcd} Proj\mathscr{F}(U) \arrow[r, "\psi_U"] \arrow[d, "\pi_U"'] & \mathbf{Proj}\mathscr{F} \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd} \tag{3}$$

In fact is easy to see that $\pi^{-1}U = Im\psi_U$, so the above diagram is also a pullback. In summary:

Definition 1. Let X be a scheme and $\mathscr{F}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then we can canonically associate to $\mathscr{F}$ a scheme $\pi : \mathbf{Proj}\mathscr{F} \longrightarrow X$ over X. For every open affine subset $U \subseteq X$ there is an open immersion $\psi_U : Proj\mathscr{F}(U) \longrightarrow \mathbf{Proj}\mathscr{F}$ with the property that the diagram (3) is a pullback and the diagram (2) commutes for any open affines $U \subseteq V$.



Lemma 1. *The morphism $\pi : \mathbf{Proj}\mathscr{F} \longrightarrow X$ is separated. If $\mathscr{F}$ is locally finitely generated as an $\mathcal{O}_X$-algebra then π is of finite type.*

Proof. Separatedness is a local property and for open affine $U \subseteq X$ the morphism $Spec\mathscr{F}(U) \longrightarrow U$ is separated (TPC,Proposition 1), so it follows that π is also separated. The same argument holds for the finite type property, using (TPC,Proposition 3). ☐

Corollary 2. *If X is noetherian and $\mathscr{F}$ is locally finitely generated as an $\mathcal{O}_X$-algebra then* $\mathbf{Proj}\mathscr{F}$ *is noetherian.*

Proof. Follows immediately from Lemma 1 and (Ex. 3.13g). ☐

Corollary 3. *If X is noetherian and $\mathscr{F}$ is locally finitely generated by $\mathscr{F}_1$ as an $\mathcal{O}_X$-algebra, then the morphism $\pi : \mathbf{Proj}\mathscr{F} \longrightarrow X$ is proper.*

Proof. For each open affine $U \subseteq X$, the morphism $\pi_U : Proj\mathscr{F}(U) \longrightarrow U$ is a projective morphism (TPC,Lemma 19) and therefore proper (H, 4.9),(TPC,Proposition 3). But since $\mathbf{Proj}\mathscr{F}$ is noetherian by Corollary 2 and properness is local (H,4.8) it follows that π is also proper. $\square$

Let X be a nonempty topological space. It is easy to check that X is irreducible if and only if every nonempty open set V is dense. We have another useful characterisation of irreducible spaces

Lemma 4. *Let X be a nonempty topological space and $\{U_\alpha\}_{\alpha\in\Lambda}$ a nonempty open cover of X by nonempty open sets U_α. Then for X to be irreducible it is necessary and sufficient that U_α be irreducible for every α, and $U_\alpha \cap U_\beta \neq \emptyset$ for every pair α, β.*

Proof. The condition is clearly necessary. To see that it is sufficient, let V be a nonempty open subset of X. To show V is dense in X, it suffices to show that $V \cap U_\alpha \neq \emptyset$ for every α. But there must exist an index γ with $V \cap U_\gamma \neq \emptyset$, in which case $V \cap U_\gamma$ is dense in U_γ, and therefore must meet the nonempty open subset $U_\alpha \cap U_\gamma \subseteq U_\gamma$ for every other index α. Therefore $U_\alpha \cap U_\gamma \cap V$ is nonempty, which shows that $V \cap U_\alpha$ is nonempty for every index α. $\square$

Definition 2. Let X be a scheme and $\mathscr{F}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Let Λ be the set of nonempty affine open subsets $U \subseteq X$ with $\mathscr{F}(U)_+ \neq 0$. We say $\mathscr{F}$ is *relevant* if it is locally an integral domain, Λ is nonempty and if for every pair $U, V \in \Lambda$ there is $W \in \Lambda$ with $W \subseteq U \cap V$.

Proposition 5. *If $\mathscr{F}$ is relevant then $\mathbf{Proj}\mathscr{F}$ is an integral scheme.*

Proof. Since $\mathscr{F}$ is relevant there is a nonempty affine open subset $W \subseteq X$ with $Proj\mathscr{F}(W)$ an integral scheme (TPC,Proposition 4). In particular $\mathbf{Proj}\mathscr{F}$ is nonempty and since we can cover it with integral schemes, it is reduced. Using Lemma 4 it suffices to show that $Im\psi_U \cap Im\psi_V$ is nonempty for every pair of nonempty open affine subsets $U, V \subseteq X$ with $\mathscr{F}(U)_+ \neq 0, \mathscr{F}(V)_+ \neq 0$. By assumption there is a nonempty affine open subset $W \subseteq U \cap V$ with $\mathscr{F}(W)_+ \neq 0$. Then $Im\psi_W \cong Proj\mathscr{F}(W)$ is nonempty and contained in $Im\psi_U \cap Im\psi_V$, as required. $\square$

2 The Sheaf Associated to a Graded Module

Throughout this section X is a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. Let $\mathfrak{GrMod}(\mathscr{S})$ denote the category of all sheaves of graded $\mathscr{S}$-modules and $\mathfrak{QcoGrMod}(\mathscr{S})$ the full subcategory of quasi-coherent sheaves of graded $\mathscr{S}$-modules (SOA,Definition 2). Note that these are precisely the sheaves of graded $\mathscr{S}$-modules that are quasi-coherent as sheaves of $\mathcal{O}_X$-modules, so in this section there is no harm in simply calling these sheaves "quasi-coherent" (SOA,Proposition 19). In this section we define a functor

$$\widetilde{-} : \mathfrak{QcoGrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathbf{Proj}\mathscr{S})$$

which is the relative version of the functor $S\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(ProjS)$ for a graded ring S. Note that $\mathfrak{QcoGrMod}(\mathscr{S})$ is an abelian category (SOA,Proposition 47).

Lemma 6. *Let X be a scheme and $U \subseteq V$ affine open subsets. If $\mathscr{S}$ is a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras, and $\mathscr{M}$ a quasi-coherent sheaf of graded $\mathscr{S}$-modules, then the following morphism of graded $\mathscr{S}(U)$-modules is an isomorphism*

$$\mathscr{M}(V) \otimes_{\mathscr{S}(V)} \mathscr{S}(U) \longrightarrow \mathscr{M}(U) \tag{4}$$

$$a \otimes b \mapsto b \cdot a|_U \tag{5}$$

Proof. Both $\mathscr{M}(V)$ and $\mathscr{S}(U)$ are graded $\mathscr{S}(V)$-modules (SOA,Proposition 40) so there is certainly a morphism of graded $\mathscr{S}(U)$-modules with $a \otimes b \mapsto b \cdot a|_U$. From (RAS,Lemma 9) we know that there are isomorphisms of $\mathcal{O}_X(U)$-modules

$$\mathscr{S}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathscr{S}(U)$$
$$\mathscr{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \cong \mathscr{M}(U)$$

So at least we have an isomorphism of abelian groups

$$\begin{aligned}\mathscr{M}(V) \otimes_{\mathscr{S}(V)} \mathscr{S}(U) &\cong \mathscr{M}(V) \otimes_{\mathscr{S}(V)} (\mathscr{S}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)) \\ &\cong (\mathscr{M}(V) \otimes_{\mathscr{S}(V)} \mathscr{S}(V)) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \\ &\cong \mathscr{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \\ &\cong \mathscr{M}(U)\end{aligned}$$

It is easily checked that this map agrees with (4), which is therefore an isomorphism. $\square$

Proposition 7. *If $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules then there is a canonical quasi-coherent sheaf of modules $\widetilde{\mathscr{M}}$ on $\mathbf{Proj}\mathscr{S}$ with the property that for every affine open subset $U \subseteq X$ there are isomorphisms*

$$(\psi'_U)_* \mathscr{M}(U)^{\sim} \cong \widetilde{\mathscr{M}}|_{Im\psi_U}$$
$$\psi_U^* \widetilde{\mathscr{M}} \cong \mathscr{M}(U)^{\sim}$$

where $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ is the canonical open immersion and $\psi'_U : Proj\mathscr{S}(U) \longrightarrow Im\psi_U$ the induced isomorphism.

Proof. Let $U \subseteq X$ be an affine open subset. Then $\mathscr{S}(U)$ is a commutative graded $\mathcal{O}_X(U)$-algebra with degree d component $\mathscr{S}_d(U)$ and $\mathscr{M}(U)$ is a graded $\mathscr{S}(U)$-module with degree n component $\mathscr{M}_n(U)$ (SOA,Proposition 40). Therefore we have a quasi-coherent sheaf of modules $\mathscr{M}(U)^{\sim}$ on $Proj\mathscr{S}(U)$. Let $\mathscr{M}_U$ denote the direct image $(\psi'_U)_*\mathscr{M}(U)^{\sim}$ where $\psi'_U : Proj\mathscr{S}(U) \cong Im\psi_U$ is induced by ψ_U. We want to glue the $\mathscr{M}_U$ over the open sets $Im\psi_U$ as U ranges over all affine open subsets of X.

Let $W \subseteq U$ be open affine subsets and $\rho_{W,U} : Proj\mathscr{S}(W) \longrightarrow Proj\mathscr{S}(U)$ the canonical open immersion. Using Lemma 6 and the fact that $\mathscr{S}$ is generated by $\mathscr{S}_1$ as a $\mathscr{S}_0$-algebra we have an isomorphism of sheaves of modules on $Proj\mathscr{S}(W)$

$$\alpha_{W,U} : \rho_{W,U}^*(\mathscr{M}(U)^{\sim}) \cong (\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{S}(W))^{\sim} \cong \mathscr{M}(W)^{\sim}$$
$$[V, m\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto (b \cdot m|_W)\dot{/}ts|_W$$

for open $T \subseteq Proj\mathscr{S}(W), V \supseteq \rho_{W,U}(T)$ sections $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ and $b, t \in \mathscr{S}(W)$ pairwise homogenous of the same degree. Let $\rho'_{W,U} : Proj\mathscr{S}(W) \cong X_{U,W}$ be the isomorphism induced by $\rho_{W,U}$ and note that $\psi''_{U,W} = \psi'_W(\rho'_{W,U})^{-1}$ where $\psi''_{U,W} : X_{U,W} \cong Im\psi_W$ is induced by ψ'_U. Therefore we have an isomorphism of sheaves of modules $\varphi_{U,W} : \mathscr{M}_U|_{Im\psi_W} \longrightarrow \mathscr{M}_W$ on $Im\psi_W$

$$\begin{aligned}\mathscr{M}_W &= (\psi'_W)_*(\mathscr{M}(W)^{\sim}) \cong (\psi'_W)_*\rho_{W,U}^*(\mathscr{M}(U)^{\sim}) \\ &\cong (\psi'_W)_*(\rho'_{W,U})_*^{-1}(\mathscr{M}(U)^{\sim}|_{X_{U,W}}) \\ &= (\psi'_W(\rho'_{W,U})^{-1})_*(\mathscr{M}(U)^{\sim}|_{X_{U,W}}) \\ &= (\psi''_{U,W})_*(\mathscr{M}(U)^{\sim}|_{X_{U,W}}) = (\psi'_U)_*(\mathscr{M}(U)^{\sim})|_{Im\psi_W} \\ &= \mathscr{M}_U|_{Im\psi_W}\end{aligned}$$

using $\alpha_{W,U}$ and (MRS,Proposition 110). Given open $V \subseteq X_{U,W}$ and sections $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ homogenous of the same degree, we have

$$\varphi_{U,W} : \mathscr{M}_U|_{Im\psi_W} \longrightarrow \mathscr{M}_W$$
$$m\dot{/}s \mapsto m|_W\dot{/}s|_W$$

Clearly $\varphi_{U,U} = 1$, and if $Q \subseteq W \subseteq U$ are open affine subsets then $\varphi_{U,Q} = \varphi_{W,Q} \circ \varphi_{U,W}|_{Im\psi_Q}$. This means that for open affine $U, V \subseteq X$ the isomorphisms $\varphi_{V,W}^{-1}\varphi_{U,W}$ for open affine $W \subseteq U \cap V$ glue together to give an isomorphism of sheaves of modules

$$\varphi_{U,V} : \mathscr{M}_U|_{Im\psi_U \cap Im\psi_V} \longrightarrow \mathscr{M}_V|_{Im\psi_U \cap Im\psi_V} \tag{6}$$

$$\varphi_{V,W} \circ \varphi_{U,V}|_{Im\psi_W} = \varphi_{U,W} \text{ for affine open } W \subseteq U \cap V \tag{7}$$

The notation is unambiguous, since this definition agrees with the earlier one in the case $V \subseteq U$. By construction these isomorphisms can be glued (GS,Proposition 1) to give a canonical sheaf of modules $\widetilde{\mathscr{M}}$ on $\mathbf{Proj}\mathscr{S}$ and a canonical isomorphism of sheaves of modules $\mu_U : \widetilde{\mathscr{M}}|_{Im\psi_U} \longrightarrow \mathscr{M}_U$ for every open affine $U \subseteq X$. These isomorphisms are compatible in the following sense: we have $\mu_V = \varphi_{U,V} \circ \mu_U$ on $Im\psi_U \cap Im\psi_V$ for any open affine $U, V \subseteq X$. It is clear that $\widetilde{\mathscr{M}}$ is quasi-coherent since the modules $\mathscr{M}(U)^\sim$ are. ☐

Example 2. For every $n \in \mathbb{Z}$ we have the quasi-coherent sheaf of graded $\mathscr{S}$-modules $\mathscr{S}(n)$, and we denote by $\mathcal{O}(n)$ the sheaf of modules $\mathscr{S}(n)^\sim$ on $\mathbf{Proj}\mathscr{S}$. It follows from (H,5.12) that $\mathcal{O}(n)$ is an invertible sheaf and moreover for every open affine $U \subseteq X$ we have a canonical isomorphism $\psi_U^*\mathcal{O}(n) \cong \mathcal{O}(n)$.

Proposition 8. *If $\beta : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism of quasi-coherent sheaves of graded $\mathscr{S}$-modules then there is a canonical morphism $\widetilde{\beta} : \widetilde{\mathscr{M}} \longrightarrow \widetilde{\mathscr{N}}$ of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *and this defines an additive functor $\widetilde{-} : \mathfrak{QcoGrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathbf{Proj}\mathscr{S})$.*

Proof. For every affine open $U \subseteq X$, $\beta_U : \mathscr{M}(U) \longrightarrow \mathscr{N}(U)$ is a morphism of graded $\mathscr{S}(U)$-modules, and therefore gives a morphism $\widetilde{\beta_U} : \mathscr{M}(U)^\sim \longrightarrow \mathscr{N}(U)^\sim$ of sheaves of modules on $Proj\mathscr{S}(U)$. Let $b_U : \mathscr{M}_U \longrightarrow \mathscr{N}_U$ be the morphism $(\psi_U')_*\widetilde{\beta_U}$ (notation of Proposition 7). We have to show that for open affine $U, V \subseteq X$ and $T = Im\psi_U \cap Im\psi_V$ the following diagram commutes

$$\begin{tikzcd}
\mathscr{M}_U|_T \arrow[r, "b_U|_T"] \arrow[d, Rightarrow] & \mathscr{N}_U|_T \arrow[d, Rightarrow] \\
\mathscr{M}_V|_T \arrow[r, "b_V|_T"'] & \mathscr{N}_V|_T
\end{tikzcd}$$

In suffices to check this on sections of the form $\dot{m/s}$, which is trivial, so the diagram commutes. Therefore there is a unique morphism of sheaves of modules $\widetilde{\beta}$ with the property that for every affine open $U \subseteq X$ the following diagram commutes (GS,Proposition 6)

$$\begin{tikzcd}
\widetilde{\mathscr{M}}|_{Im\psi_U} \arrow[r, "\widetilde{\beta}|_{Im\psi_U}"] \arrow[d, Rightarrow, "\mu_U"'] & \widetilde{\mathscr{N}}|_{Im\psi_U} \arrow[d, Rightarrow, "\mu_U"] \\
\mathscr{M}_U \arrow[r, "b_U"'] & \mathscr{N}_U
\end{tikzcd}$$

Using this unique property it is easy to check that $\widetilde{-}$ defines an additive functor. ☐

Proposition 9. *The additive functor $\widetilde{-} : \mathfrak{QcoGrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathbf{Proj}\mathscr{S})$ is exact.*

Proof. Since $\mathfrak{QcoGrMod}(\mathscr{S})$ is an abelian subcategory of $\mathfrak{GrMod}(\mathscr{S})$ (SOA,Proposition 47), a sequence is exact in the former category if and only if it is exact in the latter, which by (LC,Corollary 10) is if and only if it is exact as a sequence of sheaves of $\mathscr{S}$-modules. So suppose we have an exact sequence of quasi-coherent graded $\mathscr{S}$-modules

$$\mathscr{M}' \xrightarrow{\varphi} \mathscr{M} \xrightarrow{\psi} \mathscr{M}''$$

Since this is exact as a sequence of sheaves of $\mathscr{S}$-modules, it is exact as a sequence of sheaves of modules on X, and therefore using (MOS,Lemma 5) we see that for affine open $U \subseteq X$ the following sequence of graded $\mathscr{S}(U)$-modules is exact

$$\mathscr{M}'(U) \xrightarrow{\varphi_U} \mathscr{M}(U) \xrightarrow{\psi_U} \mathscr{M}''(U)$$

Since the functor $\widetilde{-} : \mathscr{S}(U)\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(Proj\mathscr{S}(U))$ is exact, we have an exact sequence of sheaves of modules on $Proj\mathscr{S}(U)$

$$\mathscr{M}'(U)^{\sim} \xrightarrow{\widetilde{\varphi_U}} \mathscr{M}(U)^{\sim} \xrightarrow{\widetilde{\psi_U}} \mathscr{M}''(U)^{\sim}$$

If $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ is the canonical open immersion and $\psi'_U : Proj\mathscr{S}(U) \longrightarrow Im\psi_U$ the induced isomorphism then applying $(\psi'_U)_*$ and using the natural isomorphism $(\psi'_U)_*\mathscr{M}(U)^{\sim} \cong \mathscr{M}^{\sim}|_{Im\psi_U}$ we see that the following sequence of sheaves of modules on $Im\psi_U$ is exact

$$\mathscr{M}'^{\sim}|_{Im\psi_U} \xrightarrow{\widetilde{\varphi}|_{Im\psi_U}} \mathscr{M}^{\sim}|_{Im\psi_U} \xrightarrow{\widetilde{\psi}|_{Im\psi_U}} \mathscr{M}''^{\sim}|_{Im\psi_U}$$

It now follows from (MRS,Lemma 38) that the functor $\widetilde{-}$ is exact. $\square$

Proposition 10. *Let X be a scheme and $n \geq 0$ an integer. Then* $\mathbf{Proj}\mathcal{O}_X[x_0, \dots, x_n] = \mathbb{P}^n_X$. *That is, there is a pullback diagram*

$$\begin{CD} \mathbf{Proj}\mathcal{O}_X[x_0,\dots,x_n] @>{\pi}>> X \\ @V{\zeta}VV @VVV \\ \mathbb{P}^n_{\mathbb{Z}} @>>> Spec\mathbb{Z} \end{CD} \tag{8}$$

Moreover if $\mathcal{O}(1)$ is the invertible sheaf of Proposition 7 then there is a canonical isomorphism of sheaves of modules $\zeta^*\mathcal{O}(1) \cong \mathcal{O}(1)$.

Proof. Let $\mathscr{S}$ be the commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{O}_X[x_0, \dots, x_n]$. For an affine open subset $U \subseteq X$, there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-algebras $\mathscr{S}(U) \cong \mathcal{O}_X(U)[x_0, \dots, x_n]$ (SSA,Proposition 45). The canonical morphism of graded rings $z_U : \mathbb{Z}[x_0, \dots, x_n] \longrightarrow \mathcal{O}_X(U)[x_0, \dots, x_n]$ therefore induces a morphism of schemes $\zeta_U : Proj\mathscr{S}(U) \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ fitting into the following pullback diagram (TPC,Proposition 16)

$$\begin{CD} Proj\mathscr{S}(U) @>{\pi_U}>> U \\ @V{\zeta_U}VV @VVV \\ \mathbb{P}^n_{\mathbb{Z}} @>>> Spec\mathbb{Z} \end{CD}$$

Using the naturality of (SSA,Proposition 45) in U it is easy to see that the ζ_U glue to give a morphism of schemes $\zeta : \mathbf{Proj}\mathscr{S} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$. Using our notes on the local nature of pullbacks in Section 2.3 we see that (8) is a pullback, as required.

Finally we have to show $\zeta^*\mathcal{O}(1) \cong \mathcal{O}(1)$. With the notation of Proposition 7, for affine open $U \subseteq X$ there is an isomorphism of sheaves of modules on $Im\psi_U$

$$\begin{aligned} \vartheta^U : (\zeta^*\mathcal{O}(1))|_{Im\psi_U} &\cong (\zeta_U(\psi'_U)^{-1})^*\mathcal{O}(1) \\ &\cong ((\psi'_U)^{-1})^*\zeta_U^*\mathcal{O}(1) \\ &\cong (\psi'_U)_*(\zeta_U^*\mathcal{O}(1)) \\ &\cong (\psi'_U)_*\mathcal{O}(1) \\ &\cong \mathcal{O}(1)|_{Im\psi_U} \end{aligned}$$

Using (MRS,Proposition 111), (MRS,Remark 8), (MRS,Proposition 107), (MPS,Proposition 13), and finally μ_U. We need to know the action of ϑ^U on special sections of the following form: suppose $Q \subseteq Im\psi_U$ is open and that $b, t \in \mathscr{S}(U)$ are homogenous of the same degree such that $b\dot{/}t \in \Gamma(\psi_U^{-1}Q, Proj\mathscr{S}(U))$. Denote also by $b\dot{/}t$ the corresponding section of $\mathbf{Proj}\mathscr{S}$. Suppose $a, s \in \mathbb{Z}[x_0, \ldots, x_n]$ are homogenous, with a of degree $d+1$, s of degree d, and $a\dot{/}s \in \Gamma(\psi_U^{-1}T, \mathcal{O}(1))$ for some open T with $\zeta(Q) \subseteq T$. Then we have

$$[T, a\dot{/}s] \dot{\otimes} b\dot{/}t \in \Gamma(Q, \zeta^*\mathcal{O}(1)) \tag{9}$$

$$\vartheta_Q^U\left([T, a\dot{/}s] \dot{\otimes} b\dot{/}t\right) = z_U(a)b\dot{/}z_U(s)t \tag{10}$$

The section on the right is the section of $\mathbf{Proj}\mathscr{S}$'s twisting sheaf corresponding to $z_U(a)b\dot{/}z_U(s)t \in \Gamma(\psi_U^{-1}Q, \mathcal{O}(1))$ via μ_U. To show that the isomorphisms ϑ^U glue we need only show that for affine $W \subseteq U$ we have $\vartheta^U|_{Im\psi_W} = \vartheta^W$. We can reduce immediately to sections of the form (9) for $Q \subseteq Im\psi_W$. The compatibility conditions on the morphisms μ_U, μ_W (see Proposition 7) mean that the section $[T, a\dot{/}s] \dot{\otimes} b\dot{/}t$ is the same as $[T, a|_W\dot{/}s|_W] \dot{\otimes} b|_W\dot{/}t|_W$ (the sections now being the images under μ_W instead of μ_U). But

$$\vartheta_Q^W\left([T, a|_W\dot{/}s|_W] \dot{\otimes} b|_W\dot{/}t|_W\right) = z_W(a|_W)b|_W\dot{/}z_W(s|_W)t|_W$$

and by the same argument, the section on the right is the same section of $\mathbf{Proj}\mathscr{S}$'s twisting sheaf as $z_U(a)b\dot{/}z_U(s)t$. This shows that $\vartheta^U|_{Im\psi_W} = \vartheta^W$ and therefore there is an isomorphism $\vartheta : \zeta^*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)$ of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ unique with the property that for affine open $U \subseteq X$, $\vartheta|_{Im\psi_U} = \vartheta^U$. ∎

Remark 1. In general $\mathcal{O}(1)$ is not very ample on $\mathbf{Proj}\mathscr{F}$ relative to X. See (7.10) and (Ex 7.14).

Remark 2. Taking $n = 0$ in Proposition 10 we see that the structural morphism $\mathbf{Proj}\mathcal{O}_X[x_0] \longrightarrow X$ is an isomorphism of schemes.

Definition 3. If $\mathscr{F}$ is a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ then $\mathscr{F}(n)$ denotes the tensor product $\mathscr{F} \otimes \mathcal{O}(n)$ for $n \in \mathbb{Z}$. If $\mathscr{F}$ is quasi-coherent, then so is $\mathscr{F}(n)$ for every $n \in \mathbb{Z}$.

Proposition 11. *If $\mathscr{M}, \mathscr{N}$ are quasi-coherent sheaves of graded $\mathscr{S}$-modules then $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules and there is a canonical isomorphism of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *natural in $\mathscr{M}, \mathscr{N}$*

$$\begin{gathered}\vartheta : \widetilde{\mathscr{M}} \otimes_{\mathbf{Proj}\mathscr{S}} \widetilde{\mathscr{N}} \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^{\sim} \\ m\dot{/}s \dot{\otimes} n\dot{/}t \mapsto (m \dot{\otimes} n)\dot{/}st\end{gathered}$$

Proof. We showed in (SOA,Proposition 46) that $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules. For affine open $U \subseteq X$ let $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ be the canonical open immersion and $\psi_U' : Proj\mathscr{S}(U) \longrightarrow Im\psi_U$ the induced isomorphism. Then using (SOA,Proposition 46) we see that there is an isomorphism of sheaves of modules on $Im\psi_U$

$$\begin{aligned}\vartheta^U : (\widetilde{\mathscr{M}} \otimes_{\mathbf{Proj}\mathscr{S}} \widetilde{\mathscr{N}})|_{Im\psi_U} &\cong \widetilde{\mathscr{M}}|_{Im\psi_U} \otimes \widetilde{\mathscr{N}}|_{Im\psi_U} \\ &\cong (\psi_U')_*\mathscr{M}(U)^{\sim} \otimes (\psi_U')_*\mathscr{N}(U)^{\sim} \\ &\cong (\psi_U')_*(\mathscr{M}(U)^{\sim} \otimes \mathscr{N}(U)^{\sim}) \\ &\cong (\psi_U')_*((\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{N}(U))^{\sim}) \\ &\cong (\psi_U')_*((\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})(U)^{\sim}) \\ &\cong (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^{\sim}|_{Im\psi_U}\end{aligned}$$

Note that we use the results of (MPS,Proposition 3) which means we need the standing hypothesis that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. If $Q \subseteq Im\psi_U$ is open and $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ such that $m\dot{/}s$ defines an element of $\Gamma(\psi_U^{-1}Q, \mathscr{M}(U)^{\sim})$ then identify this with a section of

$\widetilde{\mathscr{M}}(Q)$ via the canonical isomorphism μ_U of Proposition 7. Similarly suppose $n \in \mathscr{N}(U), s \in \mathscr{S}(U)$ give a section $n\dot{/}t \in \widetilde{\mathscr{N}}(Q)$. Then

$$m\dot{/}s \dot{\otimes} n\dot{/}t \in \Gamma(Q, \widetilde{\mathscr{M}} \otimes_{\mathbf{Proj}\mathscr{S}} \widetilde{\mathscr{N}}) \tag{11}$$
$$\vartheta^U_Q(m\dot{/}s \dot{\otimes} n\dot{/}t) = (m \dot{\otimes} n)\dot{/}st \tag{12}$$

To glue the morphisms ϑ^U using (GS,Corollary 5) it suffices to show that for affine open $W \subseteq U$ we have $\vartheta^U|_{Im\psi_W} = \vartheta^W$. We can reduce immediately to sections of the form (11) for $Q \subseteq Im\psi_W$, in which case we use the compatibility conditions on the morphisms μ_U, μ_W in an argument similar to the one given in the proof of Proposition 10. There is an isomorphism $\vartheta : \widetilde{\mathscr{M}} \otimes_{\mathbf{Proj}\mathscr{S}} \widetilde{\mathscr{N}} \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^{\sim}$ of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ unique with the property that for affine open $U \subseteq X$, $\vartheta|_{Im\psi_U} = \vartheta^U$. Using this unique property it is not difficult to check that ϑ is natural in both variables. $\square$

Lemma 12. *There is an equality $\mathcal{O}(0) = \widetilde{\mathscr{S}} = \mathcal{O}_{\mathbf{Proj}\mathscr{S}}$ of sheaves of modules on $\mathbf{Proj}\mathscr{S}$.*

Proof. If one inspects the glueing constructions given in (GS,Proposition 1) and our solution to (H,Ex.2.12) then it is not difficult to see that proving the equality $\mathscr{S}^{\sim} = \mathcal{O}_{\mathbf{Proj}\mathscr{S}}$ amounts to showing that for affine open subsets $U, V \subseteq X$ and open $Q \subseteq Im\psi_U \cap Im\psi_V$ the following two morphisms of abelian groups agree

$$(\varphi_{V,U})_Q, (\theta^{\#}_{U,V})_{\psi_V^{-1}Q} : \mathcal{O}_{Proj\mathscr{S}(V)}(\psi_V^{-1}Q) \longrightarrow \mathcal{O}_{Proj\mathscr{S}(U)}(\psi_U^{-1}Q)$$

where $\varphi_{V,U}$ is the isomorphism defined in Proposition 7 and $\theta_{U,V}$ is the isomorphism defined in the construction of $\mathbf{Proj}\mathscr{S}$. If $\psi_{V,U} : X_{V,U} \longrightarrow Im\psi_U \cap Im\psi_V$ denotes the isomorphism induced by $\psi_V : Proj\mathscr{F}(V) \longrightarrow \mathbf{Proj}\mathscr{S}$ then we are trying to show that $\varphi_{V,U} = (\psi_{V,U})_*\theta_{U,V}$ as morphisms of sheaves of abelian groups on $Im\psi_U \cap Im\psi_V$, so we can reduce to the case where $U \subseteq V$ and consider only special sections of the form $m\dot{/}s$. Both morphisms map this section to $m|_U\dot{/}s|_U$, so we are done. $\square$

Corollary 13. *For integers $m, n \in \mathbb{Z}$ there are canonical isomorphisms of sheaves of modules on $\mathbf{Proj}\mathscr{S}$*

$$\begin{aligned}
\tau^{m,n} : \mathcal{O}(m) \otimes \mathcal{O}(n) &\longrightarrow \mathcal{O}(m+n) && a\dot{/}b \dot{\otimes} c\dot{/}d \mapsto ac\dot{/}bd\\
\kappa^{m,n} : \mathscr{F}(m)(n) &\longrightarrow \mathscr{F}(m+n) && (m \dot{\otimes} a\dot{/}b) \dot{\otimes} c\dot{/}d \mapsto m \dot{\otimes} (ac\dot{/}bd)\\
\rho^n : (\mathscr{M}(n))^{\sim} &\longrightarrow \widetilde{\mathscr{M}}(n) && (a \cdot m)\dot{/}sb \mapsto m\dot{/}s \dot{\otimes} a\dot{/}b
\end{aligned}$$

where $\mathscr{F}$ is a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ and $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules. The latter two isomorphisms are natural in $\mathscr{F}$ and $\mathscr{M}$ respectively.

Proof. Using (MRS,Lemma 102) and Proposition 11 we have the first isomorphism of sheaves of modules

$$\tau^{m,n} : \mathcal{O}(m) \otimes \mathcal{O}(n) = \mathscr{S}(m)^{\sim} \otimes \mathscr{S}(n)^{\sim} \cong (\mathscr{S}(m) \otimes \mathscr{S}(n))^{\sim} \cong \mathscr{S}(m+n)^{\sim} = \mathcal{O}(m+n)$$

Given an open affine subset $U \subseteq X$, $Q \subseteq Im\psi_U$ and $a, b, c, d \in \mathscr{S}(U)$ homogenous with $a\dot{/}b \in \Gamma(\psi_U^{-1}Q, \mathcal{O}(m))$ respectively $c\dot{/}d \in \Gamma(\psi_U^{-1}Q, \mathcal{O}(n))$ we denote by $a\dot{/}b, c\dot{/}d$ the corresponding sections of $\mathcal{O}(m), \mathcal{O}(n)$ on Q. Then it is not hard to check that $\tau^{m,n}_Q(a\dot{/}b \dot{\otimes} c\dot{/}d) = ac\dot{/}bd$. Using $\tau^{m,n}$ we have an isomorphism of sheaves of modules

$$\kappa^{m,n} : \mathscr{F}(m)(n) = (\mathscr{F} \otimes \mathcal{O}(m)) \otimes \mathcal{O}(n) \cong \mathscr{F} \otimes (\mathcal{O}(m) \otimes \mathcal{O}(n)) \cong \mathscr{F} \otimes \mathcal{O}(m+n) = \mathscr{F}(m+n)$$

If $m \in \mathscr{F}(Q)$ then it is not hard to check that $\kappa^{m,n}_Q((m \dot{\otimes} a\dot{/}b) \dot{\otimes} c\dot{/}d) = m \dot{\otimes} ac\dot{/}bd$. Finally using (MRS,Lemma 104) we have an isomorphism of sheaves of modules

$$\rho^n : (\mathscr{M}(n))^{\sim} \cong (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{S}(n))^{\sim} \cong \widetilde{\mathscr{M}} \otimes \mathscr{S}(n)^{\sim} = \widetilde{\mathscr{M}} \otimes \mathcal{O}(n) = \widetilde{\mathscr{M}}(n)$$

If $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ are homogenous of the same degree and $a, b \in \mathscr{S}(U)$ are homogenous with a of degree $d+n$ and b of degree d such that $m\dot{/}s \in \Gamma(\psi_U^{-1}Q, \mathscr{M}(U)^\sim)$ and $a\dot{/}b \in \Gamma(\psi_U^{-1}Q, \mathcal{O}(n))$ then $(\rho_Q^n)^{-1}$ maps $m\dot{/}s \,\dot{\otimes}\, a\dot{/}b$ to $(a \cdot m)\dot{/}sb$. Naturality of the latter two isomorphisms is not difficult to check. $\square$

Proposition 14. *Let $\mathscr{M}, \mathscr{N}, \mathscr{T}$ be quasi-coherent sheaves of graded $\mathscr{S}$-modules. We claim that the following diagram of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *commutes*

$$\begin{tikzcd}
(\widetilde{\mathscr{M}} \otimes \widetilde{\mathscr{N}}) \otimes \widetilde{\mathscr{T}} \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\vartheta\otimes 1"'] & \widetilde{\mathscr{M}} \otimes (\widetilde{\mathscr{N}} \otimes \widetilde{\mathscr{T}}) \arrow[d, Rightarrow, "1\otimes\vartheta"] \\
(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^\sim \otimes \widetilde{\mathscr{T}} \arrow[d, Rightarrow, "\vartheta"'] & \widetilde{\mathscr{M}} \otimes (\mathscr{N} \otimes_{\mathscr{S}} \mathscr{T})^\sim \arrow[d, Rightarrow, "\vartheta"] \\
((\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}) \otimes_{\mathscr{S}} \mathscr{T})^\sim \arrow[r, Rightarrow, "\widetilde{\lambda}"'] & (\mathscr{M} \otimes_{\mathscr{S}} (\mathscr{N} \otimes_{\mathscr{S}} \mathscr{T}))^\sim
\end{tikzcd}$$

Proof. For open affine $U \subseteq X$ let ψ_U, ψ'_U be as in Proposition 7 and identify the sheaves of modules $(\psi'_U)_*\mathscr{M}(U)^\sim$ and $\widetilde{\mathscr{M}}|_{Im\psi_U}$ by the isomorphism μ_U given there. Similarly for $\mathscr{N}$ and $\mathscr{T}$. Then we can reduce to checking commutativity of the diagram on sections of the form

$$(m\dot{/}s \,\dot{\otimes}\, n\dot{/}t) \,\dot{\otimes}\, p\dot{/}q \in \Gamma\left(Q, (\widetilde{\mathscr{M}} \otimes \widetilde{\mathscr{N}}) \otimes \widetilde{\mathscr{T}}\right)$$

for affine open $U \subseteq X$, open $Q \subseteq Im\psi_U$ and $m \in \mathscr{M}(U), n \in \mathscr{N}(U), p \in \mathscr{N}(U)$ and $s, t, q \in \mathscr{S}(U)$. Since $\vartheta|_{Im\psi_U} = \vartheta^U$ this is not difficult to check. $\square$

Corollary 15. *For integers $n, e, d \in \mathbb{Z}$ the following diagram of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *commutes*

$$\begin{tikzcd}
(\mathcal{O}(n) \otimes \mathcal{O}(e)) \otimes \mathcal{O}(d) \arrow[rr, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\tau^{n,e}\otimes 1"'] & & \mathcal{O}(n) \otimes (\mathcal{O}(e) \otimes \mathcal{O}(d)) \arrow[d, Rightarrow, "1\otimes\tau^{e,d}"] \\
\mathcal{O}(n+e) \otimes \mathcal{O}(d) \arrow[dr, Rightarrow, "\tau^{n+e,d}"'] & & \mathcal{O}(n) \otimes \mathcal{O}(e+d) \arrow[dl, Rightarrow, "\tau^{n,e+d}"] \\
& \mathcal{O}(n+e+d) &
\end{tikzcd}$$

Proof. The same used in our Section 2.5 notes works *mutatis mutandis*, using (MRS,Lemma 103). $\square$

Corollary 16. *Let $\mathscr{F}$ be a sheaf of modules on* $\mathbf{Proj}\mathscr{S}$ *and $n, e, d \in \mathbb{Z}$ integers. Then the following diagram of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *commutes*

$$\begin{tikzcd}
(\mathscr{F}(n) \otimes \mathcal{O}(e)) \otimes \mathcal{O}(d) \arrow[rr, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\kappa^{n,e}\otimes 1"'] & & \mathscr{F}(n) \otimes (\mathcal{O}(e) \otimes \mathcal{O}(d)) \arrow[d, Rightarrow, "1\otimes\tau^{e,d}"] \\
\mathscr{F}(n+e) \otimes \mathcal{O}(d) \arrow[dr, Rightarrow, "\kappa^{n+e,d}"'] & & \mathscr{F}(n) \otimes \mathcal{O}(e+d) \arrow[dl, Rightarrow, "\kappa^{n,e+d}"] \\
& \mathscr{F}(n+e+d) &
\end{tikzcd}$$

Proof. Once again the proof given in our Section 2.5 notes works *mutatis mutandis.* $\square$

Lemma 17. *Let $e \in \mathbb{Z}$ and $n > 0$ be given. Then there is a canonical isomorphism of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$

$$\zeta : \mathcal{O}(e)^{\otimes n} \longrightarrow \mathcal{O}(ne)$$
$$f_1 \dot{/} s_1 \dot{\otimes} \cdots \dot{\otimes} f_n \dot{/} s_n \mapsto f_1 \cdots f_n \dot{/} s_1 \cdots s_n$$

Proof. The proof is by induction on n. If $n = 1$ the result is trivial, so assume $n > 1$ and the result is true for $n - 1$. Let ζ be the isomorphism

$$\mathcal{O}(e)^{\otimes n} = \mathcal{O}(e) \otimes \mathcal{O}(e)^{\otimes(n-1)} \cong \mathcal{O}(e) \otimes \mathcal{O}(ne - e) \cong \mathcal{O}(ne)$$

Using the inductive hypothesis and the explicit form of the isomorphism $\tau : \mathcal{O}(e) \otimes \mathcal{O}(ne - e) \cong \mathcal{O}_X(ne)$ one checks that we have the desired isomorphism. □

Lemma 18. *Let $e \in \mathbb{Z}$ and $n, m > 0$ be given. We claim that the following diagram of sheaves of modules on* $\mathbf{Proj}\mathscr{S}$ *commutes*

$$\begin{array}{ccc}
\mathcal{O}(e)^{\otimes m} \otimes \mathcal{O}(e)^{\otimes n} & \Longrightarrow & \mathcal{O}(e)^{\otimes(m+n)} \\
\Downarrow & & \Downarrow \\
\mathcal{O}(em) \otimes \mathcal{O}(en) & \Longrightarrow & \mathcal{O}(em + en)
\end{array}$$

Proof. After reducing to sections of the form $(f_1 \dot{/} s_1 \dot{\otimes} \cdots \dot{\otimes} f_m \dot{/} s_m) \dot{\otimes} (g_1 \dot{/} t_1 \dot{\otimes} \cdots \dot{\otimes} g_n \dot{/} t_n)$ this is straightforward. □

Lemma 19. *Let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules and $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ the structural morphism. There are canonical morphisms of sheaves of abelian groups on X natural in $\mathscr{M}$*

$$\alpha' : \mathscr{M}_0 \longrightarrow \pi_* \widetilde{\mathscr{M}}$$
$$\alpha^n : \mathscr{M}_n \longrightarrow \pi_* \widetilde{\mathscr{M}(n)} \qquad n \in \mathbb{Z}$$
$$\beta^n : \mathscr{M}_n \longrightarrow \pi_* \widetilde{\mathscr{M}}(n) \qquad n \in \mathbb{Z}$$

In particular we have a morphism of sheaves of abelian groups $\alpha^d : \mathscr{S}_d \longrightarrow \pi_* \mathcal{O}(d)$ for every $d \geq 0$. For $d = 0$ we have a morphism of sheaves of rings $\alpha^0 : \mathscr{S}_0 \longrightarrow \pi_* \mathcal{O}_{\mathbf{Proj}\mathscr{S}}$.

Proof. For affine open $U \subseteq X$ we have a morphism of abelian groups

$$\mathscr{M}_0(U) \longrightarrow \Gamma(Proj\mathscr{S}(U), \mathscr{M}(U)^{\sim}) \cong \Gamma(\pi^{-1}U, \widetilde{\mathscr{M}})$$
$$m \mapsto m \dot{/} 1 \mapsto (\mu_U)^{-1}_{Im\psi_U}(m \dot{/} 1)$$

where $\mu_U : \widetilde{\mathscr{M}}|_{Im\psi_U} \longrightarrow \mathscr{M}_U$ is the canonical isomorphism. It is not difficult to check that this morphism is natural in the affine open set U, so we can glue to obtain the required morphism of sheaves of abelian groups α'. To define α^n just replace $\mathscr{M}$ by $\mathscr{M}(n)$. Naturality of these morphisms in $\mathscr{M}$ is not difficult to check.

To define β^n for $n \in \mathbb{Z}$ we glue together the following morphisms of abelian groups

$$\mathscr{M}_n(U) \longrightarrow \Gamma(Proj\mathscr{S}(U), \mathscr{M}(n)(U)^{\sim}) \cong \Gamma(\pi^{-1}U, \widetilde{\mathscr{M}(n)}) \cong \Gamma(\pi^{-1}U, \widetilde{\mathscr{M}}(n))$$
$$m \mapsto (\rho^n \mu_U^{-1})_{Im\psi_U}(m \dot{/} 1)$$

using the isomorphism ρ^n defined in Corollary 13. Naturality in $\mathscr{M}$ is not difficult to check, using the naturality of ρ^n. □

Remark 3. Set $Y = \mathbf{Proj}\mathscr{S}$. Then for $n = 0$ we have $\mathcal{O}(n) = \mathcal{O}_Y$ and $\tau^{m,0} : \mathcal{O}(m) \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}(m)$ is the canonical isomorphism $a \,\dot\otimes\, b \mapsto b \cdot a$. Similarly $\kappa^{m,0} : \mathscr{F}(m) \otimes \mathcal{O}_Y \longrightarrow \mathscr{F}(m)$ is the canonical isomorphism. Also if $s \in \Gamma(U, \mathscr{S}_d), r \in \Gamma(U, \mathscr{S}_e)$ and $\alpha^d : \mathscr{S}_d \longrightarrow \pi_*\mathcal{O}(d)$ and $\alpha^e : \mathscr{S}_e \longrightarrow \pi_*\mathcal{O}(e)$ are as above then

$$\tau^{d,e}_{\pi^{-1}U}(\alpha^d_U(s) \,\dot\otimes\, \alpha^e_U(r)) = \alpha^{d+e}_U(sr)$$

Lemma 20. *Let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules. Then the following diagram of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ commutes for $m, n \in \mathbb{Z}$*

$$\begin{tikzcd}
\widetilde{\mathscr{M}}(m)(n) \arrow[rr, Rightarrow, "\kappa^{m,n}"] \arrow[d, Rightarrow] & & \widetilde{\mathscr{M}}(m+n) \arrow[d, Rightarrow] \\
\mathscr{M}(m)^{\sim}(n) \arrow[r, Rightarrow] & \mathscr{M}(m)(n)^{\sim} \arrow[r, Rightarrow] & \mathscr{M}(m+n)^{\sim}
\end{tikzcd}$$

Proof. After reducing to sections of the form $(m\dot{/}s \,\dot\otimes\, a\dot{/}b) \,\dot\otimes\, c\dot{/}d$ this is straightforward. $\square$

Lemma 21. *Let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules, $U \subseteq X$ an affine open subset and $n, d \in \mathbb{Z}$. Then the following diagram of sheaves of modules on $Im\psi_U$ commutes*

$$\begin{tikzcd}
\psi_*\left(\mathscr{M}(U)^{\sim}(n) \otimes \mathcal{O}(d)\right) \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \psi_*(\mathscr{M}(U)^{\sim}(n)) \otimes \psi_*\mathcal{O}(d) \arrow[d, Rightarrow] \\
\psi_*(\mathscr{M}(U)^{\sim}(n+d)) \arrow[d, Rightarrow] & & \psi_*(\mathscr{M}(U)(n)^{\sim}) \otimes \mathcal{O}(d)|_{Im\psi_U} \arrow[d, Rightarrow] \\
\psi_*(\mathscr{M}(U)(n+d)^{\sim}) \arrow[d, Rightarrow] & & \mathscr{M}(n)^{\sim}|_{Im\psi_U} \otimes \mathcal{O}(d)|_{Im\psi_U} \arrow[d, Rightarrow] \\
\mathscr{M}(n+d)^{\sim}|_{Im\psi_U} \arrow[dr, Rightarrow] & & \left(\widetilde{\mathscr{M}}(n) \otimes \mathcal{O}(d)\right)|_{Im\psi_U} \arrow[dl, Rightarrow] \\
& \widetilde{\mathscr{M}}(n+d)|_{Im\psi_U} &
\end{tikzcd}$$

where $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ is the canonical open immersion and $\psi : Proj\mathscr{S}(U) \longrightarrow Im\psi_U$ the induced isomorphism.

Proof. Once again by reduction to sections of the form $(m\dot{/}s \,\dot\otimes\, a\dot{/}b) \,\dot\otimes\, c\dot{/}d$. $\square$

Lemma 22. *Let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules. Then the following diagram of sheaves of abelian groups on X commutes*

$$\begin{tikzcd}
\mathscr{M}_n \otimes_{\mathbb{Z}} \mathscr{S}_d \arrow[r, "\chi"] \arrow[d, "\beta^n \otimes \alpha^d"'] & \mathscr{M}_{n+d} \arrow[dd, "\beta^{n+d}"] \\
\pi_*\widetilde{\mathscr{M}}(n) \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}(d) \arrow[d] & \\
\pi_*\left(\widetilde{\mathscr{M}}(n) \otimes \mathcal{O}(d)\right) \arrow[r, "\pi_*\kappa^{n,d}"'] & \pi_*\widetilde{\mathscr{M}}(n+d)
\end{tikzcd}$$

Proof. See (SGR,Section 2.3) for the definition of the tensor product of sheaves of abelian groups $-\otimes_{\mathbb{Z}}-$. We induce the morphism of sheaves of abelian groups χ using the bilinear map $\mathscr{M}_n(U)\times\mathscr{S}_d(U)\longrightarrow\mathscr{M}_{n+d}(U)$ defined by $(m,s)\mapsto s\cdot m$. The morphism $\beta^n\otimes\alpha^d$ is similarly induced by the bilinear map $(m,s)\mapsto\beta^n_U(m)\dot{\otimes}\alpha^d_U(s)$. It suffices to check commutativity on sections of the form $m\dot{\otimes}s$, which follows from Lemma 20. ☐

Definition 4. Let $\mathscr{F}$ be a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ and $U\subseteq X$ an affine open subset. Let $\psi_U: Proj\mathscr{S}(U)\longrightarrow\mathbf{Proj}\mathscr{S}$ be the canonical open immersion, $\psi'_U: Proj\mathscr{S}(U)\longrightarrow Im\psi_U$ the induced isomorphism with inverse λ. Then we denote by $\mathscr{F}_U$ the sheaf of modules $\lambda_*(\mathscr{F}|_{Im\psi_U})$ on $Proj\mathscr{S}(U)$. If $\psi:\mathscr{F}\longrightarrow\mathscr{F}'$ is a morphism of sheaves of modules then denote by ψ_U the morphism $\lambda_*(\psi|_{Im\psi_U}):\mathscr{F}_U\longrightarrow\mathscr{F}'_U$. This defines an additive functor

$$(-)_U:\mathfrak{Mod}(\mathbf{Proj}\mathscr{S})\longrightarrow\mathfrak{Mod}(Proj\mathscr{S}(U))$$

which clearly preserves quasi-coherentness.

Lemma 23. *Let $\mathscr{F}$ be a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ and $U\subseteq X$ an affine open subset. Then for $n\in\mathbb{Z}$ there is an isomorphism of sheaves of modules on $Proj\mathscr{S}(U)$ natural in $\mathscr{F}$*

$$\gamma:\mathscr{F}(n)_U\longrightarrow\mathscr{F}_U(n)$$

Proof. The isomorphism is given by the composite

$$\mathscr{F}(n)_U=\lambda_*(\mathscr{F}(n)|_{Im\psi_U})\cong\lambda_*(\mathscr{F}|_{Im\psi_U}\otimes\mathcal{O}(n)|_{Im\psi_U})\cong\mathscr{F}_U\otimes\lambda_*(\mathcal{O}(n)|_{Im\psi_U})\cong\mathscr{F}_U(n)$$

Since by construction $(\psi'_U)_*\mathcal{O}(n)\cong\mathcal{O}(n)|_{Im\psi_U}$ and therefore $\mathcal{O}(n)\cong\lambda_*(\mathcal{O}(n)|_{Im\psi_U})$. Naturality in $\mathscr{F}$ means that the following diagram commutes for a morphism $\phi:\mathscr{F}\longrightarrow\mathscr{F}'$ of sheaves of modules

$$\begin{tikzcd}
\mathscr{F}(n)_U \arrow[r, Rightarrow] \arrow[d, "\phi(n)_U"'] & \mathscr{F}_U(n) \arrow[d, "\phi_U(n)"] \\
\mathscr{F}'(n)_U \arrow[r, Rightarrow] & \mathscr{F}'_U(n)
\end{tikzcd}$$

It suffices to check commutativity on sections of the form $m\dot{\otimes}\dot{a/b}$, which is straightforward. ☐

Lemma 24. *Let $\mathscr{F}$ be a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ and $U\subseteq X$ an affine open subset. Then for $n,d\in\mathbb{Z}$ the following diagram of sheaves of modules on $Proj\mathscr{S}(U)$ commutes*

$$\begin{tikzcd}
\mathscr{F}(n)(d)_U \arrow[r, Rightarrow, "\kappa_U^{n,d}"] \arrow[d, Rightarrow] & \mathscr{F}(n+d)_U \arrow[dd, Rightarrow] \\
\mathscr{F}(n)_U(d) \arrow[d, Rightarrow] & \\
\mathscr{F}_U(n)(d) \arrow[r, Rightarrow, "\kappa^{n,d}"'] & \mathscr{F}_U(n+d)
\end{tikzcd}$$

Proof. By reduction to sections of the form $(m\dot{\otimes}\dot{a/b})\dot{\otimes}\dot{c/d}$. ☐

2.1 Quasi-Structures

See (MRS,Section 2.1) for the definition of the equivalence relation of "quasi-isomorphism" on sheaves of graded $\mathscr{S}$-modules and for the definition of *quasi-isomorphism*,*quasi-monomorphism* and *quasi-epimorphism*.

Proposition 25. *Let $\mathscr{M},\mathscr{N}$ be quasi-coherent sheaves of graded $\mathscr{S}$-modules. If $\mathscr{M}\sim\mathscr{N}$ then there is a canonical isomorphism of sheaves of modules $\mathscr{M}^\sim\cong\mathscr{N}^\sim$ on $\mathbf{Proj}\mathscr{S}$. In particular if $\mathscr{M}\sim 0$ then $\mathscr{M}^\sim=0$.*

Proof. Suppose that $\mathscr{M} \sim \mathscr{N}$ and let $\kappa : \mathscr{M}\{d\} \longrightarrow \mathscr{N}\{d\}$ be an isomorphism of sheaves of graded $\mathscr{S}$-modules for some $d \geq 0$. Then for affine open $U \subseteq X$, $\kappa_U : \mathscr{M}(U)\{d\} \longrightarrow \mathscr{N}(U)\{d\}$ is an isomorphism of graded $\mathscr{S}(U)$-modules (SOA,Lemma 48) and therefore $\mathscr{M}(U) \sim \mathscr{N}(U)$. Let $\gamma_U : \mathscr{M}(U)^{\sim} \longrightarrow \mathscr{N}(U)^{\sim}$ be the canonical isomorphism of sheaves of modules on $Proj\mathscr{S}(U)$ (MPS,Proposition 18). Denote by γ'_U the induced isomorphism $(\psi'_U)_*\gamma_U : \mathscr{M}_U \longrightarrow \mathscr{N}_U$ (notation of Proposition 7). We claim that the following diagram commutes for affine open $W \subseteq U$

$$\begin{tikzcd}
\mathscr{M}_U|_{Im\psi_W} \arrow[r, "\gamma'_U|_{Im\psi_W}"] \arrow[d, Rightarrow, "\varphi_{U,W}"'] & \mathscr{N}_U|_{Im\psi_W} \arrow[d, Rightarrow, "\varphi_{U,W}"] \\
\mathscr{M}_W \arrow[r, "\gamma'_W"'] & \mathscr{N}_W
\end{tikzcd} \tag{13}$$

Let $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ be homogenous of the same degree, assume $f \in \mathscr{S}(U)_1$ and let $Q \subseteq Im\psi_U$ be open with $(\psi_U)^{-1}Q \subseteq D_+(sf)$. It suffices to check that the diagram commutes on the section m/s of $\mathscr{M}_U(Q)$, which is straightforward. Therefore there is a unique isomorphism $\gamma : \mathscr{M}^{\sim} \longrightarrow \mathscr{N}^{\sim}$ of sheaves of modules on **Proj**$\mathscr{S}$ with the property that the following diagram commutes for every affine open $U \subseteq X$ (GS,Proposition 6)

$$\begin{tikzcd}
\widetilde{\mathscr{M}}|_{Im\psi_U} \arrow[r, "\gamma|_{Im\psi_U}"] \arrow[d, Rightarrow] & \widetilde{\mathscr{N}}|_{Im\psi_U} \arrow[d, Rightarrow] \\
\mathscr{M}_U \arrow[r, "\gamma'_U"'] & \mathscr{N}_U
\end{tikzcd}$$

which completes the proof. □

Corollary 26. *Let $\phi : \mathscr{M} \longrightarrow \mathscr{N}$ be a morphism of quasi-coherent sheaves of graded $\mathscr{S}$-modules. Then*

(i) ϕ is a quasi-monomorphism $\Longrightarrow \widetilde{\phi} : \widetilde{\mathscr{M}} \longrightarrow \widetilde{\mathscr{N}}$ is a monomorphism.

(ii) ϕ is a quasi-epimorphism $\Longrightarrow \widetilde{\phi} : \widetilde{\mathscr{M}} \longrightarrow \widetilde{\mathscr{N}}$ is an epimorphism.

(iii) ϕ is a quasi-isomorphism $\Longrightarrow \widetilde{\phi} : \widetilde{\mathscr{M}} \longrightarrow \widetilde{\mathscr{N}}$ is an isomorphism.

Proof. These statements follow immediately from Proposition 25, Proposition 9, (MRS,Lemma 100), and (SOA,Proposition 47). □

3 The Graded Module Associated to a Sheaf

If S is a graded ring generated by S_1 as an S_0-algebra then the functor $\widetilde{-} : S\mathbf{GrMod} \longrightarrow \mathfrak{Mod}(ProjS)$ has a right adjoint $\Gamma_* : \mathfrak{Mod}(ProjS) \longrightarrow S\mathbf{GrMod}$ (AAMPS,Proposition 2). In this section we study a relative version of the functor Γ_*.

Throughout this section X is a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra, and $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ the structural morphism. If $\mathscr{F}$ is a sheaf of modules on **Proj**$\mathscr{S}$ then define the following sheaf of modules on X

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} \pi_*(\mathscr{F}(n))$$

For $d \geq 0$ let $\alpha^d : \mathscr{S}_d \longrightarrow \pi_*\mathcal{O}(d)$ be the morphism of sheaves of abelian groups defined in Lemma 19 and for $m, n \in \mathbb{Z}$ let $\kappa^{m,n} : \mathscr{F}(m) \otimes \mathcal{O}(n) \longrightarrow \mathscr{F}(m+n)$ be the isomorphism defined in Corollary 13. For open $U \subseteq X$ and $s \in \Gamma(U, \mathscr{S}_d), m \in \Gamma(\pi^{-1}U, \mathscr{F}(n))$ we define the section $s \cdot m \in \Gamma(\pi^{-1}U, \mathscr{F}(d+n))$ by

$$s \cdot m = \kappa^{n,d}_{\pi^{-1}U}(m \dot{\otimes} \alpha^d_U(s)) \tag{14}$$

If $m' \in \Gamma(\pi^{-1}U, \mathscr{F}(n))$, $r \in \Gamma(U, \mathscr{S}_e)$ and 1 denotes the identity of $\Gamma(U, \mathscr{S}_0)$ then it is not difficult to check that

$$\begin{aligned} s \cdot (m + m') &= s \cdot m + s \cdot m' \\ (s + s') \cdot m &= s \cdot m + s' \cdot m \\ r \cdot (s \cdot m) &= (rs) \cdot m \\ 1 \cdot m &= m \end{aligned}$$

where we use Corollary 16 for the second last property. Let S be the coproduct of presheaves of abelian groups $\bigoplus_{d \geq 0} \mathscr{S}_d$ with the canonical structure of a presheaf of rings $\{(s_d)(t_d)\}_i = \sum_{x+y=i} s_x t_y$. The sheafification $\mathscr{S}'$ of S is a sheaf of graded rings with the subsheaf of degree d being the image of $\mathscr{S}_d \longrightarrow S \longrightarrow \mathscr{S}'$. The morphisms $\mathscr{S}_d \longrightarrow \mathscr{S}'$ induce an isomorphism of sheaves of graded rings $\mathscr{S}' \longrightarrow \mathscr{S}$.

Let $\gamma_*(\mathscr{F})$ denote the coproduct of presheaves of abelian groups $\bigoplus_{n \in \mathbb{Z}} \pi_*(\mathscr{F}(n))$ which we make into a presheaf of S-modules by defining for $s \in \Gamma(U, S)$ and $m \in \Gamma(U, \gamma_*(\mathscr{F}))$

$$(s \cdot m)_i = \sum_{d+j=i} s_d \cdot m_j$$

By definition $\Gamma_*(\mathscr{F})$ is the sheafification of $\gamma_*(\mathscr{F})$ (as a sheaf of abelian groups), so we have defined the structure of a sheaf of graded $\mathscr{S}'$-modules on $\Gamma_*(\mathscr{F})$. Using the isomorphism $\mathscr{S} \cong \mathscr{S}'$ we define $\Gamma_*(\mathscr{F})$ as a sheaf of graded $\mathscr{S}$-modules with the degree n subsheaf being the image of $\pi_*(\mathscr{F}(n)) \longrightarrow \Gamma_*(\mathscr{F})$. Note that for $s \in \Gamma(U, \mathscr{S}_d)$ and $m \in \Gamma(\pi^{-1}U, \mathscr{F}(n))$ the action of s on $\dot{m}$ is just the one given in (14).

Definition 5. Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra, and $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ the structural morphism. If $\mathscr{F}$ is a sheaf of modules on $\mathbf{Proj}\mathscr{S}$ then we have a sheaf of graded $\mathscr{S}$-modules

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \pi_*(\mathscr{F}(n))$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $\bigoplus_{n \in \mathbb{Z}} \pi_*(\phi(n))$ is a morphism of sheaves of graded $\mathscr{S}$-modules $\Gamma_*(\mathscr{F}) \longrightarrow \Gamma_*(\mathscr{G})$ and this defines an additive functor

$$\Gamma_*(-) : \mathfrak{Mod}(\mathbf{Proj}\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$$

If $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules then for $n \in \mathbb{Z}$ we have a morphism of sheaves of abelian groups $\beta^n : \mathscr{M}_n \longrightarrow \pi_* \widetilde{\mathscr{M}(n)}$ and $\eta = \bigoplus_{n \in \mathbb{Z}} \beta^n$ is a morphism $\mathscr{M} \longrightarrow \Gamma_*(\widetilde{\mathscr{M}})$ of sheaves of graded $\mathscr{S}$-modules natural in $\mathscr{M}$ (use Lemma 22 and naturality of β^n).

Lemma 27. *The functor $\Gamma_*(-) : \mathfrak{Mod}(\mathbf{Proj}\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$ is left exact.*

Proof. We set $Y = \mathbf{Proj}\mathscr{S}$ and show that $\Gamma_*(-)$ preserves kernels. Since $\mathfrak{Mod}(Y)$ and $\mathfrak{GrMod}(\mathscr{S})$ are abelian categories (MRS,Proposition 98) this implies that $\Gamma_*(-)$ preserves monomorphisms and all finite limits. For $n \in \mathbb{Z}$, the sheaf of modules $\mathcal{O}(n)$ is invertible and therefore flat, so the twisting functors $-(n) : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(Y)$ are exact. Suppose we have an exact sequence of sheaves of modules on Y

$$0 \longrightarrow \mathscr{K} \xrightarrow{k} \mathscr{F} \xrightarrow{\phi} \mathscr{G}$$

The functor $\pi_* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ has a left adjoint and therefore preserves kernels, and in the complete grothendieck abelian category $\mathfrak{Mod}(X)$ coproducts are exact, so the following sequence of sheaves of modules on X is exact

$$0 \longrightarrow \Gamma_*(\mathscr{K}) \xrightarrow{\Gamma_*(k)} \Gamma_*(\mathscr{F}) \xrightarrow{\Gamma_*(\phi)} \Gamma_*(\mathscr{G})$$

This sequence is also exact in $\mathfrak{GrMod}(\mathscr{S})$ since kernels are computed pointwise (LC,Corollary 10) so the forgetful functor $\mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathscr{S})$ reflects kernels. $\square$

Lemma 28. *Suppose $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra and let $\mathscr{F}$ be a quasi-coherent sheaf of modules on $\mathbf{Proj}\mathscr{S}$. Then $\Gamma_*(\mathscr{F})$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules and we have an additive functor*

$$\Gamma_*(-) : \mathfrak{Qco}(\mathbf{Proj}\mathscr{S}) \longrightarrow \mathfrak{QcoGrMod}(\mathscr{S})$$

Proof. If $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra then $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ is a separated morphism of finite type by Lemma 1, which therefore has the property that $\pi_*\mathscr{F}$ is quasi-coherent for any quasi-coherent sheaf of modules $\mathscr{F}$ (H, 5.8), (H, Ex.3.3a). Since $\mathcal{O}(n)$ is quasi-coherent for all $n \in \mathbb{Z}$, the sheaves $\pi_*(\mathscr{F}(n))$ are all quasi-coherent, and therefore $\Gamma_*(\mathscr{F})$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules (MOS,Proposition 25) and therefore of $\mathscr{S}$-modules (SOA,Proposition 19). $\square$

Proposition 29. *Suppose $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra and let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules. Then for affine open $U \subseteq X$ there is a canonical isomorphism of graded $\mathscr{S}(U)$-modules natural in $\mathscr{M}$*

$$\upsilon : \Gamma_*(\mathscr{M}(U)^{\sim}) \longrightarrow \Gamma_*(\widetilde{\mathscr{M}})(U)$$

Moreover the following diagram of graded $\mathscr{S}(U)$-modules commutes

$$\begin{tikzcd} \mathscr{M}(U) \arrow[r, "(\eta_{\mathscr{M}})_U"] \arrow[dr, "\eta_{\mathscr{M}(U)}"'] & \Gamma_*(\widetilde{\mathscr{M}})(U) \\ & \Gamma_*(\mathscr{M}(U)^{\sim}) \arrow[u, Rightarrow, "\upsilon"'] \end{tikzcd} \tag{15}$$

Proof. To elaborate on the notation, $\mathscr{S}(U)$ is a graded $\mathcal{O}_X(U)$-algebra and $\mathscr{M}(U)$ is a graded $\mathscr{S}(U)$-module and $\Gamma_*(\mathscr{M}(U)^{\sim})$ refers to the functor

$$\Gamma_*(-) : \mathfrak{Mod}(Proj\mathscr{S}(U)) \longrightarrow \mathscr{S}(U)\mathbf{GrMod}$$

On the other hand $\Gamma_*(\widetilde{\mathscr{M}})$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules, so $\Gamma_*(\widetilde{\mathscr{M}})(U)$ is a graded $\mathscr{S}(U)$-module (SOA,Proposition 40). If we set $Y = Proj\mathscr{S}(U)$ then we have an isomorphism of abelian groups

$$\begin{aligned} \upsilon : \Gamma_*(\mathscr{M}(U)^{\sim}) &= \bigoplus_{n\in\mathbb{Z}} \Gamma\left(Y, \mathscr{M}(U)^{\sim}(n)\right) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \Gamma\left(Y, \mathscr{M}(n)(U)^{\sim}\right) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \Gamma\left(\pi^{-1}U, \mathscr{M}(n)^{\sim}\right) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \Gamma\left(\pi^{-1}U, \mathscr{M}^{\sim}(n)\right) \\ &\cong \Gamma_*(\widetilde{\mathscr{M}})(U) \end{aligned}$$

Here we use (H,5.12(b)), the isomorphism $\mu_U : \mathscr{M}(n)|_{Im\psi_U} \longrightarrow (\psi'_U)_*(\mathscr{M}(n)(U)^{\sim})$ and ρ^n. To show υ is a morphism of graded $\mathscr{S}(U)$-modules, it suffices to show that $\upsilon(s \cdot m) = s \cdot \upsilon(m)$ for $s \in \mathscr{S}_d(U)$ and $m \in \Gamma\left(Y, \mathscr{M}(U)^{\sim}(n)\right)$, which follows from Lemma 21. Naturality in $\mathscr{M}$ is easily checked.

Let $\eta_{\mathscr{M}} : \mathscr{M} \longrightarrow \Gamma_*(\widetilde{\mathscr{M}})$ be the morphism of sheaves of graded $\mathscr{S}$-modules defined in Definition 5 and $\eta_{\mathscr{M}(U)} : \mathscr{M}(U) \longrightarrow \Gamma_*(\mathscr{M}(U)^{\sim})$ the morphism defined in (AAMPS,Proposition 2) for the graded $\mathscr{S}(U)$-module $\mathscr{M}(U)$. It is then readily checked that the diagram (15) of graded $\mathscr{S}(U)$-modules commutes. $\square$

Corollary 30. *Let X be a scheme and $\mathscr{S} = \mathcal{O}_X[x_0, \ldots, x_n]$ for $n \geq 1$. Then the morphism of graded $\mathscr{S}$-modules $\eta : \mathscr{S} \longrightarrow \Gamma_*(\widetilde{\mathscr{S}})$ is an isomorphism.*

Proof. We know that $\mathscr{S}$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras, which is locally finitely generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra and has $\mathscr{S}_0 = \mathcal{O}_X$, so $\mathscr{S}$ satisfies the hypotheses of Proposition 29 and we reduce to showing that $\mathscr{S}(U) \longrightarrow \Gamma_*(\mathscr{S}(U)^{\sim})$ is an isomorphism of graded $\mathscr{S}(U)$-modules for every affine open subset $U \subseteq X$. But there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-algebras $\mathscr{S}(U) \cong \mathcal{O}_X(U)[x_0, \ldots, x_n]$ (SSA,Proposition 6) so the claim follows from (AAMPS,Lemma 10) and (H,5.13). $\square$

Lemma 31. *Suppose $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra and let $\mathscr{F}$ be a quasi-coherent sheaf of modules on* **Proj**$\mathscr{S}$. *Let $U \subseteq X$ be an affine open subset and $\mathscr{F}_U$ the quasi-coherent sheaf of modules on $Proj\mathscr{S}(U)$ corresponding to $\mathscr{F}|_{Im\psi_U}$. Then there is a canonical isomorphism of graded $\mathscr{S}(U)$-modules natural in $\mathscr{F}$*

$$\varrho : \Gamma_*(\mathscr{F})(U) \longrightarrow \Gamma_*(\mathscr{F}_U)$$

Proof. With the notation of Definition 4, $\Gamma_*(\mathscr{F}_U)$ is a graded $\mathscr{S}(U)$-module and it follows from Lemma 28 that $\Gamma_*(\mathscr{F})$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules, so $\Gamma_*(\mathscr{F})(U)$ becomes a graded $\mathscr{S}(U)$-module in a canonical way. Using (MOS,Lemma 6) we have an isomorphism of abelian groups

$$\begin{aligned}
\Gamma_*(\mathscr{F})(U) &= \left\{ \bigoplus_{n\in\mathbb{Z}} \pi_*(\mathscr{F}(n)) \right\}(U) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \Gamma(Im\psi_U, \mathscr{F}(n)) \\
&= \bigoplus_{n\in\mathbb{Z}} \Gamma(Proj\mathscr{S}(U), \mathscr{F}(n)_U) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \Gamma(Proj\mathscr{S}(U), \mathscr{F}_U(n)) \\
&= \Gamma_*(\mathscr{F}_U)
\end{aligned}$$

It follows from Lemma 24 that this is an isomorphism of graded $\mathscr{S}(U)$-modules. Naturality in $\mathscr{F}$ follows immediately from naturality of the isomorphism in Lemma 23. $\square$

The isomorphisms of Lemma 31 and Proposition 29 are compatible in the following sense.

Lemma 32. *Suppose $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra and let $\mathscr{M}$ be a quasi-coherent sheaf of graded $\mathscr{S}$-modules, $\mathscr{F}$ a quasi-coherent sheaf of modules on* **Proj**$\mathscr{S}$ *and $U \subseteq X$ an affine subset. Let $\Phi : \mathscr{M}^{\sim} \longrightarrow \mathscr{F}$ be a morphism of sheaves of modules and $\varphi_U : \mathscr{M}(U)^{\sim} \longrightarrow \mathscr{F}_U$ the corresponding morphism of sheaves of modules on $Proj\mathscr{S}(U)$. Then the following diagram of graded $\mathscr{S}(U)$-modules commutes*

$$\begin{tikzcd}
\Gamma_*(\widetilde{\mathscr{M}})(U) \arrow[r, "\Gamma_*(\Phi)_U"] \arrow[d, Rightarrow, "\upsilon^{-1}"'] & \Gamma_*(\mathscr{F})(U) \arrow[d, Rightarrow, "\varrho"] \\
\Gamma_*(\mathscr{M}(U)^{\sim}) \arrow[r, "\Gamma_*(\varphi_U)"'] & \Gamma_*(\mathscr{F}_U)
\end{tikzcd}$$

Proof. Note that by construction $(\psi'_U)_*\mathscr{M}(U)^{\sim} \cong \mathscr{M}^{\sim}|_{Im\psi_U}$ and therefore $\mathscr{M}^{\sim}_U \cong \mathscr{M}(U)^{\sim}$. By the morphism φ_U corresponding to Φ we mean the composite of $\Phi_U : \mathscr{M}^{\sim}_U \longrightarrow \mathscr{F}_U$ with this isomorphism. The proof reduces to checking that for $n \in \mathbb{Z}$ the following diagram commutes

$$\begin{tikzcd}
\mathscr{M}^{\sim}(n)_U \arrow[r, "\Phi(n)_U"] \arrow[d, Rightarrow] & \mathscr{F}(n)_U \arrow[ddd, Rightarrow] \\
\mathscr{M}(n)^{\sim}_U \arrow[d, Rightarrow] & \\
\mathscr{M}(n)(U)^{\sim} \arrow[d, Rightarrow] & \\
\mathscr{M}(U)^{\sim}(n) \arrow[r, "\varphi_U(n)"'] & \mathscr{F}_U(n)
\end{tikzcd}$$

By reduction to the case of sections of the form $\dot{m/s} \otimes \dot{a/b}$ this is not difficult to check. $\square$

Proposition 33. *Suppose $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra. Then we have an adjoint pair of functors*

$$\begin{tikzcd}
\mathfrak{QcoGrMod}(\mathscr{S}) \arrow[r, bend left, "\tilde{-}"] & \mathfrak{Qco}(\mathbf{Proj}\mathscr{S}) \arrow[l, bend left, "\Gamma_*(-)"]
\end{tikzcd} \qquad \tilde{-} \dashv \Gamma_*(-) \tag{16}$$

For a quasi-coherent sheaf of graded $\mathscr{S}$-modules $\mathscr{M}$ the unit is the morphism $\eta : \mathscr{M} \longrightarrow \Gamma_*(\mathscr{M})$ defined above.

Proof. We already know that the morphism $\mathscr{M} \longrightarrow \Gamma_*(\mathscr{M}^{\sim})$ is natural in $\mathscr{M}$, so it suffices to show that the pair $(\mathscr{M}^{\sim}, \eta)$ is a reflection of $\mathscr{M}$ along $\Gamma_*(-)$. Suppose we are given a morphism $\phi : \mathscr{M} \longrightarrow \Gamma_*(\mathscr{G})$ of sheaves of graded $\mathscr{S}$-modules for a quasi-coherent sheaf $\mathscr{G}$ of modules on $\mathbf{Proj}\mathscr{S}$. For affine open $U \subseteq X$ we a morphism of graded $\mathscr{S}(U)$-modules $\phi_U : \mathscr{M}(U) \longrightarrow \Gamma_*(\mathscr{G})(U)$ and a pair of adjoints

$$\begin{tikzcd}
\mathscr{S}(U)\mathbf{GrMod} \arrow[r, bend left, "\tilde{-}"] & \mathfrak{Mod}(Proj\mathscr{S}(U)) \arrow[l, bend left, "\Gamma_*(-)"]
\end{tikzcd} \qquad \tilde{-} \dashv \Gamma_*(-) \tag{17}$$

Using the isomorphism of Lemma 31, the composite $\mathscr{M}(U) \longrightarrow \Gamma_*(\mathscr{G})(U) \cong \Gamma_*(\mathscr{G}_U)$ induces a morphism $\varphi_U : \mathscr{M}(U)^{\sim} \longrightarrow \mathscr{G}_U$ of sheaves of modules on $Proj\mathscr{S}(U)$. Taking the image of this morphism under the functor $(\psi'_U)_*$ and composing with the isomorphism $(\psi'_U)_*\mathscr{M}(U)^{\sim} \cong \mathscr{M}^{\sim}|_{Im\psi_U}$ we have a morphism of sheaves of modules on $Im\psi_U$

$$\Phi^U : \widetilde{\mathscr{M}}|_{Im\psi_U} \longrightarrow \mathscr{G}|_{Im\psi_U}$$
$$\dot{m/s} \mapsto (\nu\kappa^{d,-d})_Q(\phi_U(m)|_Q \dot{\otimes} \dot{1/s})$$

where $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ are homogenous of degree d such that $\dot{m/s} \in \Gamma(\psi_U^{-1}Q, \mathscr{M}(U)^{\sim})$ and $\nu : \mathscr{G}(0) \longrightarrow \mathscr{G}$ is the canonical isomorphism. Using what by now is a standard technique (see for example the proof of Proposition 11), we show that for affine open $W \subseteq U$, $\Phi^U|_{Im\psi_W} = \Phi^W$. Therefore there is a unique morphism $\Phi : \mathscr{M}^{\sim} \longrightarrow \mathscr{G}$ of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ with the property that $\Phi|_{Im\psi_U} = \Phi^U$ for every open affine $U \subseteq X$. For affine open U consider the following diagram

$$\begin{tikzcd}
 & \Gamma_*(\mathscr{G})(U) \arrow[r, Rightarrow] & \Gamma_*(\mathscr{G}_U) \\
\mathscr{M}(U) \arrow[ur, "\phi_U"] \arrow[urr] \arrow[dr, "\eta_U"'] \arrow[drr] & & \\
 & \Gamma_*(\widetilde{\mathscr{M}})(U) \arrow[uu, "\Gamma_*(\Phi)_U"] \arrow[r, Rightarrow] & \Gamma_*(\mathscr{M}(U)^{\sim}) \arrow[uu, "\Gamma_*(\varphi_U)"']
\end{tikzcd}$$

using this diagram, Lemma 32, Proposition 29 and the adjunction in (17) one shows that Φ is the unique morphism of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ making the following diagram commute

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\phi"] \arrow[d, "\eta"'] & \Gamma_*(\mathscr{G}) \\
\Gamma_*(\widetilde{\mathscr{M}}) \arrow[ur, "\Gamma_*(\Phi)"'] &
\end{tikzcd}$$

which completes the proof that we have an adjunction. The natural bijection $Hom(\mathscr{M}, \Gamma_*(\mathscr{G})) \longrightarrow Hom(\widetilde{\mathscr{M}}, \mathscr{G})$ is given by the construction $\phi \mapsto \Phi$ of the proof. $\square$

Corollary 34. *Suppose $\mathscr{S}$ is locally finitely generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra with $\mathscr{S}_0 = \mathcal{O}_X$. Then for a quasi-coherent sheaf of modules $\mathscr{G}$ on $\mathbf{Proj}\mathscr{S}$ the counit $\varepsilon : \Gamma_*(\mathscr{G})^{\sim} \longrightarrow \mathscr{G}$ is an isomorphism.*

Proof. For an affine open subset $U \subseteq X$ there is an isomorphism of sheaves of modules on $Proj\mathscr{S}(U)$

$$\Gamma_*(\mathscr{G})^{\sim}_U \cong \Gamma_*(\mathscr{G})(U)^{\sim} \cong \Gamma_*(\mathscr{G}_U)^{\sim}$$

If $\varepsilon_{\mathscr{G}} : \Gamma_*(\mathscr{G})^{\sim} \longrightarrow \mathscr{G}$ is the counit of the adjunction (16) and $\varepsilon_{\mathscr{G}_U} : \Gamma_*(\mathscr{G}_U)^{\sim} \longrightarrow \mathscr{G}_U$ is the counit of the adjunction (17) then by construction the following diagram of sheaves of modules on $Proj\mathscr{S}(U)$ commutes

$$\begin{tikzcd}
\Gamma_*(\mathscr{G}_U)^{\sim} \arrow[r, "\varepsilon_{\mathscr{G}_U}"] & \mathscr{G}_U \\
\Gamma_*(\mathscr{G})^{\sim}_U \arrow[u, Rightarrow] \arrow[ur, "(\varepsilon_{\mathscr{G}})_U"'] &
\end{tikzcd}$$

Since $\mathscr{G}_U$ is quasi-coherent, $\varepsilon_{\mathscr{G}_U}$ is an isomorphism (AAMPS,Proposition 13) and therefore $\varepsilon_{\mathscr{G}}|_{Im\psi_U}$ is an isomorphism for every affine open $U \subseteq X$. It follows that ε is an isomorphism, as required. $\square$

Corollary 35. *Let X be a scheme of finite type over a field k and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally finitely generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. If $\mathscr{M}$ is a locally quasi-finitely generated quasi-coherent sheaf of graded $\mathscr{S}$-modules then the unit $\eta : \mathscr{M} \longrightarrow \Gamma_*(\mathscr{M}^{\sim})$ is a quasi-isomorphism. That is, there is an integer $d \geq 0$ such that for every $n \geq d$ we have an isomorphism of $\mathcal{O}_X$-modules*

$$\eta_n : \mathscr{M}_n \longrightarrow \pi_*(\widetilde{\mathscr{M}(n)})$$

Proof. See (SOA,Definition 14) for the definition of *locally quasi-finitely generated*. Note that by Lemma 28, $\Gamma_*(\mathscr{M}^{\sim})$ a quasi-coherent sheaf of graded $\mathscr{S}$-modules. Since X is noetherian it is quasi-compact, so we can find an affine open cover $U_1, \ldots, U_n$ of X. By an argument similar to the one given in (MOS,Lemma 2) it suffices to show that there exists a $d \geq 0$ such that $\eta\{d\}_{U_i} = \eta_{U_i}\{d\}$ (SOA,Lemma 48) is an isomorphism of graded $\mathscr{S}(U_i)$-modules for $1 \leq i \leq n$. So it suffices to show that $\eta_{U_i} : \mathscr{M}(U_i) \longrightarrow \Gamma_*(\mathscr{M}^{\sim})(U_i)$ is a quasi-isomorphism for each i, which follows from Proposition 29 and (AEMPS,Corollary 3). $\square$

3.1 Ring Structure

In this section set $Y = \mathbf{Proj}\mathscr{S}$ and for $n \in \mathbb{Z}$ write $\widehat{\mathcal{O}}_Y(n)$ for the sheaf of modules $\mathcal{O}_Y \otimes \mathcal{O}(n)$ on Y. There is a natural isomorphism $\widehat{\mathcal{O}}_Y(n) \cong \mathcal{O}(n)$. With this distinction, we have

$$\Gamma_*(\mathcal{O}_Y) = \bigoplus_{n\in\mathbb{Z}} \pi_*(\widehat{\mathcal{O}}_Y(n))$$

For $m, n \in \mathbb{Z}$ let $\Lambda^{m,n}$ denote the following isomorphism of modules

$$\Lambda^{m,n} : \widehat{\mathcal{O}}_Y(m) \otimes \widehat{\mathcal{O}}_Y(n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n) \cong \widehat{\mathcal{O}}(m+n)$$

As above let $\gamma_*(\mathcal{O}_Y)$ denote the coproduct of presheaves of abelian groups $\bigoplus_{n\in\mathbb{Z}} \pi_*(\widehat{\mathcal{O}}_Y(n))$. For an open subset $U \subseteq X$ and $a \in \Gamma(\pi^{-1}U, \widehat{\mathcal{O}}_Y(m))$ and $b \in \Gamma(\pi^{-1}U, \widehat{\mathcal{O}}_Y(n))$ we define

$$a \cdot b = \Lambda^{m,n}_{\pi^{-1}U}(a \dot{\otimes} b) \in \Gamma(\pi^{-1}U, \widehat{\mathcal{O}}_Y(n+m))$$

If 1 denotes the element $1 \dot{\otimes} 1$ of $\Gamma(\pi^{-1}U, \widehat{\mathcal{O}}(0))$ and $c \in \Gamma(\pi^{-1}U, \widehat{\mathcal{O}}_Y(e))$ then we have

$$\begin{aligned} a(b+c) &= ab + ac \\ (a+b)c &= ac + bc \\ a(bc) &= (ab)c \\ 1a &= a1 = a \end{aligned}$$

We use the analogue of Corollary 15 for $\widehat{\mathcal{O}}$, which is easily checked by reducing to special sections. It is then not hard to check that $\gamma_*(\mathcal{O}_Y)(U)$ is a commutative ring with the product

$$(x \cdot y)_i = \sum_{m+n=i} x_m \cdot y_n = \sum_{m+n=i} \Lambda^{m,n}_{\pi^{-1}U}(x_m \dot{\otimes} y_n)$$

With this definition $\gamma_*(\mathcal{O}_Y)$ is a presheaf of rings. Therefore $\Gamma_*(\mathcal{O}_Y)$ becomes a sheaf of $\mathbb{Z}$-graded rings. It is not hard to check that $\eta : \mathscr{S} \longrightarrow \Gamma_*(\mathcal{O}_Y)$ is a morphism of sheaves of $\mathbb{Z}$-graded rings. If we let $\Gamma_*(\mathcal{O}_Y)'$ denote the graded submodule $\Gamma_*(\mathcal{O}_Y)\{0\}$ of $\Gamma_*(\mathcal{O}_Y)$ then it is not hard to check that $\Gamma_*(\mathcal{O}_Y)'$ is also a subsheaf of rings. In fact $\Gamma_*(\mathcal{O}_Y)'$ is a commutative sheaf of graded $\mathcal{O}_X$-algebras and there is an induced morphism $\eta : \mathscr{S} \longrightarrow \Gamma_*(\mathcal{O}_Y)'$ of sheaves of graded $\mathcal{O}_X$-algebras. If $\mathscr{S}$ is locally finitely generated as an $\mathcal{O}_X$-algebra then by Lemma 28 and (SOA,Lemma 48), $\Gamma_*(\mathcal{O}_Y)'$ is a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras.

Corollary 36. *Let X be a scheme of finite type over a field k and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally finitely generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. The morphism of sheaves of graded $\mathcal{O}_X$-algebras $\eta : \mathscr{S} \longrightarrow \Gamma_*(\mathcal{O}_Y)'$ is a quasi-isomorphism.*

Proof. This follows immediately from Corollary 35. □

4 Functorial Properties

Proposition 37. *Let X be a scheme and $\xi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras. Then canonically associated with ξ is an open set $G(\xi) \subseteq \mathbf{Proj}\mathscr{G}$ and a morphism of schemes $\mathbf{Proj}\xi : G(\xi) \longrightarrow \mathbf{Proj}\mathscr{F}$ over X. For every open affine subset $U \subseteq X$ we have $G(\xi) \cap Im\psi_U = \psi_U(G(\xi_U))$ and the following diagram is a pullback*

$$\begin{tikzcd} G(\xi) \arrow[r, "\mathbf{Proj}\xi"] & \mathbf{Proj}\mathscr{F} \\ G(\xi_U) \arrow[u] \arrow[r, "Proj\xi_U"'] & Proj\mathscr{F}(U) \arrow[u, "\psi_U"'] \end{tikzcd} \tag{18}$$

Proof. For an open affine $U \subseteq X$ it is clear that $\xi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is a morphism of graded $\mathcal{O}_X(U)$-algebras, inducing a morphism of schemes $Proj\xi_U : G(\xi_U) \longrightarrow Proj\mathscr{F}(U)$ over U. It is not difficult to check that given open affines $U \subseteq V$ we have

$$G(\xi_V) \cap Proj\mathscr{G}(U) = G(\xi_U) \tag{19}$$

where we identify $Proj\mathscr{G}(U)$ with an open subset of $Proj\mathscr{G}(V)$. Using our notes on the Proj construction it is immediate that the following diagram commutes

$$\begin{tikzcd} G(\xi_V) \arrow[r] & Proj\mathscr{F}(V) \\ G(\xi_U) \arrow[r] \arrow[u] & Proj\mathscr{F}(U) \arrow[u] \end{tikzcd} \tag{20}$$

Let $G(\xi)$ be the open subset $\bigcup_U \psi_U(G(\xi_U))$ of $\mathbf{Proj}\mathscr{G}$, where the union is over all affine open subsets of X. Using (19) we see that $G(\xi) \cap Im\psi_U = \psi_U(G(\xi_U))$ and (20) implies that the morphisms $\psi_U(G(\xi_U)) \cong G(\xi_U) \longrightarrow Proj\mathscr{F}(U) \longrightarrow \mathbf{Proj}\mathscr{F}$ glue to give the morphism $\mathbf{Proj}\xi$ with the desired properties (as usual it suffices to check this for open affines $U \subseteq V$, which is straightforward). $\square$

Lemma 38. *Let X be a scheme and $\xi : \mathscr{F} \longrightarrow \mathscr{G}$ and $\rho : \mathscr{G} \longrightarrow \mathscr{H}$ morphisms of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras. Then $G(\rho\xi) \subseteq G(\rho)$ and the following diagram commutes*

$$\begin{tikzcd} G(\rho\xi) \arrow[rr, "\mathbf{Proj}\rho\xi"] \arrow[dr, "(\mathbf{Proj}\rho)|_{G(\rho\xi)}"'] & & \mathbf{Proj}\mathscr{F} \\ & G(\xi) \arrow[ur, "\mathbf{Proj}\xi"'] & \end{tikzcd}$$

Moreover if $1 : \mathscr{F} \longrightarrow \mathscr{F}$ is the identity then $G(1) = \mathbf{Proj}\mathscr{F}$ and $\mathbf{Proj}1 = 1$. If $\xi : \mathscr{F} \longrightarrow \mathscr{G}$ is an isomorphism then $G(\xi) = \mathbf{Proj}\mathscr{F}$ and $\mathbf{Proj}\xi$ is an isomorphism with inverse $\mathbf{Proj}(\xi^{-1})$.

Proof. By intersecting with the open sets $Im\psi_U$ it is not hard to check that $G(\rho\xi) \subseteq G(\rho)$. It is also clear that $(\mathbf{Proj}\rho)G(\rho\xi) \subseteq G(\xi)$, so we get the induced morphism $G(\rho\xi) \longrightarrow G(\xi)$ in the diagram. To check that the above diagram commutes, cover $G(\rho\xi)$ in open subsets $\psi_U(G(\rho_U\xi_U))$ and after a little work reduce to the same result for morphisms of $Proj$ of graded rings induced by morphisms of graded rings, which we checked in our Proj Construction notes. The remaining claims of the Lemma are then easily checked. $\square$

Proposition 39. *Let X be a scheme and $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras that is an epimorphism of sheaves of modules. Then $G(\phi) = \mathbf{Proj}\mathscr{G}$ and the induced morphism $\mathbf{Proj}\mathscr{G} \longrightarrow \mathbf{Proj}\mathscr{F}$ is a closed immersion.*

Proof. This follows immediately from (H, Ex.3.12), (MOS,Lemma 2) and the local nature of the closed immersion property. $\square$

Example 3. Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras, and $\mathscr{J}$ a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals (MRS,Definition 22). Consider the following exact sequence in the abelian category $\mathfrak{GrMod}(\mathscr{S})$

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{S} \longrightarrow \mathscr{S}/\mathscr{J} \longrightarrow 0$$

Here $\mathscr{S}/\mathscr{J}$ is the sheafification of the presheaf of $\mathscr{S}$-modules $U \mapsto \mathscr{S}(U)/\mathscr{J}(U)$. For every open subset $U \subseteq X$, $\mathscr{J}(U)$ is an ideal of $\mathscr{S}(U)$, so $\mathscr{S}(U)/\mathscr{J}(U)$ is a $\mathcal{O}_X(U)$-algebra and $\mathscr{S}/\mathscr{J}$ a sheaf of $\mathcal{O}_X$-algebras, which becomes a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras with the canonical grading given in (LC,Corollary 10) and $\mathscr{S} \longrightarrow \mathscr{S}/\mathscr{J}$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras. It follows from Proposition 39 that the induced morphism of schemes $\mathbf{Proj}(\mathscr{S}/\mathscr{J}) \longrightarrow \mathbf{Proj}\mathscr{S}$ is a closed immersion. Note that for affine open $U \subseteq X$ it follows from (MOS,Lemma 5) that $(\mathscr{S}/\mathscr{J})(U) \cong \mathscr{S}(U)/\mathscr{J}(U)$ as graded $\mathcal{O}_X(U)$-algebras.

Proposition 40. *Let X be a scheme and $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2. Let $\Phi : Z \longrightarrow \mathbf{Proj}\mathscr{S}$ be the induced morphism of schemes and $\mathscr{M}$ a quasi-coherent sheaf of graded $\mathscr{S}$-modules. Then there is a canonical isomorphism of sheaves of modules on Z natural in $\mathscr{M}$*

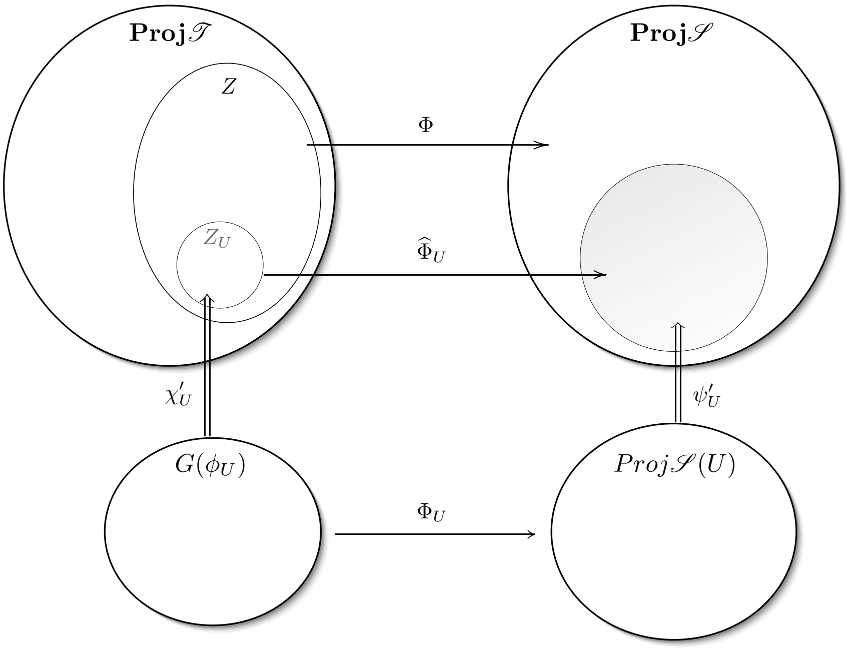
$$\vartheta : \Phi^* \widetilde{\mathscr{M}} \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^{\sim}|_Z$$
$$[T, m\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto (m \dot{\otimes} b)\dot{/}\phi_U(s)t$$

Proof. The morphism $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ induces an additive functor $- \otimes_{\mathscr{S}} \mathscr{T} : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{T})$ which is left adjoint to the restriction of scalars functor (MRS,Proposition 105). This functor maps quasi-coherent sheaves of graded $\mathscr{S}$-modules to quasi-coherent sheaves of graded $\mathscr{T}$-modules (SOA,Corollary 22),(SOA,Proposition 19), so at least the claim makes sense.

Set $Z = G(\phi)$, and for affine open $U \subseteq X$ let $\Phi_U : G(\phi_U) \longrightarrow Proj\mathscr{S}(U)$ be induced by the morphism of graded rings $\phi_U : \mathscr{S}(U) \longrightarrow \mathscr{T}(U)$. Let σ_U be the following isomorphism of sheaves of modules defined in (MPS,Proposition 6)

$$\sigma_U : \Phi_U^*(\mathscr{M}(U)^{\sim}) \longrightarrow (\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{T}(U))^{\sim}|_{G(\phi_U)}$$

It follows from (SOA,Proposition 46) that there is an isomorphism of graded $\mathscr{T}(U)$-modules $\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{T}(U) \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})(U)$ defined by $a \otimes b \mapsto a \dot{\otimes} b$. Let $\chi_U : Proj\mathscr{T}(U) \longrightarrow \mathbf{Proj}\mathscr{T}$ and $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ be the canonical open immersions, write Z_U for $\chi_U(G(\phi_U))$ and let $\widehat{\Phi}_U : Z_U \longrightarrow Im\psi_U$ be induced by Φ. These morphisms fit into the following diagram



From Proposition 7 we have isomorphisms of sheaves of modules

$$\mu_U : \mathscr{M}^{\sim}|_{Im\psi_U} \longrightarrow (\psi'_U)_* \mathscr{M}(U)^{\sim}$$
$$\mu'_U : (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^{\sim}|_{Z_U} \longrightarrow (\chi'_U)_*(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})(U)^{\sim}|_{G(\phi_U)}$$

Then there is an isomorphism of sheaves of modules on Z_U

$$\begin{aligned}\vartheta^U : (\Phi^* \widetilde{\mathscr{M}})|_{Z_U} &\cong \widehat{\Phi}_U^* (\widetilde{\mathscr{M}}|_{Im\psi_U})\\ &\cong \widehat{\Phi}_U^*((\psi_U')_* \mathscr{M}(U)^\sim)\\ &\cong (\chi_U')_* (\Phi_U^* \mathscr{M}(U)^\sim)\\ &\cong (\chi_U')_* \left((\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{T}(U))^\sim|_{G(\phi_U)} \right)\\ &\cong (\chi_U')_* \left((\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})(U)^\sim|_{G(\phi_U)} \right)\\ &\cong (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^\sim|_{Z_U}\end{aligned}$$

Using (MRS,Proposition 111), the isomorphism μ_U, (MRS,Proposition 108), σ_U, (SOA,Proposition 46) and finally μ_U'. We need to know the action of ϑ^U on special sections of the following form: suppose $Q \subseteq Z_U$ is open and that $b, t \in \mathscr{T}(U)$ are homogenous of the same degree such that $b/t \in \Gamma(\chi_U^{-1}Q, Proj\,\mathscr{T}(U))$. Denote also by b/t the corresponding section of $\mathbf{Proj}\mathscr{T}$. Suppose $m \in \mathscr{M}(U), s \in \mathscr{S}(U)$ are homogenous of the same degree with $m/s \in \Gamma(\psi_U^{-1}T, \mathscr{M}(U)^\sim)$ for some open T with $\Phi(Q) \subseteq T \subseteq Im\psi_U$. Denote also by m/s the corresponding section of $\widetilde{\mathscr{M}}$ under μ_U. Then we have

$$[T, m/s] \dot\otimes b/t \in \Gamma(Q, \Phi^* \widetilde{\mathscr{M}}) \tag{21}$$

$$\vartheta_Q^U \left([T, m/s] \dot\otimes b/t \right) = (m \dot\otimes b)/\phi_U(s)t \tag{22}$$

Where we use μ_U' to identify $(m \dot\otimes b)/\phi_U(s)t$ with a section of $(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^\sim$. To show that the isomorphisms ϑ^U glue we need only show that for affine $W \subseteq U$ we have $\vartheta^U|_{Z_W} = \vartheta^W$. We can reduce immediately to sections of the form (21) for $Q \subseteq Z_W$ and $T \subseteq Im\psi_W$. The compatibility conditions on the morphisms μ_U, μ_W and μ_U', μ_W' (see Proposition 7) mean that the section $[T, m/s] \dot\otimes b/t$ is the same as $[T, m|_W/s|_W] \dot\otimes b|_W/t|_W$ (the sections being images under μ_U and μ_W respectively). But

$$\vartheta_Q^W \left([T, m|_W/s|_W] \dot\otimes b|_W/t|_W \right) = (m|_W \dot\otimes b|_W)/\phi_W(s|_W)t|_W$$

by the same argument, the section on the right is the section of $(\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^\sim$ corresponding to $(m \dot\otimes b)/\phi_U(s)t$. This shows that $\vartheta^U|_{Z_W} = \vartheta^W$ and therefore (GS,Corollary 5) there is an isomorphism $\vartheta : \Phi^* \mathscr{M}^\sim \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{T})^\sim|_Z$ of sheaves of modules on Z unique with the property that for affine open $U \subseteq X$, $\vartheta|_{Z_U} = \vartheta^U$. By reducing to sections of the form (21) naturality in $\mathscr{M}$ is not difficult to check. $\square$

Proposition 41. *Let X be a scheme and $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2. Let $\Phi : Z \longrightarrow \mathbf{Proj}\mathscr{S}$ be the induced morphism of schemes. Then for $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of modules on Z*

$$\vartheta : \Phi^* \mathcal{O}(n) \longrightarrow \mathcal{O}(n)|_Z$$
$$[T, a/s] \dot\otimes b/t \mapsto \phi_U(a)b/\phi_U(s)t$$

Proof. Using Proposition 40 and (MRS,Lemma 106) we have an isomorphism of sheaves of modules

$$\Phi^* \mathcal{O}(n) = \Phi^* \mathscr{S}(n)^\sim \cong (\mathscr{S}(n) \otimes_{\mathscr{S}} \mathscr{T})^\sim|_Z \cong \mathscr{T}(n)^\sim|_Z = \mathcal{O}(n)|_Z$$

With the notation of Proposition 40 let $U \subseteq X$ be an affine open subset and supose $Q \subseteq Z_U$ is open and that $b, t \in \mathscr{T}(U)$ are homogenous of the same degree such that $b/t \in \Gamma(\chi_U^{-1}Q, Proj\,\mathscr{T}(U))$. Denote also by b/t the corresponding section of $\mathbf{Proj}\mathscr{T}$. Suppose $a, s \in \mathscr{S}(U)$ are homogenous, with a of degree $d + n$, s of degree d, and $a/s \in \Gamma(\psi_U^{-1}T, \mathcal{O}(n))$ for some open T with $\Phi(Q) \subseteq T \subseteq Im\psi_U$. Denote also by a/s the corresponding section of the sheaf $\mathcal{O}(n)$ on $\mathbf{Proj}\mathscr{S}$. Then we have

$$[T, a/s] \dot\otimes b/t \in \Gamma(Q, \Phi^* \mathcal{O}(n)) \tag{23}$$

$$\sigma_Q \left([T, a/s] \dot\otimes b/t \right) = \phi_U(a)b/\phi_U(s)t \tag{24}$$

which completes the proof. $\square$

Proposition 42. *Let X be a scheme and $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2. Let $\Phi : Z \longrightarrow$ **Proj**$\mathscr{S}$ be the induced morphism of schemes and $\mathscr{N}$ a quasi-coherent sheaf of graded $\mathscr{T}$-modules. Then there is a canonical isomorphism of sheaves of modules on* **Proj**$\mathscr{S}$ *natural in $\mathscr{N}$*

$$\vartheta : ({}_{\mathscr{S}}\mathscr{N})^{\sim} \longrightarrow \Phi_* \widetilde{\mathscr{N}}|_Z$$
$$m \dot{/} s \mapsto m / \dot{\phi_U}(s)$$

Proof. The morphism $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ induces a restriction of scalars functor ${}_{\mathscr{S}}(-) : \mathfrak{GrMod}(\mathscr{T}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$ which is right adjoint to the extension of scalars functor (MRS,Proposition 105). This functor maps quasi-coherent sheaves of graded $\mathscr{T}$-modules to quasi-coherent sheaves of graded $\mathscr{S}$-modules (SOA,Proposition 19), so at least the claim makes sense. From Proposition 7 we have isomorphisms of sheaves of modules

$$\mu_U : ({}_{\mathscr{S}}\mathscr{N})^{\sim}|_{Im\psi_U} \longrightarrow (\psi'_U)_*({}_{\mathscr{S}(U)}\mathscr{N}(U)^{\sim})$$
$$\mu'_U : \mathscr{N}^{\sim}|_{Z_U} \longrightarrow (\chi'_U)_*(\mathscr{N}(U)^{\sim}|_{G(\phi_U)})$$

With the notation of Proposition 40 let $U \subseteq X$ be an affine open subset and define an isomorphism of sheaves of modules on $Im\psi_U$

$$\begin{aligned}
\vartheta^U : ({}_{\mathscr{S}}\mathscr{N})^{\sim}|_{Im\psi_U} &\cong (\psi'_U)_*({}_{\mathscr{S}(U)}\mathscr{N}(U)^{\sim}) \\
&\cong (\psi'_U)_*(\Phi_U)_*(\mathscr{N}(U)^{\sim}|_{G(\phi_U)}) \\
&= (\widehat{\Phi}_U)_*(\chi'_U)_*(\mathscr{N}(U)^{\sim}|_{G(\phi_U)}) \\
&\cong (\widehat{\Phi}_U)_*(\widetilde{\mathscr{N}}|_{Z_U}) \\
&= \left(\Phi_*(\widetilde{\mathscr{N}}|_Z)\right)|_{Im\psi_U}
\end{aligned}$$

Using μ_U, (MPS,Proposition 7) and μ'_U. Suppose $Q \subseteq Im\psi_U$ is open and that $n \in \mathscr{N}(U), s \in \mathscr{S}(U)$ are homogenous of the same degree such that $n \dot{/} s \in \Gamma(\psi_U^{-1}Q, \mathscr{N}(U)^{\sim})$. Denote also by $n \dot{/} s$ the corresponding section of $({}_{\mathscr{S}}\mathscr{N})^{\sim}$. Then we have

$$n \dot{/} s \in \Gamma(Q, ({}_{\mathscr{S}}\mathscr{N})^{\sim}) \tag{25}$$
$$\vartheta^U_Q(n \dot{/} s) = n / \dot{\phi_U}(s) \tag{26}$$

As above we check that for affine open $W \subseteq U$, $\vartheta^U|_{Im\psi_W} = \vartheta^W$, which means that there is a unique isomorphism of sheaves of modules $\vartheta : ({}_{\mathscr{S}}\mathscr{N})^{\sim} \longrightarrow \Phi_*\widetilde{\mathscr{N}}|_Z$ with the property that $\vartheta|_{Im\psi_U} = \vartheta^U$. Naturality in $\mathscr{N}$ is not difficult to check. $\square$

5 Ideal Sheaves and Closed Subchemes

Definition 6. Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. Let $\mathscr{J}$ be a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals (MRS,Definition 22). By Proposition 9 the functor $\tilde{-} : \mathfrak{QcoGrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathbf{Proj}\mathscr{S})$ is exact, so the induced morphism of sheaves of modules on **Proj**$\mathscr{S}$ is a monomorphism

$$i : \widetilde{\mathscr{J}} \longrightarrow \widetilde{\mathscr{S}} = \mathcal{O}_{\mathbf{Proj}\mathscr{S}}$$

Therefore we can identify $\mathscr{J}^{\sim}$ with the quasi-coherent sheaf of ideals on **Proj**$\mathscr{S}$ given by the image of i. Note that for an open subset $U \subseteq X$, $\mathscr{J}(U)$ is an ideal of the $\mathcal{O}_X(U)$-algebra $\mathscr{S}(U)$, and moreover if U is *affine* then $\mathscr{J}(U)$ is a homogenous ideal of the graded $\mathcal{O}_X(U)$-algebra $\mathscr{S}(U)$.

Lemma 43. *Let X be a scheme and $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2. If $\mathscr{K}$ is the kernel of ϕ then the ideal sheaf of the induced morphism of schemes $\Phi : Z \longrightarrow \mathbf{Proj}\mathscr{S}$ is $\mathscr{K}^\sim$.*

Proof. Considering ϕ as a morphism of sheaves of graded $\mathscr{S}$-modules, the kernel of ϕ in the category $\mathfrak{GrMod}(\mathscr{S})$ is the normal kernel of a morphism of sheaves of modules $\mathscr{K}(U) = Ker\phi_U$ which acquires the structure of a sheaf of graded $\mathscr{S}$-modules (LC,Corollary 10). In fact $\mathscr{K}$ is a sheaf of homogenous $\mathscr{S}$-ideals and we have an exact sequence of sheaves of modules on $\mathbf{Proj}\mathscr{S}$

$$0 \longrightarrow \widetilde{\mathscr{K}} \longrightarrow \mathcal{O}_{\mathbf{Proj}\mathscr{S}} \longrightarrow \widetilde{\mathscr{T}}$$

Let $\vartheta : \widetilde{\mathscr{T}} \longrightarrow \Phi_*(\mathcal{O}_{\mathbf{Proj}\mathscr{T}}|_Z)$ be the isomorphism of sheaves of modules on $\mathbf{Proj}\mathscr{S}$ given in Proposition 42. By definition the ideal sheaf of Φ is the kernel of the morphism $\Phi^\# : \mathcal{O}_{\mathbf{Proj}\mathscr{S}} \longrightarrow \Phi_*(\mathcal{O}_{\mathbf{Proj}\mathscr{T}}|_Z)$ of sheaves of modules on $\mathbf{Proj}\mathscr{S}$. So to complete the proof we need only show that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{Proj}\mathscr{S}} \arrow[r, "\widetilde{\phi}"] \arrow[dr, "\Phi^\#"'] & \widetilde{\mathscr{T}} \arrow[d, Rightarrow, "\vartheta"] \\
& \Phi_*(\mathcal{O}_{\mathbf{Proj}\mathscr{T}}|_Z)
\end{tikzcd}$$

With the notation of Proposition 40 we can reduce to sections of the form $\dot{a/s} \in \Gamma(Q, \mathcal{O}_{\mathbf{Proj}\mathscr{S}})$ where $U \subseteq X$ is affine, $Q \subseteq Im\psi_U$ open and $a, s \in \mathscr{S}(U)$ homogenous of the same degree. In that case commutativity of the diagram is easily checked, so the proof is complete. $\square$

Corollary 44. *Let X be a scheme, $\mathscr{S}$ a sheaf of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2 and $\mathscr{J}$ a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals. Then the ideal sheaf of the closed immersion $\mathbf{Proj}(\mathscr{S}/\mathscr{J}) \longrightarrow \mathbf{Proj}\mathscr{S}$ is $\mathscr{J}^\sim$.*

Proof. See Example 3 for the definition of the commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{S}/\mathscr{J}$. It is easy to see that if $\mathscr{S}$ is locally generated (resp. locally finitely generated) by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra then so is $\mathscr{S}/\mathscr{J}$. Since the morphism of sheaves of graded $\mathcal{O}_X$-algebras $\mathscr{S} \longrightarrow \mathscr{S}/\mathscr{J}$ has kernel $\mathscr{J}$, the result follows at once from Lemma 43. $\square$

Proposition 45. *Let X be a scheme and $\mathscr{S} = \mathcal{O}_X[x_0, \ldots, x_n]$ for $n \geq 1$. If $Y \longrightarrow \mathbb{P}^n_X$ is a closed immersion then there is a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals $\mathscr{J}$ such that Y is the closed subscheme determined by $\mathscr{J}$.*

Proof. We know that $\mathscr{S}$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras, which is locally finitely generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra and has $\mathscr{S}_0 = \mathcal{O}_X$, and moreover $\mathbf{Proj}\mathscr{S} = \mathbb{P}^n_X$ by Proposition 10. Let $\mathscr{F}$ be the ideal sheaf of a closed immersion $j : Y \longrightarrow \mathbb{P}^n_X$. The functor $\Gamma_*(-) : \mathfrak{Mod}(\mathbf{Proj}\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$ is left exact and preserves quasi-coherentness by Lemma 28, so we have a monomorphism of quasi-coherent sheaves of graded $\mathscr{S}$-modules $\Gamma_*(\mathscr{F}) \longrightarrow \Gamma_*(\widetilde{\mathscr{S}})$. By Corollary 30 the unit $\eta : \mathscr{S} \longrightarrow \Gamma_*(\mathscr{S}^\sim)$ is an isomorphism, and we let $\mathscr{J}$ be the quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals fitting into the following commutative diagram

$$\begin{tikzcd}
\Gamma_*(\mathscr{F}) \arrow[r] & \Gamma_*(\widetilde{\mathscr{S}}) \\
\mathscr{J} \arrow[u, Rightarrow] \arrow[r] & \mathscr{S} \arrow[u, Rightarrow]
\end{tikzcd}$$

Since $\mathscr{F}$ is quasi-coherent, we can apply the functor $\widetilde{-} : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{Mod}(\mathbf{Proj}\mathscr{S})$ and use

Corollary 34 to obtain a commutative diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r] & \mathcal{O}_{\mathbf{Proj}\mathscr{S}} \\
\Gamma_*(\mathscr{F})^\sim \arrow[r] \arrow[u, Rightarrow] & \Gamma_*(\widetilde{\mathscr{S}})^\sim \arrow[u, Rightarrow] \\
\mathscr{J}^\sim \arrow[r] \arrow[u, Rightarrow] & \mathscr{S}^\sim \arrow[u, Rightarrow]
\end{tikzcd}$$

Using the adjunction of Proposition 33 we see that the vertical morphism on the right is the identity, so $\mathscr{F} = \mathscr{J}^\sim$ as sheaves of ideals on $\mathbf{Proj}\mathscr{S}$. It now follows from Corollary 44 and (SI,Theorem 1) that $Y \longrightarrow \mathbb{P}^n_X$ is the same closed subscheme as $\mathbf{Proj}(\mathscr{S}/\mathscr{J}) \longrightarrow \mathbb{P}^n_X$, which completes the proof. □

Corollary 46. *Let X be a scheme. Then any projective morphism $Z \longrightarrow X$ is isomorphic as a scheme over X to the structural morphism $\mathbf{Proj}\mathscr{S} \longrightarrow X$ for some commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{S}$ locally finitely generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra.*

Proof. This follows immediately from Proposition 45. □

6 The Duple Embedding

Definition 7. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{S}$ a sheaf of graded $\mathcal{O}_X$-algebras. For $d > 0$ let $\mathscr{S}^{(d)}$ be the subsheaf of modules of $\mathscr{S}$ given by the internal direct sum $\oplus_{n\geq 0}\mathscr{S}_{nd}$. Using the explicit description in (MRS,Lemma 3) it is not difficult to check that for all open $U \subseteq X$, $\mathscr{S}^{(d)}(U)$ is a subalgebra of $\mathscr{S}(U)$, and therefore $\mathscr{S}^{(d)}$ is a sheaf of $\mathcal{O}_X$-algebras, and the inclusions give a monomorphism of algebras $\mathscr{S}^{(d)} \longrightarrow \mathscr{S}$. With the grading $\mathscr{S}^{(d)}_n = \mathscr{S}_{nd}$ it is clear that $\mathscr{S}^{(d)}$ is a sheaf of graded $\mathcal{O}_X$-algebras.

Let $\mathscr{S}^{[d]}$ be the same sheaf of algebras but with the grading $\mathscr{S}^{[d]}_0 = \mathscr{S}_0, \mathscr{S}^{[d]}_d = \mathscr{S}_d, \ldots$ with vanishing graded pieces in degrees not divisible by d. This is a sheaf of graded $\mathcal{O}_X$-algebras, and in this case $\varphi : \mathscr{S}^{[d]} \longrightarrow \mathscr{S}$ is a morphism of sheaves of graded algebras (for $\mathscr{S}^{(d)}$ this is clearly not true). If $\mathscr{S}$ is commutative or quasi-coherent (for X a scheme) then the same is true of $\mathscr{S}^{(d)}$ and $\mathscr{S}^{[d]}$ (MOS,Lemma 1), (MOS,Proposition 25).

If $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras then $\oplus_{n\geq 0}\phi_{nd}$ defines morphisms of sheaves of graded $\mathcal{O}_X$-algebras $\phi^{[d]} : \mathscr{S}^{[d]} \longrightarrow \mathscr{T}^{[d]}$ and $\phi^{(d)} : \mathscr{S}^{(d)} \longrightarrow \mathscr{T}^{(d)}$, so this construction defines functors

$$(-)^{[d]}, (-)^{(d)} : \mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(X)$$
$$(-)^{[d]}, (-)^{(d)} : \mathfrak{GrAlg}(X) \longrightarrow \mathfrak{GrAlg}(X)$$

With this definition it is clear that the morphism $\mathscr{S}^{[d]} \longrightarrow \mathscr{S}$ is natural in $\mathscr{S}$.

Lemma 47. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. For any open affine $U \subseteq X$ and $d > 0$ we have equalities of graded $\mathcal{O}_X(U)$-algebras*

$$\mathscr{S}^{[d]}(U) = \mathscr{S}(U)^{[d]} \qquad \mathscr{S}^{(d)}(U) = \mathscr{S}(U)^{(d)}$$

Moreover if $\psi : \mathscr{S} \longrightarrow \mathscr{T}$ is a morphism of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras then $(\psi^{[d]})_U = (\psi_U)^{[d]}$ and $(\psi^{(d)})_U = (\psi_U)^{(d)}$.

Proof. Both $\mathscr{S}^{(d)}(U)$ and $\mathscr{S}(U)^{(d)}$ are $\mathcal{O}_X(U)$-subalgebras of $\mathscr{S}(U)$, and using the argument of (SOA,Proposition 40) we see that both are the internal direct sum of the submodules $\mathscr{S}_{nd}(U) \subseteq \mathscr{S}(U)$ for $n \geq 0$. Therefore they are equal as algebras, and they clearly have the same grading. The same argument holds for $\mathscr{S}^{[d]}$. □

Lemma 48. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. Then for any $d > 0$, $\mathscr{S}^{(d)}$ is locally generated by $\mathscr{S}_1^{(d)}$ as an $\mathscr{S}_0^{(d)}$-algebra.*

Proof. This follows directly from (GRM,Lemma 10) and Lemma 47. □

Proposition 49. *Let X be a scheme and $\mathscr{S}$ be a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. For $d > 0$ the morphism $\varphi : \mathscr{S}^{[d]} \longrightarrow \mathscr{S}$ induces an isomorphism of X-schemes $\Phi : \mathbf{Proj}\mathscr{S} \longrightarrow \mathbf{Proj}\mathscr{S}^{[d]}$ natural in $\mathscr{S}$.*

Proof. Let $\Phi : G(\varphi) \longrightarrow \mathbf{Proj}\mathscr{S}^{[d]}$ be the morphism of schemes over X given by Proposition 37. For affine open U the morphism $\varphi_U : \mathscr{S}(U)^{[d]} \longrightarrow \mathscr{S}(U)$ is the one defined in (PM,Section 2). Therefore $G(\varphi) \cap Im\psi_U = Im\psi_U$ for all affine open U, and therefore $G(\varphi) = \mathbf{Proj}\mathscr{S}$. For every open affine U we have a pullback diagram

$$\begin{tikzcd}
\mathbf{Proj}\mathscr{S} \arrow[r, "\Phi"] & \mathbf{Proj}\mathscr{S}^{[d]} \\
Proj\mathscr{S}(U) \arrow[u] \arrow[r, "\Phi_U"'] & Proj\mathscr{S}(U)^{[d]} \arrow[u]
\end{tikzcd}$$

It follows from (TPC,Proposition 9) that the bottom row is an isomorphism. This shows that Φ is locally an isomorphism, and is therefore an isomorphism. Naturality of this isomorphism means that for a morphism $\psi : \mathscr{S} \longrightarrow \mathscr{T}$ of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras the isomorphism $\mathbf{Proj}\mathscr{T} \cong \mathbf{Proj}\mathscr{T}^{[d]}$ identifies $G(\psi)$ and $G(\psi^{[d]})$ and the following diagram commutes

$$\begin{tikzcd}
G(\psi) \arrow[r, "\Psi"] \arrow[d, Rightarrow, no head] & \mathbf{Proj}\mathscr{S} \arrow[d, Rightarrow, no head] \\
G(\psi^{[d]}) \arrow[r, "\Psi^{[d]}"'] & \mathbf{Proj}\mathscr{S}^{[d]}
\end{tikzcd}$$

these claims follow directly from naturality of (TPC,Proposition 9). □

Definition 8. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{S}$ a sheaf of graded $\mathcal{O}_X$-algebras. For $e > 0$ let $\mathscr{S}|e$ denote the same sheaf of $\mathcal{O}_X$-algebras with the *inflated* grading

$$\mathscr{S}|e = \mathscr{S}_0 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathscr{S}_1 \oplus 0 \oplus \cdots$$

That is, $(\mathscr{S}|e)_0 = \mathscr{S}_0, (\mathscr{S}|e)_e = \mathscr{S}_1, (\mathscr{S}|e)_{2e} = \mathscr{S}_2$ and so on. This is clearly a sheaf of graded $\mathcal{O}_X$-algebras. If $\mathscr{S}$ is commutative or quasi-coherent then the same is true of $\mathscr{S}|e$. The important example is the equality $\mathscr{S}^{(d)}|d = \mathscr{S}^{[d]}$ of sheaves of graded $\mathcal{O}_X$-algebras. If $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras then the same morphism is also a morphism of sheaves of graded $\mathcal{O}_X$-algebras $\phi|e : \mathscr{S}|e \longrightarrow \mathscr{T}|e$.

Lemma 50. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. For any open affine $U \subseteq X$ and $e > 0$ we have an equality of graded $\mathcal{O}_X(U)$-algebras $(\mathscr{S}|e)(U) = \mathscr{S}(U)|e$.*

Lemma 51. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. For $e > 0$ there is an* equality $\mathbf{Proj}\mathscr{S} = \mathbf{Proj}\mathscr{S}|e$ *of schemes over X natural in $\mathscr{S}$.*

Proof. It follows from (PM,Section 2) and Lemma 50 that the schemes $\mathbf{Proj}\mathscr{S}, \mathbf{Proj}\mathscr{S}|e$ are constructed by gluing the same schemes over U for every affine open U. One checks the "patches" $\theta_{U,V}$ are also the same, and then it is clear that $\mathbf{Proj}\mathscr{S} = \mathbf{Proj}\mathscr{S}|e$ as schemes over X. This equality is natural in the sense that if $\phi : \mathscr{S} \longrightarrow \mathscr{T}$ is a morphism of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras then $G(\phi) = G(\phi|e)$ and the induced morphisms $G(\phi) \longrightarrow \mathbf{Proj}\mathscr{S}$ and $G(\phi|e) \longrightarrow \mathbf{Proj}\mathscr{S}|e$ are the same. □

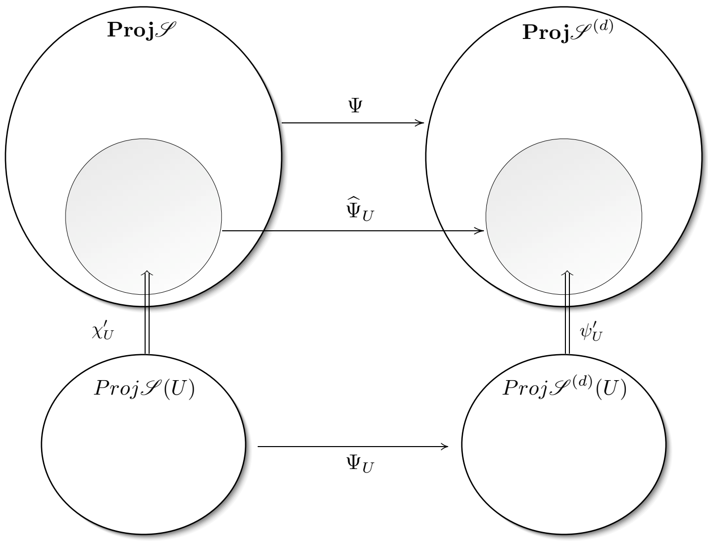
Corollary 52. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. For $d > 0$ there is a canonical isomorphism $\Psi : \mathbf{Proj}\mathscr{S} \longrightarrow \mathbf{Proj}\mathscr{S}^{(d)}$ of schemes over X natural in $\mathscr{S}$. If $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra then for $n \geq 1$ there is a canonical isomorphism of sheaves of modules on $\mathbf{Proj}\mathscr{S}$*

$$\begin{gathered}\zeta : \Psi^*\mathcal{O}(n) \longrightarrow \mathcal{O}(nd)\\ [W, a\dot{/}s] \dot{\otimes} b\dot{/}t \mapsto ab\dot{/}st\end{gathered}$$

Proof. Combine the isomorphism of Proposition 49 with the equality of Lemma 51. Naturality follows from the naturality of these two morphisms, and means that for a morphism $\theta : \mathscr{S} \longrightarrow \mathscr{T}$ of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras the isomorphism $\mathbf{Proj}\mathscr{T} \cong \mathbf{Proj}\mathscr{T}^{(d)}$ identifies $G(\theta)$ and $G(\theta^{(d)})$, while the following diagram commutes

$$\begin{tikzcd} G(\theta) \arrow[r, "\Theta"] \arrow[d, Rightarrow] & \mathbf{Proj}\mathscr{S} \arrow[d, Rightarrow] \\ G(\theta^{(d)}) \arrow[r, "\Theta^{(d)}"'] & \mathbf{Proj}\mathscr{S}^{(d)} \end{tikzcd}$$

Now assume that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. Let $U \subseteq X$ be an affine open subset with canonical open immersions $\chi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ and $\psi_U : Proj\mathscr{S}^{(d)}(U) \longrightarrow \mathbf{Proj}\mathscr{S}^{(d)}$ and induced isomorphism $\widehat{\Phi}_U : Im\chi_U \longrightarrow Im\psi_U$. These morphisms fit into the following diagram



There is an isomorphism of sheaves of modules on $Im\chi_U$

$$\begin{aligned}\zeta^U : (\Psi^*\mathcal{O}(n))|_{Im\chi_U} &\cong \widehat{\Psi}_U^*(\mathcal{O}(n)|_{Im\psi_U})\\ &\cong \widehat{\Psi}_U^*((\psi'_U)_*\mathcal{O}(n))\\ &\cong (\chi'_U)_*\Psi_U^*(\mathcal{O}(n))\\ &\cong (\chi'_U)_*\mathcal{O}(nd)\\ &\cong \mathcal{O}(nd)|_{Im\chi_U}\end{aligned}$$

Using (MRS,Proposition 111), (MRS,Proposition 108) and (PM,Corollary 13). Suppose $Q \subseteq Im\chi_U$ is open and that $b, t \in \mathscr{S}(U)$ are homogenous of the same degree with $b\dot{/}t \in \Gamma(\chi_U^{-1}Q, Proj\mathscr{T}(U))$. Identify this with a section of $\mathbf{Proj}\mathscr{S}$ in the usual way. Suppose $Im\psi_U \supseteq W \supseteq \Psi(Q)$ is open and

$a \in \mathscr{S}_{md+nd}(U), s \in \mathscr{S}_{md}(U)$ for some $m \geq 0$ with $a\dot{/}s \in \Gamma(\psi_U^{-1}W, \mathcal{O}(n))$. Identify this with a section of the sheaf $\mathcal{O}(n)$ on $\mathbf{Proj}\mathscr{S}^{(d)}$. Then

$$[W, a\dot{/}s] \dot{\otimes} b\dot{/}t \in \Gamma(Q, \Psi^*\mathcal{O}(n)) \tag{27}$$

$$\zeta_Q^U([W, a\dot{/}s] \dot{\otimes} b\dot{/}t) = ab\dot{/}st \tag{28}$$

Using the technique of Proposition 10 and Proposition 40 we check that $\zeta^U|_{Im\chi_W} = \zeta^W$ for affine open $W \subseteq U$ and therefore the isomorphism ζ^U glue to give an isomorphism $\zeta : \Psi^*\mathcal{O}(n) \longrightarrow \mathcal{O}(nd)$ unique with $\zeta|_{Im\chi_U} = \zeta^U$ for affine open $U \subseteq X$, which completes the proof. $\square$

Proposition 53. *Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. Let $\mathscr{S}'$ be the sheaf of graded $\mathcal{O}_X$-algebras defined by $\mathscr{S}'_0 = \mathcal{O}_X, \mathscr{S}'_d = \mathscr{S}_d$ for $d > 0$. Then the canonical morphism $\mathscr{S}' \longrightarrow \mathscr{S}$ induces an isomorphism of X-schemes $\mathbf{Proj}\mathscr{S} \longrightarrow \mathbf{Proj}\mathscr{S}'$ natural in $\mathscr{S}$.*

Proof. We make the coproduct of sheaves of $\mathcal{O}_X$-modules $\mathscr{S}' = \mathcal{O}_X \oplus \bigoplus_{d\geq 1} \mathscr{S}_d$ into a sheaf of graded $\mathcal{O}_X$-algebras in the canonical way (see our note on Constructing Sheaves of Graded Algebras). In fact $\mathscr{S}'$ is a commutative quasi-coherent (MOS,Proposition 25) sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra with $\mathscr{S}_0 = \mathcal{O}_X$. Let $\varphi : \mathscr{S}' \longrightarrow \mathscr{S}$ be the morphism of sheaves of $\mathcal{O}_X$-modules induced by $\mathcal{O}_X \longrightarrow \mathscr{S}$ and the inclusions $\mathscr{S}_d \longrightarrow \mathscr{S}$. This is a morphism of sheaves of graded $\mathcal{O}_X$-algebras. For an affine open subset $U \subseteq X$ the morphism $\varphi_U : \mathscr{S}'(U) \longrightarrow \mathscr{S}(U)$ is the morphism defined in (TPC,Proposition 11). Therefore $G(\varphi) \cap Im\psi_U = Im\psi_U$ and $G(\varphi) = \mathbf{Proj}\mathscr{S}$. For every affine open U we have a pullback diagram

$$\begin{tikzcd}
\mathbf{Proj}\mathscr{S} \arrow[r, "\Phi"] & \mathbf{Proj}\mathscr{S}' \\
Proj\mathscr{S}(U) \arrow[u] \arrow[r, "\Phi_U"'] & Proj\mathscr{S}(U)' \arrow[u]
\end{tikzcd}$$

Since Φ_U is an isomorphism for every affine open U, it follows that Φ is also an isomorphism.

Let $\psi : \mathscr{S} \longrightarrow \mathscr{T}$ be a morphism of commutative quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras, where $\mathscr{T}$ is also locally generated by $\mathscr{T}_1$ as an $\mathcal{O}_X$-algebra. There is an induced morphism of sheaves of graded $\mathcal{O}_X$-algebras $\psi' = 1 \oplus \bigoplus_{d\geq 1} \psi_d : \mathscr{S}' \longrightarrow \mathscr{T}'$. Naturality means that the isomorphism $\mathbf{Proj}\mathscr{T} \cong \mathbf{Proj}\mathscr{T}'$ identifies $G(\psi)$ and $G(\psi')$ and that the following diagram commutes

$$\begin{tikzcd}
G(\psi) \arrow[r, "\Psi"] \arrow[d, Rightarrow, no head] & \mathbf{Proj}\mathscr{S} \arrow[d, Rightarrow, no head] \\
G(\psi') \arrow[r, "\Psi'"'] & \mathbf{Proj}\mathscr{S}'
\end{tikzcd}$$

Both these claims follow from naturality of (TPC,Proposition 11), so the proof is complete. $\square$

Corollary 54. *Let X be a scheme and $\varphi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2. Let $\Phi : G(\varphi) \longrightarrow \mathbf{Proj}\mathscr{S}$ be the induced morphism of schemes. Then*

(i) If φ is a quasi-epimorphism then $G(\varphi) = \mathbf{Proj}\mathscr{T}$ and Φ is a closed immersion.

(ii) If φ is a quasi-isomorphism then Φ is an isomorphism.

Proof. Suppose that φ is a quasi-epimorphism. Then for every affine open subset $U \subseteq X$ the morphism of graded $\mathcal{O}_X(U)$-algebras $\varphi_U : \mathscr{S}(U) \longrightarrow \mathscr{T}(U)$ is a quasi-epimorphism (SOA,Proposition 49) so it follows from (TPC,Corollary 12)(i) that $G(\varphi_U) = Proj\mathscr{T}(U)$ and $Proj\mathscr{T}(U) \longrightarrow Proj\mathscr{S}(U)$ is a closed immersion. Therefore $G(\varphi) = \mathbf{Proj}\mathscr{T}$ and since (18) is a pullback for every affine open $U \subseteq X$ we see that Φ is a closed immersion. (ii) follows in the same way from (TPC,Corollary 12)(ii). $\square$

Proposition 55. *Let X be a scheme and $\varphi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras with $\mathscr{S}_0 = \mathcal{O}_X$ and $\mathscr{T}_0 = \mathcal{O}_X$. If φ is a quasi-isomorphism then there is an integer $E > 0$ such that for all $e \geq E$, $\varphi^{(e)} : \mathscr{S}^{(e)} \longrightarrow \mathscr{T}^{(e)}$ is an isomorphism of sheaves of graded $\mathcal{O}_X$-algebras.*

Proof. Using (MRS,Lemma 99) this is straightforward. $\square$

Corollary 56. *Let X be a scheme, $\mathscr{S}$ a sheaf of graded $\mathcal{O}_X$-algebras satisfying the conditions of Section 2 and $\mathscr{J}$ a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals. If $d \geq 0$ and $\mathscr{J}_{\geq d} = \bigoplus_{n \geq d} \mathscr{J}_n$ then there is a canonical isomorphism $\mathbf{Proj}(\mathscr{S}/\mathscr{J}) \longrightarrow \mathbf{Proj}(\mathscr{S}/\mathscr{J}_{\geq d})$ of schemes over $\mathbf{Proj}\mathscr{S}$.*

Proof. We are in the situation of Example 3. It is not hard to check that $\mathscr{J}_{\geq d}$ is a quasi-coherent sheaf of homogenous $\mathscr{S}$-ideals with $\mathscr{J}_{\geq d} \leq \mathscr{J}$. The inclusion $i : \mathscr{J}_{\geq d} \longrightarrow \mathscr{J}$ is a quasi-isomorphism of sheaves of graded $\mathscr{S}$-modules since $i\{d\}$ is the identity. Therefore $\mathscr{J}^{\sim} = \mathscr{J}_{\geq d}{}^{\sim}$ as quasi-coherent sheaves of ideals on $\mathbf{Proj}\mathscr{S}$ by Corollary 26. It now follows from Corollary 44 and (SI,Theorem 1) that $\mathbf{Proj}(\mathscr{S}/\mathscr{J}) \longrightarrow \mathbf{Proj}\mathscr{S}$ and $\mathbf{Proj}(\mathscr{S}/\mathscr{J}_{\geq d}) \longrightarrow \mathbf{Proj}\mathscr{S}$ determine the same closed subscheme of $\mathbf{Proj}\mathscr{S}$, so we have the desired isomorphism of schemes over $\mathbf{Proj}\mathscr{S}$. $\square$

Corollary 57. *Let X be a scheme. Then any projective morphism $Z \longrightarrow X$ is isomorphic as a scheme over X to the structural morphism $\mathbf{Proj}\mathscr{S} \longrightarrow X$ for some commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{S}$ locally finitely generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra with $\mathscr{S}_0 = \mathcal{O}_X$.*

Proof. If $Z \longrightarrow X$ is a projective morphism then by Proposition 45 it is X-isomorphic to $\mathbf{Proj}\mathscr{S}$ where $\mathscr{S} = \mathcal{O}_X[x_0, \ldots, x_n]/\mathscr{J}$ for some $n \geq 1$ and quasi-coherent sheaf of homogenous $\mathcal{O}_X[x_0, \ldots, x_n]$-ideals $\mathscr{J}$. By Corollary 56 we can replace $\mathscr{J}$ by $\mathscr{J}_{\geq 1}$ and therefore assume that $\mathscr{J}_0 = 0$. It follows that $\mathscr{S}$ is a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{S}$ locally finitely generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra with $\mathscr{S}_0 = \mathcal{O}_X$, as required. $\square$

7 Twisting With Invertible Sheaves

If $\mathscr{S}$ is a sheaf of graded $\mathcal{O}_X$-algebras, then there is another kind of "twisting" we can apply to $\mathscr{S}$, controlled by another sheaf of modules $\mathscr{L}$. As a sheaf of modules, this twisted algebra $\mathscr{S}_{(\mathscr{L})}$ is the coproduct

$$\mathscr{S}_{(\mathscr{L})} = \bigoplus_{d \geq 0} \mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$$

See (TSGA,Definition 1) for the definition and basic properties of this twisted algebra. The next result defines an analogue of the morphism of sheaves of graded algebras $\mathscr{S}^{(d)} \longrightarrow \mathscr{S}$ given above.

Proposition 58. *Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras and $\mathscr{L}$ an invertible sheaf. Let $U \subseteq X$ be an affine open subset such that $\mathscr{L}|_U$ is free. Then there is a canonical isomorphism $\Phi_U : Proj\mathscr{S}_{(\mathscr{L})}(U) \cong Proj\mathscr{S}(U)$ of schemes over U. If $V \subseteq U$ is another affine open set then the following diagram commutes*

$$\begin{tikzcd}
Proj\mathscr{S}_{(\mathscr{L})}(U) \arrow[r, Rightarrow, "\Phi_U"] & Proj\mathscr{S}(U) \\
Proj\mathscr{S}_{(\mathscr{L})}(V) \arrow[u] \arrow[r, Rightarrow, "\Phi_V"'] & Proj\mathscr{S}(V) \arrow[u]
\end{tikzcd} \qquad (29)$$

Proof. We know from (TSGA,Definition 1) that $\mathscr{S}_{(\mathscr{L})}$ is a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras, so the claim at least makes sense (see also (SOA,Proposition 40)). Let $\eta \in \mathscr{L}(U)$ be a $\mathcal{O}_X(U)$-basis, and define a morphism of graded $\mathcal{O}_X(U)$-algebras $\phi_U : \mathscr{S}(U) \longrightarrow \mathscr{S}_{(\mathscr{L})}(U)$ as follows

$$\phi_U(s) = \sum_{d \geq 0} s_d \dot{\otimes} \eta^{\otimes d} = s_0 \dot{\otimes} 1 + s_1 \dot{\otimes} \eta + s_2 \dot{\otimes} \eta^{\otimes 2} + \cdots$$

This is an isomorphism, since for every $d \geq 0$ the restriction to graded pieces $\mathscr{S}_d(U) \longrightarrow (\mathscr{S}_{(\mathscr{L})})_d(U)$ is given by $s \mapsto s \dot{\otimes} \eta^{\otimes d}$, which is also the action of the following isomorphism

$$\begin{aligned} \mathscr{S}_d|_U &\cong \mathscr{S}_d|_U \otimes (\mathcal{O}_X|_U)^{\otimes d} \\ &\cong \mathscr{S}_d|_U \otimes (\mathscr{L}|_U)^{\otimes d} \\ &\cong \mathscr{S}_d|_U \otimes (\mathscr{L}^{\otimes d})|_U \\ &\cong (\mathscr{S}_d \otimes \mathscr{L}^{\otimes d})|_U \end{aligned}$$

Therefore ϕ_U induces an isomorphism $\Phi_U : Proj\mathscr{S}_{(\mathscr{L})}(U) \longrightarrow Proj\mathscr{S}(U)$ of schemes over U. We have to show this isomorphism does not depend on the basis η chosen. If η' is another basis, then there is a unit $f \in \mathcal{O}_X(U)$ with $\eta' = f \cdot \eta$, in which case the morphism ϕ'_U defined using η' has the form $\phi'_U(s) = \sum_{d \geq 0} f^d \cdot (s_d \dot{\otimes} \eta^{\otimes d})$. It follows from (TPC,Lemma 8) that the induced morphisms of schemes are the same. To check commutativity of the diagram above, just note that if $V \subseteq U$ is affine then $\eta|_V$ is a $\mathcal{O}_X(V)$-basis for $\mathscr{L}(V)$. □

Proposition 59. *Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras and $\mathscr{L}$ an invertible sheaf. Let $\mathscr{S}_{(\mathscr{L})}$ be the twisted sheaf of graded $\mathcal{O}_X$-algebras. We claim there is a canonical isomorphism $\Phi : \mathbf{Proj}\mathscr{S}_{(\mathscr{L})} \cong \mathbf{Proj}\mathscr{S}$ of schemes over X, with the property that for any affine open $U \subseteq X$ with $\mathscr{L}|_U$ free, the following diagram commutes*

$$\begin{tikzcd} \mathbf{Proj}\mathscr{S}_{(\mathscr{L})} \arrow[r, Rightarrow, "\Phi"] & \mathbf{Proj}\mathscr{S} \\ Proj\mathscr{S}_{(\mathscr{L})}(U) \arrow[u] \arrow[r, Rightarrow, "\Phi_U"'] & Proj\mathscr{S}(U) \arrow[u] \end{tikzcd} \tag{30}$$

Proof. Consider the cover of X given by the collection of all affine open subsets $U \subseteq X$ such that $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. For each such U we have a morphism of schemes

$$Im\psi_U \cong Proj\mathscr{S}_{(\mathscr{L})}(U) \longrightarrow Proj\mathscr{S}(U) \longrightarrow Proj\mathscr{S}$$

It follows from commutativity of the diagram (29) that these morphisms can be glued, to give a morphism of schemes $\Phi : \mathbf{Proj}\mathscr{S}_{(\mathscr{L})} \longrightarrow \mathbf{Proj}\mathscr{S}$ over X making (30) commute. To show Φ is an isomorphism, it is enough to show that (30) is a pullback for every U, and this is straightforward. □

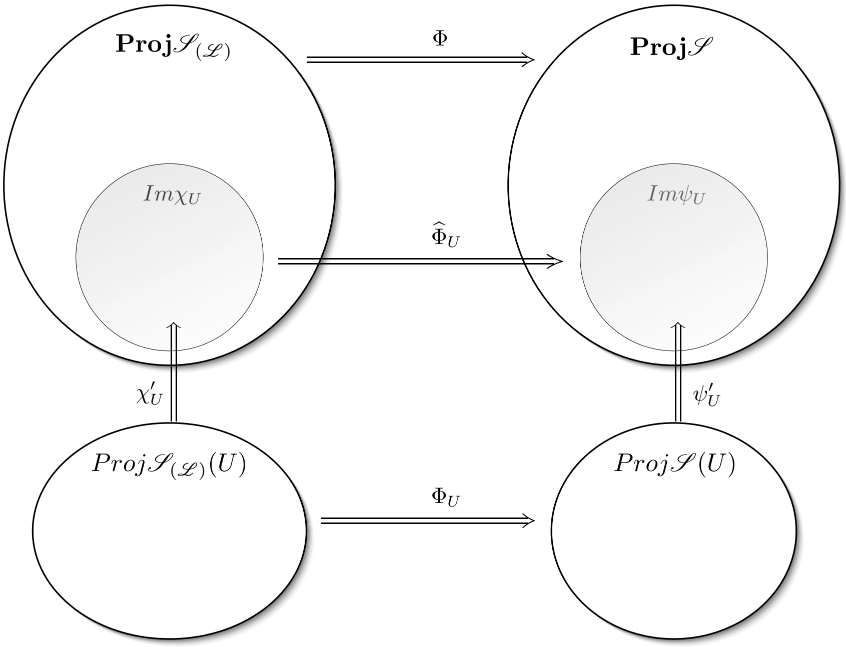
Lemma 60. *Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra, and $\mathscr{L}$ an invertible sheaf. Then $\mathscr{S}_{(\mathscr{L})}$ is also locally generated by $(\mathscr{S}_{(\mathscr{L})})_1$ as an $(\mathscr{S}_{(\mathscr{L})})_0$-algebra.*

Proof. For affine open $U \subseteq X$ we use (MOS,Lemma 7) and (MOS,Lemma 40) to see that for $d > 0$ every element of $(\mathscr{S}_d \otimes \mathscr{L}^{\otimes d})(U)$ is a sum of elements of the form $s \dot{\otimes} (l_1 \dot{\otimes} \cdots \dot{\otimes} l_d)$ for $s \in \mathscr{S}_d(U), l_i \in \mathscr{L}(U)$. Then it follows from the proof of (SOA,Proposition 40) that $\mathscr{S}_{(\mathscr{L})}(U)$ is generated as an abelian group by elements of the form $s \dot{\otimes} r$ for $s \in \mathscr{S}_0(U), r \in \mathcal{O}_X(U)$ and $s \dot{\otimes} (l_1 \dot{\otimes} \cdots \dot{\otimes} l_h)$ for $h \geq 1, s \in \mathscr{S}_h(U)$ and $l_i \in \mathscr{L}(U)$. But by assumption $\mathscr{S}(U)$ is generated by $\mathscr{S}_1(U)$ as an $\mathscr{S}_0(U)$-algebra, so the definition of multiplication in $\mathscr{S}_{(\mathscr{L})}(U)$ makes it clear that $\mathscr{S}_{(\mathscr{L})}(U)$ is generated by $(\mathscr{S}_{(\mathscr{L})})_1(U)$ as an $(\mathscr{S}_{(\mathscr{L})})_0(U)$-algebra. □

Proposition 61. *Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra, and $\mathscr{L}$ an invertible sheaf. Let $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ and $\omega : \mathbf{Proj}\mathscr{S}_{(\mathscr{L})} \longrightarrow X$ be the structural morphisms and $\Phi : \mathbf{Proj}\mathscr{S}_{(\mathscr{L})} \longrightarrow \mathbf{Proj}\mathscr{S}$ the canonical isomorphism. Then there is a canonical isomorphism of sheaves of modules on* $\mathbf{Proj}\mathscr{S}_{(\mathscr{L})}$

$$\mathcal{O}(1) \cong \Phi^*\mathcal{O}(1) \otimes \omega^*\mathscr{L}$$

Proof. Let $\mathfrak{U}$ be the collection of affine open subsets $U \subseteq X$ with $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. For each such U let $\psi_U : Proj\mathscr{S}(U) \longrightarrow \mathbf{Proj}\mathscr{S}$ and $\chi_U : Proj\mathscr{S}_{(\mathscr{L})}(U) \longrightarrow \mathbf{Proj}\mathscr{S}_{(\mathscr{L})}$ be the canonical open immersions, and let $\widehat{\Phi}_U : Im\chi_U \longrightarrow Im\psi_U$ be the isomorphism of schemes induced by Φ, as in the following diagram



Let $\mu_U : \mathcal{O}(1)|_{Im\psi_U} \longrightarrow (\psi'_U)_*\mathcal{O}(1)$ and $\mu'_U : \mathcal{O}(1)_{Im\chi_U} \longrightarrow (\chi'_U)_*\mathcal{O}(1)$ be the isomorphisms of Proposition 7. Pick an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$ and let η be the corresponding $\mathcal{O}_X(U)$-basis of $\mathscr{L}(U)$. Then there is an isomorphism of sheaves of modules on $Im\chi_U$

$$\begin{aligned}
\vartheta^{U,\eta} : (\Phi^*\mathcal{O}(1))|_{Im\chi_U} &\cong \widehat{\Phi}_U^*(\mathcal{O}(1)|_{Im\psi_U}) \\
&\cong \widehat{\Phi}_U^*((\psi'_U)_*\mathcal{O}(1)) \\
&\cong (\chi'_U)_*\Phi_U^*\mathcal{O}(1) \\
&\cong (\chi'_U)_*\mathcal{O}(1) \\
&\cong \mathcal{O}(1)|_{Im\chi_U}
\end{aligned}$$

Using (MRS,Proposition 111), the isomorphism μ_U, (MRS,Proposition 108), (H, 5.12c) and finally μ'_U. Suppose $Q \subseteq Im\chi_U$ is open and that $b, t \in \mathscr{S}_{(\mathscr{L})}(U)$ are homogenous of the same degree with $\chi_U^{-1}Q \subseteq D_+(t)$. Denote by $b\dot{/}t$ the element of $\Gamma(Q, \mathbf{Proj}\mathscr{S}_{(\mathscr{L})})$ corresponding to the section $b\dot{/}t$ of $Proj\mathscr{S}_{(\mathscr{L})}(U)$. Suppose $a, s \in \mathscr{S}(U)$ are homogenous, with a of degree $d+1$, s of degree d, and $a\dot{/}s \in \Gamma(\psi_U^{-1}\Phi(Q), \mathcal{O}(1))$. Denote also by $a\dot{/}s$ the section of $\mathbf{Proj}\mathscr{S}$'s twisting sheaf $\mathcal{O}(1)$ mapping to $a\dot{/}s$ under $(\mu_U)_{\Phi(Q)}$. Then we have

$$[\Phi(Q), a\dot{/}s] \dot{\otimes} b\dot{/}t \in \Gamma(Q, \Phi^*\mathcal{O}(1)) \tag{31}$$

$$\vartheta_Q^{U,\eta}\left([\Phi(Q), a\dot{/}s] \dot{\otimes} b\dot{/}t\right) = \phi_U(a)b\dot{/}\phi_U(s)t = (a \dot{\otimes} \eta^{\otimes(d+1)})b\dot{/}(s \dot{\otimes} \eta^{\otimes d})t \tag{32}$$

Using (MRS,Proposition 111), (MRS,Proposition 108) and (MPS,Proposition 13). To use this last result, we need the hypothesis that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra. The section on the right is the section of $\mathbf{Proj}\mathscr{S}_{(\mathscr{L})}$'s twisting sheaf corresponding to $\phi_U(a)b\dot{/}\phi_U(s)t \in \Gamma(\chi_U^{-1}Q, \mathcal{O}(1))$ via μ'_U. The morphism $\phi_U : \mathscr{S}(U) \longrightarrow \mathscr{S}_{(\mathscr{L})}(U)$ is defined relative to the basis η of $\mathscr{L}(U)$.

The second isomorphism of interest is

$$\begin{aligned}\varrho^{U,\eta} : (\omega^* \mathscr{L})|_{Im\chi_U} &\cong \omega_U^*(\mathscr{L}|_U) \\ &\cong \omega_U^*(\mathcal{O}_X|_U) \\ &\cong \mathcal{O}_{\mathbf{Proj}\mathscr{S}_{(\mathscr{L})}}|_{Im\chi_U}\end{aligned}$$

Where $\omega_U : Im\chi_U \longrightarrow U$ is induced by ω and we use (MRS,Proposition 111), the basis η and (MRS,Proposition 109). Suppose $Q \subseteq Im\xi_U$ as before, $\dot{c/q} \in \Gamma(Q, \mathbf{Proj}\mathscr{S}_{(\mathscr{L})})$ for homogenous $c, q \in \mathscr{S}_{(\mathscr{L})}(U)$ in the same sense as above, and suppose T is open with $\omega(Q) \subseteq T \subseteq U$ and say $x, y \in \mathcal{O}_X(U)$. Then we have

$$[U, r \cdot \eta] \dot{\otimes} \dot{c/q} \in \Gamma(Q, \omega^* \mathscr{L}) \tag{33}$$

$$\varrho^{U,\eta}\left([T, \dot{x/y} \cdot \eta|_T] \dot{\otimes} \dot{c/q}\right) = \dot{xc/yq} \tag{34}$$

Combining these we have an isomorphism of sheaves of modules on $Im\chi_U$

$$\begin{aligned}\Lambda^{U,\eta} : (\Phi^* \mathcal{O}(1) \otimes \omega^* \mathscr{L})\,|_{Im\chi_U} &\cong (\Phi^* \mathcal{O}(1))|_{Im\chi_U} \otimes (\omega^* \mathscr{L})|_{Im\chi_U} \\ &\cong \mathcal{O}(1)|_{Im\chi_U} \otimes \mathcal{O}_{\mathbf{Proj}\mathscr{S}_{(\mathscr{L})}}|_{Im\chi_U} \\ &\cong \mathcal{O}(1)|_{Im\chi_U}\end{aligned}$$

We claim that $\Lambda^{U,\eta} = \Lambda^{U,\xi}$ for any other basis ξ of $\mathscr{L}(U)$ corresponding to another isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. We can reduce to checking they agree on sections which are tensor products of sections in (31) and (33). In this case

$$\Lambda_Q^{U,\eta}\left(([\Phi(Q), \dot{a/s}] \dot{\otimes} \dot{b/t}) \dot{\otimes} ([T, \dot{x/y} \cdot \eta|_T] \dot{\otimes} \dot{c/q})\right) = \dot{xc(a \dot{\otimes} \eta^{\otimes(d+1)})b/yq(s \dot{\otimes} \eta^{\otimes d})t}$$

If $\eta = u \cdot \xi$ for a unit $u \in \mathcal{O}_X(U)$ then $\dot{x/y} \cdot \eta|_T = \dot{ux/y} \cdot \xi|_T$ and

$$\Lambda_Q^{U,\xi}\left(([\Phi(Q), \dot{a/s}] \dot{\otimes} \dot{b/t}) \dot{\otimes} ([T, \dot{ux/y} \cdot \xi|_T] \dot{\otimes} \dot{c/q})\right) = \dot{uxc(a \dot{\otimes} \xi^{\otimes(d+1)})b/yq(s \dot{\otimes} \xi^{\otimes d})t}$$

Replace the single u in the numerator by u^{d+1}, and insert u^d in the denominator. We can then move these inside the terms $a \dot{\otimes} \xi^{\otimes(d+1)}$ and use $u^{d+1}\xi^{\otimes(d+1)} = \eta^{\otimes(d+1)}$ to show $\Lambda_Q^{U,\eta}$ and $\Lambda_Q^{U,\xi}$ agree on this section. Therefore $\Lambda^{U,\eta} = \Lambda^{U,\xi}$ is independent of the chosen basis and we denote it by Λ^U.

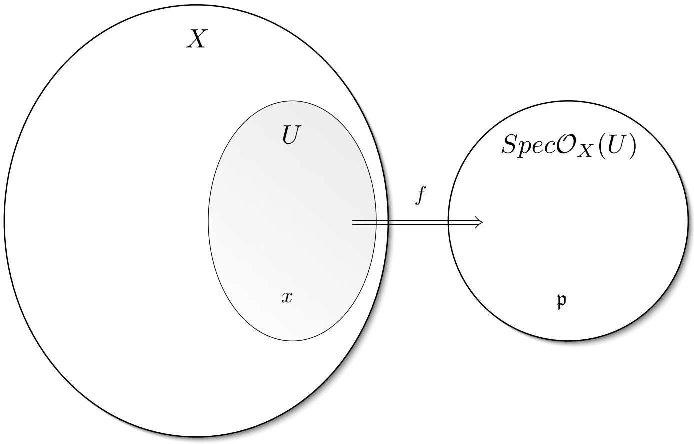
The open sets $Im\chi_U$ for $U \in \mathfrak{U}$ form an open cover of $\mathbf{Proj}\mathscr{S}_{(\mathscr{L})}$ and to complete the proof we need only show that the isomorphisms Λ^U can be glued. It suffices to show that $\Lambda^U|_{Im\chi_W} = \Lambda^W$ for affine open $W \subseteq U$. If η is a basis for $\mathscr{L}(U)$ corresponding to an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$ then $\eta|_W$ is such a basis for $\mathscr{L}(W)$, so we need to show that $\Lambda^{U,\eta}|_{Im\chi_W} = \Lambda^{W,\eta|_W}$, which is not difficult to check by reduction to the case of the special sections above (one must use the open immersion $Proj\mathscr{S}_{(\mathscr{L})}(W) \longrightarrow Proj\mathscr{S}_{(\mathscr{L})}(U)$, compatibility μ_U with the morphism $\varphi_{U,W}$ of Proposition 7 and be careful about where all the special sections live). Therefore there is a canonical isomorphism of sheaves of modules

$$\Lambda : \Phi^* \mathcal{O}(1) \otimes \omega^* \mathscr{L} \longrightarrow \mathcal{O}(1)$$

unique with the property that for $U \in \mathfrak{U}$ we have $\Lambda|_{Im\chi_U} = \Lambda^U$. $\square$

Lemma 62. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. If $U \subseteq X$ is affine and $x \in U$ corresponds to $\mathfrak{p} \in Spec\mathcal{O}_X(U)$ then there is an isomorphism of abelian groups $\mathscr{F}_x \cong \mathscr{F}(U)_\mathfrak{p}$ compatible with the ring isomorphism $\mathcal{O}_{X,x} \cong \mathcal{O}_X(U)_\mathfrak{p}$.*

Proof. We are in the situation of the following diagram



There is a canonical ring isomorphism $\mathcal{O}_X(U)_\mathfrak{p} \cong \mathcal{O}_{X,x}$ given by mapping a/s to $(X_s, a|_{X_s} s^{-1})$. Similarly the isomorphism $\mathscr{F}(U)_\mathfrak{p} \cong \mathscr{F}_x$ maps m/s to $(X_s, s^{-1} \cdot m|_{X_s})$. It is clear that this is an isomorphism of abelian groups compatible with the given ring isomorphism. $\square$

Lemma 63. *Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf generated by global sections $\{s_i\}_{i\in I}$. Then $\mathscr{F}$ can be generated by a finite subset of the s_i.*

Proof. By assumption there is an epimorphism $\phi : \bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathscr{F}$ where the ith component is the morphism $\phi_i : \mathcal{O}_X \longrightarrow \mathscr{F}$ corresponding to s_i. Since X is noetherian, we can find a finite affine open cover $U_1, \ldots, U_n$ of X. For each $1 \le j \le n$ we have an epimorphism $\bigoplus_{i\in I} \mathcal{O}_X|_{U_j} \longrightarrow \mathscr{F}|_{U_j}$ and therefore an epimorphism of $\mathcal{O}_X(U_j)$-modules

$$\phi' : \bigoplus_{i\in I} \mathcal{O}_X(U_j) \longrightarrow \mathscr{F}(U_j)$$

which shows that $\mathscr{F}(U_j)$ is generated as an $\mathcal{O}_X(U_j)$-module by the $s_i|_{U_j}$. But $\mathcal{O}_X(U_j)$ is noetherian and $\mathscr{F}(U_j)$ is finitely generated, therefore noetherian, so we can generate $\mathscr{F}(U_j)$ with a finite subset of the $s_i|_{U_j}$. Taking the corresponding global sections for $1 \le j \le n$ gives a finite generating set for $\mathscr{F}$ by Lemma 62. $\square$

Proposition 64. *Let X be a noetherian scheme with ample invertible sheaf $\mathscr{L}$, and let $\mathscr{S}$ be a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally finitely generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra with $\mathscr{S}_0 = \mathcal{O}_X$. Then $\pi : \mathbf{Proj}\mathscr{S} \longrightarrow X$ is projective and $\mathcal{O}(1) \otimes \pi^*(\mathscr{L}^{\otimes n})$ is a very ample invertible sheaf on $\mathbf{Proj}\mathscr{S}$ for some $n > 0$.*

Proof. It follows from the hypothesis that for every affine open $U \subseteq X$, $\mathscr{S}_1(U)$ is a finitely generated $\mathcal{O}_X(U)$-module, and therefore $\mathscr{S}_1$ is coherent and for some $n > 0$, $\mathscr{S}_1 \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. By Lemma 63 we can assume the generating set $s_0, \ldots, s_N$ is finite with $N \ge 1$. Let $\mathscr{E}$ be the invertible sheaf $\mathscr{L}^{\otimes n}$ and identify $\mathscr{S}_1 \otimes \mathscr{E}$ with $(\mathscr{S}_{(\mathscr{E})})_1$. Then the sections s_i induce a morphism of sheaves of graded $\mathcal{O}_X$-algebras $\phi : \mathcal{O}_X[x_0, \ldots, x_N] \longrightarrow \mathscr{S}_{(\mathscr{E})}$ with $\phi_X(x_i) = s_i$ (SSA,Corollary 38). By construction for every affine open subset $U \subseteq X$ the sections $s_i|_U$ generate $(\mathscr{S}_{(\mathscr{E})})_1(U)$ as a $\mathcal{O}_X(U)$-module (use the argument of Lemma 63) and therefore by (SSA,Proposition 45) and the assumptions on $\mathscr{S}$ the morphism ϕ_U is surjective. Therefore ϕ is an epimorphism of sheaves of modules (MOS,Lemma 2), and the induced morphism $\Phi : \mathbf{Proj}\mathscr{S}_{(\mathscr{E})} \longrightarrow \mathbf{Proj}\mathcal{O}_X[x_0, \ldots, x_N]$ is a closed immersion of schemes over X by Proposition 39. But by Proposition 10 the latter scheme is $\mathbb{P}^N_X$ and therefore by Proposition 41 and Proposition 10 the invertible sheaf $\mathcal{O}(1)$ on $\mathbf{Proj}\mathscr{S}_{(\mathscr{E})}$ is very ample relative to X.

Therefore if $\Psi : \mathbf{Proj}\mathscr{S}_{(\mathscr{E})} \longrightarrow \mathbf{Proj}\mathscr{S}$ is the canonical isomorphism of schemes over X defined in Proposition 59 then the sheaf of modules $\Phi_*\mathcal{O}(1)$ is very ample relative to X. Then using Proposition 61 we have an isomorphism of sheaves of modules

$$\begin{aligned}\Phi_*\mathcal{O}(1) &\cong \Phi_*(\Phi^*\mathcal{O}(1)\otimes\omega^*\mathscr{E})\\ &\cong (\Phi_*\Phi^*\mathcal{O}(1))\otimes(\Phi_*\omega^*\mathscr{E})\\ &\cong \mathcal{O}(1)\otimes(\omega\Phi^{-1})^*\mathscr{E}\\ &= \mathcal{O}(1)\otimes\pi^*(\mathscr{E})\end{aligned}$$

and therefore $\mathcal{O}(1)\otimes\pi^*(\mathscr{L}^{\otimes n})$ is very ample relative to X, as required. By Corollary 3 the morphism π is proper, and therefore projective, since we showed in our Section 2.5 notes that a proper morphism into a noetherian scheme is projective if and only if it admits a very ample sheaf. ☐

8 Projective Space Bundles

Definition 9. Let X be a noetherian scheme, and let $\mathscr{E}$ be a locally free coherent sheaf. We define the associated *projective space bundle* $\mathbb{P}(\mathscr{E})$ to be $\mathbf{Proj}\mathbb{S}(\mathscr{E})$, which comes with a separated morphism $\pi:\mathbb{P}(\mathscr{E})\longrightarrow X$ and twisting sheaf $\mathcal{O}(1)$.

The next result shows that $\mathbb{P}(\mathscr{E})$ is locally a projective space over an affine scheme.

Lemma 65. *Let X be a noetherian scheme and $\mathscr{E}$ a locally free coherent sheaf. If $\mathscr{E}|_U$ is free of rank $n+1$ for a nonempty affine open set $U\subseteq X$ and $n\geq 0$ then there is a pullback diagram*

$$\begin{tikzcd} \mathbb{P}^n_U \arrow[r] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd}$$

In particular there is an isomorphism of schemes $\pi^{-1}U\cong\mathbb{P}^n_U$ over U for every affine open subset $U\subseteq X$.

Proof. By $\mathbb{P}^n_U$ we mean $\mathbb{P}^n_{\mathcal{O}_X(U)} = Proj\mathcal{O}_X(U)[x_0,\dots,x_n]$ together with the canonical morphisms $\mathbb{P}^n_{\mathcal{O}_X(U)}\longrightarrow\mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathcal{O}_X(U)}\longrightarrow Spec\mathcal{O}_X(U)\cong U$. By construction of $\mathbb{P}(\mathscr{E})$ we have the following pullback

$$\begin{tikzcd} Proj\mathbb{S}(\mathscr{E})(U) \arrow[r] \arrow[d, "\pi_U"'] & \mathbb{P}(\mathscr{E}) \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd}$$

So to complete the proof it suffices to show that $\mathbb{S}(\mathscr{E})(U)\cong\mathcal{O}_X(U)[x_0,\dots,x_n]$ as graded $\mathcal{O}_X(U)$-algebras, and this follows from (SSA,Lemma 47), (SSA,Proposition 45) and (SSA,Proposition 13). ☐

Let us now summarise our results with all the strongest hypotheses. Let X be a noetherian scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras locally finitely generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra with $\mathscr{S}_0=\mathcal{O}_X$. Then

- There is a noetherian scheme $\mathbf{Proj}\mathscr{S}$ together with a proper morphism $\pi:\mathbf{Proj}\mathscr{S}\longrightarrow X$ and a canonical twisting sheaf $\mathcal{O}(1)$.
- If X admits an ample invertible sheaf $\mathscr{L}$ then π is projective and $\mathcal{O}(1)\otimes\pi^*(\mathscr{L}^{\otimes n})$ is a very ample invertible sheaf on $\mathbf{Proj}\mathscr{S}$ relative to X.