The Proj Construction

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1 Basic Properties

Definition 1. Let A be a ring. A *graded A-algebra* is an A-algebra R which is also a graded ring in such a way that if $r \in R_d$ then $ar \in R_d$ for all $a \in A$. That is, R_d is an A-submodule of R for all $d \geq 0$. Equivalently a graded A-algebra is a morphism of graded rings $A \longrightarrow R$ where we grade A by setting $A_0 = A, A_n = 0$ for $n > 0$. A *morphism* of graded A-algebras is a morphism of A-algebras which preserves grade. Equivalently, this is a morphism of graded rings which is also a morphism of A-modules, or a morphism of graded rings $R \longrightarrow S$ making the following diagram commute

$$\begin{tikzcd} R \arrow[rr] & & S \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Let S be a graded A-algebra, and let $X = ProjS$. Then there is a canonical morphism of rings $\theta : A \longrightarrow \Gamma(X, ProjS)$ defined by

$$\theta(a)(\mathfrak{p}) = a/1 \in S_{(\mathfrak{p})}$$

Where as usual, a denotes the element $a \cdot 1$ of S. By Ex 2.4 this induces a morphism of schemes $f : ProjS \longrightarrow SpecA$. If $\gamma : A \longrightarrow S$ gives the A-algebra structure on S, then an element $a \in A$ is mapped to the maximal ideal of $\mathcal{O}_{ProjS,\mathfrak{p}}$ via $A \longrightarrow \Gamma(X, ProjS) \longrightarrow \mathcal{O}_{ProjS,\mathfrak{p}}$ if and only if $a \in \gamma^{-1}\mathfrak{p}$. We denote this prime ideal by $A \cap \mathfrak{p}$. Let $\kappa_{\mathfrak{p}} : A_{f(\mathfrak{p})} \longrightarrow S_{(\mathfrak{p})}$ be defined by $\kappa_{\mathfrak{p}}(a/s) = a/s$. Then

$$f : ProjS \longrightarrow SpecA$$
$$f(\mathfrak{p}) = \mathfrak{p} \cap A$$
$$f_U^{\#}(t)(\mathfrak{p}) = \kappa_{\mathfrak{p}}(t(f(\mathfrak{p})))$$

So the projective space over any graded A-algebra is a scheme over A. In particular there is a canonical morphism of schemes $\mathbb{P}^n_A \longrightarrow SpecA$. Notice that if $e \in S_+$ is homogenous then the composite of the open immersion $Spec(S_{(e)}) \cong D_+(e) \longrightarrow ProjS$ with f is the morphism of schemes corresponding to the canonical map $A \longrightarrow S_{(e)}$ defined by $a \mapsto a/1$.

Proposition 1. *If S is a graded A-algebra then the morphism $f : ProjS \longrightarrow SpecA$ is separated. In particular $ProjS$ is separated for any graded ring S.*

Proof. The open sets $D_+(a)$ for homogenous $a \in S_+$ are an open cover of $ProjS$. Therefore the open sets $D_+(a) \times_A D_+(b)$ are an open cover of $ProjS \times_A ProjS$, and the inverse image of this open set under the diagonal $ProjS \longrightarrow ProjS \times_A ProjS$ is clearly $D_+(a) \cap D_+(b) = D_+(ab)$. Since the property of being a closed immersion is local on the base, we reduce to showing that the morphism $D_+(ab) \longrightarrow D_+(a) \times_A D_+(b)$ is a closed immersion. Since we know that this gives the canonical morphisms $D_+(ab) \longrightarrow D_+(a), D_+(ab) \longrightarrow D_+(b)$ when composed with the projections, the corresponding morphism of A-algebras is the morphism $\alpha : S_{(a)} \otimes_A S_{(b)} \longrightarrow S_{(ab)}$ induced by the canonical morphisms $S_{(a)} \longrightarrow S_{(ab)}, S_{(b)} \longrightarrow S_{(ab)}$. Since α is clearly surjective, the proof is complete. □

Proposition 2. *If S is a noetherian graded ring then $ProjS$ is a noetherian scheme.*

Proof. For any homogenous $f \in S_+$ the ring $S_{(f)}$ is noetherian (GRM,Corollary 15). Since S is noetherian the ideal S_+ admits a finite set of homogenous generators $f_1, \dots, f_p$ and the schemes $D_+(f_i) \cong SpecS_{(f_i)}$ cover $ProjS$. Therefore $ProjS$ is noetherian. □

Proposition 3. *If S is a finitely generated graded A-algebra, then the scheme $ProjS$ is of finite type over $SpecA$.*

Proof. The hypothesis imply that S_0 is a finitely generated A-algebra, and that S is a finitely generated S_0-algebra. Therefore S_+ is a finitely generated ideal (GRM,Corollary 8). So using the argument of Proposition 2 we reduce to showing that for homogenous $f \in S_+$, $S_{(f)}$ is a finitely generated A-algebra. By (GRM,Proposition 14) it suffices to show that $S^{(d)}$ is a finitely generated A-algebra, which follows from (GRM,Lemma 11). □

Proposition 4. *If S is a graded domain with $S_+ \neq 0$ then $ProjS$ is an integral scheme.*

Proof. By (3.1) it is enough to show that $ProjS$ is reduced and irreducible. By assumption the zero ideal 0 is a homogenous prime ideal, so $ProjS$ has a generic point and is therefore irreducible. The scheme $ProjS$ is covered by open subsets isomorphic to schemes $Spec(S_{(f)})$ for nonzero homogenous $f \in S_+$, and it is clear that $S_{(f)}$ is an integral domain. This shows that $ProjS$ is reduced and therefore integral. □

Corollary 5. *Let A be a ring and consider the scheme $\mathbb{P}^n_A$ for $n \geq 1$. Then*

(i) The morphism $\mathbb{P}^n_A \longrightarrow SpecA$ is separated and of finite type.

(ii) If A is noetherian then so is $\mathbb{P}^n_A$.

(iii) If A is an integral domain, then $\mathbb{P}^n_A$ is an integral scheme.

2 Functorial Properties

Let $\varphi : S \longrightarrow T$ be a morphism of graded rings, $G(\varphi)$ the open subset $\{\mathfrak{p} \in ProjT \,|\, \mathfrak{p} \not\supseteq \varphi(S_+)\}$ and induce the following morphism of schemes using Ex 2.14:

$$\Phi : G(\varphi) \longrightarrow ProjS$$
$$\Phi(\mathfrak{p}) = \varphi^{-1}\mathfrak{p}$$
$$\Phi^{\#}_V(s)(\mathfrak{p}) = \varphi_{(\mathfrak{p})}(s(\varphi^{-1}\mathfrak{p}))$$

Here $\varphi_{(\mathfrak{p})} : S_{(\varphi^{-1}\mathfrak{p})} \longrightarrow T_{(\mathfrak{p})}$ is defined by $a/s \mapsto \varphi(a)/\varphi(s)$. If φ is an isomorphism of graded rings with inverse ψ then clearly $G(\varphi) = ProjT$, $G(\psi) = ProjS$ and one checks that if $\Psi : ProjS \longrightarrow ProjT$ is induced by ψ then $\Phi\Psi = 1$ and $\Psi\Phi = 1$ so that $ProjS \cong ProjT$. More generally, let $\varphi : S \longrightarrow T$ and $\psi : T \longrightarrow Z$ be morphisms of graded rings. Then $G(\psi\varphi) \subseteq G(\psi)$. Let $\Phi : G(\varphi) \longrightarrow ProjS$ and $\Psi : G(\psi) \longrightarrow ProjT$ be induced as above, and let Ω be induced by the composite $\psi\varphi$. Then $\Psi G(\psi\varphi) \subseteq G(\varphi)$ and it is readily checked that the following diagram of schemes commutes:

$$\begin{tikzcd} ProjS & & G(\psi\varphi) \arrow[ll, "\Omega"'] \arrow[dl, "\Psi|_{G(\psi\varphi)}"] \\ & G(\varphi) \arrow[ul, "\Phi"] & \end{tikzcd}$$

If $\varphi : S \longrightarrow T$ is a morphism of graded A-algebras for some ring A, then $\Phi : G(\varphi) \longrightarrow ProjS$ is a morphism of schemes over A, where $G(\varphi), ProjS$ are A-schemes in the canonical way. Also if $\varphi : S \longrightarrow S$ is the identity, clearly $G(\varphi) = ProjS$ and Φ is the identity.

For any graded ring S and homogenous $f \in S_+$ there is an isomorphism of schemes $Spec(S_{(f)}) \cong D_+(f)$. We claim that this isomorphism is natural with respect to morphisms of graded rings, in the following sense.

Lemma 6. *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings and $\Phi : G(\varphi) \longrightarrow ProjS$ the induced morphism of schemes. If $f \in S_+$ is homogenous then $\Phi^{-1}D_+(f) = D_+(\varphi(f))$ and the following diagram commutes:*

$$\begin{tikzcd} Spec(T_{(\varphi(f))}) \arrow[r] & Spec(S_{(f)}) \\ D_+(\varphi(f)) \arrow[r] \arrow[u, Rightarrow] & D_+(f) \arrow[u, Rightarrow] \end{tikzcd} \tag{1}$$

where the top morphism corresponds to $\varphi_{(f)} : S_{(f)} \longrightarrow T_{(\varphi(f))}$ defined by $s/f^n \mapsto \varphi(s)/\varphi(f)^n$, and the bottom morphism is the restriction of Φ.

Proof. By (H, *Ex*.2.4) it suffices to check that the diagram commutes on global sections of $Spec(S_{(f)})$, which follows immediately from the definition of Φ and the explicit form of the vertical isomorphisms given in our Section 2.2 notes. $\square$

Lemma 7. *Let S be a graded ring and $f, g \in S_+$ homogenous elements. Then the following diagram commutes*

$$\begin{tikzcd} Spec(S_{(f)}) \arrow[r, Rightarrow] & D_+(f) \\ Spec(S_{(fg)}) \arrow[r, Rightarrow] \arrow[u] & D_+(fg) \arrow[u] \end{tikzcd}$$

where the vertical morphism on the left corresponds to the ring morphism $S_{(f)} \longrightarrow S_{(fg)}$ defined by $s/f^n \mapsto sg^n/(fg)^n$.

Proof. By (H, *Ex*.2.4) it suffices to check that the diagram commutes on global sections of $Spec(S_{(f)})$. But this is easy to check, using the explicit form of the isomorphisms given in our Section 2.2 notes. $\square$

Let H be a graded ring and suppose $f \in H_0$. Then $\varphi_f(h) = \sum_{d \geq 0} f^d h_d$ defines a morphism of graded rings $\varphi_f : H \longrightarrow H$. If f is a unit with inverse g, then φ_f is an isomorphism with inverse φ_g. If f is a unit, then it is easy to see that for a homogenous prime ideal $\mathfrak{p}$, $\varphi_f^{-1}\mathfrak{p} = \mathfrak{p}$ and the induced morphism $(\varphi_f)_{(\mathfrak{p})} : H_{(\mathfrak{p})} \longrightarrow H_{(\mathfrak{p})}$ is the identity. Therefore φ_f induces the identity morphism $ProjH \longrightarrow ProjH$. This simple observation has the following important consequence

Lemma 8. *Let $\varphi : S \longrightarrow T$ be a morphism of graded rings and suppose $f \in T_0$ is a unit. Let $\varphi_f : S \longrightarrow T$ be the following morphism of graded rings*

$$\varphi_f(s) = \sum_{d \geq 0} f^d \varphi(s_d)$$

Then $G(\varphi) = G(\varphi_f)$ and the induced morphisms $G(\varphi) \longrightarrow ProjS$ are the same.

Proposition 9. *Let S be a graded ring and $e > 0$. The morphism of graded rings $\varphi : S^{[e]} \longrightarrow S$ induces an isomorphism of schemes $\Phi : ProjS \longrightarrow ProjS^{[e]}$ natural in S.*

Proof. We defined the graded ring $S^{[e]}$ in (GRM,Definition 6). The inclusion $\varphi : S^{[e]} \longrightarrow S$ is a morphism of graded rings. It is clear that $G(\varphi) = ProjS$, so we get a morphism of schemes $\Phi : ProjS \longrightarrow ProjS^{[e]}$. If $\mathfrak{a}$ is a homogenous ideal of S then $\mathfrak{a} \cap S^{[e]}$ is homogenous ideal of $S^{[e]}$, and it is not hard to see that for a homogenous prime ideal $\mathfrak{p}$ we have $\mathfrak{p} \supseteq \mathfrak{a}$ if and only if $\mathfrak{p} \cap S^{[e]} \supseteq \mathfrak{a} \cap S^{[e]}$. In particular this shows that Φ is injective.

To see that it is surjective, let $\mathfrak{q}'$ be a homogenous prime ideal of $S^{[e]}$. Then the ideal $\mathfrak{q}$ generated by $\mathfrak{q}'$ in S is homogenous and it is not difficult to check that $\mathfrak{q} \cap S^{[e]} = \mathfrak{q}'$. We claim that the homogenous ideal $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is prime. If $a \in S_m$ and $b \in S_n$ are such that $ab \in \mathfrak{p}$ then $(ab)^{ke} \in \mathfrak{q}'$ for some $k > 0$ and therefore either $a^{ke} \in \mathfrak{q}'$ or $b^{ke} \in \mathfrak{q}'$, which shows that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Clearly $\mathfrak{p} \cap S^{[e]} = \mathfrak{q}'$ and if $\mathfrak{q}' \not\supseteq S_+^{[e]}$ then $\mathfrak{p} \not\supseteq S_+$, which shows that Φ is surjective. Our arguments in the previous paragraph also imply that $\Phi(V(\mathfrak{a})) = V(\mathfrak{a} \cap S^{[e]})$ for any homogenous ideal $\mathfrak{a}$, so Φ is in fact a homeomorphism.

For homogenous $f \in S_+$ we have $D_+(f) = D_+(f^e)$ and it is clear that for $f \in S_+^{[e]}$ we have $\Phi(D_+(f)) = D_+(f)$. The induced morphism of rings $S_{(f)}^{[e]} \longrightarrow S_{(f)}$ is the morphism $\varphi_{(f)}$ which is clearly an isomorphism. This proves that $\Phi : ProjS \longrightarrow ProjS^{[e]}$ is an isomorphism.

This isomorphism is natural in S in the following sense. Suppose $\psi : S \longrightarrow T$ is a morphism of graded rings, $\psi^{[e]} : S^{[e]} \longrightarrow T^{[e]}$ the induced morphism of graded rings. It is not hard to check that the isomorphism $ProjT \cong ProjT^{[e]}$ identifies the open subsets $G(\psi)$ and $G(\psi^{[e]})$ and that the following diagram commutes

$$\begin{array}{ccc} G(\psi) & \xrightarrow{\Psi} & ProjS \\ \Downarrow & & \Downarrow \\ G(\psi^{[e]}) & \xrightarrow[\Psi^{[e]}]{} & ProjS^{[e]} \end{array}$$

which completes the proof. ∎

Definition 2. Let S be a graded ring and $e > 0$. Let $S|e$ denote the graded ring with the same ring structure as S, but with the *inflated* grading

$$S|e = S_0 \oplus 0 \oplus \cdots \oplus 0 \oplus S_1 \oplus 0 \oplus \cdots$$

so $(S|e)_0 = S_0, (S|e)_e = S_1, (S|e)_{2e} = S_2$ and so on. It is not hard to see that an ideal $\mathfrak{a} \subseteq S$ is homogenous in S iff. it is homogenous in $S|e$, so as topological spaces we have $ProjS = Proj(S|e)$. Moreover for a homogenous prime ideal $\mathfrak{p}$ we have an *equality* of rings $S_{(\mathfrak{p})} = (S|e)_{(\mathfrak{p})}$, and therefore the sheaves of rings are also the same. Hence we have an equality of schemes $ProjS = Proj(S|e)$.

Example 1. For a graded ring S and $e > 0$ it is clear that $S^{(e)}|e = S^{[e]}$. Therefore as schemes there is an equality $ProjS^{(e)} = ProjS^{[e]}$.

Corollary 10. *Let S be a graded ring and $e > 0$. There is a canonical isomorphism of schemes $\Psi : ProjS \longrightarrow ProjS^{(e)}$ natural in S.*

Proof. The isomorphism is the equality $ProjS^{(e)} = ProjS^{[e]}$ followed by the isomorphism of Proposition 9. Naturality in S means that for a morphism of graded rings $\theta : S \longrightarrow T$ with induced morphism of graded rings $\theta^{(e)} : S^{(e)} \longrightarrow T^{(e)}$ the isomorphism $ProjS \cong ProjS^{(e)}$ identifies $G(\theta)$ and $G(\theta^{(e)})$ and the following diagram commutes

$$\begin{tikzcd} G(\theta) \arrow[r, "\Theta"] \arrow[d, Rightarrow] & ProjS \arrow[d, Rightarrow] \\ G(\theta^{(e)}) \arrow[r, "\Theta^{(e)}"'] & ProjS^{(e)} \end{tikzcd}$$

This follows easily from the naturality of Proposition 9. □

Proposition 11. *Let S be a graded A-algebra generated by S_1 as an A-algebra. Let S' be the graded A-algebra defined by $S'_0 = A$, $S'_d = S_d$ for $d > 0$. Then the canonical morphism of graded A-algebras $S' \longrightarrow S$ induces an isomorphism of A-schemes $ProjS' \longrightarrow ProjS$ natural in S.*

Proof. We make the direct sum $S' = A \oplus \bigoplus_{d \geq 1} S_d$ into a graded A-algebra in the canonical way (see our note on Constructing Graded Algebras). Let $\varphi : S' \longrightarrow S$ be induced by $A \longrightarrow S$ and all the inclusions $S_d \longrightarrow A$. This is a morphism of graded A-algebras. It is clear that $G(\varphi) = ProjS$, so let $\Phi : ProjS \longrightarrow ProjS'$ be the induced morphism of schemes. It follows from (GRM,Lemma 17) that Φ is injective and from (GRM,Proposition 18), (GRM,Lemma 17) that Φ is surjective. For a homogenous ideal $\mathfrak{a}$ of S we have $\Phi(V(\mathfrak{a})) = V(\varphi^{-1}\mathfrak{a})$ (GRM,Lemma 16) so Φ is a homeomorphism. To show that Φ is an isomorphism of schemes it suffices by Lemma 1 to show that for $d > 0$ and $f \in S_d$ the following ring morphism is bijective

$$\varphi_{(f)} : S'_{(f)} \longrightarrow S_{(f)}$$
$$a/f^n \mapsto a/f^n$$

This map is clearly surjective. To see that it is injective, suppose a/f^n maps to zero in $S_{(f)}$. If $n > 0$ then $a \in S_{nd}$ and $f^k a = 0$ for some $k \geq 0$ implies that $a/f^n = 0$ in $S'_{(f)}$. If $n = 0$ and $a \in A$ then we can write $a/1 = af/f$ and reduce to the case where $n > 0$. This shows that $\varphi_{(f)}$ is an isomorphism and completes the proof that Φ is an isomorphism of schemes.

To prove naturality, let $\psi : S \longrightarrow T$ be a morphism of graded A-algebras, where T is also generated by T_1 as an A-algebra. There is an induced morphism of graded A-algebras $\psi' = 1 \oplus \bigoplus_{d \geq 1} \psi_d : S' \longrightarrow T'$ making the following diagram commute

$$\begin{tikzcd} S \arrow[r, "\psi"] & T \\ S' \arrow[u] \arrow[r, "\psi'"'] & T' \arrow[u] \end{tikzcd}$$

We claim that the isomorphisms $\Phi_S : ProjS \longrightarrow ProjS'$ and $\Phi_T : ProjT \longrightarrow ProjT'$ are natural in the sense that Φ_T identifies the open subsets $G(\psi)$ and $G(\psi')$ and the following diagram commutes

$$\begin{tikzcd} G(\psi) \arrow[r, "\Psi"] \arrow[d, Rightarrow] & ProjS \arrow[d, Rightarrow, "\Phi_S"] \\ G(\psi') \arrow[r, "\Psi'"'] & ProjS' \end{tikzcd}$$

The details are easily checked. □

Definition 3. Let $\varphi : S \longrightarrow T$ be a morphism of graded rings. We say that φ is a *quasi-monomorphism*, *quasi-epimorphism* or *quasi-isomorphism* if it has this property as a morphism of graded S-modules (GRM,Definition 10).

Corollary 12. *Let $\varphi : S \longrightarrow T$ be a morphism of graded A-algebras where S is generated by S_1 as an A-algebra and T is generated by T_1 as an A-algebra. If $\Phi : G(\varphi) \longrightarrow ProjS$ is the induced morphism of schemes then*

(i) If φ is a quasi-epimorphism then $G(\varphi) = ProjT$ and Φ is a closed immersion.

(ii) If further φ is a quasi-isomorphism then Φ is an isomorphism.

Proof. Using Proposition 11 we can assume that $S_0 = A$ and $T_0 = A$. If φ is a quasi-epimorphism (resp. quasi-isomorphism) then for some $e > 0$ the morphism $\varphi^{(e)} : S^{(e)} \longrightarrow T^{(e)}$ is surjective (resp. bijective) so by Corollary 10 we can reduce to proving that if φ is surjective then Φ is a closed immersion, and further that if φ is an isomorphism then so is Φ. But we proved the former claim in our solution to (H,Ex.3.12) and the latter claim is trivial. $\square$

Remark 1. Note that if S is a graded A-algebra and S' denotes the graded A-algebra $S'_0 = A, S'_d = S_d$ for $d > 0$ defined in Proposition 11 then the canonical morphism of graded A-algebras $\varphi : S' \longrightarrow S$ is trivially a quasi-isomorphism.

Proposition 13. *Let $\varphi : S \longrightarrow T$ be a quasi-isomorphism of graded A-algebras with $S_0 = A$ and $T_0 = A$. Then there is an integer $E > 0$ such that for all $e \geq E$, $\varphi^{(e)} : S^{(e)} \longrightarrow T^{(e)}$ is an isomorphism of graded A-algebras.*

Proof. Let $E > 0$ be large enough that $\varphi_n : S_n \longrightarrow T_n$ is bijective for all $n \geq E$. Then it is not hard to see that $\varphi^{(e)} : S^{(e)} \longrightarrow T^{(e)}$ is an isomorphism for $e \geq E$. $\square$

3 Products

Lemma 14. *Let A, B be rings, S a graded A-algebra, and T a graded B-algebra. Let $\psi : A \longrightarrow B$ be a morphism of rings and $\varphi : S \longrightarrow T$ a morphism of graded rings making the following diagram commute:*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & B \arrow[d] \\ S \arrow[r, "\varphi"'] & T \end{tikzcd} \tag{2}$$

Then the following diagram commutes, where $\Psi = Spec(\psi)$ and the vertical morphisms are canonical:

$$\begin{tikzcd} G(\varphi) \arrow[r, "\Phi"] \arrow[d] & ProjS \arrow[d] \\ SpecB \arrow[r, "\Psi"'] & SpecA \end{tikzcd} \tag{3}$$

Proof. The open sets $D_+(f) \subseteq ProjS$ for homogenous $f \in S_+$ are an open cover, so the open sets $D_+(\varphi(f)) = \Phi^{-1}D_+(f)$ for homogenous $f \in S_+$ must be an open cover of $G(\varphi)$. So it suffices to show that the two legs of the diagram agree when composed with $Spec(T_{(\varphi(f))}) \longrightarrow G(\varphi)$ for every homogenous $f \in S_+$. Using the commutativity of (1), this reduces to checking commutativity of the following diagram of rings, where the vertical morphisms are canonical:

$$\begin{tikzcd} T_{(\varphi(f))} & S_{(f)} \arrow[l, "\varphi_{(f)}"'] \\ B \arrow[u] & A \arrow[l, "\varphi"] \arrow[u] \end{tikzcd} \tag{4}$$

But this square is trivially commutative, completing the proof. Alternatively, use the fact that morphisms into $SpecA$ are in bijection with ring morphisms out of A and just check the diagram on global sections. $\square$

Proposition 15. *Let A, B be rings, S a graded A-algebra, $\psi : A \longrightarrow B$ a morphism of rings. Then $T = S \otimes_A B$ is a graded B-algebra and $ProjT = ProjS \times_A SpecB$.*

Proof. Consider A, B as graded rings in the canonical way. Then S and B become graded A-modules, so the tensor product $S \otimes_A B$ becomes a graded B-algebra (GRM,Section 6) with grading

$$(S \otimes_A B)_n = \left\{ \sum_i s_i \otimes b_i \,|\, s_i \in S_n, b_i \in B \right\}$$

The map $\varphi : S \longrightarrow T$ defined by $\varphi(s) = s \otimes 1$ is a morphism of graded rings, which makes the diagram (2) commute. Since T_+ is generated as a B-module by $\varphi(S_+)$ it follows that $G(\varphi) = ProjT$ and we get a morphism of schemes $\Phi : ProjT \longrightarrow ProjS$ fitting into a commutative diagram

$$\begin{tikzcd} ProjT \arrow[r, "\Phi"] \arrow[d] & ProjS \arrow[d] \\ SpecB \arrow[r, "\Psi"'] & SpecA \end{tikzcd} \tag{5}$$

We have to show that this diagram is a product of schemes over $SpecA$. The open sets $D_+(f)$ for homogenous $f \in S_+$ cover $ProjS$ and by our earlier notes on the local nature of products, it suffices to show that $D_+(\varphi(f)) = \Phi^{-1} D_+(f)$ is a product for $D_+(f)$ and $SpecB$ over $SpecA$ for all homogenous $f \in S_+$. Using commutativity of (1) we reduce this to showing that the following diagram is a pullback:

$$\begin{tikzcd} Spec(T_{(\varphi(f))}) \arrow[r] \arrow[d] & Spec(S_{(f)}) \arrow[d] \\ Spec(B) \arrow[r] & Spec(A) \end{tikzcd}$$

Which is equivalent to showing that (4) is a pushout of rings. The map $S_f \times B \longrightarrow T_{\varphi(f)}$ defined by $(s/f^n, b) \mapsto (s \otimes b)/\varphi(f)^n$ is well-defined and A-bilinear, so induces $S_f \otimes_A B \longrightarrow T_{\varphi(f)}$, which is easily checked to be a morphism of rings. The canonical map $S \longrightarrow S_f$ is a morphism of A-modules, so there is a well-defined morphism $T = S \otimes_A B \longrightarrow S_f \otimes_A B$ defined by $s \otimes b \mapsto s/1 \otimes b$. This is a morphism of rings mapping multiples of $\varphi(f) = f \otimes 1$ to units, so it induces a morphism of rings $T_{\varphi(f)} \longrightarrow S_f \otimes_A B$ defined by $(s \otimes b)/\varphi(f)^n \mapsto s/f^n \otimes b$. Since we have already constructed the inverse, this is an isomorphism of rings $T_{\varphi(f)} \cong S_f \otimes_A B$.

The ring S_f is a $\mathbb{Z}$-graded ring, and thus can be considered as a graded A-module. The ring $S_f \otimes_A B$ is therefore also $\mathbb{Z}$-graded. Putting the canonical $\mathbb{Z}$-grading on $T_{\varphi(f)}$ it is easily checked that $T_{\varphi(f)} \cong S_f \otimes_A B$ is an isomorphism of $\mathbb{Z}$-graded rings, which restricts to give an isomorphism of the degree zero subrings $T_{(\varphi(f))} \cong (S_f \otimes_A B)_0$. We have to show that this latter ring is isomorphic as a ring to $S_{(f)} \otimes_A B$.

Let $\alpha : S_f \otimes_{\mathbb{Z}} B \longrightarrow S_f \otimes_A B$ be canonical (GRM,Section 6). The kernel of α is the abelian group P' generated by elements $(a \cdot x) \otimes b - x \otimes (a \cdot b)$ where x is homogenous. The morphism of groups $S_{(f)} \otimes_{\mathbb{Z}} B \longrightarrow S_f \otimes_{\mathbb{Z}} B$ is injective since $S_{(f)}$ is a direct summand of S_f and tensor products preserve colimits. Therefore the group $S_{(f)} \otimes_{\mathbb{Z}} B$ is isomorphic to its image in $S_f \otimes_{\mathbb{Z}} B$, which is mapped by α onto $(S_f \otimes_A B)_0$. So there is an isomorphism of abelian groups $(S_{(f)} \otimes_{\mathbb{Z}} B)/P'' \cong (S_f \otimes_A B)_0$ where $P'' = P' \cap (S_{(f)} \otimes_{\mathbb{Z}} B)$. But since $S_{(f)} \otimes_{\mathbb{Z}} B$ can be identified with the degree zero subring of $S_f \otimes_{\mathbb{Z}} B$, P'' is generated as an abelian group by elements $(a \cdot x) \otimes b - x \otimes (a \cdot b)$ where x is homogenous of degree zero, that is, $x \in S_{(f)}$. Hence there is an isomorphism of abelian groups

$$S_{(f)} \otimes_A B \cong (S_{(f)} \otimes_{\mathbb{Z}} B)/P'' \cong (S_f \otimes_A B)_0$$
$$s/f^n \otimes b \mapsto s/f^n \otimes b$$

So finally we have an isomorphism of rings $T_{(\varphi(f))} \cong S_{(f)} \otimes_A B$ defined by $(s \otimes b)/\varphi(f)^n \mapsto s/f^n \otimes b$. Using this isomorphism one checks easily that (4) is a pushout, completing the proof. ∎

Let $\psi : A \longrightarrow B$ be a morphism of rings ($n \geq 0$) and $\varphi : A[x_0, \dots, x_n] \longrightarrow B[x_0, \dots, x_n]$ the morphism of graded rings induced by ψ. Then it is not difficult to check that the following diagram is a pushout of rings:

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d] & B \arrow[d] \\
A[x_0, \dots, x_n] \arrow[r, "\varphi"'] & B[x_0, \dots, x_n]
\end{tikzcd}$$

By uniqueness of the pushout, there is an isomorphism of rings

$$\begin{aligned}
B[x_0, \dots, x_n] &\longrightarrow B \otimes_A A[x_0, \dots, x_n] \\
f(x_0, \dots, x_n) &\mapsto f(1 \otimes x_0, \dots, 1 \otimes x_n) \\
\sum_\alpha f(\alpha) x_0^{\alpha_0} \cdots x_n^{\alpha_n} &\mapsto \sum_\alpha f(\alpha) \otimes x_0^{\alpha_0} \cdots x_n^{\alpha_n}
\end{aligned}$$

This is the morphism of B-algebras induced by $x_i \mapsto 1 \otimes x_i$. Similarly $x_i \mapsto x_i \otimes 1$ induces an isomorphism of rings $\theta : B[x_0, \dots, x_n] \longrightarrow A[x_0, \dots, x_n] \otimes_A B$. If we give the tensor product the canonical graded structure, these are isomorphisms of graded rings. Note that $\theta\varphi$ is just the canonical map of $A[x_0, \dots, x_n]$ into the tensor product. Now let $n \geq 1$ and $T = A[x_0, \dots, x_n] \otimes_A B$. Then $G(\varphi) = \mathbb{P}^n_B$ and $G(\theta) = G(\theta\varphi) = ProjT$. Let $\Omega : ProjT \longrightarrow \mathbb{P}^n_A$ be induced by $\theta\varphi$. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{P}^n_B \arrow[drr, bend left, "\Phi"] \arrow[ddr] & & \\
& ProjT \arrow[ul, Rightarrow, "\Theta"'] \arrow[r, "\Omega"] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\
& SpecB \arrow[r, "\Psi"'] & SpecA
\end{tikzcd} \tag{6}$$

All unmarked morphisms are the canonical ones, and by the previous Proposition the square is a pullback. The top triangle commutes by the results of the previous section. To check commutativity of the left triangle, we need only check the two legs give the same morphism on global sections (morphisms into $SpecB$ are in bijection with ring morphisms out of B). An element $b \in B$ determines the global section $\mathfrak{q} \mapsto b/1$ of $\mathbb{P}^n_B$ and $\mathfrak{p} \mapsto (1 \otimes b)/1$ of $ProjT$. These clearly correspond under the localisations of θ, so (6) commutes. Hence the outside square is a pullback, and we have proved the following result.

Proposition 16. *Let $\psi : A \longrightarrow B$ be a morphism of rings and for $n \geq 1$ let $\varphi : A[x_0, \dots, x_n] \longrightarrow B[x_0, \dots, x_n]$ be the corresponding morphism of graded rings. Then the following diagram is a pullback:*

$$\begin{tikzcd}
\mathbb{P}^n_B \arrow[r, "\Phi"] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\
SpecB \arrow[r, "\Psi"'] & SpecA
\end{tikzcd}$$

In other words, $\mathbb{P}^n_B = SpecB \times_{SpecA} \mathbb{P}^n_A$. In particular, for any ring A we have $\mathbb{P}^n_A = SpecA \times \mathbb{P}^n_{\mathbb{Z}}$.

One could probably run all the proofs with $n = 0$, but since in that case $\mathbb{P}^n_A \longrightarrow SpecA$ and $\mathbb{P}^n_B \longrightarrow SpecB$ are isomorphisms, the diagram in the proposition is trivially a pullback.

4 Linear Morphisms

Let A be a ring $n, m \geq 0$ and $f_0(x_0, \dots, x_n), \dots, f_m(x_0, \dots, x_n) \in A[x_0, \dots, x_n]$ homogenous polynomials of degree 1. Then we define

$$\begin{aligned} \phi : A[y_0, \dots, y_m] &\longrightarrow A[x_0, \dots, x_n] \\ \phi(y_i) &= f_i(x_0, \dots, x_n) \end{aligned}$$

This is a morphism of graded A-algebras, hence induces a morphism of schemes $\Phi : G(\phi) \longrightarrow \mathbb{P}^m_A$. Here

$$\begin{aligned} G(\phi) &= \{\mathfrak{p} \in \mathbb{P}^n_A \,|\, \mathfrak{p} \not\supseteq \phi(S_+)\} \\ &= \{\mathfrak{p} \in \mathbb{P}^n_A \,|\, \mathfrak{p} \not\supseteq \{f_0, \dots, f_m\}\} \\ &= \bigcup_{i=0}^{m} D_+(f_i) \end{aligned}$$

For example, suppose $m \geq 1$ and $0 \leq i \leq m$ and let $\phi_i : A[x_0, \dots, x_m] \longrightarrow A[x_0, \dots, x_{m-1}]$ be the morphism of graded A-algebras determined by the following assignments: $x_0 \mapsto x_0, \dots, x_{i-1} \mapsto x_{i-1}, x_{i+1} \mapsto x_i, \dots, x_m \mapsto x_{m-1}$ and $x_i \mapsto 0$. Then clearly $G(\phi_i) = \mathbb{P}^{m-1}_A$ and so we have a morphism of schemes

$$\Phi_i : \mathbb{P}^{m-1}_A \longrightarrow \mathbb{P}^m_A$$

The morphism ϕ_i is surjective, so by Ex 3.12 the morphism Φ_i is a closed immersion whose image is the closed set $V(Ker\phi) = V((x_i)) = \mathbb{P}^m_A - D_+(x_i)$. So for $m \geq 1$ there are $m+1$ closed immersions of $\mathbb{P}^{m-1}_A$ in $\mathbb{P}^m_A$. Notice that the following diagram also commutes for any $0 \leq i \leq m$:

$$\begin{tikzcd} \mathbb{P}^{m-1}_A \arrow[rr, "\Phi_i"] \arrow[dr] & & \mathbb{P}^m_A \arrow[dl] \\ & SpecA & \end{tikzcd}$$

In particular there are two closed immersions $SpecA \cong \mathbb{P}^0_A \longrightarrow \mathbb{P}^1_A$. Also note that for $n \geq 1$ and $0 \leq i \leq n$ the open set $D_+(x_i)$ of $\mathbb{P}^n_A$ is isomorphic to $SpecA[x_0, \dots, x_n]_{(x_i)}$ and thus to $SpecA[x_1, \dots, x_n]$. In fact these are isomorphisms of schemes over $SpecA$.

Another example are automorphisms arising from invertible matrices. Fix $n \geq 1$ and let $A = (a_{ij})$ be an invertible $(n+1) \times (n+1)$ matrix over a field k. For convenience we use the indices $0 \leq i, j \leq n$. Then the map $x_i \mapsto \sum a_{ij} x_j$ determines a k-algebra automorphism

$$\varphi_A : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$$

which gives rise to an automorphism of schemes over k, $\Phi_A = Proj\varphi_A : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$.

5 Projective Morphisms

Definition 4. Let Y be a scheme and $n \geq 0$. A *projective n-space* over Y is a pullback $Y \times_{Spec\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$. This consists of two morphisms $Z \longrightarrow Y, Z \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ making the following diagram a pullback:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbb{P}^n_{\mathbb{Z}} \arrow[r] & Spec\mathbb{Z} \end{tikzcd}$$

Two projective n-spaces over Y are canonically isomorphic as schemes over Y and $\mathbb{P}^n_{\mathbb{Z}}$, and we denote any projective n-space over Y by $\mathbb{P}^n_Y$. A morphism $X \longrightarrow Y$ is *projective* if it factors as a closed immersion $X \longrightarrow \mathbb{P}^n_Y$ followed by the projection $\mathbb{P}^n_Y \longrightarrow Y$ for some $n \geq 1$ and projective n-space $\mathbb{P}^n_Y$. This definition is independent of the projective n-space chosen, in the sense that if it factors through one projection via a closed immersion, then it factors through all of them by a closed immersion. We refer to this situation by saying that $X \longrightarrow Y$ *factors via a closed immersion through projective n-space.*

For a ring A and $n \geq 0$ the scheme $ProjA[x_0, \ldots, x_n]$ is a projective n-space over $SpecA$ with the canonical morphisms, so the notation $\mathbb{P}^n_A$ is unambiguous. Since the morphism $\mathbb{P}^0_{\mathbb{Z}} \longrightarrow Spec\mathbb{Z}$ is an isomorphism, projective 0-spaces over Y correspond to isomorphisms $Z \longrightarrow Y$, in the sense that if $Z \longrightarrow Y, Z \longrightarrow \mathbb{P}^0_{\mathbb{Z}}$ is a projective 0-space then $Z \longrightarrow Y$ is an isomorphism, and any isomorphism $Z \longrightarrow Y$ can be paired with $Z \longrightarrow Y \longrightarrow Spec\mathbb{Z} \longrightarrow \mathbb{P}^0_{\mathbb{Z}}$ to make a projective 0-space. For any ring A and $n \geq 1$ there are $n+1$ closed immersions $\mathbb{P}^{n-1}_A \longrightarrow \mathbb{P}^n_A$ of schemes over $SpecA$. We can extend this to projective space over any scheme:

Lemma 17. *For any scheme Y there are $n+1$ canonical closed immersions $\mathbb{P}^{n-1}_Y \longrightarrow \mathbb{P}^n_Y$ of schemes over Y for any $n \geq 1$.*

Proof. Let $\mathbb{P}^{n-1}_Y$ and $\mathbb{P}^n_Y$ be any projective $n-1$ (resp. n)-spaces over Y. We induce a morphism $\mathbb{P}^{n-1}_Y \longrightarrow \mathbb{P}^n_Y$ into the pullback so that the following diagram commutes:

$$\begin{tikzcd}
& \mathbb{P}^{n-1}_{\mathbb{Z}} \arrow[ddl] \arrow[dr] & & & \mathbb{P}^{n-1}_Y \arrow[lll] \arrow[dr] \arrow[ddl] & \\
& & Spec\mathbb{Z} & & & Y \arrow[lll] \\
\mathbb{P}^n_{\mathbb{Z}} \arrow[urr] & & & \mathbb{P}^n_Y \arrow[lll] \arrow[urr] & &
\end{tikzcd}$$

Where for $0 \leq i \leq n$ we let $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ be the ith closed immersion. Using the usual pullback argument, we see that $\mathbb{P}^{n-1}_Y \longrightarrow \mathbb{P}^n_Y$ is the pullback of $\mathbb{P}^{n-1}_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$, and since closed immersions are stable under pullback the proof is complete. □

Lemma 18. *If for some $n \geq 0$ a morphism $X \longrightarrow Y$ factors via a closed immersion through projective n-space, then it factors via a closed immersion through projective m-space for any $m \geq n$.*

The lemma implies that we could just as well define a projective morphism $X \longrightarrow Y$ to be a morphism factoring via a closed immersion through a projection $\mathbb{P}^n_Y \longrightarrow Y$ for some $n \geq 0$, since if $X \longrightarrow Y$ factors via a closed immersion through projective 0-space, then it factors through projective 1-space by a closed immersion, and is thus projective. So our final definition is:

Definition 5. A morphsim $X \longrightarrow Y$ is *projective* if it factors via a closed immersion through projective n-space over Y for some $n \geq 1$ (equivalently, some $n \geq 0$). A morphism $X \longrightarrow Y$ is *quasi-projective* if it factors via an immersion through projective n-space for some $n \geq 1$ (equivalently, some $n \geq 0$).

Notice that a morphism $X \longrightarrow Y$ factors through projective 0-space over Y iff. it is a closed immersion. In particular any closed immersion is projective.

Lemma 19. *Let A be a ring and S a graded A-algebra finitely generated as an A-algebra by S_1. Then the structural morphism $ProjS \longrightarrow SpecA$ is projective.*

Proof. By hypothesis we can find a surjective morphism of graded A-algebras $A[x_0, \ldots, x_n] \longrightarrow S$ with $n \geq 1$, so the induced morphism $ProjS \longrightarrow \mathbb{P}^n_A$ is a closed immersion of schemes over A, as required. □

Let $f : X \longrightarrow Y$ be a morphism of schemes, fix an integer $n \geq 1$ and pullbacks $\mathbb{P}^n_X, \mathbb{P}^n_Y$. Then the morphisms $\mathbb{P}^n_X \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_X \longrightarrow X \longrightarrow Y$ induce a morphism $\mathbb{P}^n_f : \mathbb{P}^n_X \longrightarrow \mathbb{P}^n_Y$ which is the unique morphism of schemes over $\mathbb{P}^n_{\mathbb{Z}}$ fitting into the following commutative diagram

$$\begin{tikzcd} \mathbb{P}^n_X \arrow[r, "\mathbb{P}^n_f"] \arrow[d] & \mathbb{P}^n_Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In fact using standard properties of pullbacks it is not hard to see that this is a pullback diagram. Clearly $\mathbb{P}^n_1 = 1$ and if we have a morphism $g : Y \longrightarrow Z$ and fix a pullback $\mathbb{P}^n_Z$ then $\mathbb{P}^n_g \mathbb{P}^n_f = \mathbb{P}^n_{gf}$. If $X = SpecA, Y = SpecB$ and f is induced by a ring morphism $\psi : A \longrightarrow B$, and if we choose $\mathbb{P}^n_A, \mathbb{P}^n_B$ as our pullbacks, then $\mathbb{P}^n_f$ is the morphism induced by the canonical ring morphism $A[x_0, \ldots, x_n] \longrightarrow B[x_0, \ldots, x_n]$ as in Proposition 16.

6 Dimensions of Schemes

We want to translate some results about varieties into results about schemes. To be specific, recall the following (proved in $I, 3.4$):

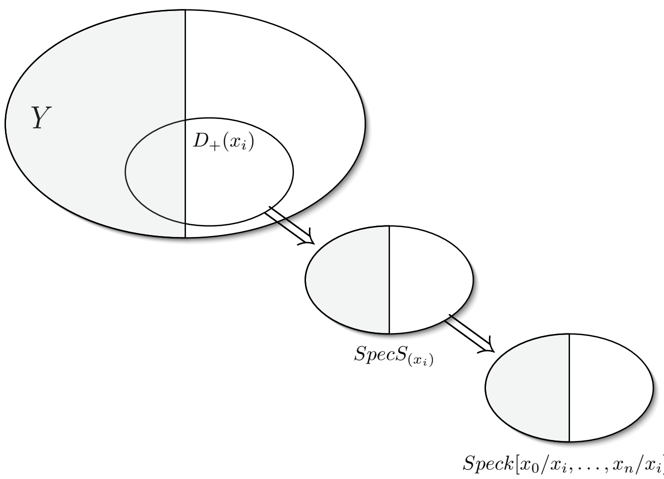
Proposition 20. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety and $0 \leq i \leq n$ such that $Y_i = Y \cap U_i \neq \emptyset$, where $U_i \cong \mathbb{A}^n$ is the open set $x_i \neq 0$. Then Y_i corresponds to an affine variety in $\mathbb{A}^n$ and $S(Y)_{(x_i)} \cong A(Y_i)$, where $S(Y) = k[x_0, \ldots, x_n]/I(Y)$ and $A(Y_i) = k[y_1, \ldots, y_n]/I(Y_i)$.*

Proof. In outline: let $S = k[x_0, \ldots, x_n]$. Then $k[y_1, \ldots, y_n] \cong S_{(x_i)}$ and this isomorphism identifies $I(Y_i)$ with the prime ideal $I(Y)S_{x_i} \cap S_{(x_i)}$ (which is denoted $I(Y)S_{(x_i)}$ in the proof of $I, 3.4$). It follows that

$$A(Y_i) = k[y_1, \ldots, y_n]/I(Y_i) \cong S_{(x_i)}/I(Y)S_{(x_i)}) \cong (S/I(Y))_{(x_i)} = S(Y)_{(x_i)}$$

$\square$

Let k be a field, and for $n \geq 1$ set $S = k[x_0, \ldots, x_n]$ and consider the the projective space $\mathbb{P}^n_k = ProjS$. There are $n+1$ open affine subsets $D_+(x_i) \cong SpecS_{(x_i)} \cong Speck[x_0/x_i, \ldots, x_n/x_i]$ which cover $\mathbb{P}^n_k$. We know from our divisor notes that the closed irreducible subsets of $\mathbb{P}^n_k$ are in bijection with the homogenous primes of S other than S_+. Let $Y \subseteq \mathbb{P}^n_k$ be a closed irreducible subset, and suppose $D_+(x_i) \cap Y \neq \emptyset$. Then $I(Y) \in D_+(x_i)$ and $Y \cap D_+(x_i)$ is the closure of $I(Y)$. The prime $I(Y)$ corresponds to the prime $I(Y)S_{(x_i)} = I(Y)S_{x_i} \cap S_{(x_i)}$ of $SpecS_{(x_i)}$. Let $Y_i \subseteq Speck[x_0/x_i, \ldots, x_n/x_i]$ be the closed irreducible subset corresponding to $Y \cap D_+(x_i)$. Then $I(Y_i)$ is the prime ideal of $k[x_0/x_i, \ldots, x_n/x_i]$ corresponding to $I(Y)S_{(x_i)}$.



Let $A(Y_i)$ denote $k[x_0/x_i, \dots, x_n/x_i]/I(Y_i)$ and $S(Y)$ denote $S/I(Y)$. Then immediately we have an isomorphism of k-algebras $A(Y_i) \cong S_{(x_i)}/I(Y)S_{(x_i)}$. It follows from the next Lemma that $S_{(x_i)}/I(Y)S_{(x_i)} \cong (S/I(Y))_{(x_i)}$. So finally we have an isomorphism of k-algebras $A(Y_i) \cong S(Y)_{(x_i)}$.

Lemma 21. *Let A be a ring and S a graded A-algebra. If I is a homogenous ideal of S and $x \in S$ homogenous, then there is a canonical isomorphism of A-algebras*

$$\alpha : (S/I)_{(x+I)} \longrightarrow S_{(x)}/(IS_x \cap S_{(x)})$$
$$(f+I)/(x^n+I) \mapsto f/x^n + IS_x \cap S_{(x)}$$

Proof. It is easy to check the given map is an isomorphism of A-algebras. □

We proved in our Chapter 1, Section 3 notes that for a graded domain S and nonzero $x \in S_1$ there is an isomorphism of S_0-algebras $S_x \cong S_{(x)}[z, z^{-1}]$ (the ring of Laurent polynomials). As in the proof of Ex $I, 2.6$, this implies that for the case $S = k[x_0, \dots, x_n]$ we have an isomorphism of k-algebras

$$S(Y)_{x_i} \cong S(Y)_{(x_i)}[z, z^{-1}] \cong A(Y_i)[z]_z$$

It follows that the quotient field of $S(Y)$ is k-isomorphic to the quotient field of $A(Y_i)[z]$. Since both $S(Y)$ and $A(Y_i)[z]$ are finitely generated domains over the field k, this implies that $dimS(Y) = dimA(Y_i)[z] = dimA(Y_i) + 1$.

Proposition 22. *Let $S = k[x_0, \dots, x_n]$ for a field k and $n \geq 1$. If $Y \subseteq \mathbb{P}^n_k = ProjS$ is an irreducible closed subset then*

$$dimS(Y) = 1 + dimY$$

where $I(Y)$ is the homogenous prime associated with Y and $S(Y) = S/I(Y)$.

Proof. The previous discussion shows that Y is covered by open subsets homeomorphic to schemes of the form $SpecA(Y_i)$ where $dimA(Y_i) = dimS(Y) - 1$. The claim now follows from $I, Ex1.10$ and the fact that for a ring A, $dimA = dim(SpecA)$. □

Since $dimS(Y) = coht.I(Y)$, another way to state the result is $dimY = coht.I(Y) - 1$. This is in contrast to the affine case, where for an irreducible closed Y we would have $dimY = coht.I(Y)$.

7 Points of Projective Space

Let $S = k[x_0, \dots, x_n]$ where k is a field and $n \geq 1$. If k is algebraically closed, then we know from our Varieties as Schemes notes that the closed points of $\mathbb{P}^n_k = ProjS$ are homeomorphic to the variety $\mathbb{P}^n$ defined in Chapter 1. Corresponding to a point $P = (a_0, \dots, a_n)$ with $a_i \neq 0$ is the homogenous prime ideal $I(P) = (a_i x_0 - a_0 x_i, \dots, a_i x_n - a_n x_i)$. In fact, we can still use this idea of "points" when k is not algebraically closed. Throughout this section k is an arbitrary field and $S = k[x_0, \dots, x_n]$ for fixed $n \geq 1$.

Lemma 23. *Given a nonzero tuple $P = (a_0, \dots, a_n) \in k^{n+1}$ with say $a_i \neq 0$, $I(P) = (a_i x_0 - a_0 x_i, \dots, a_i x_n - a_n x_i)$ is a homogenous prime ideal of S of height $n - 1$.*

Proof. First we show that the ideal $I(P)$ does not depend on i. That is, if also $a_j \neq 0$ then we claim $I(P) = (a_j x_0 - a_0 x_j, \dots, a_j x_n - a_n x_j)$. This follows easily from the following formula for any $0 \leq k \leq n$

$$a_j x_k - a_k x_j = a_j/a_i(a_i x_k - a_k x_i) - a_k/a_i(a_i x_j - a_j x_i)$$

Since $I(P)$ is generated by linear polynomials, it is a homogenous prime ideal of height $n-1$ by our Linear Variety notes. □

It is clear that if $P = \lambda Q$ for some nonzero $\lambda \in k$ then $I(P) = I(Q)$, as one would expect for points of projective space. Since $ht.S_+ = n$ it is clear that $I(P) \in ProjS$, so we have associated points of $\mathbb{P}^n_k$ with tuples of k^{n+1}.

Lemma 24. *Given a nonzero tuple $P = (a_0, \dots, a_n) \in k^{n+1}$ there is an automorphism $\Phi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$ of schemes over k mapping P to $(1, 0, \dots, 0)$.*

Proof. Suppose that $a_i \neq 0$ and define an automorphism of k-algebras $\psi : k[x_0, \dots, x_n] \longrightarrow k[x_0, \dots, x_n]$ by

$$\begin{aligned} x_0 &\mapsto x_i \\ x_1 &\mapsto a_i x_0 - a_0 x_i \\ &\vdots \\ x_n &\mapsto a_i x_n - a_n x_i \end{aligned}$$

It is not hard to see that this induces a k-automorphism of $\mathbb{P}^n_k$ identifying P and $(1, 0, \dots, 0)$ (that is, identifying the homogenous prime ideals $I(P)$ and $(x_1, \dots, x_n)$). $\square$