Tensor, Exterior and Symmetric Algebras

Daniel Murfet

May 16, 2006

Throughout this note R is a commutative ring, all modules are left R-modules. If we say a ring is noncommutative, we mean it is not necessarily commutative. Unless otherwise specified, all rings are noncommutative (except for R). If A is a ring then the *center* of A is the set of all $x \in A$ with $xy = yx$ for all $y \in A$.

Contents

1 Definitions

Definition 1. A R-algebra is a ring morphism $\phi : R \longrightarrow A$ where A is a ring and the image of ϕ is contained in the center of A. This is equivalent to A being an R-module and a ring, with $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$, via the identification of $r \cdot 1$ and $\phi(r)$. A morphism of R-algebras is a ring morphism making the appropriate diagram commute, or equivalently a ring morphism which is also an R-module morphism. In this section R**nAlg** will denote the category of these R-algebras. We use R**Alg** to denote the category of *commutative* R-algebras.

A *graded* ring is a ring A together with a set of subgroups $A_d, d \geq 0$ such that $A = \oplus_{d \geq 0} A_d$ as an abelian group, and $st \in A_{d+e}$ for all $s \in A_d, t \in A_e$. We also require that $1 \in A_0$ (if A is commutative this follows from the other axioms). A morphism of graded rings is a ring morphism which preserves degree.

A *graded R-module* is an R-module M together with a set of subgroups $M_n, n \in \mathbb{Z}$ such that $M = \oplus_{n \in \mathbb{Z}} M_n$ and each M_n is an R-submodule of M. A morphism of graded R-modules is a morphism of R-modules which preserves degree.

A *graded R-algebra* is an R-algebra A which is also a graded ring, in such a way that the image of the structural morphism $R \longrightarrow A$ is contained in A_0. Equivalently, A is a graded ring and a R-algebra and all the graded pieces $A_d, d \geq 0$ are R-submodules. A morphism of graded R-algebras is an R-algebra morphism which preserves degree. Let R**GrnAlg** denote the category of all graded R-algebras, and R**GrAlg** the full subcategory of commutative graded R-algebras.

Let A be an R-algebra. An R-*subalgebra* is a subring of A which is also an R-submodule. The subalgebra *generated* by a subset $S \subseteq A$ is the intersection of all subalgebras containing S. This consists of all polynomials in the elements of S with coefficients in R.

Let A be a graded ring. We call elements of A_d *homogenous of degree* d. If $f \in A$ then we can write f uniquely as a sum of homogenous elements, and we denote by f_d the degree d component. A two-sided ideal I in A is *homogenous* if whenever $f \in I$ we have $f_d \in I$ for all $d \geq 0$. Let $S \subseteq A$ be a subset (possibly empty). The *two-sided ideal* generated by S is the intersection of all two-sided ideals containing S. This is a two-sided ideal denoted by (S). In fact (S) consists of sums of elements of the form asb with $s \in S$ and $a, b \in A$. We say a two-sided ideal I is *generated* by a subset $S \subseteq I$ if $I = (S)$. Given elements $f, g \in A$ it is clear that

$$\begin{aligned}(f+g)_d &= f_d + g_d \\ (fg)_d &= \sum_{i+j=d} f_i g_j\end{aligned}$$

Lemma 1. *Let A be a graded ring. A two-sided ideal I is homogenous if and only if it is generated by homogenous elements.*

Proof. It is clear that if I is homogenous, it is generated by the homogenous components of all its elements. Conversely, if I is generated by a set of homogenous elements $\{f_i\}_{i \in I}$ with $f_i \in A_{n_i}$ then suppose $g \in I$, say $g = a_1 f_1 b_1 + \cdots + a_n f_n b_n$. Then $g_d = (a_1 f_1 b_1)_d + \cdots + (a_n f_b b_n)_d$ so it suffices to consider afb for some f homogenous of degree n. We have

$$(afb)_d = \sum_{i+j=d} a_i (fb)_j = \sum_{i+j=d} a_i f b_{j-n}$$

with the convention that $b_k = 0$ for $k < 0$. So clearly $(afb)_d \in I$, as required. □

Let A be a ring and I a two-sided ideal. Then the quotient A/I is a ring and $A \longrightarrow A/I$ is a ring morphism. If A is graded and I homogenous, then A/I is a graded ring, where we define

$$(A/I)_d = \{f + I \,|\, f \in A_d\}$$

and the morphism $A \longrightarrow A/I$ is a morphism of graded rings. If A is an R-algebra and I a two-sided ideal, then I is also an R-submodule, so A/I is an R-algebra and $A \longrightarrow A/I$ is a morphism of R-algebras. If A is a graded R-algebra, and I is a two-sided homogenous ideal, then I is also an R-submodule, so A/I is a graded R-algebra with the above grading, and $A \longrightarrow A/I$ is a morphism of graded R-algebras. If $\{I_i\}$ is a nonempty collection of two-sided ideals which are homogenous (resp. R-submodules) then the intersection $\bigcap_i I_i$ is a two-sided ideal with the same property.

Definition 2. Let $M_1, \ldots, M_n$ be R-modules and A an abelian group. A function $f : M_1 \times \cdots \times M_n \longrightarrow A$ is *multilinear* if it is linear in each variable

$$f(m_1, \ldots, m_i + m_i', \ldots, m_n) = f(m_1, \ldots, m_i, \ldots, m_n) + f(m_1, \ldots, m_i', \ldots, m_n)$$

and

$$f(m_1, \ldots, am_i, \ldots, m_n) = f(m_1, \ldots, am_j, \ldots, m_n)$$

for any i, j and $a \in R$. If A is also an R-module and if f satisfies the following additional condition

$$f(m_1, \ldots, am_i, \ldots, m_n) = a \cdot f(m_1, \ldots, m_n)$$

for any i and $a \in R$, then we say that f is a *multilinear form.*

Definition 3. Let $M_1, \ldots, M_n$ be R-modules, $n \geq 2$. A *tensor product* for the M_i is an abelian group G together with a multilinear map $\gamma : M_1 \times \cdots \times M_n \longrightarrow G$ with the following property: if H is an abelian group and $g : M_1 \times \cdots \times M_n \longrightarrow H$ a multilinear map, there is a unique morphism of abelian groups $\theta : G \longrightarrow H$ such that $\theta\gamma = g$.

The tensor product exists: let G be the free abelian group on the set $M_1 \times \cdots \times M_n$ (= functions from this set into $\mathbb{Z}$ with finite support and pointwise operations) and let H be the quotient by the submodule generated by the elements $(m_1, \ldots, m_i + m_i', \ldots, m_n) - (m_1, \ldots, m_i, \ldots, m_n) -$

$(m_1, \dots, m'_i, \dots, m_n)$ and $(m_1, \dots, am_i, \dots, m_n) - (m_1, \dots, am_j, \dots, m_n)$ (with the usual convention observed in all our thinking where the quotient of the zero module is just the identity). Let γ be the canonical set map $M_1 \times \cdots \times M_n \longrightarrow G$ composed with the projection $G \longrightarrow H$. This pair (H, γ) has the required property. We denote H by $M_1 \otimes \cdots \otimes M_n$ (sometimes we put the subscript R in) and denote the image of $(m_1, \dots, m_n)$ in H by $m_1 \otimes \cdots \otimes m_n$. These elements generate H as an abelian group. In fact any element of H can be written as a sum of elements of the form $m_1 \otimes \cdots \otimes m_n$.

By definition for any $a \in R$,

$$(am_1) \otimes \cdots \otimes m_n = m_1 \otimes (am_2) \otimes \cdots \otimes m_n = \cdots = m_1 \otimes \cdots \otimes (am_n)$$

Given $a \in R$ define a multilinear morphism $M_1 \times \cdots \times M_n \longrightarrow M_1 \otimes \cdots \otimes M_n$ by $(m_1, \dots, m_n) \mapsto (am_1) \otimes \cdots \otimes m_n$. This induces an abelian group endomorphism of $M_1 \otimes \cdots \otimes M_n$ which sends $m_1 \otimes \cdots \otimes m_n$ to $(am_1) \otimes \cdots \otimes m_n$. Denote this by φ_a. This action makes $M_1 \otimes \cdots \otimes M_n$ into an R-module. For one of the generators we have

$$a \cdot (m_1 \otimes \cdots \otimes m_n) = (am_1) \otimes \cdots \otimes m_n$$

If $\varphi_i : M_i \longrightarrow M'_i$ are morphisms of R-modules $1 \le i \le n$ then we induce a morphism of R-modules that acts as follows on the generators

$$\begin{aligned} \varphi_1 \otimes \cdots \otimes \varphi_n : M_1 \otimes \cdots \otimes M_n &\longrightarrow M'_1 \otimes \cdots \otimes M'_n \\ m_1 \otimes \cdots \otimes m_n &\mapsto \varphi_1(m_1) \otimes \cdots \otimes \varphi_n(m_n) \end{aligned}$$

We have now defined for $n \ge 2$ a covariant functor $- \otimes \cdots \otimes - : R\mathbf{Mod} \times \cdots \times R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ which is additive in each variable (meaning the partial functor, which is defined on objects in the obvious way and on morphisms by shoving identities everywhere else, is additive $R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$). Note it is *not* an additive functor from the product category, it is only additive in each variable.

Lemma 2. *Let $M_1, \dots, M_n$ be R-modules for $n \ge 2$. Then there is a canonical isomorphism of R-modules natural in all variables*

$$\begin{aligned} M_1 \otimes \cdots \otimes M_n &\cong M_1 \otimes (M_2 \otimes (M_3 \otimes \cdots \otimes (M_{n-1} \otimes M_n) \cdots) \\ m_1 \otimes \cdots \otimes m_n &\mapsto m_1 \otimes (m_2 \otimes \cdots \otimes (m_{n-1} \otimes m_n) \cdots) \end{aligned}$$

Proof. We proceed by induction on n. In the case $n = 2$ our multilinear tensor product is just the usual one, and they both behave the same with respect to morphisms in their variables. For $n \ge 3$ let $\Phi : M_2 \otimes \cdots \otimes M_n \longrightarrow M_2 \otimes (M_3 \otimes \cdots \otimes (M_{n-1} \otimes M_n) \cdots)$ be the isomorphism provided by the inductive hypothesis. By tensoring with M_1 it suffices to show that there is an isomorphism of R-modules natural in all variables

$$M_1 \otimes (M_2 \otimes \cdots \otimes M_n) \cong M_1 \otimes \cdots \otimes M_n$$

Define a bilinear map from $M_1 \times (M_2 \otimes \cdots \otimes M_n)$ as follows: for each $m \in M_1$ define a multilinear map $M_2 \times \cdots \times M_n \longrightarrow M_1 \otimes \cdots \otimes M_n$ by attaching m on the left. This induces a morphism φ_m of out the tensor product, and we define $(m, x) \mapsto \varphi_m(x)$. One checks this is a tensor product, which completes the proof. □

Lemma 3. *Suppose R is nonzero and let $M_1, \dots, M_r$ be finite free R-modules of ranks $n_i \ge 1$. If $\{x_{i1}, \dots, x_{in_i}\}$ is a basis for M_i then the elements $x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{rj_r}$ with $1 \le j_s \le n_s$ give a basis for $M_1 \otimes \cdots \otimes M_n$, which therefore has rank $n_1 n_2 \cdots n_r$.*

Proof. By induction on $r \ge 2$. The case $r = 2$ is easily checked. For $r > 2$ we use the fact that $M_1 \otimes (M_2 \otimes \cdots M_n) \cong M_1 \otimes \cdots \otimes M_n$. □

For an R-module M and $n \geq 0$ we define

$$M^{\otimes n} = \begin{cases} R & n = 0 \\ M & n = 1 \\ M \otimes \cdots \otimes M & n > 1 \;\; (n \text{ factors}) \end{cases}$$

Let $\varphi : M \longrightarrow N$ be a morphism of R-modules. For $n \geq 0$ we define a morphism of R-modules $\varphi^{\otimes n} : M^{\otimes n} \longrightarrow N^{\otimes n}$ by

$$\varphi^{\otimes n} = \begin{cases} 1_R & n = 0 \\ \varphi & n = 1 \\ \varphi \otimes \cdots \otimes \varphi & n > 1 \;\; (n \text{ factors}) \end{cases}$$

For $n \geq 0$ this defines a functor $-^{\otimes n} : R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$ (not additive). Clearly $-^{\otimes 1}$ is the identity functor and $-^{\otimes 0}$ maps every module to R and every morphism to the identity.

Lemma 4. *Let M, A be R-modules. Then for $n \geq 1$ there is a canonical bijection between multilinear forms $f : M^n \longrightarrow A$ and the set $Hom_R(M^{\otimes n}, A)$.*

Proof. Corresponding to any multilinear form $f : M^n \longrightarrow A$ is the induced morphism of abelian groups $M^{\otimes n} \longrightarrow A$, which is also clearly a morphism of R-modules. This is a bijective correspondence. $\square$

Proposition 5. *For R-modules M, N and integers $r, s \geq 1$ there is a canonical isomorphism of R-modules $M^{\otimes r} \otimes N^{\otimes s} \longrightarrow M \otimes \cdots \otimes M \otimes N \otimes \cdots \otimes N$ (with r copies of M and s copies of N) which is defined by*

$$(m_1 \otimes \cdots \otimes m_r) \otimes (n_1 \otimes \cdots \otimes n_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s$$

Proof. Let K denote the tensor product $M \otimes \cdots \otimes M \otimes N \otimes \cdots \otimes N$. For $(m_1, \ldots, m_r) \in M^r$ define a multilinear morphism $M^s \longrightarrow K$ by $(n_1, \ldots, n_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s$ which induces a morphism of R-modules $\varphi_{m_1,\ldots,m_r} : M^{\otimes s} \longrightarrow K$. Consider the R-module $Hom_R(M^{\otimes s}, K)$. The assignment $(m_1, \ldots, m_r) \mapsto \varphi_{m_1,\ldots,m_r}$ is multilinear, and it induces a morphism of R-modules $\Phi : M^{\otimes r} \longrightarrow Hom_R(M^{\otimes s}, K)$ with $m_1 \otimes \cdots \otimes m_r \mapsto \varphi_{m_1,\ldots,m_r}$. There is an induced morphism of R-modules

$$\begin{gathered} M^{\otimes r} \otimes N^{\otimes s} \longrightarrow K \\ p \otimes q \mapsto \Phi(p)(q) \\ (m_1 \otimes \cdots \otimes m_r) \otimes (n_1 \otimes \cdots \otimes n_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s \end{gathered}$$

One defines the inverse $K \longrightarrow M^{\otimes r} \otimes N^{\otimes s}$ easily, so the proof is complete. $\square$

Proposition 6. *For an R-module M and integers $r, s \geq 0$ there is a canonical isomorphism of R-modules $\kappa_{r,s} : M^{\otimes r} \otimes M^{\otimes s} \longrightarrow M^{\otimes (r+s)}$ which for $r, s \geq 1$ is defined by*

$$(m_1 \otimes \cdots \otimes m_r) \otimes (n_1 \otimes \cdots \otimes n_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s$$

Moreover for any integers $i, j, k \geq 0$ the following diagram commutes

$$\begin{tikzcd}
(M^{\otimes i} \otimes M^{\otimes j}) \otimes M^{\otimes k} \arrow[rr, Rightarrow] \arrow[d, Rightarrow, "\kappa_{i,j} \otimes 1"'] & & M^{\otimes i} \otimes (M^{\otimes j} \otimes M^{\otimes k}) \arrow[d, Rightarrow, "1 \otimes \kappa_{j,k}"] \\
M^{\otimes (i+j)} \otimes M^{\otimes k} \arrow[dr, Rightarrow, "\kappa_{i+j,k}"'] & & M^{\otimes i} \otimes M^{\otimes (j+k)} \arrow[dl, Rightarrow, "\kappa_{i,j+k}"] \\
& M^{\otimes (i+j+k)} &
\end{tikzcd}$$

Proof. For $r = 0$ we define $\kappa_{r,s}$ to be the canonical isomorphism of R-modules $R \otimes M^{\otimes s} \cong M^{\otimes s}$, and similarly if $s = 0$. For $r, s \geq 1$ we have the canonical isomorphism of Proposition 5. We have to show that the pentagon commutes. One settles the case where one of the i, j, k is zero separately, and otherwise it suffices to prove commutativity on elements of the form

$$((m_1 \otimes \cdots \otimes m_i) \otimes (n_1 \otimes \cdots \otimes n_j)) \otimes (t_1 \otimes \cdots \otimes t_k)$$

and this is straightforward to check. $\square$

2 The Tensor Algebra

For any R-module M we define the *tensor algebra* $T(M)$ to be the R-module

$$T(M) = \bigoplus_{i=0}^{\infty} M^{\otimes i} = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$$

Given sequences $\underline{m} = (m_i)_{i \geq 0}$ and $\underline{n} = (n_i)_{i \geq 0}$ we define their product by

$$(\underline{m} \cdot \underline{n})_k = \sum_{r+s=k} \kappa_{r,s}(m_r \otimes n_s)$$

Where $r, s, k \geq 0$. We have to check this makes $T(M)$ into a graded R-algebra. First we check associativity, using commutativity of the above pentagon

$$\begin{aligned}
((\underline{m} \cdot \underline{n}) \cdot \underline{l})_k &= \sum_{r+s=k} ((\underline{m} \cdot \underline{n})_r \otimes \underline{l}_s) \\
&= \sum_{r+s=k} \kappa_{r,s} \left(\sum_{i+j=r} \kappa_{i,j}(m_i \otimes n_j) \otimes l_s \right) \\
&= \sum_{i+j+s=k} \kappa_{i+j,s}(\kappa_{i,j}(m_i \otimes n_j) \otimes l_s) \\
&= \sum_{i+j+s=k} \kappa_{i,j+s}(m_i \otimes \kappa_{j,s}(n_j \otimes l_s)) \\
&= (\underline{m} \cdot (\underline{n} \cdot \underline{l}))_k
\end{aligned}$$

The sequence $(1, 0, \ldots)$ serves as an identity, and it is not hard to check right and left distributivity. Hence $T(M)$ is a graded R-algebra with the graded piece of degree $n \geq 0$ being the subgroup $M^{\otimes n}$, which we denote by $T^n(M)$. We make the following notes:

- The map $R \longrightarrow T(M)$ defined by $r \mapsto (r, 0, \ldots)$ is a ring morphism, which gives a ring isomorphism of R with its image $T^0(M)$. We denote a sequence $(r, 0, \ldots)$ simply by r.

- The map $M \longrightarrow T(M)$ defined by $m \mapsto (0, m, 0, \ldots)$ is a morphism of R-modules, which gives an isomorphism of R-modules of M with its image $T^1(M)$. We denote a sequence $(0, m, 0, \ldots)$ simply by m.

- For $n \geq 2$ the map $M^{\otimes n} \longrightarrow T(M)$ defined by $m_1 \otimes \cdots \otimes m_n \mapsto (0, \ldots, m_1 \otimes \cdots \otimes m_n, 0, \ldots)$ is a morphism of R-modules, which gives an isomorphism of R-modules of $M^{\otimes n}$ with its image $T^n(M)$. We denote this sequence simply by $m_1 \otimes \cdots \otimes m_n$. Notice that $m_1 \otimes \cdots \otimes m_n = m_1 m_2 \cdots m_n$ is the product of the m_i in the ring $T(M)$. In particular $(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m$ for $m, n \geq 1$.

- $T(M)$ is generated as an R-algebra by $T^1(M)$

Proposition 7. *We have the following properties of the tensor algebra* *(a) Given any morphism of R-modules $\varphi : M \longrightarrow N$ there is a unique morphism of R-algebras $T(\varphi) : T(M) \longrightarrow T(N)$ making the following diagram of R-modules commute:*

$$\begin{tikzcd} T(M) \arrow[r] & T(N) \\ M \arrow[u] \arrow[r, "\varphi"'] & N \arrow[u] \end{tikzcd}$$

This defines a functor $T : R\mathbf{Mod} \longrightarrow R\mathbf{nAlg}$. Also note that $T(\varphi)$ is a morphism of graded R-algebras, so the construction also defines a functor $T : R\mathbf{Mod} \longrightarrow R\mathbf{GrnAlg}$.

(b) Let $\varphi : M \longrightarrow S$ be a morphism of R-modules, where S is an R-algebra. Then there is a unique morphism of R-algebras $\Phi : T(M) \longrightarrow S$ making the following diagram commute

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & S \\ T(M) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Moreover if S is a graded R-algebra and $Im\varphi \subseteq S_1$, then Φ is a morphism of graded R-algebras.

(c) The functor $T : R\mathbf{Mod} \longrightarrow R\mathbf{nAlg}$ is left adjoint to the forgetful functor $F : R\mathbf{nAlg} \longrightarrow R\mathbf{Mod}$. The unit of the adjunction is the natural transformation $\eta : 1 \longrightarrow FT$ given by the canonical morphism of R-modules $\eta_M : M \longrightarrow T(M)$.

(d) The functor $T : R\mathbf{Mod} \longrightarrow R\mathbf{GrnAlg}$ is left adjoint to functor $(-)_1 : R\mathbf{GrnAlg} \longrightarrow R\mathbf{Mod}$ which maps a graded R-algebra to its degree 1 component. The unit of the adjunction is the natural transformation $1 \longrightarrow (-)_1 T$ given by the isomorphism $M \longrightarrow T^1(M)$.

Proof. (a) We define $T(\varphi) = \bigoplus_{i=0}^{\infty} \varphi^{\otimes i} = 1_R \oplus \varphi \oplus \varphi^{\otimes 2} \oplus \cdots$. This is clearly a morphism of R-modules making the required diagram commute. It is not hard to check it is also a morphism of graded rings. It is the *unique* morphism of R-algebras making the diagram commute, since if $\psi : T(M) \longrightarrow T(N)$ maps m to $\varphi(m)$ then it also maps $m_1 \cdots m_n = m_1 \otimes \cdots \otimes m_n$ to $\varphi(m_1) \cdots \varphi(m_n) = \varphi^{\otimes n}(m_1 \otimes \cdots \otimes m_n)$.

(b) For $n \geq 2$ induce a morphism of R-modules $M^{\otimes n} \longrightarrow S$ with $m_1 \otimes \cdots \otimes m_n \mapsto \varphi(m_1) \cdots \varphi(m_n)$. Together with $R \longrightarrow S$ and $\varphi : M \longrightarrow S$ this induces a morphism of R-modules $T(M) \longrightarrow S$ which clearly makes the diagram commute. It is easy to check this is a morphism of rings, and Φ is clearly the unique morphism of R-algebras making the diagram commute.

(c) and (d) both follow immediately from (b). $\square$

Lemma 8. *There is a canonical isomorphism of graded R-algebras*

$$\beta : R[x] \longrightarrow T(R)$$
$$x \mapsto (0, 1, \ldots)$$

Proof. It is not hard to see that $T(R)$ is a commutative R-algebra, so such a morphism of graded R-algebras certainly exists. Since $T(R) = R \oplus R \oplus (R \otimes R) \oplus \cdots$ we have to be careful to distinguish the identity $(1, 0, \ldots)$ and the element $(0, 1, \ldots)$ which is the image of x under β. It is easy to see that β is an isomorphism. $\square$

3 The Exterior Algebra

Let M be an R-module and let I be the two-sided ideal generated by the subset $S = \{x \otimes x \,|\, x \in M\}$ of $T(M)$. Note that $(x + y) \otimes (x + y) = x \otimes x + y \otimes x + x \otimes y + y \otimes y$ so $x \otimes y + y \otimes x \in I$ for any $x, y \in M$. Then $\wedge M = T(M)/I$ is a graded R-algebra called the *exterior algebra* and $T(M) \longrightarrow \wedge M$ is a morphism of graded R-algebras. For $d \geq 0$ the graded piece of $\wedge M$ is the image of $T^d(M)$, which we denote by $\wedge^d M$.

- The proof of Lemma 1 makes it clear that every nonzero homogenous element of I has degree ≥ 2. So the ring morphism $R \longrightarrow T(M) \longrightarrow \wedge M$ and gives an isomorphism of R with $\wedge^0 M$. Also the morphism of R-modules $M \longrightarrow T(M) \longrightarrow \wedge M$ gives an isomorphism of M with $\wedge^1 M$. As usual we denote the image of $r \in R$ and $m \in M$ by r, m respectively.
- For $n \geq 2$ the map $M^{\otimes n} \longrightarrow T(M) \longrightarrow \wedge M$ is not necessarily injective, but it is a morphism of R-modules whose image is equal to $\wedge^n M$. For elements $m_1, \ldots, m_n \in M$ we denote the image of $m_1 \otimes \cdots \otimes m_n$ in $\wedge M$ by $m_1 \wedge \cdots \wedge m_n$. Every element of $\wedge^n M$ can be written as a nontrivial sum of such elements.
- Given any elements $u, v \in \wedge M$ we denote their product by $u \wedge v$. It is clear that this notation agrees with one just defined for elements $m_i \in M$. Clearly for $m, n \in M$ we have $m \wedge n = -(n \wedge m)$, and it is not hard to see that for any $u \in \wedge^r M$ and $v \in \wedge^s M$ we have $u \wedge v = (-1)^{rs} v \wedge u$.

Clearly if $M = 0$ then the ring morphisms $R \longrightarrow T(M)$ and $R \longrightarrow \wedge M$ are isomorphisms. Let $n \geq 2$ and suppose $m_1, \ldots, m_n \in M$. If $m_s = m_{s+1}$ for some $1 \leq s \leq n-1$ then it is clear that $m_1 \otimes \cdots \otimes m_n \in I$. Since $x \otimes y + y \otimes x \in I$ for all $x, y \in M$ it also follows that if we interchange the order of two of the m_i then their sum is in I. That is,

$$m_1 \otimes \cdots \otimes m_s \otimes m_{s+1} \otimes \cdots \otimes m_n + m_1 \otimes \cdots \otimes m_{s+1} \otimes m_s \otimes \cdots \otimes m_n \in I \tag{1}$$

So clearly if $m_i = m_j$ for some $i \neq j$ then we can interchange positions until we have adjacent elements equal, so that $m_i \otimes \cdots \otimes m_n \in I$. Hence if $x, y \in M$ and $m_1, \ldots, m_n$ are any elements of M $(n \geq 1)$ then

$$\begin{aligned}(x+y) \otimes m_1 \otimes \cdots \otimes m_n \otimes (x+y) &= x \otimes m_1 \otimes \cdots \otimes m_n \otimes y \\ &\quad + x \otimes m_1 \otimes \cdots \otimes m_n \otimes x \\ &\quad + y \otimes m_1 \otimes \cdots \otimes m_n \otimes x \\ &\quad + y \otimes m_1 \otimes \cdots \otimes m_n \otimes y\end{aligned}$$

So $x \otimes m_1 \otimes \cdots \otimes m_n \otimes y + y \otimes m_1 \otimes \cdots \otimes m_n \otimes x \in I$. It follows that if $m_1, \ldots, m_n \in M$ with $n \geq 2$ then when we interchange *any* m_i with m_j, $i \neq j$ we change the sign of the element of $\wedge M$.

For $n \geq 2$ let I_n denote the submodule $I \cap M^{\otimes n}$ of $M^{\otimes n}$. Every element of I_2 is an R-linear combination of elements $x \otimes x, x \in M$. From Lemma 1 we know that any element of I_n for $n \geq 3$ can be written as an R-linear combination of terms of the following form: $m_1 \otimes \cdots \otimes m_{n-2} \otimes x \otimes x$, $x \otimes x \otimes m_1 \otimes \cdots \otimes m_{n-2}$ and if $n \geq 4$ also $m_1 \otimes m_s \otimes x \otimes x \otimes n_1 \otimes \cdots \otimes n_t$ with $s + t = n - 2$. This proves

Lemma 9. *Let M be an R-module. Then for $n \geq 2$ the kernel of the canonical epimorphism $M^{\otimes n} \longrightarrow \wedge^n M$ is the R-submodule I_n generated by elements of the form*

$$m_1 \otimes \cdots \otimes x \otimes x \otimes \cdots \otimes m_n$$

That is, an element $m_1 \otimes m_2 \otimes \cdots \otimes m_n$ of $M^{\otimes n}$ with $m_i = m_{i+1}$ for some $1 \leq i \leq n-1$. Therefore there is a canonical isomorphism of R-modules $\wedge^n M \cong M^{\otimes n}/I_n$.

Lemma 10. *Let M be an R-module. Then for $n \geq 2$ distinct elements $m_1, \ldots, m_n \in M$ and any permutation $\sigma \in S_n$ we have $m_{\sigma(1)} \wedge \cdots \wedge m_{\sigma(n)} = sgn(\sigma) m_1 \wedge \cdots \wedge m_n$.*

Proof. It suffices to check this in the case where σ is a transposition: that is, it switches two distinct indices. But we have already observed that the sign of an element of $\wedge^n M$ alternates under a transposition, so the proof is complete. □

Lemma 11. *If M is a finitely generated R-module then so is $\wedge^d M$ for $d \geq 0$.*

In the category of commutative R-algebras, the tensor product plays the role of a coproduct. Unfortunately this fails for noncommutative R-algebras, but the tensor product does satisfy a weaker universal property. To recover the tensor product as a coproduct, we have to restrict to the *super* R-algebras.

Lemma 12. *Let G, H be R-algebras. Then the tensor product $G \otimes_R H$ is an R-algebra with the following universal property: if $\varphi : G \longrightarrow T, \psi : H \longrightarrow T$ are morphisms of R-algebras such that $\varphi(g)$ commutes with $\psi(h)$ for all $g \in G, h \in H$, then there is a unique morphism of R-algebras $\theta : G \otimes_R H \longrightarrow T$ making the following diagram commute*

$$\begin{tikzcd}
G \arrow[d] \arrow[dr, "\varphi"] & \\
G \otimes_R H \arrow[r, "\theta"'] & T \\
H \arrow[u] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Proof. We define multiplication on the R-module $G \otimes_R H$ as follows: for $g \in G, h \in H$ define an R-bilinear map $G \times H \longrightarrow G \otimes_R H$ by $(g', h') \mapsto gg' \otimes hh'$. This induces a morphism of R-modules $\theta_{g,h} : G \otimes_R H \longrightarrow G \otimes_R H$, and the assignment $(g, h) \mapsto \theta_{g,h}$ defines an R-bilinear map $G \times H \longrightarrow End_R(G \otimes_R H)$, which lifts to a morphism of R-modules $\Theta : G \otimes_R H \longrightarrow End_R(G \otimes_R H)$ with $\Theta(g \otimes h)(g' \otimes h') = gg' \otimes hh'$. We define the ring structure on $G \otimes_R H$ by $ab = \Theta(a)(b)$. This gives the structure of an R-algebra with identity $1 \otimes 1$. The maps $G \longrightarrow G \otimes_R H$ and $H \longrightarrow G \otimes_R H$ given by $g \mapsto g \otimes 1$ and $h \mapsto 1 \otimes h$ are morphisms of R-algebras.

If ψ, φ are given, then induce θ using the R-bilinear map $(g, h) \mapsto \varphi(g)\psi(h)$. This gives a morphism of R-algebras $\theta : G \otimes_R H \longrightarrow T$ with $\theta(g \otimes h) = \varphi(g)\psi(h)$, which has the required property. □

In the same way that the tensor algebra is the simplest R-algebra associated to a module, and the symmetric algebra is the simplest *commutative* R-algebra associated to a module, the exterior algebra is the simplest *super* R-algebra associated to a module.

Definition 4. A *super R-algebra* is a graded R-algebra A with the following two properties (i) if $a \in A_m, b \in A_n$ then $ab = (-1)^{mn} ba$ and (ii) if a is homogenous of odd degree, then $a^2 = 0$. A morphism of super R-algebras is a morphism of graded R-algebras. Let R**sAlg** denote the category of super R-algebras.

Lemma 13. *Let M, N be graded R-modules. Then $M \otimes_R N$ is a graded R-module with*

$$(M \otimes_R N)_p = \{\sum m_i \otimes n_i \,|\, deg(m_i) + deg(n_i) = p\} \qquad p \in \mathbb{Z}$$

For $n \in \mathbb{Z}$ there is an isomorphism of R-modules $(M \otimes_R N)_p \cong \bigoplus_{m+n=p} M_m \otimes_R N_n$. If M, N are positive then so is $M \otimes_R N$.

Proof. Since tensor products commute with coproducts and the subgroups M_n, N_n are also submodules, we have

$$\begin{aligned} M \otimes_R N &= (\oplus_m M_m) \otimes_R (\oplus_n N_n) \\ &= \bigoplus_p \bigoplus_{m+n=p} M_m \otimes_R N_n \end{aligned}$$

The rest of the proof is clear. □

Lemma 14. *Let A, B be super R-algebras. Then the tensor product $A \otimes_R B$ is a super R-algebra with multiplication defined for homogenous elements by*

$$(a \otimes b)(a' \otimes b') = (-1)^{deg(b)deg(a')} aa' \otimes bb'$$

*Together with the canonical morphisms $A \longrightarrow A \otimes_R B, B \longrightarrow A \otimes_R B$ this is a coproduct in the category R**sAlg**.*

Proof. We give $A \otimes_R B$ the canonical structure of a graded R-module (with all negative grades zero). Let homogenous elements $a \in A_m, b \in B_n$ be given with $m, n \geq 0$. For integers $i, j \geq 0$ we define an R-bilinear map $A_i \times B_j \longrightarrow A \otimes_R B$ by $(a', b') \mapsto (-1)^{ni} aa' \otimes bb'$ which induces an R-linear map $\theta^{i,j}_{a,b} : A_i \otimes_R B_j \longrightarrow A \otimes_R B$. The morphisms induced by all pairs i, j give an R-linear morphism $\theta_{a,b} : A \otimes_R B \longrightarrow A \otimes_R B$ with $\theta_{a,b}(a' \otimes b') = (-1)^{deg(b)deg(a')} aa' \otimes bb'$ for homogenous elements a', b'. The assignment $(a, b) \mapsto \theta_{a,b}$ gives an R-bilinear map $A_m \times B_n \longrightarrow End_R(A \otimes_R B)$ which induces an R-linear map $\Theta^{m,n} : A_m \otimes_R B_n \longrightarrow End_R(A \otimes_R B)$ and therefore an R-linear map $\Theta : A \otimes_R B \longrightarrow End_R(A \otimes_R B)$ with $\Theta(a \otimes b)(a' \otimes b') = (-1)^{deg(b)deg(a')} aa' \otimes bb'$ for homogenous elements a, a', b, b'. We define multiplication on $A \otimes_R B$ by $gh = \Theta(g)(h)$. It is not hard to check this is a super R-algebra with identity $1 \otimes 1$ and the canonical grading.

The maps $A \longrightarrow A \otimes_R B, B \longrightarrow A \otimes_R B$ defined by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ are clearly morphisms of super R-algebras. Suppose we are given super R-algebra morphisms $\varphi : A \longrightarrow T$ and $\psi : B \longrightarrow T$. Induce a morphism of R-modules $\theta : A \otimes_R B \longrightarrow T$ with $\theta(a \otimes b) = \varphi(a)\psi(b)$ in the normal way. This is a morphism of rings since T is a super algebra, and it is clearly preserves degree. It is not hard to check it has the required properties. $\square$

Proposition 15. *We have the following properties of the exterior algebra*

(a) Given any morphism of R-modules $\varphi : M \longrightarrow N$ there is a unique morphism of R-algebras $\wedge\varphi : \wedge M \longrightarrow \wedge N$ making the following diagram of R-modules commute

$$\begin{tikzcd} \wedge M \arrow[r, "\wedge\varphi"] & \wedge N \\ M \arrow[u] \arrow[r, "\varphi"'] & N \arrow[u] \end{tikzcd}$$

This defines a functor $\bigwedge : R\mathbf{Mod} \longrightarrow R\mathbf{nAlg}$. Also note that $\wedge\varphi$ is a morphism of graded R-algebras.

(b) Let $\varphi : M \longrightarrow S$ be a morphism of R-modules, where S is an R-algebra and $\varphi(x)^2 = 0$ for all $x \in M$. Then there is a unique morphism of R-algebras $\Phi : \wedge M \longrightarrow S$ making the following diagram commute

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & S \\ \wedge M \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Moreover if S is a graded R-algebra and $Im\varphi \subseteq S_1$, then Φ is a morphism of graded R-algebras.

(c) If $R \longrightarrow S$ is any morphism of commutative rings and M an R-module, there is an isomorphism of graded S-algebras natural in M

$$S \otimes_R \bigwedge_R M \cong \bigwedge_S (S \otimes_R M)$$

(d) Let M be an R-module. Then $\wedge M$ is a super R-algebra, and if $\varphi : M \longrightarrow S_1$ is a morphism of R-modules, where M is an R-module and S is a super R-algebra, then there is a unique morphism of super R-algebras $\Phi : \wedge M \longrightarrow S$ making the following diagram commute

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & S_1 \\ (\wedge M)_1 \arrow[ur, "\Phi_1"'] & \end{tikzcd}$$

So the functor $\bigwedge : R\mathbf{Mod} \longrightarrow R\mathbf{sAlg}$ is left adjoint to the functor $(-)_1 : R\mathbf{sAlg} \longrightarrow R\mathbf{Mod}$ which maps a super R-algebra to its degree 1 component.

(e) *For any R-modules M, N there is a canonical isomorphism of super R-algebras $\wedge(M \oplus N) \cong \wedge M \otimes \wedge N$.*

Proof. (a) The morphism of graded R-algebras $T(\varphi) : T(M) \longrightarrow T(N)$ maps $x \otimes x$ to $\varphi(x) \otimes \varphi(x)$, so it is easy to see that we induce a morphism of graded R-algebras $\wedge M \longrightarrow \wedge N$ with the required property. (b) Using Proposition 7 we induce a morphism of R-algebras $T(M) \longrightarrow S$. By assumption this maps the generators of I to zero, so it induces a morphism of R-algebras $\wedge M \longrightarrow S$ with the required properties.

(c) Tensoring with S gives an S-linear map $S \otimes_R M \longrightarrow S \otimes_R \wedge_R M$. But $S \otimes_R \wedge_R M$ is a graded S-algebra, so we obtain a morphism of graded S-algebras $\phi : \wedge_S(S \otimes_R M) \longrightarrow S \otimes_R \wedge_R M$. On the other hand $m \mapsto 1 \otimes m$ gives an R-linear map $M \longrightarrow S \otimes_R M$, which composes with $S \otimes_R M \longrightarrow \wedge_S(S \otimes_R M)$ to give an R-linear map $M \longrightarrow \wedge_S(S \otimes_R M)$. Since $\wedge_S(S \otimes_R M)$ is a graded S-algebra it is also a graded R-algbera, so induces a morphism of graded R-algebras $\psi : \wedge_R M \longrightarrow \wedge_S(S \otimes_R M)$. The map $S \times \wedge_R M \longrightarrow \wedge_S(S \otimes_R M)$ given by $(s, a) \mapsto s \cdot \psi(a)$ is R-bilinear, so we get a morphism of graded S-algebras $\theta : S \otimes_R \wedge_R M \longrightarrow \wedge_S(S \otimes_R M)$. It is not hard to check θ, ϕ are mutually inverse, and that this isomorphism is natural in morphisms of R-modules $M \longrightarrow M'$.

(d) We have already checked that $\wedge M$ satisfies the first property of a super algebra. For the second, any element of $\wedge^i M$ is a sum of elements of the form $m_1 \wedge \cdots \wedge m_i$. When you square such an element, the diagonals (i.e. squares of the summands) all disappear since they have repeated m_j. The other terms cancel in pairs since i is odd. If $\alpha : M \longrightarrow M'$ is a morphism of R-modules, then $\wedge\alpha$ is a morphism of super R-algebras, so $\bigwedge$ gives a functor $R\mathbf{Mod} \longrightarrow R\mathbf{sAlg}$. The canonical morphism $M \longrightarrow (\wedge M)_1$ is clearly a natural transformation $1 \longrightarrow (-)_1 \bigwedge$. So to show $\bigwedge$ is left adjoint to $(-)_1$ we just apply (b).

(e) Follows immediately from (d) and Lemma 14. Explicitly if $i : \wedge M \longrightarrow \wedge(M \oplus N), j : \wedge N \longrightarrow \wedge(M \oplus N)$ are $\wedge$ of the inclusions, then the isomorphism $\wedge M \otimes \wedge N \longrightarrow \wedge(M \otimes N)$ is defined by $g \otimes h \mapsto i(g)j(h)$. ∎

Definition 5. If $\varphi : M \longrightarrow N$ is a morphism of R-modules then we have a morphism of graded R-algebras $\wedge\varphi : \wedge M \longrightarrow \wedge N$. There is an induced morphism of R-modules $\wedge^a M \longrightarrow \wedge^a N$ for $a \geq 1$ defined by $m_1 \wedge \cdots \wedge m_a \mapsto \varphi(m_1) \wedge \cdots \wedge \varphi(m_a)$. This defines the functor $\wedge^a(-) : R\mathbf{Mod} \longrightarrow R\mathbf{Mod}$. It is not difficult to see that this functor preserves epimorphisms.

Corollary 16. *Let R be a nonzero commutative ring, and let M be an R-module. Then for any multiplicatively closed subset $T \subseteq R$ there is a canonical isomorphism of graded $T^{-1}R$-algebras natural in M*

$$T^{-1}\left(\bigwedge_R M\right) \cong \bigwedge_{T^{-1}R}(T^{-1}M)$$
$$(m_1 \wedge \cdots \wedge m_r)/(t_1 \cdots t_r) \mapsto (m_1/t_1) \wedge \cdots \wedge (m_r/t_r)$$

For $d \geq 0$ there is a canonical isomorphism of $T^{-1}R$-modules $T^{-1}(\bigwedge_R^d M) \cong \bigwedge_{T^{-1}R}^d(T^{-1}M)$ natural in M.

Corollary 17. *Let R be a commutative ring, M an R-module and $\mathfrak{p}$ a prime ideal of R. Then for $d \geq 0$ there is a canonical isomorphism of $R_\mathfrak{p}$-modules $(\bigwedge_R^d M)_\mathfrak{p} \cong \bigwedge_{R_\mathfrak{p}}^d M_\mathfrak{p}$ natural in M.*

Definition 6. Let M be an R-module. For $n \geq 2$ we say a multilinear map $f : M \times \cdots \times M \longrightarrow A$ from the n-fold product into an abelian group A is *alternating* if $f(m_1, \ldots, m_n) = 0$ whenever $m_i = m_j$ for $i \neq j$. The canonical map

$$\gamma : M \times \cdots \times M \longrightarrow \wedge^n M$$
$$(m_1, \ldots, m_n) \mapsto m_1 \wedge \cdots \wedge m_n$$

is clearly an alternating multilinear form. This is the universal alternating multilinear form, in the sense that if A is an R-module and f an alternating multilinear form as above, then there is a *unique* morphism of R-modules $\theta : \wedge^n M \longrightarrow A$ such that $\theta\gamma = f$. In particular this implies that

$$\theta(m_1 \wedge \cdots \wedge m_n) = f(m_1, \ldots, m_n)$$

Lemma 18. *Let M, A be R-modules. Then for $n \geq 1$ there is a canonical bijection between the set of alternating multilinear forms $f : M^n \longrightarrow A$ and the set $Hom_R(\wedge^n M, A)$.*

Proof. We have already associated with an alternating mutlilinear form $f : M^n \longrightarrow A$ a morphism of R-modules $\theta : \wedge^n M \longrightarrow A$. Conversely, composing an R-morphism morphism $\wedge^n M \longrightarrow A$ with the canonical map $\gamma : M^n \longrightarrow \wedge^n M$ certainly gives an alternating multilinear form. These two maps are inverse to one another, so the proof is complete. □

3.1 Dimension of the Exterior Powers

In this section we assume that R is a nonzero commutative ring. Let M be a free R-module of finite rank $p \geq 1$ and let $\{x_1, \ldots, x_p\}$ be a basis. Then since the tensor product commutes with coproducts, for $n \geq 2$ the elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with $1 \leq i_t \leq p$ form a basis of $M^{\otimes n}$. We put the lexicographic order on these basis elements, so $x_{i_1} \otimes \cdots \otimes x_{i_n} < x_{j_1} \otimes \cdots \otimes x_{j_n}$ iff. $i_1 < j_1$, or $i_1 = j_1$ and $i_2 < j_2$, and so on. We also write $(i_1, \ldots, i_n) < (j_1, \ldots, j_n)$. By writing elements of M as linear combinations of the basis elements x_i and expanding these tensor products, we can write elements of I_n as linear combinations of certain combinations of basis elements. To illustrate we give some low dimension cases:

- $(n = 2, p = 2)$ Put $x = a_1x_1 + a_2x_2$. Then
$$x \otimes x = a_1^2(x_1 \otimes x_1) + a_1a_2(x_1 \otimes x_2 + x_2 \otimes x_1) + a_2^2(x_2 \otimes x_2)$$
We do not care about the terms involving basis elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with some x_i repeated, so denote the sum of terms involving such elements by C henceforth.

- $(n = 3, p = 3)$ Put $x = a_1x_1 + a_2x_2 + a_3x_3$ and $z = b_1x_1 + b_2x_2 + b_3x_3$. Then
$$\begin{aligned} x \otimes x \otimes z &= a_1a_2b_3(x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3) \\ &\quad + a_1a_3b_2(x_1 \otimes x_3 \otimes x_2 + x_3 \otimes x_1 \otimes x_2) \\ &\quad + a_2a_3b_1(x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_2 \otimes x_1) + C \end{aligned}$$
and
$$\begin{aligned} z \otimes x \otimes x &= b_1a_2a_3(x_1 \otimes x_2 \otimes x_3 + x_1 \otimes x_3 \otimes x_2) \\ &\quad + b_2a_1a_3(x_2 \otimes x_1 \otimes x_3 + x_2 \otimes x_3 \otimes x_1) \\ &\quad + b_3a_2a_1(x_3 \otimes x_2 \otimes x_1 + x_3 \otimes x_1 \otimes x_2) + C \end{aligned}$$

Clearly if $n > p$ then every basis element has some x_i occurring twice, so the whole thing is swallowed into C. For general $p \geq 1$ and $n \geq 2$ the module $M^{\otimes n}$ has rank p^n and we partition all these basis elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$ into classes:

Class C This class contains all basis elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with some x_i repeated. We have already seen that these elements belongs to I_n.

Class S If $n > p$ then every basis element belongs to C. Otherwise if $n \leq p$ for every subset $S \subseteq \{1, \ldots, p\}$ of n elements we say that $x_{i_1} \otimes \cdots \otimes x_{i_n}$ is of class S if no x_i is repeated, and $i_t \in S$ for $1 \leq t \leq n$. So that $i_1, \ldots, i_n$ is a permutation of S.

If $n > p$ then every basis element is of class C so clearly $I_n = M^{\otimes n}$. If on the other hand $n \leq p$ consider an element $m_1 \otimes \cdots \otimes m_s \otimes x \otimes x \otimes n_1 \otimes \cdots \otimes n_t$ with $s + t = n - 2$. It is clear what we mean if $t = 0$ or $s = 0$. Write $x = a_1x_1 + \cdots + a_px_p$ and similary write all the m_i, n_i and x as linear combinations of the basis x_i. Expanding, we get a linear combination of basis elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$. Ignore the ones of class C. We collect the remaining terms which share the same indices (that is, collect them according to the subset $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, p\}$). Fix one particular subset S. There are $n!$ basis elements in this class. This is an even number, and we can pair up the basis elements so that in the expansion the members of a pair share the same coefficient, since for example $x_{i_1} \otimes \cdots x_{i_s} \otimes x_1 \otimes x_2 \otimes x_{j_1} \otimes \cdots \otimes x_{j_t}$ has the same coefficient as the basis element with the center x_2, x_1. This proves the following:

Lemma 19. *Let M be a free R-module of rank $p \geq 1$ with basis $\{x_1, \dots, x_p\}$. For $n \geq 2$ the following basis elements of $M^{\otimes n}$ belong to I_n and together they generate it as an R-module:*

(i) $y_1 \otimes \cdots \otimes y_n$ *with* $y_i \in \{x_1, \dots, x_p\}$ *and some* x_i *repeated.*

(ii) $y_1 \otimes \cdots \otimes y_s \otimes y_{s+1} \otimes \cdots \otimes y_n + y_1 \otimes \cdots y_{s+1} \otimes y_s \otimes \cdots \otimes y_n$ *with* $y_i \in \{x_1, \dots, x_p\}$ *and no* x_i *repeated.*

Be warned that these elements do *not* in general form a basis for I_n, although many authors believe they do. For example, in the case $n = 3, p = 3$ the following elements are linearly dependent in $M^{\otimes 3}$ (subtract the last three from the first three):

$$\begin{gathered} x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 \\ x_1 \otimes x_3 \otimes x_2 + x_3 \otimes x_1 \otimes x_2 \\ x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_2 \otimes x_1 \\ x_1 \otimes x_2 \otimes x_3 + x_1 \otimes x_3 \otimes x_2 \\ x_2 \otimes x_1 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 \\ x_3 \otimes x_1 \otimes x_2 + x_3 \otimes x_2 \otimes x_1 \end{gathered}$$

In fact if $S \subseteq \{1, \dots, p\}$ is any subset of n-elements, and if we choose any two adjacent positions, then swapping these positions partitions the permutations of S into pairs. For example the first three rows above correspond to swapping the first two positions, while the last three rows swap the last two. Since every permutation occurs this way, of course everything cancels.

Let $n, p \geq 2$ be integers and put the lexicographic order on sequences $i_1, \dots, i_n$ taken from $\{1, \dots, p\}$ with no element repeated. These sequences are therefore arranged in a chain, with $p, p-1, \dots, p-n$ at the top and $1, 2, \dots, n$ at the bottom. Define the following operation on a sequence

> $(*)$ Given a sequence $i_1, \dots, i_n$ find the first $1 \leq s \leq n-1$ with $i_s > i_{s+1}$. If no such s exists our sequence is of the form $i_1 < \cdots < i_n$ and we do nothing. Otherwise swap i_s and i_{s+1}.

A sequence is invariant under this operation iff. it is of the form $i_1 < \cdots < i_n$ and otherwise the result is strictly lower in the chain of sequences.

Lemma 20. *We make the following claims about the operation $(*)$:*

(i) If you begin with a sequence $i_1, \dots, i_n$ and continually apply the operation $(*)$, the output terminates at the sequence consisting of $i_1, \dots, i_n$ arranged in strictly ascending order.

(ii) Take a sequence $S_0 : i_1, \dots, i_n$ and swap any two positions to obtain $T_0 : i_1, \dots, i_{q+1}, i_q, \dots, i_n$. Apply the operation $(*)$ to S_0 to produce S_1 and to T_0 to produce T_1 and recursively define S_n, T_n. Then for some $n \geq 0$ we have $S_n \neq T_n$ and either $S_{n+1} = T_n$ or $T_{n+1} = S_n$. If n is the least such integer and $n \geq 1$ then S_i, T_i are not in ascending order for $0 \leq i < n$.

Proof. (i) By induction on n. If $n = 2$ then this is trivial. If $n \geq 3$ let $i_1, \dots, i_n$ be a sequence and let i_q be the largest occurring integer. By the inductive hypothesis, applying $(*)$ first arranges everything to the left of i_q in ascending order, then shifts i_q to the right. It then orders everything to the left of i_q in ascending order and shifts i_q to the right, and so on, until i_q reaches the last position, and then $(*)$ arranges the whole sequence in ascending order.

(ii) Wlog we may assume $i_q > i_{q+1}$. If q is the first position with $i_q > i_{q+1}$ then the result is trivial with $n = 0$. Otherwise recursively applying $(*)$ to S_0 and T_0 will first rearrange $i_1, \dots, i_{q-1}$ to be in ascending order. Say this happens at stage $n \geq 1$. Then S_n and T_n agree up to position $q - 1$. Let a be the $q - 1$th integer in both sequences. There are two cases:

(i) If $a < i_q$ then it is clear that applying $(*)$ to S_n gives T_n, so we are done.

(ii) If $a > i_q$ then also $a > i_{q+1}$. So S_{n+2} and T_{n+2} look like $\dots, i_q, i_{q+1}, a, \dots$ and $\dots, i_{q+1}, i_q, a, \dots$ respectively, where they agree to the first $q - 2$ places.

If we continue to apply $(*)$ then we either terminate in case (i) or continue to move integers across to the right of i_q, i_{q+1} and i_{q+1}, i_q until there is nothing left. If this happens at stage m then clearly applying $(*)$ to S_m gives T_m, which completes the proof. This proof finds the smallest $n \geq 0$ with the desired property, and the result is always $S_{n+1} = T_n$, where S was the sequence with $i_q > i_{q+1}$ (may have originally been T before we renamed). The proof shows that n is smallest with the property that applying $(*)$ to S_n consists of swapping i_q, i_{q+1}. So if $n \geq 1$ is clear that S_i cannot be ascending for $0 \leq i \leq n$. If any T_i were ascending with $0 \leq i < n$ then T_{n-1} would be ascending. But the proof shows that $T_{n-1} \longrightarrow T_n$ is either swapping two integers to the left of i_{q+1}, i_q, or it is $\ldots, i_{q+1}, a, i_q, \ldots \longrightarrow \ldots, i_{q+1}, i_q, a, \ldots$ where $a > i_q$. In either case T_{n-1} is clearly not ascending (although T_n may be). $\square$

Now assume $p \geq 2$ and $2 \leq n \leq p$. The lexicographic order on sequences of n elements from $\{1, \ldots, p\}$ with no element repeated orders the corresponding basis elements of $M^{\otimes n}$. This is a total order, so we can arrange the basis elements in a chain. We describe an algorithm for changing the basis of $M^{\otimes n}$. We do nothing to any basis element $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with some x_i repeated. These basis elements belong to I_n. Now consider the basis elements $x_{i_1} \otimes \cdots \otimes x_{i_n}$ in which no x_i is repeated. Denote by $\sigma(x_{i_1} \otimes \cdots \otimes x_{i_n})$ the basis element corresponding to the sequence obtained by applying $(*)$ to the sequence $i_1, \ldots, i_n$. So if $i_1 < \cdots < i_n$ we do nothing, and otherwise we simply interchange x_{i_s} and $x_{i_{s+1}}$ where s is the first integer with $i_s > i_{s+1}$. Proceed as follows:

1. Begin with the highest basis element $x_{i_1} \otimes \cdots \otimes x_{i_n}$ under the lexicographic order (for example $x_3 \otimes x_2 \otimes x_1$ if $n = p = 3$). Clearly $i_1 > i_2$. In the basis replace $x_{i_1} \otimes \cdots \otimes x_{i_n}$ by

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} + x_{i_2} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} \in I$$

   These elements are still a basis for $M^{\otimes n}$.

2. Consider the next highest basis element under the lexicographic order. If it has the form $i_1 < \cdots < i_n$ then do nothing. Otherwise replace it by

$$x_{i_1} \otimes \cdots \otimes x_{i_n} + \sigma(x_{i_1} \otimes \cdots \otimes x_{i_n}) \in I \tag{2}$$

   That is, apply the operation $(*)$ to the sequence $i_1, \ldots, i_n$ and add the corresponding basis element. Proceeding in this way down the chain, we either do nothing, or add to a basis element another basis element from further "down" the list (i.e. one that hasn't been touched yet) so at each stage we still have a basis.

At the end of this process we have a final basis of p^n elements for $M^{\otimes n}$, divided into three types:

1. $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with a repeated integer in $i_1, \ldots, i_n$.

2. $x_{i_1} \otimes \cdots \otimes x_{i_n} + \sigma(x_{i_1} \otimes \cdots \otimes x_{i_n})$ with no repeats in $i_1, \ldots, i_n$ and $i_s > i_{s+1}$ for some $1 \leq s \leq n-1$.

3. $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with $i_1 < \cdots < i_n$.

Type 1 and 2 generators belong to I_n. We claim that every generator of I_n of the form (ii) in Lemma 19 is a linear combination of the Type 2 basis elements. Let $w = u + v$ be a generator, where $u = x_{i_1} \otimes \cdots \otimes x_{i_n}$ and $v = x_{i_1} \otimes \cdots x_{i_{q+1}} \otimes x_{i_q} \otimes \cdots \otimes x_n$, and wlog assume that $i_q > i_{q+1}$. There are two cases

(i) If q is the first position with $i_q > i_{q+1}$ then $x_{i_1} \otimes \cdots \otimes x_{i_n} + x_{i_1} \otimes \cdots x_{i_{q+1}} \otimes x_{i_q} \otimes \cdots \otimes x_n$ is a basis element of Type 2.

(ii) Write $\sigma^0(u), \sigma^0(v)$ for u, v and $\sigma^n(u), \sigma^n(v)$ for the result of applying σ n times. Assume q is not the first position with $i_q > i_{q+1}$. Then by Lemma 20 there exists $n \geq 1$ with $\sigma^{n+1}(u) = \sigma^n(v)$. Moreover we can assume $\sigma^i(u)$ is not a basis element of Type 3 for $0 \leq i \leq n$ and $\sigma^i(v)$ is not a basis element of Type 3 for $0 \leq i < n$. So $\sigma^n(u) + \sigma^n(v)$ is a basis element of Type 2 and $\sigma^i(u) + \sigma^{i+1}(u), \sigma^i(v) + \sigma^{i+1}(v)$ are basis elements of Type 2 for $0 \leq i < n$. Proceeding recursively

$$\begin{aligned}
u &= (u + \sigma(u)) + (v + \sigma(v)) - (\sigma(u) + \sigma(v)) \\
&= (u + \sigma(u)) + (v + \sigma(v)) - (\sigma(u) + \sigma^2(u)) - (\sigma(v) + \sigma^2(v)) + (\sigma^2(u) + \sigma^2(v)) \\
&\vdots \\
&= \cdots \pm (\sigma^n(u) + \sigma^n(v))
\end{aligned}$$

Each summand in the final expression is $\pm$ a basis element of Type 2, so we are done.

Therefore the basis elements of Type 1 and 2 give a basis for the R-submodule I_n of $M^{\otimes n}$ and consequently the residues of the basis elements of Type 3 give a basis for $M^{\otimes n}/I_n$. Hence

Proposition 21. *Let M be a free R-module of finite rank $p \geq 1$ and let $\{x_1, \ldots, x_p\}$ be a basis. Then for $n \geq 0$ the R-module $\wedge^n M$ is isomorphic to $M^{\otimes n}/I_n$ and is therefore free of finite rank*

$$rank_R(\wedge^n M) = \begin{cases} \binom{p}{n} & 0 \leq n \leq p \\ 0 & n > p \end{cases}$$

For $1 \leq n \leq p$ a basis for $\wedge^n M$ is given by $x_{i_1} \wedge \cdots \wedge x_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq p$.

Proof. If $n = 0$ then $\wedge^n M$ is isomorphic to R, which has rank 1. If $n = 1$ then $\wedge^1 M$ is isomorphic to M and the images of the x_i give a basis. If $n > p$ then $I_n = M^{\otimes n}$ so $\wedge^n M = 0$. So we can reduce to the case where $p \geq 2$ and $2 \leq n \leq p$. In this case the above shows that the elements $x_{i_1} \wedge \cdots \wedge x_{i_n}$ with $i_1 < \cdots < i_n$ are a basis of $\wedge^n M$. Finding a strictly ascending sequence $i_1, \ldots, i_n$ in the set $\{1, \ldots, p\}$ consists of choosing n elements, so clearly there are $\binom{p}{n}$ of them. $\square$

Definition 7. Let M be a free R-modules of finite rank $p \geq 1$. Given a morphism of R-modules $\varphi : M \longrightarrow M$ the morphism $\wedge^p \varphi : \wedge^p M \longrightarrow \wedge^p M$ is a morphism of free R-modules of rank 1. It is therefore of the form $x \mapsto \lambda \cdot x$ for some unique $\lambda \in R$, which we call the *determinant* of φ.

Lemma 22. *Let φ be an endomorphism of a free R-module M of finite rank $p \geq 1$. Let $\{x_1, \ldots, x_p\}$ be a basis of M and $A = (a_{ij})$ the matrix of φ relative to this basis. Then*

$$det(\varphi) = \sum_{\sigma \in S_p} sgn(\sigma) a_{1\sigma(1)} \cdots a_{p\sigma(p)}$$

Proof. If $p = 1$ then the only permutation is the identity and the result is trivial, so assume $p \geq 2$. Then $x_1 \wedge \cdots \wedge x_p$ is a basis for $\wedge^p M$ and we have

$$\begin{aligned}
(\wedge^p \varphi)(x_1 \wedge \cdots \wedge x_p) &= \varphi(x_1) \wedge \cdots \wedge \varphi(x_p) \\
&= (a_{11}x_1 + \cdots + a_{p1}x_p) \wedge \cdots \wedge (a_{p1}x_1 + \cdots + a_{pp}x_p)
\end{aligned}$$

when we expand this out, any terms with repeated x_i's will vanish, and we are left with a sum over the permutations of $\{1, \ldots, p\}$. Rewriting each basis element in terms of $x_1 \wedge \cdots \wedge x_p$ we pick up the sign of the corresponding permuation by Lemma 10, so the proof is complete. $\square$

3.2 Bilinear Forms

Throughout this section all rings are commutative, and R denotes an arbitrary commutative ring unless there is some indication otherwise.

Definition 8. Let M, N, K be R-modules. A *bilinear form* or *bilinear pairing* $f : M \times N \longrightarrow K$ is a multilinear function (see Definition 2) with the further property that for $a \in R, m \in M, n \in N$ we have $f(am, n) = f(m, an) = af(m, n)$. There is a canonical bijection between such bilinear pairings and morphisms of R-modules $M \otimes N \longrightarrow K$. If $f : M \times N \longrightarrow K$ is a bilinear pairing then so is the twisted pairing $g : N \times M \longrightarrow K$ defined by $g(n, m) = f(m, n)$.

Lemma 23. *Let M, N be R-modules. There is a canonical bijection between bilinear pairings $f : M \times N \longrightarrow R$ and morphisms of R-modules $M \longrightarrow N^\vee$, which associates f with the morphism*

$$F : M \longrightarrow N^\vee$$
$$F(m)(n) = f(m, n)$$

Proof. Here $N^\vee$ denotes the dual module $Hom_R(N, R)$. We already know there is a bijection between bilinear pairings $M \times N \longrightarrow R$ and morphisms of R-modules $M \otimes N \longrightarrow R$. But there is a canonical bijection

$$Hom_R(M \otimes N, R) \cong Hom_R(M, Hom_R(N, R)) = Hom_R(M, N^\vee)$$

which clearly maps f to the required morphism. $\square$

Example 1. Let M be an R-module. The map $ev : M^\vee \times M \longrightarrow R$ defined by $(\nu, m) \mapsto \nu(m)$ is a bilinear pairing, which corresponds to the identity $M^\vee \longrightarrow M^\vee$. This is called the *evaluation pairing.*

Definition 9. Let M, N be R-modules. We say a bilinear pairing $f : M \times N \longrightarrow R$ is *perfect* if the corresponding morphism of R-modules $M \longrightarrow N^\vee$ is an isomorphism.

Remark 1. Let R be a nonzero ring, M a free R-module of positive finite rank n. A basis $\{x_1, \ldots, x_n\}$ for M is a collection of morphisms $x_i : R \longrightarrow M$ which form a coproduct in $R\mathbf{Mod}$. Corresponding to each of these injections into the coproduct is a projection $x_i^* : M \longrightarrow R$, and these form a basis for the dual module $M^\vee = Hom_R(M, R)$. The map $x_i \mapsto x_i^*$ defines an isomorphism of R-modules $M \cong M^\vee$.

Remark 2. If R is nonzero and M, N are free of finite rank, then the existence of a perfect pairing $f : M \times N \longrightarrow R$ implies that $rank(M) = rank(N)$.

Lemma 24. *Let k be a field, M, N free k-modules of the same finite rank. Then a bilinear pairing $f : M \times N \longrightarrow R$ is perfect if and only if $f(m, n) = 0$ for all $n \in N$ implies $m = 0$ and $f(m, n) = 0$ for all $m \in M$ implies $n = 0$.*

Proof. Suppose that the bilinear pairing $f : M \times N \longrightarrow R$ is perfect. If $M, N = 0$ the result is trivial, so assume otherwise and let $y_1, \ldots, y_n$ be a basis of N with dual basis $y_1^*, \ldots, y_n^*$ for $N^\vee$. By assumption f gives rise to an isomorphism $F : M \longrightarrow N^\vee$. If $f(m, n) = 0$ for all $n \in N$ then $F(m) = 0$ and consequently $m = 0$. On the other hand if $F(m)(n) = f(m, n) = 0$ for all $m \in M$, then in particular since F is surjective we have $y_i^*(n) = 0$ for $1 \le i \le n$. It follows that $n = 0$, as required (this direction works over an arbitrary nonzero ring).

Conversely, let $f : M \times N \longrightarrow R$ be a bilinear pairing satisfying the given condition. Then the corresponding morphism $F : M \longrightarrow N^\vee$ is certainly injective, and by a rank calculation it must be surjective as well. This shows that f is perfect, and completes the proof. $\square$

Remark 3. If R is not a field then Lemma 24 does not necessarily hold, which is why in Definition 9 one has to be careful to generalise the right property of a perfect pairing of vector spaces (the isomorphism $M \longrightarrow N^\vee$ is what we really need). For example take $M = N = R = \mathbb{Z}$. Then the map $f : M \times N \longrightarrow R$ defined by $f(a, b) = 2ab$ satisfies the condition of the Lemma, but the corresponding morphism is $\mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $n \mapsto 2n$, which is certainly not an isomorphism (this example was kindly poined out to me by Laurent Busé).

Lemma 25. *Let R be a nonzero ring, M, N free R-modules of the same positive finite rank n and $f : M \times N \longrightarrow R$ a bilinear pairing. Then f is perfect if and only if there exists bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of M, N respectively, such that $f(x_i, y_j) = \delta_{ij}$ for $1 \le i, j \le n$.*

Proof. Suppose that f is perfect, and let $\{y_1, \ldots, y_n\}$ be a basis of N. Let x_i be the image of y_i under the isomorphism $N \cong N^\vee \cong M$ determined by f and the chosen basis on N. Then x_i is the unique element of M with $f(x_i, y_j) = \delta_{ij}$ for all $1 \le j \le n$. The basis $\{x_1, \dots, x_n\}$ has the property that

$$f(x_i, y_j) = \delta_{ij} \quad 1 \le i, j \le n$$

as required. Conversely suppose such a pair of bases exist for M, N. We have to show that the morphism $F : M \longrightarrow N^\vee$ defined by $F(m)(n) = f(m, n)$ is an isomorphism. But this is clear, since $F(x_i) = y_i^*$, so the proof is complete. $\square$

Corollary 26. *Let R be a nonzero ring, M, N free R-modules of the same positive finite rank n. Then a bilinear pairing $f : M \times N \longrightarrow R$ is perfect if and only if the twisted pairing $g : N \times M \longrightarrow R$ is perfect.*

Definition 10. Let R be a nonzero ring, M, N free R-modules of positive finite rank and $f : M \times N \longrightarrow R$ a perfect pairing. Given a basis $\{y_1, \dots, y_n\}$ of N we call the basis $\{x_1, \dots, x_n\}$ constructed in Lemma 25 the f-*dual basis* of $\{y_1, \dots, y_n\}$. It has the property that

$$f(x_i, y_j) = \delta_{ij} \quad 1 \le i, j \le n$$

Using the twisted pairing $g : N \times M \longrightarrow R$ we can associate with any basis $\{x_1, \dots, x_n\}$ of M a g-dual basis $\{y_1, \dots, y_n\}$ of N, with the property that $f(x_i, y_j) = \delta_{ij}$ for $1 \le i, j \le n$. It is clear that these two assignments are mutually inverse, so that the perfect pairing f sets up a bijection between the set of all bases of M and the set of all bases of N.

Example 2. Let R be a nonzero ring, M a free R-module of positive finite rank n. Then $M^\vee$ is a free R-module of rank n, and the following map is clearly a bilinear form

$$f : M \times M^\vee \longrightarrow R$$
$$f(a, \nu) = \nu(a)$$

To see that it is perfect, choose a basis $x_1, \dots, x_n$ of M, and let $x_1^*, \dots, x_n^*$ be the dual basis. Then $f(x_i, x_j^*) = \delta_{ij}$, so by Lemma 25, the pairing f is perfect. This corresponds to a canonical isomorphism of R-modules $\psi : M \longrightarrow (M^\vee)^\vee$ defined by $\psi(m)(\nu) = \nu(m)$. This is clearly natural in M.

Example 3. Let A be a nonzero ring, M, N free A-modules of the same finite rank $n \ge 1$ and choose bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ for M, N respectively. Then $M \otimes_A N$ is free of rank n^2 on the basis $x_i \otimes y_j$ and we define a morphism of A-modules $M \otimes_A N \longrightarrow A$ by $x_i \otimes y_j \mapsto \delta_{ij}$. This corresponds to a perfect pairing $\alpha : M \times N \longrightarrow A$ with $\alpha(\sum_i \alpha_i x_i, \sum_j \beta_j y_j) = \sum_i (\alpha_i \beta_i)$.

Lemma 27. *Let M, N, K be R-modules, $M' \subseteq M, N' \subseteq N$ submodules and $f : M \times N \longrightarrow K$ a bilinear pairing with the property that $f(m, n) = 0$ whenever $m \in M'$* or *$n \in N'$. Then there is a unique bilinear pairing $f' : (M/M') \times (N/N') \longrightarrow K$ making the following diagram commute*

$$\begin{tikzcd}
(M/M') \times (N/N') \arrow[r, "f'"] & K \\
M \times N \arrow[u] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof. Define $f'(m + M', n + N') = f(m, n)$. One checks that this is a well-defined bilinear pairing, and it clearly possesses the stated uniqueness property. $\square$

Lemma 28. *Let M, N, K be R-modules, $m, n \ge 1$ integers and $f : M^{\otimes m} \times N^{\otimes n} \longrightarrow K$ a bilinear pairing with the property that $f(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n) = 0$ whenever $x_i = x_j$ for some $i \neq j$ or $y_i = y_j$ for some $i \neq j$. Then there is a unique bilinear pairing $F : \wedge^m M \times \wedge^n N \longrightarrow K$ with the property that*

$$F(x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_n) = f(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n)$$

for all $x_1, \dots, x_m \in M$ and $y_1, \dots, y_n \in N$.

Proof. Let I_m, J_n be the R-submodules of $M^{\otimes m}, N^{\otimes n}$ respectively given by the kernels of $M^{\otimes m} \longrightarrow \wedge^m M, N^{\otimes n} \longrightarrow \wedge^n N$. Then the condition on f implies that $f(m,n) = 0$ whenever $m \in I_m$ or $n \in I_n$, so by Lemma 27 and Lemma 9 we have a unique bilinear pairing $F : \wedge^m M \times \wedge^n N \longrightarrow K$ making the following diagram commute

$$\begin{tikzcd} \wedge^m M \times \wedge^n N \arrow[r, "F"] & K \\ M^{\otimes m} \times N^{\otimes n} \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

which is what we wanted to show. ☐

Proposition 29. *Let M, N be R-modules and $B : M \times N \longrightarrow R$ a bilinear pairing. Then there are canonical bilinear pairings for every $n \geq 1$*

$$B^{\otimes n} : M^{\otimes n} \times N^{\otimes n} \longrightarrow R, \quad \wedge^n(B) : \wedge^n M \times \wedge^n N \longrightarrow R$$

with the following properties for $x_1, \dots, x_n \in M$ and $y_1, \dots, y_n \in N$

$$B^{\otimes n}(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \prod_{i=1}^n B(x_i, y_i)$$

$$\wedge^n(B)(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n B(x_i, y_{\sigma(i)}) = det(B(x_i, y_j))$$

Proof. The product $\prod_{i=1}^n B(x_i, y_i)$ is clearly mutlilinear, so there is an induced morphism out of the tensor product $M \otimes \cdots \otimes M \otimes N \otimes \cdots \otimes N$. Composed with the canonical isomorphism of R-modules defined in Proposition 5 we have the desired bilinear pairing $B^{\otimes n}$.

The determinant $det(B(x_i, y_j))$ is multilinear, so as before we obtain an induced bilinear pairing

$$T : M^{\otimes n} \times N^{\otimes n} \longrightarrow R$$
$$T(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = det(B(x_i, y_j))$$

This satisfies the condition of Lemma 28, which produces the desired bilinear pairing $\wedge^n B$. ☐

Example 4. Let M be an R-module. The evaluation pairing $ev : M^\vee \times M \longrightarrow R$ induces bilinear pairings $ev^{\otimes n} : (M^\vee)^{\otimes n} \times M^{\otimes n} \longrightarrow R$ and $\wedge^n ev : \wedge^n(M^\vee) \times \wedge^n M \longrightarrow R$ with

$$ev^{\otimes n}(\nu_1 \otimes \cdots \otimes \nu_n, m_1 \otimes \cdots \otimes m_n) = \prod_{i=1}^n \nu_i(m_i)$$

$$\wedge^n(ev)(\nu_1 \wedge \cdots \wedge \nu_n, m_1 \wedge \cdots \wedge m_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n \nu_i(m_{\sigma(i)})$$

By Lemma 23 these correspond to canonical morphisms of R-modules $\alpha : (M^\vee)^{\otimes n} \longrightarrow (M^{\otimes n})^\vee$ and $\beta : \wedge^n(M^\vee) \longrightarrow (\wedge^n M)^\vee$ with

$$\alpha(\nu_1 \otimes \cdots \otimes \nu_n)(m_1 \otimes \cdots \otimes m_n) = \prod_{i=1}^n \nu_i(m_i)$$

$$\beta(\nu_1 \wedge \cdots \wedge \nu_n)(m_1 \wedge \cdots \wedge m_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n \nu_i(m_{\sigma(i)})$$

for $\nu_1, \dots, \nu_n \in M^\vee$ and $m_1, \dots, m_n \in M$.

Proposition 30. *Let R be a nonzero ring, M a free R-module of positive finite rank r. Then the bilinear pairings*

$$ev^{\otimes n} : (M^\vee)^{\otimes n} \times M^{\otimes n} \longrightarrow R, \quad \wedge^n ev : \wedge^n(M^\vee) \times \wedge^n M \longrightarrow R$$

are perfect. If $\{x_i\}$ is a basis of M with dual basis $\{x_i^*\}$ of $M^\vee$ then the dual basis to $\{x_{i_1} \otimes \cdots \otimes x_{i_n}\}$ is $\{x_{i_1}^* \otimes \cdots \otimes x_{i_n}^*\}$ (with $1 \le i_j \le r$) and the dual basis to $\{x_{i_1} \wedge \cdots \wedge x_{i_n}\}$ is $\{x_{i_1}^* \wedge \cdots \wedge x_{i_n}^*\}$ (where $1 \le i_1 < \cdots < i_n \le r$).

Proof. Let $\{x_i\}$ be a basis of M with dual basis $\{x_i^*\}$. Then we know that the given sets of elements are bases for the respective modules, so it suffices by Lemma 25 to check that they behave correctly with respect to the given bilinear pairings, which is easy to do. □

Corollary 31. *Let R be a nonzero ring, M a free R-module of positive finite rank. Then for $n \ge 1$ the canonical morphisms of R-modules*

$$\alpha : (M^\vee)^{\otimes n} \longrightarrow (M^{\otimes n})^\vee$$
$$\beta : \wedge^n(M^\vee) \longrightarrow (\wedge^n M)^\vee$$

defined in Example 4 are isomorphisms.

Proof. This is an immediate consequence of Proposition 30 and the definition of a perfect pairing. □

3.3 Other Properties

Let R be a commutative ring, M an R-module. The product in the graded R-algebra $\bigwedge M$ induces a morphism of R-modules for $a, b \ge 1$

$$\bigwedge^a M \otimes \bigwedge^b M \longrightarrow \bigwedge^{a+b} M$$
$$(m_1 \wedge \cdots \wedge m_a) \otimes (n_1 \wedge \cdots \wedge n_b) \mapsto m_1 \wedge \cdots \wedge m_a \wedge n_1 \wedge \cdots \wedge n_b$$

3.3.1 The determinant formula

Let R be a nonzero commutative ring, and suppose we have an exact sequence of free R-modules

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

Identify L with a submodule of M and assume that L, N are free of finite ranks $a, b \ge 1$ respectively, so that M is free of rank $a + b$. Find a basis $x_1, \ldots, x_a, x_{a+1}, \ldots, x_{a+b}$ of M with $x_1, \ldots, x_a$ a basis of L and the images $\psi(x_{a+1}), \ldots, \psi(x_{a+b})$ a basis of N. We set $y_i = x_{a+i}$ for $1 \le i \le b$ and write $X = x_1 \wedge x_2 \wedge \cdots \wedge x_a \in \wedge^a L$ and $Y = y_1 \wedge \cdots \wedge y_b \in \wedge^b N$. The module $\wedge^a L \otimes \wedge^b M$ is free of rank $\binom{a+b}{b}$ with basis

$$X \otimes (x_{i_1} \wedge \cdots \wedge x_{i_b}) \text{ where } 1 \le i_1 < \cdots < i_b \le a + b$$

Tensoring the canonical morphism of R-modules $\wedge^a L \longrightarrow \wedge^a M$ with $\wedge^b M$ and then composing with the product $\wedge^a M \otimes \wedge^b M \longrightarrow \wedge^{a+b} M$ gives an epimorphism of R-modules

$$\theta : \bigwedge^a L \otimes \bigwedge^b M \longrightarrow \bigwedge^{a+b} M$$
$$X \otimes (x_{i_1} \wedge \cdots \wedge x_{i_b}) \mapsto x_1 \wedge \cdots \wedge x_a \wedge x_{i_1} \wedge \cdots \wedge x_{i_b}$$

Tensoring the canonical morphism of R-modules $\wedge^b M \longrightarrow \wedge^b N$ with $\wedge^a L$ gives an epimorphism of R-modules

$$\rho : \bigwedge^a L \otimes \bigwedge^b M \longrightarrow \bigwedge^a L \otimes \bigwedge^b N$$
$$X \otimes (x_{i_1} \wedge \cdots \wedge x_{i_b}) \mapsto X \otimes (\psi(x_{i_1}) \wedge \cdots \wedge \psi(x_{i_b}))$$

Proposition 32. *Let R be a nonzero commutative ring and suppose we have an exact sequence of free R-modules*

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

where L, N are free of nonzero finite ranks a, b respectively. Then there is a unique isomorphism of R-modules $\kappa : \bigwedge^a L \otimes \bigwedge^b N \longrightarrow \bigwedge^{a+b} M$ fitting into the following commutative diagram

$$\begin{tikzcd}
 & \bigwedge^a L \otimes \bigwedge^b N \arrow[dd, Rightarrow, "\kappa"] \\
\bigwedge^a L \otimes \bigwedge^b M \arrow[ur, "\rho"] \arrow[dr, "\theta"'] & \\
 & \bigwedge^{a+b} M
\end{tikzcd}$$

Proof. Since ρ, θ are both epimorphisms, it suffices to show that they have the same kernel. Let $\alpha = \sum_{i_1 < \cdots < i_b} \alpha_{i_1,\ldots,i_b} \cdot (X \otimes (x_{i_1} \wedge \cdots \wedge x_{i_b}))$ be an arbitrary element of $\wedge^a L \otimes \wedge^b M$. Then

$$\theta(\alpha) = \sum_{i_1 < \cdots < i_b} \alpha_{i_1,\ldots,i_b} \cdot (x_1 \wedge \cdots \wedge x_a \wedge x_{i_1} \wedge \cdots \wedge x_{i_b})$$

Since elements of degree one in $\bigwedge M$ have square zero, any summand with $i_1 \leq a$ is zero. The only summand left is $\alpha_{a+1,\ldots,a+b} \cdot (x_1 \wedge \cdots \wedge x_a \wedge x_{a+1} \wedge \cdots \wedge x_{a+b})$. Therefore $\theta(\alpha) = 0$ if and only if $\alpha_{a+1,\ldots,a+b} = 0$. On the other hand,

$$\begin{aligned}
\rho(\alpha) &= \sum_{i_1 < \cdots < i_b} \alpha_{i_1,\ldots,i_b} \cdot (X \otimes (\psi(x_{i_1}) \wedge \cdots \wedge \psi(x_{i_b}))) \\
&= \alpha_{a+1,\ldots,a+b} \cdot (X \otimes (\psi(y_1) \wedge \cdots \wedge \psi(y_b)))
\end{aligned}$$

Since by definition ψ is zero on L. Therefore $\rho(\alpha) = 0$ if and only if $\alpha_{a+1,\ldots,a+b} = 0$, which completes the proof. $\square$

4 The Symmetric Algebra

In this section R is a commutative ring. Let M be an R-module and $T(M)$ the tensor algebra. Let S be the following subset of $T(M)$: $\{x \otimes y - y \otimes x \,|\, x, y \in M\}$. Let I be the two sided ideal generated by S. Then I is homogenous and so $S(M) = T(M)/I$ is a graded R-algebra, and the projection $T(M) \longrightarrow S(M)$ is a morphism of graded R-algebras. For $n \geq 0$ we denote the n-th homogenous piece by $S^n(M)$. This is an R-submodule of $S(M)$. It is not difficult to see that $S(M)$ is a commutative R-algebra.

- The proof of Lemma 1 makes it clear that every nonzero homogenous element of I has degree ≥ 2. So the ring morphism $R \longrightarrow T(M) \longrightarrow S(M)$ and gives an isomorphism of R with $S^0(M)$. Also the morphism of R-modules $M \longrightarrow T(M) \longrightarrow S(M)$ gives an isomorphism of M with $S^1(M)$. As usual we denote the image of $r \in R$ and $m \in M$ by r, m respectively.

- For $n \geq 2$ the map $M^{\otimes n} \longrightarrow T(M) \longrightarrow S(M)$ is not necessarily injective, but it is a morphism of R-modules whose image is equal to $S^n(M)$. For elements $m_1, \ldots, m_n \in M$ we denote the image of $m_1 \otimes \cdots \otimes m_n$ in $S(M)$ by $m_1 \cdots m_n$.

Proposition 33. *We have the following properties of the symmetric algebra* *(a) Given any morphism of R-modules $\varphi : M \longrightarrow N$ there is a unique morphism of commutative R-algebras $S(\varphi) : S(M) \longrightarrow S(N)$ making the following diagram of R-modules commute*

$$\begin{tikzcd} S(M) \arrow[r] & S(N) \\ M \arrow[u] \arrow[r, "\varphi"'] & N \arrow[u] \end{tikzcd}$$

This defines a functor $S : R\mathbf{Mod} \longrightarrow R\mathbf{Alg}$. Also note that $S(\varphi)$ is a morphism of graded R-algebras, so the construction also defines a functor $S : R\mathbf{Mod} \longrightarrow R\mathbf{GrAlg}$.

(b) Let $\varphi : M \longrightarrow B$ be a morphism of R-modules, where B is a commutative R-algebra. Then there is a unique morphism of R-algebras $\Phi : S(M) \longrightarrow B$ making the following diagram commute

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & B \\ S(M) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Moreover if B is a graded R-algebra and $Im\varphi \subseteq B_1$ then Φ is a morphism of graded R-algebras.

(c) The functor $S : R\mathbf{Mod} \longrightarrow R\mathbf{Alg}$ is left adjoint to the forgetful functor $F : R\mathbf{Alg} \longrightarrow R\mathbf{Mod}$. The unit of the adjunction is the natural transformation $\eta : 1 \longrightarrow FS$ given by the canonical morphism of R-modules $\eta_M : M \longrightarrow S(M)$.

(d) Let M, N be R-modules. Then there is a canonical isomorphism of R-algebras $S(M) \otimes_R S(N) \longrightarrow S(M \oplus N)$ defined for $m \in M, n \in N$ by $m \otimes n \mapsto (m, 0) \otimes (n, 0)$.

(e) The functor $S : R\mathbf{Mod} \longrightarrow R\mathbf{GrAlg}$ is left adjoint to the functor $(-)_1 : R\mathbf{GrAlg} \longrightarrow R\mathbf{Mod}$ which maps a commutative graded R-algebra to its degree 1 component. The unit of the adjunction is the natural transformation $1 \longrightarrow (-)_1 S$ given by the isomorphism $M \longrightarrow S^1(M)$.

Proof. (a) The morphism of graded R-algebras $T(\varphi) : T(M) \longrightarrow T(N)$ maps $x \otimes y - y \otimes x$ to $\varphi(x) \otimes \varphi(y) - \varphi(y) \otimes \varphi(x)$, so it is easy to see that we induce a morphism of commutative graded R-algebras $S(M) \longrightarrow S(N)$ with the required property. (b) Follows easily from the results for $T(M)$. (c) and (e) follow from (b) and (d) follows from (c). □

Lemma 34. *Let F be a free R-module of rank $n \geq 1$. Then for any basis $f_1, \ldots, f_n$ there is a canonical isomorphism of graded R-algebras*

$$\beta : R[x_1, \ldots, x_n] \longrightarrow S(F)$$
$$x_i \mapsto f_i$$

Proof. Let $g : F \longrightarrow R[x_1, \ldots, x_n]$ be the morphism of R-modules with $f_i \mapsto x_i$. Using the uniqueness property of reflections (see our Borceaux notes) it suffices to show that the pair $(g, R[x_1, \ldots, x_n])$ is a reflection of F along the forgetful functor $R\mathbf{Alg} \longrightarrow R\mathbf{Mod}$. Let $\varphi : F \longrightarrow B$ be a morphism of R-modules, where B is a commutative R-algebra, and let $\Phi : R[x_1, \ldots, x_n] \longrightarrow B$ be the R-algebra morphism induced by the tuple $(\varphi(f_1), \ldots, \varphi(f_n))$. This is clearly unique with the property that $\Phi g = \varphi$, as required. □

The following diagram summarises the constructions of this note. In this diagram the adjacent arrows represent adjoint functors, with the left adjoint on the left.