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Chern-Simons theory as an example of a TQFT

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Literature: E. Witten
„Quantum Field Theory and the Jones polynomial"

Outline: – Basics of Chern-Simons theory
– Relation to knot theory

1. Chern-Simons theory

Example of a 2+1D TQFT (2+1D refers to a metric, so 3D may be more appropriate)

Ingredients:

- Closed oriented 3-manifold (framed $\leftrightarrow$ triv. of tangent bdl.)
- A Lie group G (compact, simple simply-connected)
- A Lie(G)-valued connection 1-form A

Jargon:

Physics: A is a gauge field

Mathematics: Principal G-bundle $P \xrightarrow{\pi} M$

This data is used to define an action functional

$$S[A] = \frac{k}{4\pi} \int_M d^3x \, \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

where the level $k \in \mathbb{Z}$ is a parameter.
(quantization explained later)

The associated path integral measure

$$\mathcal{D}\mu[A] = e^{iS[A]}\mathcal{D}A$$

„defines" G_k Chern–Simons theory (physically)

<u>Remark:</u> Formally, $S[A]$ only depends on the topology of $M \longrightarrow$ TQFT
(in practice, gauge fixing requires a metric...)

<u>Def:</u> A gauge transformation is defined by

$$A \longmapsto A^g = g^{-1}Ag + g^{-1}dg$$

automorphism of $P \xrightarrow{\pi} M$

<u>Def:</u> Covariant derivative D and curvature F are defined by $D = d + A$ and $F = D^2 = dA + A\wedge A$

<u>Lemma:</u> Under gauge transformations,

$$D \longmapsto g^{-1}Dg \quad \text{and} \quad F \longmapsto g^{-1}Fg$$

<u>Lemma:</u> $\mathcal{D}\mu[A]$ is invariant under gauge transformations

<u>Proof:</u> An explicit calculation gives rise to

$$\delta S[A] = S[A^g] - S[A]$$

$$= \underbrace{-\frac{k}{12\pi}\int_M tr\left(g^{-1}dg\right)^3}_{\sim \text{ winding number } (\pi_3(G) = \mathbb{Z})} \in 2\pi k\mathbb{Z}$$

Lemma: The stationary points of $S[A]$ are flat connections ($F=0$)

Wilson loops

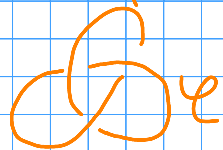
Def: For every closed path $\mathcal{C}$ and representation ρ of $\mathrm{Lie}(G)$ we define the Wilson loop (holonomy)

$$W_{\mathcal{C}}[A|\rho] = \mathrm{tr}\Big[\underbrace{\mathcal{P} e^{\oint_{\mathcal{C}} \rho(A)}}_{\text{path ordered exponential}}\Big]$$

Lemma: $W_{\mathcal{C}}[A|\rho]$ is gauge invariant

Expectation value: $$\langle W_{\mathcal{C}}[A|\rho]\rangle = \int \underbrace{\mathcal{D}A}_{\substack{\text{connections}\\ \text{modulo gauge}\\ \text{transformations}}} W_{\mathcal{C}}[A|\rho]\, e^{iS[A]}$$

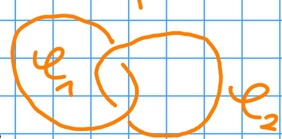
trefoil



Note:
- $\mathcal{C}$ can be complicated, e.g. a knot
- The definition can be extended to links

$$(*) \quad \langle W_{\mathcal{C}_1}[A|\rho_1] \cdots W_{\mathcal{C}_n}[A|\rho_n]\rangle$$

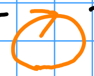
Hopf link

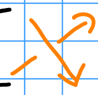
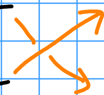
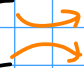


- The value will depend on a choice of <u>framing</u>

Claim: $(*)$ gives rise to the Jones polynomial for $G = SU(2)$ and ρ = fundamental and to the Kauffman polynomial for $G = SO(N)$ and ρ = fundamental.

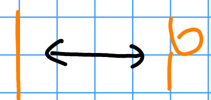
Def: Jones polynomial V for oriented links

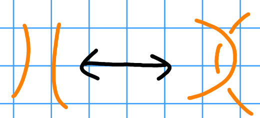
i) $V[$  $] = 1$

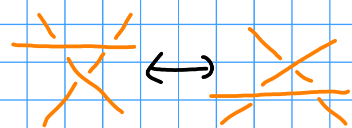
ii) $t^{-1} V[$  $] - t\, V[$  $] = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\, V[$  $]$

Remark: These relations can also be deduced from $SU(2)_k$ Chern-Simons theory (with $t = e^{\frac{2\pi i}{k+2}}$)

Theorem: The Jones polynomial is a link invariant, i.e. it is invariant under the Reidemeister moves

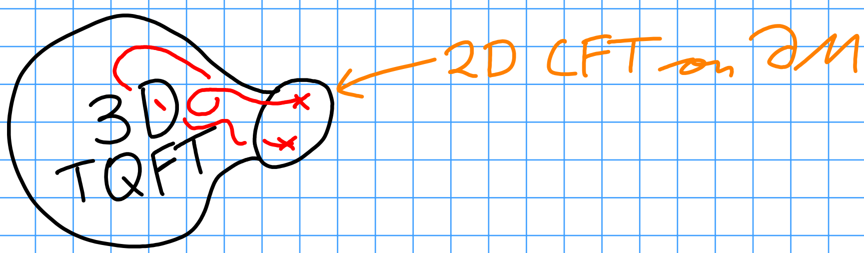
i) 

ii) 

iii) 

Remark: Chern-Simons theory also has links to

- CFT / WZW models
- Affine Kac-Moody algebras
- Quantum groups $\mathcal{U}_q(\mathfrak{g})$ (q root of unity)



Side calculation:

Before we start the actual calculation, we write

$$\mathrm{tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]$$
$$= \mathrm{tr}\left[A\wedge F - \frac{1}{3}A\wedge A\wedge A\right]$$

Now one finds

$$\mathrm{tr}\left[A^g\wedge F^g\right] = \mathrm{tr}\left[\left(g^{-1}Ag + g^{-1}dg\right)\wedge g^{-1}Fg\right]$$
$$= \mathrm{tr}\left[A\wedge F + dg\,g^{-1}\wedge F\right]$$

$$-\frac{1}{3}\mathrm{tr}\left(A^g\wedge A^g\wedge A^g\right) = -\frac{1}{3}\mathrm{tr}\left[\left(g^{-1}Ag + g^{-1}dg\right)^3\right]$$
$$= \mathrm{tr}\left[-\frac{1}{3}A\wedge A\wedge A - A\wedge dg\,g^{-1}\wedge dg\,g^{-1} - A\wedge A\wedge dg\,g^{-1} - \frac{1}{3}\left(g^{-1}dg\right)^3\right]$$

We hence conclude that

$$\delta S[A] = S[A^g] - S[A]$$
$$= \frac{k}{4\pi}\int_M \mathrm{tr}\left[dg\,g^{-1}\wedge F - A\wedge dg\,g^{-1}\wedge dg\,g^{-1} - A\wedge A\wedge dg\,g^{-1} - \frac{1}{3}\left(g^{-1}dg\right)^3\right]$$
$$= \frac{k}{4\pi}\int_M d\,\mathrm{tr}\left[-dg\,g^{-1}\wedge A\right] \longrightarrow \text{vanishes if } \partial M = \phi$$
$$-\frac{k}{12\pi}\int_M \mathrm{tr}\left(g^{-1}dg\right)^3$$

Indeed: $d\,\mathrm{tr}\left[-dg\,g^{-1}\wedge A\right] = \mathrm{tr}\left[-g^{-1}dg\,g^{-1}dg\,A + dg\,g^{-1}\wedge dA\right]$

Def: Kauffman bracket for links

i) $\langle \bigcirc \rangle = 1$

ii) $\langle \bigcirc \sqcup L \rangle = -(A^2 + A^{-2}) \langle L \rangle$

iii) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

Def: The Jones polynomial is defined by

$$V[L] = (-A)^{-w(L)} \langle L \rangle$$

where $w(L)$ is the writhe of L

(# pos crossings − # negative crossings)