Spectral Sequences

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In this note we give a minimal presentation of spectral sequences following EGA. We cover essentially only that part of the theory needed in algebraic geometry. In Section 2 we start with a filtration of a complex, and show how the various pieces of a spectral sequence arise. Applying these observations in Section 3 to two natural filtrations of the total complex of a bicomplex, we deduce two spectral sequences and discover that their first three pages are not mysterious at all: they consist of very natural invariants of the bicomplex, arranged in the obvious way. With this background we can study the Grothendieck spectral sequence in Section 4.

Contents

1 Definitions

As usual we assume that our abelian categories all come with canonical structures, allowing us to define cokernels, kernels, images and direct sums in a canonical way. If we have two subobjects X, Y of an object A then $X \subseteq Y$ means that X precedes Y as a subobject (i.e. the morphism $X \longrightarrow A$ factors through $Y \longrightarrow A$). We use the notation and conventions of our notes on Derived Functors (DF) and Abelian Categories (AC). In particular we tend to denote the differential of any complex X by $\partial^n : X^n \longrightarrow X^{n+1}$. The definitions in this section follow EGA III Ch.0 §11.1.

Definition 1. Let $\mathcal{A}$ be an abelian category and X an object of $\mathcal{A}$. A *filtration* (or *decreasing filtration*) or X is a sequence of subobjects of X

$$\cdots \supseteq F^0(X) \supseteq F^1(X) \supseteq \cdots \supseteq F^p(X) \supseteq \cdots$$

If they exist, we write $inf(F^p(X))$ for the intersection $\cap_p F^p(X)$ and $sup(F^p(X))$ for the union $\cup_p F^p(X)$. We say that the filtration is *separated* if $inf(F^p(X)) = 0$ and *coseparated* or *exhaustive* if $sup(F^p(X)) = X$. We say that the filtration is *discrete* if there exists $p \in \mathbb{Z}$ with $F^p(X) = 0$, and *codiscrete* if there exists $p \in \mathbb{Z}$ with $F^p(X) = X$.

Definition 2. Let $\mathcal{A}$ be an abelian category and $a \geq 0$ an integer. A *spectral sequence in $\mathcal{A}$ starting on page a* consists of the following elements:

(a) An object E_r^{pq} of $\mathcal{A}$ for every $p, q \in \mathbb{Z}$ and $r \geq a$.

(b) A morphism $d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,q-r+1}$ for $p, q \in \mathbb{Z}$ and $r \geq a$ such that $d_r^{p+r,q-r+1} d_r^{pq} = 0$. If we set $Z_{r+1}(E_r^{pq}) = Ker(d_r^{pq}), B_{r+1}(E_r^{pq}) = Im(d_r^{p-r,q+r-1})$ then

$$B_{r+1}(E_r^{pq}) \subseteq Z_{r+1}(E_r^{pq}) \subseteq E_r^{pq}$$

(c) An isomorphism $\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \longrightarrow E_{r+1}^{pq}$ for $p, q \in \mathbb{Z}$ and $r \geq a$.

Before continuing, let us introduce some notation. For $k \geq r+1$ one defines by recursion on k subobjects $B_k(E_r^{pq})$ and $Z_k(E_r^{pq})$ of E_r^{pq} as the inverse image, by the canonical morphism $E_r^{pq} \longrightarrow E_r^{pq}/B_{r+1}(E_r^{pq})$, of the subobject of the quotient identified by α_r^{pq} with the subobjects $B_k(E_{r+1}^{pq})$ and $Z_k(E_{r+1}^{pq})$ respectively. Clearly $B_k(E_r^{pq}) \subseteq Z_k(E_r^{pq})$ and for $k \geq r+1$ we deduce a pullback

$$\begin{tikzcd}
B_k(E_r^{pq}) \arrow[r] \arrow[d] & B_k(E_{r+1}^{pq}) \arrow[d] \\
Z_k(E_r^{pq}) \arrow[r] & Z_k(E_{r+1}^{pq})
\end{tikzcd}$$

with horizontal epimorphisms and vertical monomorphisms. From (AC,Lemma 35) we infer that the induced morphism on the cokernels $Z_k(E_r^{pq})/B_k(E_r^{pq}) \longrightarrow Z_k(E_{r+1}^{pq})/B_k(E_{r+1}^{pq})$ is an isomorphism. We can therefore recursively define a canonical isomorphism

$$Z_k(E_r^{pq})/B_k(E_r^{pq}) \longrightarrow E_k^{pq} \qquad k \geq r+1, r \geq a \tag{1}$$

If we set $B_r(E_r^{pq}) = 0$ and $Z_r(E_r^{pq}) = E_r^{pq}$ then we have inclusions

$$\begin{aligned} 0 = B_r(E_r^{pq}) \subseteq B_{r+1}(E_r^{pq}) \subseteq B_{r+2}(E_r^{pq}) \subseteq \cdots \\ \cdots \subseteq Z_{r+2}(E_r^{pq}) \subseteq Z_{r+1}(E_r^{pq}) \subseteq Z_r(E_r^{pq}) = E_r^{pq} \end{aligned} \tag{2}$$

We now return to enumerating the defining data of a spectral sequence:

(d) Two subobjects $B_\infty(E_a^{pq})$ and $Z_\infty(E_a^{pq})$ of E_a^{pq} such that $B_\infty(E_a^{pq}) \subseteq Z_\infty(E_a^{pq})$ and for $k \geq a$ we have $B_k(E_a^{pq}) \subseteq B_\infty(E_a^{pq})$ and $Z_\infty(E_a^{pq}) \subseteq Z_k(E_a^{pq})$. We define $E_\infty^{pq} = Z_\infty(E_a^{pq})/B_\infty(E_a^{pq})$.

(e) A family $\{E^n\}_{n \in \mathbb{Z}}$ of objects of $\mathcal{A}$, each with a filtration $\{F^p(E^n)\}_{p \in \mathbb{Z}}$. We define $gr_p(E^n) = F^p(E^n)/F^{p+1}(E^n)$ for $p, n \in \mathbb{Z}$.

(f) For each pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ an isomorphism $\beta^{pq} : E_\infty^{pq} \longrightarrow gr_p(E^{p+q})$.

The family of objects $\{E^n\}_{n \in \mathbb{Z}}$, without the associated filtrations, is called the *limit* of the spectral sequence. We usually just write E or (E_r^{pq}, E^n) for the spectral sequence, with all other data implicit, and write $E_r^{pq} \Rightarrow E^{p+q}$ to represent the fact that the spectral sequence converges to the family $\{E^n\}_{n \in \mathbb{Z}}$.

Definition 3. Let $\mathcal{A}$ be an abelian category. A spectral sequence $E = (E_r^{pq}, E^n)$ is *weakly convergent* if $B_\infty(E_a^{pq}) = sup_k(B_k(E_a^{pq}))$ and $Z_\infty(E_a^{pq}) = inf_k(Z_k(E_a^{pq}))$. We say that the spectral sequence is *regular* if it is weakly convergent and also

(1) For each pair (p, q) the decreasing sequence $\{Z_k(E_a^{pq})\}_{k \geq a}$ stabilises: that is, we have $Z_k(E_a^{pq}) = Z_{k+1}(E_a^{pq})$ for all sufficiently large k. In this case it follows that $Z_\infty(E_a^{pq}) = Z_k(E_a^{pq})$ for all sufficiently large k (the bound depending on p, q).

(2) For each $n \in \mathbb{Z}$ the filtration $\{F^p(E^n)\}_{p \in \mathbb{Z}}$ is discrete and exhaustive.

We say that the spectral sequence E is *coregular* if it is weakly convergent and also

(3) For each pair (p, q) the increasing sequence $\{B_k(E_a^{pq})\}_{k \geq a}$ stablises. In this case it follows that $B_\infty(E_a^{pq}) = B_k(E_a^{pq})$ for all sufficiently large k (the bound depending on p, q).

(4) For each $n \in \mathbb{Z}$ the filtration $\{F^p(E^n)\}_{p \in \mathbb{Z}}$ is codiscrete.

We say that E is *biregular* if it is both regular and coregular, or equivalently if the following conditions are satisfied

(a) For each pair (p, q) the sequences $\{Z_k(E_a^{pq})\}_{k \geq a}$ and $\{B_k(E_a^{pq})\}_{k \geq a}$ both stabilise and therefore $Z_\infty(E_a^{pq}) = Z_k(E_a^{pq})$ and $B_\infty(E_a^{pq}) = B_k(E_a^{pq})$ for all sufficiently large k (therefore $E_\infty^{pq} \cong E_k^{pq}$).

(b) For each $n \in \mathbb{Z}$ the filtration $\{F^p(E^n)\}_{p \in \mathbb{Z}}$ is discrete and codiscrete, and therefore finite.

Most spectral sequences we will encounter will be biregular.

Remark 1. Let E be a spectral sequence, and suppose that for some $r \geq a$ and $p, q \in \mathbb{Z}$ we have $E_r^{pq} = 0$. It follows from (1) and (2) that the entry of every subsequent page of the spectral sequence is also zero: that is, $E_k^{pq} = 0$ for $k \geq r$. Further, if we fix some $s \geq a$ then for all sufficiently large k we have $Z_k(E_s^{pq}) = B_k(E_s^{pq}) = 0$. In particular we must have $E_\infty^{pq} = 0$.

Definition 4. Let $\mathcal{A}$ be an abelian category and E a biregular spectral sequence in $\mathcal{A}$ starting on page a. We say that E *degenerates on page* $r \geq a$ if for every $p, q \in \mathbb{Z}$ the morphism d_r^{pq} is zero. That is, all the morphisms on the rth page are zero. It follows that we have a chain of canonical isomorphisms

$$E_r^{pq} \cong E_{r+1}^{pq} \cong E_{r+2}^{pq} \cong \cdots$$

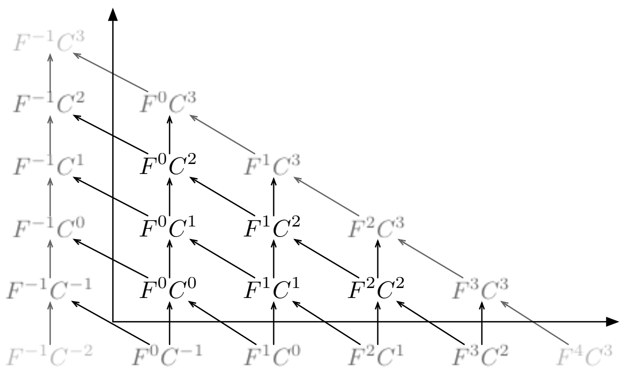
and since E is biregular this eventually stabilises to E_∞^{pq}. That is, for every $k \geq r$ we have an isomorphism $E_k^{pq} \cong E_\infty^{pq}$. Finding a degenerate page of the spectral sequence is one of the most common ways to extract useful information.

2 The Spectral Sequence of a Filtration

Throughout this section let $\mathcal{A}$ be an abelian category. Let C be a complex in $\mathcal{A}$ and suppose we have a decreasing filtration $\{F^p(C)\}_{p \in \mathbb{Z}}$ of C. That is, a sequence of subobjects

$$\cdots \supseteq F^{p-1}(C) \supseteq F^p(C) \supseteq F^{p+1}(C) \supseteq \cdots$$

For $p, q \in \mathbb{Z}$ we place the object $F^pC^{p+q} = F^p(C)^{p+q}$ at position (p, q) to form the following commutative diagram in $\mathcal{A}$



in which all diagonal morphisms are monomorphisms. In other words, we "spread out" the object C^n along the diagonal $p + q = n$. To make the construction of the spectral sequence more transparent, we will allow ourselves to informally talk about "elements" of C^n and its filtrations F^pC^n. The reader may be comforted, however, by our assurance that any occurrences of such informal notation will be accompanied by the equivalent formal statement.

For each $n \in \mathbb{Z}$ we have a filtration $\cdots \supseteq F^{p-1}C^n \supseteq F^pC^n \supseteq F^{p+1}C^n \supseteq \cdots$ of the object C^n, and we think of elements of C^n further down the filtration as being "closer to zero". The filtration of the complex C then allows us to talk about elements $x \in C^n$ which are "close to being cocycles". We know that if $x \in F^pC^n$ then $\partial(x) \in F^pC^{n+1}$, and the higher the value of $r \geq 0$ for which $\partial(x) \in F^{p+r}C^{n+1}$ then the closer x is to being a cocycle. Formally, we define for $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}$

$$A_r^{p,q} = F^pC^{p+q} \cap \partial^{-1}(F^{p+r}C^{p+q+1})$$

It is clear that $A_r^{p,q} = F^pC^{p+q}$ for $r \leq 0$ and we have therefore the following sequence of inclusions

$$F^pC^{p+q} = A_0^{p,q} \supseteq A_1^{p,q} \supseteq \cdots \supseteq A_r^{p,q} \supseteq A_{r+1}^{p,q} \supseteq \cdots \supseteq Ker(\partial) \cap F^pC^{p+q}$$

For $p, q, r \in \mathbb{Z}$ the elements of $A_{r-1}^{p-r+1,q+r-2}$ map under ∂ into F^pC^{p+q} and we denote the image by $\ddot{A}_r^{p,q}$ (the index of r instead of $r-1$ is to make life easier later on). That is,

$$\ddot{A}_r^{p,q} = \partial(A_{r-1}^{p-r+1,q+r-2})$$

For $r \leq 1$ we have $\ddot{A}_r^{p,q} = \partial(F^{p-r+1}C^{p+q-1})$ and we have therefore a sequence

$$Ker(\partial) \cap F^pC^{p+q} \supseteq \ddot{A}_1^{p,q} \supseteq \ddot{A}_2^{p,q} \supseteq \cdots \supseteq \ddot{A}_r^{p,q} \supseteq \ddot{A}_{r+1}^{p,q} \supseteq \cdots$$

This defines for $r \in \mathbb{Z}$ two subobjects $A_r^{p,q}, \ddot{A}_r^{p,q}$ of F^pC^{p+q}. We can now introduce "approximate" cocycles and coboundaries for $p, q \in \mathbb{Z}$ and $r \geq 0$

$$Z_r^{p,q} = \frac{A_r^{p,q} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}} \qquad \text{"approximate cocycles"}$$

$$B_r^{p,q} = \frac{\ddot{A}_r^{p,q} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}} \qquad \text{"approximate coboundaries"}$$

Note that if the filtration is trivial (i.e. $F^1(C) = C$ and $F^p(C) = 0$ for $p \neq 1$) then for $r = 1$ the objects $Z_1^{0,q}$ and $B_1^{0,q}$ are the just the kernel of $\partial : C^q \longrightarrow C^{q+1}$ and image of $\partial : C^{q-1} \longrightarrow C^q$ respectively; that is, the usual cocycles and coboundaries. Both $Z_r^{p,q}$ and $B_r^{p,q}$ are canonically subobjects of $F^pC^{p+q}/F^{p+1}C^{p+q}$ and $B_r^{p,q} \subseteq Z_r^{p,q}$. We can therefore define

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} \cong \frac{A_r^{p,q} + F^{p+1}C^{p+q}}{\ddot{A}_r^{p,q} + F^{p+1}C^{p+q}} \qquad \text{"approximate cohomology"}$$

The idea is that as r is made large, the approximate cocycles and coboundaries of degree r approach the real cocycles and coboundaries, and therefore $E_r^{p,q}$ approaches something related to the cohomology $H^{p+q}(C)$ (in fact, it will approach the quotients of a certain filtration of this cohomology object).

Lemma 1. *Suppose we have two subobjects Y, F of an object X. Then the canonical morphism $F \longrightarrow F + Y \longrightarrow (F + Y)/Y$ is an epimorphism.*

Proof. We can realise $F + Y$ as the image of $F \oplus Y \longrightarrow X$, and the projection $F \oplus Y \longrightarrow F$ is the cokernel of $Y \longrightarrow F \oplus Y$, so we deduce a morphism $F \longrightarrow (F + Y)/Y$ fitting into a commutative diagram

$$\begin{tikzcd} F \arrow[r] & (F+Y)/Y \\ F \oplus Y \arrow[u] \arrow[r] & F + Y \arrow[u] \end{tikzcd}$$

It is therefore clear that $F \longrightarrow (F + Y)/Y$ is an epimorphism, and one checks that this is actually the composite $F \longrightarrow F + Y \longrightarrow (F + Y)/Y$. $\square$

We have already defined the objects $E_r^{p,q}$ of the spectral sequence, and next we define the morphisms $d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,q-r+1}$. Observe that by Lemma 1 we have a canonical epimorphism

$$A_r^{p,q} \longrightarrow A_r^{p,q} + F^{p+1}C^{p+q} \longrightarrow \frac{A_r^{p,q} + F^{p+1}C^{p+q}}{\ddot{A}_r^{p,q} + F^{p+1}C^{p+q}} \cong E_r^{p,q}$$

and the differential induces a canonical morphism $\partial : A_r^{pq} \longrightarrow A_r^{p+r,q-r+1}$ making the following diagram commute

$$\begin{tikzcd}
C^{p+q} \arrow[r, "\partial"] & C^{p+q+1} \\
A_r^{pq} \arrow[u] \arrow[r, "\partial"'] & A_r^{p+r,q-r+1} \arrow[u]
\end{tikzcd}$$

We make the following claim:

Lemma 2. *There is a unique morphism $d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,q-r+1}$ making the following diagram commute*

$$\begin{tikzcd}
A_r^{pq} \arrow[r, "\partial"] \arrow[d] & A_r^{p+r,q-r+1} \arrow[d] \\
E_r^{pq} \arrow[r, "d_r^{pq}"'] & E_r^{p+r,q-r+1}
\end{tikzcd}$$

Proof. Uniqueness is trivial since the vertical morphisms are epimorphisms, so it suffices to show existence. Even on the level of modules over a ring this is a fairly delicate technical question, so we proceed by reducing to this case by the full embedding theorem (see Mitchell VI Theorem 7.2). That is, we assume $\mathcal{A} = R\mathbf{Mod}$ for some (noncommutative) ring R, in which case the claim is reasonably straightforward to check. $\square$

It is clear by construction that $d_r^{p+r,q-r+1} d_r^{pq} = 0$, so we have defined the elements $(a), (b)$ of a spectral sequence. Thus we obtain objects $Z_{r+1}(E_r^{pq})$ and $B_{r+1}(E_r^{pq})$ and we desire a canonical isomorphism $\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \longrightarrow E_{r+1}^{pq}$. We have by definition a commutative diagram

$$\begin{tikzcd}
E_r^{p-r,q+r-1} \arrow[r, "d_r^{p-r,q+r-1}"] \arrow[d, Rightarrow] & E_r^{pq} \arrow[r, "d_r^{pq}"] \arrow[d, Rightarrow] & E_r^{p+r,q-r+1} \arrow[d, Rightarrow] \\
\frac{A_r^{p-r,q+r-1} + F^{p-r+1}C^{p+q-1}}{\ddot{A}_r^{p-r,q+r-1} + F^{p-r+1}C^{p+q-1}} \arrow[r] & \frac{A_r^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_r^{pq} + F^{p+1}C^{p+q}} \arrow[r] & \frac{A_r^{p+r,q-r+1} + F^{p+r+1}C^{p+q+1}}{\ddot{A}_r^{p+r,q-r+1} + F^{p+r+1}C^{p+q+1}}
\end{tikzcd}$$

By applying the full embedding theorem to reduce to the case of modules over a ring (or otherwise) one checks that

$$Z_{r+1}(E_r^{pq}) = Ker(d_r^{p,q}) = \frac{A_{r+1}^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_r^{pq} + F^{p+1}C^{p+q}}$$

$$B_{r+1}(E_r^{pq}) = Im(d_r^{p-r,q+r-1}) = \frac{\ddot{A}_{r+1}^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_r^{pq} + F^{p+1}C^{p+q}}$$

From which we deduce the required canonical isomorphism

$$\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \longrightarrow \frac{A_{r+1}^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_{r+1}^{pq} + F^{p+1}C^{p+q}} \cong E_{r+1}^{pq}$$

Once again using the full embedding theorem one checks that for $k \geq r+1$ we have

$$Z_k(E_r^{pq}) = \frac{A_k^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_r^{pq} + F^{p+1}C^{p+q}}$$

$$B_k(E_r^{pq}) = \frac{\ddot{A}_k^{pq} + F^{p+1}C^{p+q}}{\ddot{A}_r^{pq} + F^{p+1}C^{p+q}}$$

And we define part (d) of the data of a spectral sequence as follows

$$\begin{aligned} Z_\infty(E_0^{pq}) &= \frac{Ker(\partial)\cap F^pC^{p+q} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}} \\ B_\infty(E_0^{pq}) &= \frac{Im(\partial)\cap F^pC^{p+q} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}} \end{aligned}$$

It is clear that $B_k(E_0^{pq}) \subseteq B_\infty(E_0^{pq})$ and $Z_\infty(E_0^{pq}) \subseteq Z_k(E_0^{pq})$ for $k \geq 0$. By definition we have

$$E_\infty^{pq} = Z_\infty(E_0^{pq})/B_\infty(E_0^{pq}) \cong \frac{Ker(\partial)\cap F^pC^{p+q}}{Ker(\partial)\cap F^pC^{p+q}\cap(Im(\partial)\cap F^pC^{p+q} + F^{p+1}C^{p+q})} \tag{3}$$

To define part (e) of a spectral sequence, we set $E^n = H^n(C)$. For each $p \in \mathbb{Z}$ the inclusion $F^p(C) \longrightarrow C$ induces a morphism $H^n(F^p(C)) \longrightarrow H^n(C)$ and we denote the image of this morphism by $F^p(E^n)$. This defines a filtration on each E^n. One checks that for $p, q \in \mathbb{Z}$ we have

$$F^p(E^{p+q}) = \frac{Ker(\partial)\cap F^pC^{p+q} + Im(\partial)}{Im(\partial)}$$

and therefore

$$\begin{aligned} gr_p(E^{p+q}) = F^p(E^{p+q})/F^{p+1}(E^{p+q}) &\cong \frac{Ker(\partial)\cap F^pC^{p+q} + Im(\partial)}{Ker(\partial)\cap F^{p+1}C^{p+q} + Im(\partial)} \\ &\cong \frac{Ker(\partial)\cap F^pC^{p+q}}{Ker(\partial)\cap F^pC^{p+q}\cap(Ker(\partial)\cap F^{p+1}C^{p+q} + Im(\partial))} \end{aligned} \tag{4}$$

Using an embedding theorem (or otherwise) one checks that the denominators in (3) and (4) are the same, so we have a canonical isomorphism

$$\beta^{pq} : E_\infty^{pq} \longrightarrow gr_p(E^{p+q})$$

which completes the definition of our spectral sequence. We make one observation about the zero page of this spectral sequence.

Lemma 3. *There is a canonical isomorphism $E_0^{pq} \cong F^pC^{p+q}/F^{p+1}C^{p+q}$ which is natural, in the sense that the following diagram commutes*

$$\begin{tikzcd} E_0^{pq} \arrow[r] \arrow[d, "d_0^{pq}"'] & \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}} \arrow[d] \\ E_0^{p,q+1} \arrow[r] & \frac{F^pC^{p+q+1}}{F^{p+1}C^{p+q+1}} \end{tikzcd}$$

Proof. We already know that $E_0^{pq} \cong (A_0^{pq} + F^{p+1}C^{p+q})/(\ddot{A}_0^{pq} + F^{p+1}C^{p+q})$ so the existence of a canonical isomorphism follows from the fact that $A_0^{pq} = F^pC^{p+q}$ and $\ddot{A}_0^{p,q} \subseteq F^{p+1}C^{p+q}$. Naturality follows from the definition of d_0^{pq} given in Lemma 2. □

In summary

Proposition 4. *Let $\mathcal{A}$ be an abelian category and C a complex in $\mathcal{A}$ with a decreasing filtration $\{F^p(C)\}_{p\in\mathbb{Z}}$. Then there is a canonical spectral sequence (E_r^{pq}, E^n) starting on page zero, with*

$$\begin{aligned} E_0^{pq} &= F^pC^{p+q}/F^{p+1}C^{p+q} \\ E^n &= H^n(C) \end{aligned}$$

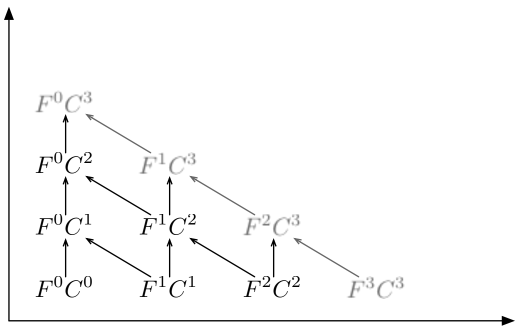
In other words, $E_r^{pq} \Longrightarrow H^{p+q}(C)$.

2.1 First Quadrant Filtration

In this section we describe the convergence properties of the above spectral sequence in an important special case. Throughout this section let $\mathcal{A}$ be an abelian category and C a complex in $\mathcal{A}$. Suppose that $C^i = 0$ for $i < 0$ and that the filtration on C^n for $n \geq 0$ has the following form

$$C^n = \cdots = F^{-1}C^n = F^0C^n \supseteq \cdots \supseteq F^nC^n \supseteq F^{n+1}C^n = \cdots = 0$$

That is, $F^jC^n = 0$ for $j > n$ and $F^jC^n = C^n$ for $j \leq 0$. Graphically, this means that when we place the filtered pieces of C in a diagram, everything essentially lives in the first quadrant



Let $p, q \in \mathbb{Z}$ be given with $p + q \geq 0$. For $r > q + 1$ we have $F^{p+r}C^{p+q+1} = 0$ and therefore $A_r^{pq} = F^pC^{p+q} \cap Ker(\partial)$. For $r \geq p + 1$ we have $A_{r-1}^{p-r+1,q+r-2} = \partial^{-1}(F^pC^{p+q})$ and therefore

$$\ddot{A}_r^{pq} = Im(\partial) \cap F^pC^{p+q}$$

If we define

$$A_\infty^{pq} = Ker(\partial) \cap F^pC^{p+q}$$
$$\ddot{A}_\infty^{pq} = Im(\partial) \cap F^pC^{p+q}$$

Then we have inclusions

$$F^pC^{p+q} = A_0^{pq} \supseteq A_1^{pq} \supseteq \cdots \supseteq A_{q+1}^{pq} \supseteq A_{q+2}^{pq} = A_\infty^{pq}$$
$$\partial(F^pC^{p+q-1}) = \ddot{A}_1^{pq} \supseteq \ddot{A}_2^{pq} \supseteq \cdots \supseteq \ddot{A}_{p+1}^{pq} = \ddot{A}_\infty^{pq}$$

In particular the objects

$$Z_k(E_0^{pq}) = \frac{A_k^{pq} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}}$$
$$B_k(E_0^{pq}) = \frac{\ddot{A}_k^{pq} + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}}$$

stabilise to $Z_\infty(E_0^{pq})$ and $B_\infty(E_0^{pq})$ for sufficiently large k. In particular the spectral sequence E_r^{pq} corresponding to C and its filtration is weakly convergent. It is also a first quadrant spectral sequence, in the sense that $E_r^{pq} = 0$ unless $p \geq 0, q \geq 0$. One checks that the spectral sequence is biregular in the sense of Definition 3.

Remark 2. There is nothing special about the requirement $C^i = 0$ for $i < 0$ throughout the above. That is to say, if $N \leq 0$ is any integer such that $C^i = 0$ for $i < N$ with the filtration on C defined in the appropriate way (so as to fit into the region $p \geq N, q \geq 0$ of the plane), then the corresponding spectral sequence is also biregular, and satisfies $E_r^{pq} = 0$ unless $p \geq N, q \geq 0$.

3 The Spectral Sequences of a Double Complex

Throughout this section let $\mathcal{A}$ be a cocomplete abelian category. See (DTC,Definition 32) for the definition of a *bicomplex* in $\mathcal{A}$ and (DTC,Definition 33) for the definition of the *totalisation* of a bicomplex.

Let C be a bicomplex in $\mathcal{A}$ with totalisation $Tot(C)$. We define two decreasing filtrations on the complex $Tot(C)$, which corresponds to filtering the bicomplex by rows and columns respectively. Graphically, our bicomplex has the form

$$\begin{tikzcd}
 & \vphantom{C} & \vphantom{C} & \vphantom{C} & \\
{} \arrow[r, dotted] & C^{(i-1)(j+1)} \arrow[u, dotted] \arrow[r] & C^{i(j+1)} \arrow[u, dotted] \arrow[r, "\partial_1^{i(j+1)}"] & C^{(i+1)(j+1)} \arrow[u, dotted] \arrow[r, dotted] & {} \\
{} \arrow[r, dotted] & C^{(i-1)j} \arrow[u] \arrow[r] & C^{ij} \arrow[u, "\partial_2^{ij}"] \arrow[r, "\partial_1^{ij}"'] & C^{(i+1)j} \arrow[u, "\partial_2^{(i+1)j}"'] \arrow[r, dotted] & {} \\
{} \arrow[r, dotted] & C^{(i-1)(j-1)} \arrow[u] \arrow[r] & C^{i(j-1)} \arrow[u] \arrow[r] & C^{(i+1)(j-1)} \arrow[u] \arrow[r, dotted] & {} \\
 & {} \arrow[u, dotted] & {} \arrow[u, dotted] & {} \arrow[u, dotted] &
\end{tikzcd} \tag{5}$$

At each position we can define the "vertical" and "horizontal" cohomology by

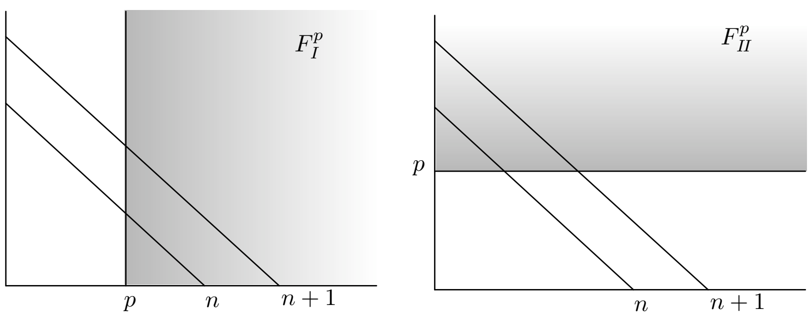
$$H_I^{ij}(C) = \frac{Ker(\partial_2^{ij})}{Im(\partial_2^{i(j-1)})}$$
$$H_{II}^{ij}(C) = \frac{Ker(\partial_1^{ij})}{Im(\partial_1^{(i-1)j})}$$

and these become complexes $H_I^{\bullet,j}(C)$ and $H_{II}^{i,\bullet}(C)$ in the obvious way. This yields at each position two cohomology objects $H^p(H_I^{\bullet,q}(C))$ and $H^q(H_{II}^{p,\bullet}(C))$. While these are not necessarily equal, they "converge" to the same limit in a sense we will now make precise.

Now we define two filtrations $\{F_I^p(Tot(C))\}_{p\in\mathbb{Z}}$ and $\{F_{II}^p(Tot(C))\}_{p\in\mathbb{Z}}$ on the complex $Tot(C)$. By definition $Tot(C)^n = \oplus_{i+j=n} C^{ij}$ and we set

$$F_I^p(Tot(C))^n = \bigoplus_{r\geq p} C^{r,n-r} \qquad \partial^n u_r = u_{r+1}\partial_1^{r,n-r} + (-1)^r u_r \partial_2^{r,n-r}$$
$$F_{II}^p(Tot(C))^n = \bigoplus_{r\geq p} C^{n-r,r} \qquad \partial^n u_r = u_r\partial_1^{n-r,r} + (-1)^{n-r} u_{r+1} \partial_2^{n-r,r}$$

where the u_r denote injections into the respective coproducts. These are complexes in $\mathcal{A}$ that admit canonical monomorphisms into $Tot(C)$.



We obtain by Proposition 4 spectral sequences $'E$ and $''E$ corresponding to the filtrations $\{F_I^p(Tot(C))\}_{p\in\mathbb{Z}}$ and $\{F_{II}^p(Tot(C))\}_{p\in\mathbb{Z}}$ respectively. Both converge to the cohomology $H(Tot(C))$. In fact the zero page of both these spectral sequences is just the original bicomplex.

Lemma 5. *The zero pages $'E_0$ and $''E_0$ are canonically isomorphic to the following diagrams respectively (up to some sign factors)*

$$\begin{tikzcd}
C^{0,3} & & & \\
C^{0,2} \arrow[u] & C^{1,2} & & \\
C^{0,1} \arrow[u] & C^{1,1} \arrow[u] & C^{2,1} & \\
C^{0,0} \arrow[u] & C^{1,0} \arrow[u] & C^{2,0} \arrow[u] & C^{3,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{3,0} & & & \\
C^{2,0} \arrow[u] & C^{2,1} & & \\
C^{1,0} \arrow[u] & C^{1,1} \arrow[u] & C^{1,2} & \\
C^{0,0} \arrow[u] & C^{0,1} \arrow[u] & C^{0,2} \arrow[u] & C^{0,3}
\end{tikzcd}$$

To be precise, we claim that for $p,q\in\mathbb{Z}$ there are canonical isomorphisms

$$'E_0^{p,q} \longrightarrow C^{p,q}, \quad ''E_0^{p,q} \longrightarrow C^{q,p}$$

making the following diagrams commute

$$\begin{tikzcd}
'E_0^{p,q} \arrow[r] \arrow[d, "'d_0^{p,q}"'] & C^{p,q} \arrow[d, "(-1)^p\partial_2^{p,q}"] \\
'E_0^{p,q+1} \arrow[r] & C^{p,q+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
''E_0^{p,q} \arrow[r] \arrow[d, "''d_0^{p,q}"'] & C^{q,p} \arrow[d, "\partial_1^{q,p}"] \\
''E_0^{p,q+1} \arrow[r] & C^{q+1,p}
\end{tikzcd}$$

Proof. We have a canonical isomorphism

$$\begin{aligned}
'E_0^{p,q} &\cong \frac{F_I^p Tot(C)^{p+q}}{F_I^{p+1} Tot(C)^{p+q}} \\
&= \frac{\oplus_{r\geq p} C^{r,p+q-r}}{\oplus_{r\geq p+1} C^{r,p+q-r}} \\
&\cong C^{p,q}
\end{aligned}$$

and similarly $''E_0^{p,q}\cong C^{q,p}$. One checks using Lemma 3 that these isomorphisms are natural in the way described. $\square$

Lemma 6. *The first pages $'E_1$ and $''E_1$ are canonically isomorphic to the following diagrams respectively*

$$\begin{tikzcd}
H_I^{03}(C) & & & \\
H_I^{02}(C) \arrow[r] & H_I^{12}(C) & & \\
H_I^{01}(C) \arrow[r] & H_I^{11}(C) \arrow[r] & H_I^{21}(C) & \\
H_I^{00}(C) \arrow[r] & H_I^{10}(C) \arrow[r] & H_I^{20}(C) \arrow[r] & H_I^{30}(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
H_{II}^{30}(C) & & & \\
H_{II}^{20}(C) \arrow[r] & H_{II}^{21}(C) & & \\
H_{II}^{10}(C) \arrow[r] & H_{II}^{11}(C) \arrow[r] & H_{II}^{12}(C) & \\
H_{II}^{00}(C) \arrow[r] & H_{II}^{01}(C) \arrow[r] & H_{II}^{02}(C) \arrow[r] & H_{II}^{03}(C)
\end{tikzcd}$$

To be precise, we claim that for $p,q\in\mathbb{Z}$ there are canonical isomorphisms

$$'E_1^{pq} \longrightarrow H_I^{pq}(C), \quad ''E_1^{pq} \longrightarrow H_{II}^{qp}(C)$$

making the following diagrams commute

$$\begin{tikzcd} {'E_1^{pq}} \arrow[r, "{'d_1^{pq}}"] \arrow[d] & {'E_1^{p+1,q}} \arrow[d] \\ H_I^{pq}(C) \arrow[r] & H_I^{p+1,q}(C) \end{tikzcd} \qquad \begin{tikzcd} {''E_1^{pq}} \arrow[r, "{(-1)^q {''d_1^{pq}}}"] \arrow[d] & {''E_1^{p+1,q}} \arrow[d] \\ H_{II}^{qp}(C) \arrow[r] & H_{II}^{q,p+1}(C) \end{tikzcd} \tag{6}$$

Proof. From Lemma 5 we deduce a commutative diagram

$$\begin{tikzcd} {'E_0^{p,q-1}} \arrow[r, "{'d_0^{p,q-1}}"] \arrow[d, Rightarrow, no head] & {'E_0^{pq}} \arrow[r, "{'d_0^{pq}}"] \arrow[d, Rightarrow, no head] & {'E_0^{p,q+1}} \arrow[d, Rightarrow, no head] \\ C^{p,q-1} \arrow[r, "{(-1)^p \partial_2^{p,q-1}}"'] & C^{pq} \arrow[r, "{(-1)^p \partial_2^{pq}}"'] & C^{p,q+1} \end{tikzcd}$$

and therefore by definition of a spectral sequence a canonical isomorphism $'E_1^{pq} \cong H_I^{pq}(C)$. Similarly we obtain a canonical isomorphism $''E_1^{pq} \cong H_{II}^{qp}(C)$. For the naturality statements we would like to use an embedding theorem, but this would be dishonest because the definiton of $Tot(C)$ involves infinite coproducts which are not necessarily preserved by our embedding.

First we define a canonical morphism $\alpha : {'A_1^{pq}} \longrightarrow H_I^{pq}(C)$. By definition

$$'A_1^{pq} = \oplus_{r \geq p} C^{r,p+q-r} \cap \partial^{-1}\left(\oplus_{r \geq p+1} C^{r,p+q+1-r}\right)$$

Informally, a sequence $(a_r)_{r \geq p}$ in $'A_1^{pq}$ must have $\partial_2^{pq}(a_p) = 0$, because $\partial((a_r)_{r \geq p})$ belongs to $\oplus_{r \geq p+1} C^{r,p+q+1-r}$. That is, $a_q \in Ker(\partial_2^{pq})$, so we obtain a canonical morphism $'A_1^{pq} \longrightarrow Ker(\partial_2^{pq})$. Formally, we have a commutative diagram

$$\begin{tikzcd} {'A_1^{pq}} \arrow[r] \arrow[d] & \oplus_{r \geq p} C^{r,p+q-r} \arrow[r] \arrow[d, "\partial^{p+q}"] & C^{pq} \arrow[d, "{(-1)^p \partial_2^{pq}}"] \\ \oplus_{r \geq p+1} C^{r,p+q+1-r} \arrow[r] & \oplus_{r \geq p} C^{r,p+q+1-r} \arrow[r] & C^{p,q+1} \end{tikzcd}$$

From which we deduce that the image of $'A_1^{pq} \longrightarrow C^{pq}$ is contained in $Ker(\partial_2^{pq})$, so we obtain the desired morphism $\alpha : {'A_1^{pq}} \longrightarrow Ker(\partial_2^{pq}) \longrightarrow H_I^{pq}(C)$. We claim this is equal to the composite $'A_1^{pq} \longrightarrow {'E_1^{pq}} \longrightarrow H_I^{pq}(C)$. The verification is not difficult, but it is tedious since it involves unfolding the definitions of all these morphisms. We omit the details. In fact one can check that

$$'A_1^{pq} = Ker(\partial_2^{pq}) \oplus \bigoplus_{r \geq p+1} C^{r,p+q-r} \tag{7}$$

so α is just the projection onto the first factor followed by the canonical quotient. We are now prepared to check commutativity of the diagrams in (6). To show that the first diagram commutes, we compose with $'A_1^{pq} \longrightarrow {'E_1^{pq}}$ and reduce to showing that the following diagram commutes

$$\begin{tikzcd} {'A_1^{pq}} \arrow[r] \arrow[d, "\alpha"'] & {'A_1^{p+1,q}} \arrow[d, "\alpha"] \\ H_I^{pq}(C) \arrow[r] & H_I^{p+1,q}(C) \end{tikzcd} \tag{8}$$

The reader may find the diagram below helpful in the following discussion. Note that we are not claiming all faces of this diagram commute.

$$
\begin{tikzcd}
 & Tot(C)^{p+q} \arrow[d] \arrow[drrr] & & Ker(\partial_2^{pq}) \arrow[rr] \arrow[d] \arrow[dr] & & H_I^{pq}(C) \arrow[dd] \\
{'A_1^{pq}} \arrow[ur] \arrow[urrr] \arrow[rr] \arrow[dd] & & \oplus_{r\geq p} C^{r,p+q-r} \arrow[ul] \arrow[rr] & & C^{pq} \arrow[dd] & \\
 & Tot(C)^{p+q+1} \arrow[drrr] & & Ker(\partial_2^{p+1,q}) \arrow[rr] \arrow[dr] & & H_I^{p+1,q}(C) \\
{'A_1^{p+1,q}} \arrow[ur] \arrow[urrr] \arrow[rr] & & \oplus_{r\geq p+1} C^{r,p+q+1-r} \arrow[ul] \arrow[rr] & & C^{p+1,q} &
\end{tikzcd}
$$

We check that (8) commutes by writing $'A_1^{pq}$ as the coproduct (7) and checking on components. It is straightforward to check that both ways around the square agree on $Ker(\partial_2^{pq})$. Given $r \geq p+1$ the composite $'A_1^{pq} \longrightarrow H_I^{pq}(C) \longrightarrow H_I^{p+1,q}(C)$ always vanishes on $C^{r,p+q-r}$, and we have to show the other way around also vanishes. If $r > p+1$ this is easy to check. For $r = p+1$ the morphism $C^{p+1,q-1} \longrightarrow {'A_1^{pq}} \longrightarrow {'A_1^{p+1,q}} \longrightarrow Ker(\partial_2^{p+1,q})$ factors through $Im(\partial_2^{p+1,q-1})$, so when we compose with the quotient $Ker(\partial_2^{p+1,q}) \longrightarrow H_I^{p+1,q}(C)$ we get zero. Therefore (8) commutes, and consequently so does the first diagram of (6). One checks commutativity of the second diagram of (6) in much the same way. $\square$

Lemma 7. *For $p, q \in \mathbb{Z}$ there are canonical isomorphisms*

$$'E_2^{pq} \cong H^p(H_I^{\bullet,q}(C)) \quad \text{and} \quad ''E_2^{pq} \cong H^p(H_{II}^{q,\bullet}(C))$$

Proof. By definition of a spectral sequence the entries on the second page are canonically isomorphic to the cohomology of the first page, so the claims are an immediate consequence of Lemma 6. $\square$

In summary

Proposition 8. *Let $\mathcal{A}$ be a cocomplete abelian category and C a bicomplex in $\mathcal{A}$. There are canonical spectral sequences $'E$ and $''E$ with*

$$
\begin{aligned}
'E_0^{pq} &= C^{pq}, \quad 'E_1^{pq} = H_I^{pq}(C), \quad 'E_2^{pq} = H^p(H_I^{\bullet,q}(C)) \\
''E_0^{pq} &= C^{qp}, \quad ''E_1^{pq} = H_{II}^{qp}(C), \quad ''E_2^{pq} = H^p(H_{II}^{q,\bullet}(C))
\end{aligned}
$$

Both spectral sequences $'E_r^{pq}$ and $''E_r^{pq}$ converge to $H^{p+q}(Tot(C))$.

Example 1. Let $\mathcal{A}$ be a cocomplete abelian category and C a bicomplex in $\mathcal{A}$. Assume that C is a first quadrant bicomplex: that is, $C^{ij} = 0$ unless $i \geq 0, j \geq 0$. Then $Tot(C)^n = 0$ for $n < 0$ and the filtrations $F_I^p(Tot(C)), F_{II}^p(Tot(C))$ have the form of the filtration in Section 2.1. That is,

$$
\begin{aligned}
Tot(C)^n = \cdots = F_I^0(Tot(C))^n \supseteq \cdots \supseteq F_I^n(Tot(C))^n \supseteq F_I^{n+1}(Tot(C))^n = \cdots = 0 \\
Tot(C)^n = \cdots = F_{II}^0(Tot(C))^n \supseteq \cdots \supseteq F_{II}^n(Tot(C))^n \supseteq F_{II}^{n+1}(Tot(C))^n = \cdots = 0
\end{aligned}
$$

It follows that the spectral sequences $'E$ and $''E$ are biregular first quadrant spectral sequences. More generally, if there exists an integer $N \leq 0$ such that $C^{ij} = 0$ unless $i \geq N, j \geq 0$ then the corresponding spectral sequences $'E$ and $''E$ are biregular and zero outside of $p \geq N, q \geq 0$.

Example 2. Let $\mathcal{A}$ be a cocomplete abelian category and C a bicomplex in $\mathcal{A}$. Assume that the nonzero terms of C are all concentrated in a horizontal strip: that is, assume that there exists an integer $N \geq 0$ such that $C^{ij} = 0$ unless $0 \leq j \leq N$. The complex $Tot(C)$ is no longer necessarily bounded, but the filtration $F_{II}^p(Tot(C))$ is of the form

$$Tot(C)^n = F_{II}^0(Tot(C))^n \supseteq \cdots \supseteq F_{II}^N(Tot(C)) \supseteq F_{II}^{N+1}(Tot(C)) = 0$$

Let $''E$ be the spectral sequence derived from this filtration. This is not necessarily a first quadrant spectral sequence, but it is straightforward to check that it is biregular.

For a trivial example of how this spectral sequence might be useful, suppose that the rows of C are exact. Using Proposition 8 it is clear that the second page of the spectral sequence $''E$ vanishes. Therefore by Remark 4 every subsequent page vanishes, and in particular $E_\infty^{pq} = 0$ for $p, q \in \mathbb{Z}$. It follows that $H^n(Tot(C)) = 0$, so the complex $Tot(C)$ is exact.

4 The Grothendieck Spectral Sequence

Throughout this section let $\mathcal{A}$ be an abelian category.

Definition 5. Given a complex C in $\mathcal{A}$ an *injective resolution* of C is a commutative diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & I^{p,1} \arrow[r] \arrow[u] & I^{p+1,1} \arrow[r] \arrow[u] & I^{p+2,1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & I^{p,0} \arrow[r] \arrow[u] & I^{p+1,0} \arrow[r] \arrow[u] & I^{p+2,0} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^p \arrow[r] \arrow[u] & C^{p+1} \arrow[r] \arrow[u] & C^{p+2} \arrow[r] \arrow[u] & \cdots
\end{tikzcd} \tag{9}$$

in which the rows are complexes and each column is an injective resolution. That is, $C^p \longrightarrow I^{p,\bullet}$ is an injective resolution in $\mathcal{A}$. To be precise, the injective resolution is the upper halfplane bicomplex I together with the morphism of complexes $C \longrightarrow I^{\bullet,0}$. For each $p \in \mathbb{Z}$ we have complexes

$$0 \longrightarrow Z^p(C) \longrightarrow Z^p(I^{\bullet,0}) \longrightarrow Z^p(I^{\bullet,1}) \longrightarrow \cdots$$
$$0 \longrightarrow B^p(C) \longrightarrow B^p(I^{\bullet,0}) \longrightarrow B^p(I^{\bullet,1}) \longrightarrow \cdots$$
$$0 \longrightarrow H^p(C) \longrightarrow H^p(I^{\bullet,0}) \longrightarrow H^p(I^{\bullet,1}) \longrightarrow \cdots$$

and we say that the injective resolution (9) is *fully injective* if for each $p \in \mathbb{Z}$ these complexes are all injective resolutions.

Lemma 9. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then every complex C in $\mathcal{A}$ has a fully injective resolution.*

Proof. For each $n \in \mathbb{Z}$ we have short exact sequences

$$\begin{gathered} 0 \longrightarrow Z^n(C) \longrightarrow C^n \longrightarrow B^{n+1}(C) \longrightarrow 0 \\ 0 \longrightarrow B^n(C) \longrightarrow Z^n(C) \longrightarrow H^n(C) \longrightarrow 0 \end{gathered} \tag{10}$$

Choose for each $n \in \mathbb{Z}$ injective resolutions of $H^n(C)$ and $B^n(C)$. By (DF,Corollary 40) we can find an injective resolution of $Z^n(C)$ fitting into a short exact sequence with the original two resolutions. Applying this result again we construct an injective resolution of C^n. Placing all these injective resolutions in the columns of a bicomplex, we have the desired fully injective resolution. ☐

Remark 3. In the context of Lemma 9 if there is a set of integers N with $C^i = 0$ for $i \in N$ then we can clearly arrange for the fully injective resolution I to have zero columns for any index $i \in N$.

Definition 6. Let $\mathcal{A}$ be an abelian category with enough injectives, and C a complex in $\mathcal{A}$. We say that a fully injective resolution of C is *normal* if it is constructed as in Lemma 9 from an initial choice of injective resolutions for $H^n(C)$ and $B^n(C)$. A normal, fully injective resolution is called a *Cartan-Eilenberg resolution*.

Theorem 10 (Grothendieck spectral sequence). *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be additive functors between abelian categories where $\mathcal{A}, \mathcal{B}$ have enough injectives and $\mathcal{C}$ is cocomplete, and suppose that F sends injectives to G-acyclics. Then for any object $A \in \mathcal{A}$ there is a biregular first quadrant spectral sequence E starting on page zero, such that*

$$E_2^{pq} = R^pG(R^qF(A)) \Longrightarrow R^{p+q}(GF)(A)$$

Proof. Let us be clear on several points in the statement. Given an injective object $I \in \mathcal{A}$ we require that $F(I)$ be right G-acyclic in the sense of (DF,Definition 14). Fix assignments of injective resolutions to $\mathcal{A}, \mathcal{B}$ and calculate all right derived functors with respect to these choices. We claim that there exists a (noncanonical) spectral sequence (E_r^{pq}, E) starting on page zero, together with isomorphisms $E_2^{pq} \cong R^pG(R^qF(A))$ for $p, q \geq 0$ and $E^n \cong R^n(GF)(A)$ for $n \geq 0$.

Let the complex C be an injective resolution of A, and let the bicomplex I be a fully injective resolution of the complex FC (with $I^{p,q} = 0$ unless $p, q \geq 0$). That is, we have a commutative diagram with exact columns

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & F(C^0) \arrow[r] \arrow[u] & F(C^1) \arrow[r] \arrow[u] & F(C^2) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We can apply the results of Section 3 to the bicomplex GI to obtain two canonical filtrations

$$F_I^p(Tot(GI))^n = \bigoplus_{r \geq p} G(I^{r,n-r})$$
$$F_{II}^p(Tot(GI))^n = \bigoplus_{r \geq p} G(I^{n-r,r})$$

and spectral sequences ${}'E_r^{pq}, {}''E_r^{pq}$ both converging to the cohomology of $Tot(GI)$. By Example 1 these are both biregular first quadrant spectral sequences. We have a canonical isomorphism

$${}'E_2^{pq} \cong H^p(H_I^{\bullet,q}(GI)) \cong H^p(R^qG(FC^\bullet))$$

But C is a complex of injectives and F sends injectives to G-acyclics, so for $q > 0$ the complex $R^qG(FC^\bullet)$ is zero, and for $q = 0$ it is canonically isomorphic to GFC. In other words, we have

$${}'E_2^{pq} = \begin{cases} 0 & q > 0 \\ R^p(GF)(A) & q = 0 \end{cases}$$

Since all the morphisms on page two and beyond are zero, we deduce canonical isomorphisms

$${}'E_2^{pq} \cong {}'E_3^{pq} \cong \cdots$$

But ${}'E$ is biregular, so eventually this sequence stabilises to ${}'E_\infty^{pq}$, so there is an isomorphism ${}'E_2^{pq} \cong' E_\infty^{pq}$, and therefore ${}'E_\infty^{pq} \cong R^p(GF)(A)$ for $p \geq 0$ and $q = 0$. All the filter quotients

$F^p(E^n)/F^{p+1}(E^n)$ except for the first one vanish, because ${}'E_\infty^{pq}$ vanishes off the line $p \geq 0, q = 0$. We have therefore an isomorphism for every $n \geq 0$

$$H^n Tot(GI) \cong {}'E_\infty^{n0} \cong R^n(GF)(A)$$

Now we turn to the second spectral sequence ${}''E$. Since the resolution I of $F(C)$ is fully injective, we have injective resolutions

$$\begin{gathered} 0 \longrightarrow Z^p(FC) \longrightarrow Z^{p,0} \longrightarrow Z^{p,1} \longrightarrow \cdots \\ 0 \longrightarrow B^p(FC) \longrightarrow B^{p,0} \longrightarrow B^{p,1} \longrightarrow \cdots \\ 0 \longrightarrow H^p(FC) \longrightarrow H_{II}^{p,0}(I) \longrightarrow H_{II}^{p,1}(I) \longrightarrow \cdots \end{gathered}$$

and exact sequences

$$\begin{gathered} 0 \longrightarrow Z^{pq} \longrightarrow I^{pq} \longrightarrow B^{p+1,q} \longrightarrow 0 \\ 0 \longrightarrow B^{pq} \longrightarrow Z^{pq} \longrightarrow H^{pq} \longrightarrow 0 \end{gathered}$$

which must be split exact since everything in sight is injective. It follows that the image under G of these sequences is exact, from which we deduce that $Z^{pq}(GI) = G(Z^{pq}), B^{pq}(GI) = G(B^{pq})$ and most importantly $H_{II}^{pq}(GI) \cong G(H_{II}^{pq}(I))$. In fact this is a canonical isomorphism of complexes $H_{II}^{p,\bullet}(GI) \cong G(H_{II}^{p,\bullet}(I))$. But then

$${}''E_2^{pq} \cong H^p(H_{II}^{q,\bullet}(GI)) \cong H^p G(H_{II}^{q,\bullet}(I))$$

But $H_{II}^{q,\bullet}(I)$ is an injective resolution of $H^q(FC) \cong R^qF(A)$, so there is a canonical isomorphism $H^pG(H_{II}^{q,\bullet}(I)) \cong R^pG(R^qF(A))$. So finally we have a canonical isomorphism for $p, q \geq 0$

$${}''E_2^{pq} \cong R^pG(R^qF(A))$$

Setting $E = {}''E$ we have a spectral sequence starting on page zero, whose second page has all its entries isomorphic to $R^pG(R^qF(A))$, and which converges to $H^{p+q}(Tot(GI))$ which we know is isomorphic to $R^{p+q}(GF)(A)$, so the proof is complete. □

4.1 Examples

4.1.1 The Leray Spectral Sequence

Corollary 11 (Leray spectral sequence). *Let $f : X \longrightarrow Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf of abelian groups on X. Then there is a biregular first quadrant spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_*(\mathscr{F})) \Longrightarrow H^{p+q}(X, \mathscr{F})$$

Proof. In Theorem 10 we take $\mathcal{A} = \mathfrak{Ab}(X), \mathcal{B} = \mathfrak{Ab}(Y)$ and $\mathcal{C} = \mathbf{Ab}$. The functor F is f_* and the functor G is $\Gamma(X, -)$. Then F sends injectives to flasque sheaves, which are certainly acyclic for G. We obtain a biregular first quadrant spectral sequence E starting on page zero of the desired form. □

4.1.2 The Local-to-Global Ext Spectral Sequence

Lemma 12. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{I}$ sheaves of modules with $\mathscr{F}$ flat and $\mathscr{I}$ injective. Then $\mathscr{H}om(\mathscr{F}, \mathscr{I})$ is injective.*

Proof. We have a canonical natural equivalence (MRS,Proposition 76)

$$Hom(-, \mathscr{H}om(\mathscr{F}, \mathscr{I})) \cong Hom(- \otimes \mathscr{F}, \mathscr{I})$$

The right hand side is the composite of two exact functors, therefore exact, so the left hand side is exact as well. □

Lemma 13. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{I}$ sheaves of modules with $\mathscr{I}$ injective. Then $\mathscr{H}om(\mathscr{F}, \mathscr{I})$ is acyclic for $\Gamma(X, -)$. That is, $H^i(X, \mathscr{H}om(\mathscr{F}, \mathscr{I})) = 0$ for $i > 0$.*

Proof. We can find an exact sequence $0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{F} \longrightarrow 0$ with $\mathscr{Y}$ flat (DCOS,Lemma 47). Applying the exact functor $\mathscr{H}om(-, \mathscr{I})$ we have a short exact sequence

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{I}) \longrightarrow \mathscr{H}om(\mathscr{Y}, \mathscr{I}) \longrightarrow \mathscr{H}om(\mathscr{X}, \mathscr{I}) \longrightarrow 0$$

in which the middle term is injective by Lemma 12. From the corresponding long exact cohomology sequence we deduce $H^1(X, \mathscr{H}om(\mathscr{F}, \mathscr{I})) = 0$ and an isomorphism for $i > 1$

$$H^i(X, \mathscr{H}om(\mathscr{F}, \mathscr{I})) \cong H^{i-1}(X, \mathscr{H}om(\mathscr{X}, \mathscr{I}))$$

The claim is now easily checked. $\square$

We can now establish the *local-to-global Ext spectral sequence.*

Proposition 14. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ sheaves of modules. There is a biregular first quadrant spectral sequence E starting on page zero, such that*

$$E_2^{pq} = H^p(X, \mathscr{E}xt^q(\mathscr{F}, \mathscr{G})) \Longrightarrow Ext^{p+q}(\mathscr{F}, \mathscr{G})$$

Proof. In Theorem 10 we take $\mathcal{A} = \mathcal{B} = \mathfrak{Mod}(X)$ and $\mathcal{C} = \mathbf{Ab}$. The functor F is $\mathscr{H}om(\mathscr{F}, -)$ and the functor G is $\Gamma(X, -)$. Then F sends injectives to G-acyclics by Lemma 13. We obtain a biregular first quadrant spectral sequence E starting on page zero of the desired form. $\square$

5 Hyperderived Functors

Throughout this section let $\mathcal{A}$ be an abelian category.

Definition 7. A *homotopy* $\Sigma : \varphi \longrightarrow \psi$ between two morphisms of bicomplexes $\varphi, \psi : C \longrightarrow D$ is a collection of morphisms

$$\Sigma_1^{ij} : C^{ij} \longrightarrow D^{i-1,j}, \qquad \Sigma_2^{ij} : C^{ij} \longrightarrow D^{i,j-1}$$

such that $\psi - \varphi = \partial_1 \Sigma_1 + \Sigma_1 \partial_1 + \partial_2 \Sigma_2 + \Sigma_2 \partial_2$ and $\Sigma_1 \partial_2 + \partial_2 \Sigma_1 = 0$, $\Sigma_2 \partial_1 + \partial_1 \Sigma_2 = 0$. That is, for every $i, j \in \mathbb{Z}$ we have

$$\begin{aligned}
&\psi^{ij} - \varphi^{ij} = \partial_1^{i-1,j} \Sigma_1^{ij} + \Sigma_1^{i+1,j} \partial_1^{ij} + \partial_2^{i,j-1} \Sigma_2^{ij} + \Sigma_2^{i,j+1} \partial_2^{ij} \\
&\Sigma_1^{ij} \partial_2^{i,j-1} = \partial_2^{i-1,j-1} \Sigma_1^{i,j-1} \\
&\Sigma_2^{i+1,j} \partial_1^{ij} = \partial_1^{i,j-1} \Sigma_2^{ij}
\end{aligned}$$

as in the diagram

$$\begin{tikzcd}
 & D^{i-1,j+1} \arrow[rr] & & D^{i,j+1} \arrow[rr] & & D^{i+1,j+1} \\
C^{i-1,j+1} \arrow[rr] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & & C^{i,j+1} \arrow[rr] \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & & C^{i+1,j+1} \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & \\
 & D^{i-1,j} \arrow[rr] \arrow[uu] & & D^{ij} \arrow[rr] \arrow[uu] & & D^{i+1,j} \arrow[uu] \\
C^{i-1,j} \arrow[rr] \arrow[uu] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & & C^{ij} \arrow[rr] \arrow[uu] \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & & C^{i+1,j} \arrow[uu] \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr] & \\
 & D^{i-1,j-1} \arrow[rr] \arrow[uu] & & D^{i,j-1} \arrow[rr] \arrow[uu] & & D^{i+1,j-1} \arrow[uu] \\
C^{i-1,j-1} \arrow[rr] \arrow[uu] \arrow[ur, bend left] \arrow[ur, bend right] & & C^{i,j-1} \arrow[rr] \arrow[uu] \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] & & C^{i+1,j-1} \arrow[uu] \arrow[ul] \arrow[ur, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

We say that φ, ψ are *homotopic* and write $\varphi \simeq \psi$ if there is a homotopy $\Sigma : \varphi \longrightarrow \psi$. If Σ is such a homotopy, thn the morphisms $-\Sigma$ (meaning $-\Sigma_1^{ij}$ and $-\Sigma_2^{ij}$) define a homotopy $-\Sigma : \psi \longrightarrow \varphi$, so there is a bijection between homotopies $\varphi \longrightarrow \psi$ and homotopies $\psi \longrightarrow \varphi$. Homotopy defines an equivalence relation on the set of morphisms of bicomplexes $C \longrightarrow D$, and this relation is compatible with composition in the usual sense.

Remark 4. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. This induces an additive functor $F : \mathbf{C}^2(A) \longrightarrow \mathbf{C}^2(B)$ between the categories of bicomplexes. It is clear that if $\Sigma : \varphi \longrightarrow \psi$ is a homotopy of morphisms of bicomplexes $\varphi, \psi : C \longrightarrow D$ then there is a homotopy $F(\Sigma) : F(\varphi) \longrightarrow F(\psi)$.

Lemma 15. *Suppose that $\mathcal{A}$ is cocomplete, and let $\varphi, \psi : C \longrightarrow D$ be morphisms of bicomplexes. Given a homotopy $\Sigma : \varphi \longrightarrow \psi$ there is a homotopy $Tot(\Sigma) : Tot(\varphi) \longrightarrow Tot(\psi)$ of the induced morphisms of the totalisations $Tot(\varphi), Tot(\psi) : Tot(C) \longrightarrow Tot(D)$, defined by*

$$Tot(\Sigma)^n u_{ij} = u_{i-1,j}\Sigma_1^{ij} + (-1)^i u_{i,j-1}\Sigma_2^{ij}$$

As one would expect, a morphism of complexes lifts to a morphism of the fully injective resolutions which is unique up to homotopy. To prove this, we have to first recall some basic facts about injective resolutions.

Lemma 16. *Suppose that $\mathcal{A}$ has enough injectives and that we have a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\varphi"] \arrow[d, "f'"'] & A \arrow[r, "\psi"] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & B' \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & B'' \arrow[r] & 0
\end{tikzcd}$$

together with injective resolutions $\varepsilon' : A' \longrightarrow I', \varepsilon'' : A'' \longrightarrow I''$ and $\eta' : B' \longrightarrow J', \eta'' : B'' \longrightarrow J''$. Let $\varepsilon : A \longrightarrow I, \eta : B \longrightarrow J$ be injective resolutions induced as in (DF,Corollary 40). Given morphisms of complexes $F : I' \longrightarrow J', F'' : I'' \longrightarrow J''$ lifting f', f'' respectively, there is a morphism $F : I \longrightarrow J$ lifting f and giving a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r] \arrow[d, "F'"'] & I \arrow[r] \arrow[d, "F"'] & I'' \arrow[r] \arrow[d, "F''"'] & 0 \\
0 \arrow[r] & J' \arrow[r] & J \arrow[r] & J'' \arrow[r] & 0
\end{tikzcd}$$

If G', G, G'' is another triple of morphisms lifting f', f, f'' with the same property and if $\Sigma' : F' \longrightarrow G', \Sigma'' : F'' \longrightarrow G''$ are homotopies, then there exists a homotopy $\Sigma : F \longrightarrow G$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & (I')^n \arrow[r] \arrow[d, "(\Sigma')^n"'] & I^n \arrow[r] \arrow[d, "\Sigma^n"'] & (I'')^n \arrow[r] \arrow[d, "(\Sigma'')^n"'] & 0 \\
0 \arrow[r] & (J')^{n-1} \arrow[r] & J^{n-1} \arrow[r] & (J'')^{n-1} \arrow[r] & 0
\end{tikzcd}$$

Proof. Everything apart from the last claim about homotopies was checked as part of the proof of (DF,Theorem 41). For the proof of the dual, which is given in some detail, see (DF,Theorem 34). It remains to prove the statement about homotopies.

By hypothesis the resolutions I, J are produced by (DF,Corollary 40), so there are morphisms $\sigma_I, \lambda_{I,n}$ and $\sigma_J, \lambda_{J,n}$ satisfying some relations listed in the proof of (DF,Corollary 40). We need to define for each $n \geq 1$ a morphism

$$\Sigma^n = \begin{pmatrix} \Sigma'_n & t_n \\ 0 & \Sigma''_n \end{pmatrix} : I^n \longrightarrow J^{n-1}$$

satisfying various properties. First one writes down exactly what these properties mean for the morphism t_n, and then one constructs $t_0, t_1, \ldots$ recursively. $\square$

Proposition 17. *Suppose that $\mathcal{A}$ has enough injectives, and let $\varphi : C \longrightarrow D$ be a morphism of complexes. Let I, J be Cartan-Eilenberg resolutions of C, D respectively. Then there is an induced morphism of bicomplexes $\Phi : I \longrightarrow J$ making the following diagram commute*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\varphi"'] & I^{\bullet,0} \arrow[d, "\Phi^{\bullet,0}"] \\ D \arrow[r] & J^{\bullet,0} \end{tikzcd}$$

If $\varphi, \psi : C \longrightarrow D$ are homotopic morphisms of complexes then any two lifts $\Phi, \Psi : I \longrightarrow J$ are also homotopic.

Proof. By assumption the resolutions I, J are constructed by first choosing resolutions for $B^n(C), H^n(C)$ and $B^n(D), H^n(D)$, then inducing resolutions of $Z^n(C), C^n$ as in (DF,Corollary 40). By an argument we verified in the proof of (DF,Theorem 41) (for the dual result, which is explained in detail, see the proof of (DF,Theorem 34)) the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B^n(C) \arrow[r] \arrow[d] & Z^n(C) \arrow[r] \arrow[d] & H^n(C) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B^n(D) \arrow[r] & Z^n(D) \arrow[r] & H^n(D) \arrow[r] & 0 \end{tikzcd}$$

can be lifted to a commutative diagram with exact rows on the chosen injective resolutions of each object. Applying this again to the exact sequences $0 \longrightarrow Z^n(-) \longrightarrow (-)^n \longrightarrow B^{n+1}(-) \longrightarrow 0$ we deduce the existence of the desired morphism of bicomplexes.

It remains to prove the statement about lifting homotopies. Let $\Sigma : \varphi \longrightarrow \psi$ be a homotopy. The morphism $\Sigma^n : D^n \longrightarrow C^{n-1}$ lifts to a morphism of complexes $I^{n,\bullet} \longrightarrow J^{n-1,\bullet}$. Taking these together for all $n \in \mathbb{Z}$ we have for $i \in \mathbb{Z}, j \geq 0$ a morphism $S^{ij} : I^{ij} \longrightarrow I^{i-1,j}$ and we define

$$\Theta^{ij} = \Phi^{ij} + S^{i+1,j}\partial_1^{ij} + \partial_1^{i-1,j}S^{ij}$$

This is a morphism of bicomplexes $\Theta : I \longrightarrow J$ lifting $\psi : C \longrightarrow D$. The S^{ij} clearly define a homotopy $\Phi \longrightarrow \Theta$, so we can reduce to proving the following statement: given a morphism of complexes $\varphi : C \longrightarrow D$ any two lifts $\Phi, \Psi : I \longrightarrow J$ are homotopic.

For each $n \in \mathbb{Z}$ we have short exact sequences

$$\begin{gathered} 0 \longrightarrow Z^n(C) \longrightarrow C^n \longrightarrow B^{n+1}(C) \longrightarrow 0 \\ 0 \longrightarrow B^n(C) \longrightarrow Z^n(C) \longrightarrow H^n(C) \longrightarrow 0 \end{gathered} \tag{11}$$

and similarly for D. The morphism of complexes φ induces a morphism of these short exact sequences, and Φ, Ψ induce morphisms of the short exact sequences of resolutions lifting this original morphism of short exact sequences. We can find homotopies connecting the two lifts of $B^n(C) \longrightarrow B^n(D)$ and $H^n(C) \longrightarrow H^n(D)$, and therefore by the last statement of Lemma 16 a homotopy of the lifts of $Z^n(C) \longrightarrow Z^n(D)$ compatible with the first two. Repeating this process with the first exact sequence of (11) we obtain a special homotopy $\Sigma_2^{n,\bullet} : \Phi^{n,\bullet} \longrightarrow \Psi^{n,\bullet}$ of the lifts of $\varphi^n : C^n \longrightarrow D^n$. The construction means that $\Sigma_2^{i+1,j}\partial_1^{ij} = \partial_1^{i,j-1}\Sigma_2^{ij}$ so if we set $\Sigma_1 = 0$ then Σ is a homotopy $\Phi \longrightarrow \Psi$ as required. $\square$

Definition 8 (Hyperderived functor). Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories where $\mathcal{A}$ has enough injectives and $\mathcal{B}$ is cocomplete, and let $\mathcal{I}$ be an assignment of Cartan-Eilenberg resolutions to the complexes in $\mathcal{A}$. Given a complex C in $\mathcal{A}$ we define for $n \in \mathbb{Z}$

$${}^h\mathbb{R}^n_{\mathcal{I}}F(C) = H^n(Tot(FI))$$

where I is the chosen Cartan-Eilenberg resolution of C. A morphism of complexes $\varphi : C \longrightarrow D$ induces a morphism of the resolutions $\Phi : I \longrightarrow J$ and therefore a morphism

$$\begin{gathered} {}^h\mathbb{R}^n_{\mathcal{I}}F(\varphi) : {}^h\mathbb{R}^n_{\mathcal{I}}F(C) \longrightarrow {}^h\mathbb{R}^n_{\mathcal{I}}F(D) \\ {}^h\mathbb{R}^n_{\mathcal{I}}F(\varphi) = H^n(Tot(F\Phi)) \end{gathered}$$

which by Lemma 17 does not depend on the choice of morphism $I \longrightarrow J$ lifting φ. This defines an additive functor

$$^h\mathbb{R}^n_{\mathcal{I}}F(-) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{B}$$

called the n-th *hyperderived functor* of F. If necessary, we denote the complex $Tot(FI)$ itself by $^h\mathbb{R}_{\mathcal{I}}F(C)$. As usual, we drop the subscript $\mathcal{I}$ if there is no chance of confusion. The functor $^h\mathbb{R}^nF(-)$ is independent of the choice of resolutions up to canonical natural equivalence.

Remark 5. We write $^h\mathbb{R}^nF(-)$ for the hyperderived functor in order to distinguish it from the *derived* functor $\mathbb{R}F$ which we define in our notes on Derived Categories (DTC2).

Proposition 18. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories where $\mathcal{A}$ has enough injectives and $\mathcal{B}$ is cocomplete. Given a complex C in $\mathcal{A}$ there are spectral sequences $'E, ''E$ starting on page zero, with*

$$'E_2^{pq} = H^p(R^qF(C)) \Longrightarrow {}^h\mathbb{R}^{p+q}F(C)$$
$$''E_2^{pq} = R^pF(H^q(C)) \Longrightarrow {}^h\mathbb{R}^{p+q}F(C)$$

If C is bounded below then we can arrange for both spectral sequences to be biregular.

Proof. Fix an assignment of Cartan-Eilenberg resolutions $\mathcal{I}$ which we use to calculate the hyperderived functor. If I is the assigned resolution of C then the bicomplex FI yields canonical spectral sequences $'E,''E$ in $\mathcal{B}$ by Proposition 8, which converge to $H^{p+q}(Tot(FI)) = {}^h\mathbb{R}^{p+q}F(C)$. Observe that these are not necessarily first quadrant spectral sequences.

For $p, q \in \mathbb{Z}$ we have a canonical isomorphism $'E_2^{pq} \cong H^p(H_I^{\bullet,q}(FI))$. Since $I^{pq} = 0$ for $q < 0$ we deduce a canonical isomorphism

$$'E_2^{pq} \cong \begin{cases} 0 & q < 0 \\ H^p(R^qF(C)) & q \geq 0 \end{cases}$$

where $R^qF(C)$ denotes the image of the complex C under the additive functor $R^qF(-)$. As in the proof of Theorem 10 we have a canonical isomorphism of complexes $H_{II}^{p,\bullet}(FI) \cong F(H_{II}^{p,\bullet}(I))$, and therefore

$$''E_2^{pq} \cong H^p(H_{II}^{q,\bullet}(FI)) \cong H^pF(H_{II}^{q,\bullet}(I))$$

But $H^{q,\bullet}(I)$ is an injective resolution of $H^q(C)$, so there is a canonical isomorphism

$$''E_2^{pq} \cong \begin{cases} 0 & p < 0 \\ R^pF(H^q(C)) & p \geq 0 \end{cases}$$

Observe in particular that $'E_2$ lives in the first and second quadrants, while $''E_2$ lives in the first and fourth. If C is bounded below then we can arrange for the Cartan-Eilenberg resolution to be bounded in the same way, in which case it follows from Example 1 that both spectral sequences are biregular. $\square$