Special Sheaves of Algebras

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1 Introduction

In this note “ring” means a not necessarily commutative ring. If \( A \) is a commutative ring then an \( A \)-algebra is a ring morphism \( A \to B \) whose image is contained in the center of \( B \). We allow noncommutative sheaves of rings, but if we say \((X, O_X)\) is a ringed space then we mean \( O_X \) is a sheaf of commutative rings. Throughout this note \((X, O_X)\) is a ringed space. Associated to this ringed space are the following categories:

\[ \text{Mod}(X), \text{GrMod}(X), \text{Alg}(X), \text{nAlg}(X), \text{GrAlg}(X), \text{GrnAlg}(X) \]

We show that the forgetful functors \( \text{Alg}(X) \to \text{Mod}(X) \) and \( \text{nAlg}(X) \to \text{Mod}(X) \) have left adjoints. If \( A \) is a nonzero commutative ring, the forgetful functors \( A \text{Alg} \to A \text{Mod} \) and \( A \text{nAlg} \to A \text{Mod} \) have left adjoints given by the symmetric algebra and tensor algebra constructions respectively.

2 Sheaves of Tensor Algebras

Let \( \mathcal{F} \) be a sheaf of \( O_X \)-modules, and for an open set \( U \) let \( P(U) \) be the \( O_X(U) \)-algebra given by the tensor algebra \( T(\mathcal{F}(U)) \). That is,

\[ P(U) = O_X(U) \oplus \mathcal{F}(U) \oplus \mathcal{F}(U) \otimes \mathcal{F}(U) \oplus \cdots \]

For an inclusion \( V \subseteq U \) let \( \rho : O_X(U) \to O_X(V) \) and \( \eta : \mathcal{F}(U) \to \mathcal{F}(V) \) be the morphisms of abelian groups given by restriction. For \( n \geq 2 \) we define a multilinear map

\[ \mathcal{F}(U) \times \cdots \times \mathcal{F}(U) \to \mathcal{F}(V) \otimes \cdots \otimes \mathcal{F}(V) \]

\[ (m_1, \ldots, m_n) \mapsto m_1|V \otimes \cdots \otimes m_n|V \]

Let \( \eta^\otimes n \) denote the induced morphism of abelian groups \( \mathcal{F}(U)^{\otimes n} \to \mathcal{F}(V)^{\otimes n} \). Then \( \rho \oplus \eta \oplus \eta^{\otimes 2} \oplus \cdots \) gives a morphism of abelian groups \( P(U) \to P(V) \) compatible with the module structures.
and the ring morphism $O_X(U) \longrightarrow O_X(V)$. It is then readily seen that $P$ is a presheaf of $O_X$-algebras, and we let $T(\mathcal{F})$ be the sheaf of $O_X$-algebras given by the sheafification. The morphism of presheaves of modules $\mathcal{F} \longrightarrow P$ given pointwise by the canonical injection $\mathcal{F}(U) \longrightarrow T(\mathcal{F}(U))$ composes with $P \longrightarrow T(\mathcal{F})$ to give a monomorphism of sheaves of $O_X$-modules $\mathcal{F} \longrightarrow T(\mathcal{F})$.

If $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves of modules, whose associated presheaves of tensor algebras are $P, Q$ respectively, then we define a morphism of presheaves of $O_X$-algebras $\phi' : P \longrightarrow Q$ by $\phi'_U = T(\phi_U)$. That is, $\phi'_U = 1 \oplus \phi_U \oplus \phi_U^{\otimes 2} \oplus \phi_U^{\otimes 3} \oplus \cdots$.

Let $T(\phi) : T(\mathcal{F}) \longrightarrow T(\mathcal{G})$ denote the morphism of sheaves of $O_X$-algebras given by the sheafification of $\phi'$. This defines a functor

$$T(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Alg}(X)$$

Note that the following diagram of sheaves of modules commutes

$$\xymatrix{ T(\mathcal{F}) \ar[r]^{T(\phi)} \ar@{<->}[d] & T(\mathcal{G}) \ar@{<->}[d] \\ \mathcal{F} \ar[r]^-{\phi} & \mathcal{G}}$$

For $d \geq 0$ let $P_d$ denote the sub-presheaf of $O_X$-modules of $P$ given by $P_d(U) = T^d(\mathcal{F}(U))$, which is the submodule of $T(\mathcal{F}(U))$ given by the isomorphic copy of $\mathcal{F}(U)^{\otimes d}$. In particular there are isomorphisms of presheaves of modules $P_0 \cong O_X$ and $P_1 \cong \mathcal{F}$. By construction the induced morphism $\bigoplus_{d \geq 0} P_d \longrightarrow P$ is an isomorphism (coproduct of presheaves of modules) and $P_dP_e \subseteq P_{d+e}, 1 \in P_0(X)$. Let $T^d(\mathcal{F})$ denote the submodule of $T(\mathcal{F})$ given by the image of $aP_d \longrightarrow aP = T(\mathcal{F})$. Then $T(\mathcal{F})$ together with the submodules $T^d(\mathcal{F})$ is a sheaf of graded $O_X$-algebras. Note that $T^1(\mathcal{F})$ is the image of the monomorphism $\mathcal{F} \longrightarrow T(\mathcal{F})$ and $T^0(\mathcal{F})$ is the image of the canonical morphism of sheaves of algebras $O_X \longrightarrow T(\mathcal{F})$ (this latter morphism is also a monomorphism of sheaves of modules, so $T^0(\mathcal{F}) \cong O_X$ and $T^1(\mathcal{F}) \cong \mathcal{F}$ as sheaves of modules). More generally for $d \geq 0$ there is a canonical monomorphism of sheaves of modules $\mathcal{F}^{\otimes d} \longrightarrow T(\mathcal{F})$ whose image is $T^d(\mathcal{F})$ so we have (in the category $\mathfrak{Mod}(X)$)

$$T(\mathcal{F}) \cong \bigoplus_{d \geq 0} T^d(\mathcal{F}) \cong \bigoplus_{d \geq 0} \mathcal{F}^{\otimes d}$$

and the product in $T(\mathcal{F})$ is described by commutativity of the following diagram for $d, e \geq 0$

$$\xymatrix{ \mathcal{F}^{\otimes d} \otimes \mathcal{F}^{\otimes e} \ar[r] \ar[d] & T(\mathcal{F}) \otimes T(\mathcal{F}) \ar[d] \\ \mathcal{F}^{\otimes (d+e)} \ar[r] & T(\mathcal{F}) \ar[ur] }$$

It is clear that if $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves of modules, then $T(\phi)$ is a morphism of sheaves of graded $O_X$-algebras, so we also have a functor

$$T(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrAlg}(X)$$

As usual, given $r \in O_X(U)$ we also write $r$ for the corresponding element of $T^0(\mathcal{F})(U)$. Similarly if $a \in \mathcal{F}(U)$ we write $a$ for the corresponding element of $T^1(\mathcal{F})(U)$. For $n > 1$ and $a_1, \ldots, a_n \in \mathcal{F}(U)$ we write $a_1 \otimes \cdots \otimes a_n$ for the element of $T^n(\mathcal{F})(U)$. In this notation, for a morphism of sheaves of modules $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ we have

$$T(\phi)_U(r + a_{11} \otimes a_{22} + \cdots + a_{hh}) = r + \phi_U(a_{11}) + \phi_U(a_{22}) + \cdots + \phi_U(a_{hh})$$
Proposition 1. If \( q \in \mathbb{T}(\mathcal{F})(U) \) then for every \( x \in U \) there is an open neighborhood \( x \in V \subseteq U \) such that \( q|_V = q_1 + \cdots + q_k \) where each \( q_k \) has the form

\[
q_k = r + a_{11} + a_{21} \hat{\otimes} a_{22} + \cdots + a_{hh} \hat{\otimes} \cdots \hat{\otimes} a_{hh}
\]

where \( r \in \mathcal{O}_X(V) \) and \( a_{ij} \in \mathcal{F}(V) \).

Proof. This follows immediately from the fact that \( \mathbb{T}(\mathcal{F}) \) is the sheafification of the presheaf \( P \) defined above. \( \square \)

Proposition 2. The functor \( \mathbb{T}(-) : \text{Mod}(X) \rightarrow \text{nAlg}(X) \) is left adjoint to the forgetful functor \( \text{nAlg}(X) \rightarrow \text{Mod}(X) \). The unit of the adjunction is given for a sheaf of modules \( \mathcal{F} \) by the canonical morphism \( \mathcal{F} \rightarrow \mathbb{T} \mathcal{F} \).

Proof. Let \( \mathcal{I} \) be a sheaf of \( \mathcal{O}_X \)-algebras and \( \phi : \mathcal{F} \rightarrow \mathcal{I} \) a morphism of sheaves of modules. We have to show there exists a unique morphism of sheaves of algebras \( \Phi : \mathbb{T}(\mathcal{F}) \rightarrow \mathcal{I} \) making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{I} \\
\mathbb{T}(\mathcal{F}) & \xrightarrow{\Phi} & \mathcal{I}
\end{array}
\]

We use the results of our notes on Tensor, Exterior and Symmetric algebras (TES) in what follows. For every open set \( U \) there is a unique morphism of \( \mathcal{O}_X(U) \)-algebras \( \Phi_U : \mathbb{T}(\mathcal{F}(U)) \rightarrow \mathcal{I}(U) \) making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{I}(U) \\
\mathbb{T}(\mathcal{F}(U)) & \xrightarrow{\Phi_U} & \mathcal{I}(U)
\end{array}
\]

This defines a morphism of presheaves of \( \mathcal{O}_X \)-algebras \( \Phi' : P \rightarrow \mathcal{I} \), which induces a morphism of sheaves of \( \mathcal{O}_X \)-algebras \( \Phi : \mathbb{T}(\mathcal{F}) \rightarrow \mathcal{I} \) with the required property. \( \square \)

Proposition 3. The functor \( \mathbb{T}(-) : \text{Mod}(X) \rightarrow \text{GrnAlg}(X) \) is left adjoint to the functor \( (-)_1 : \text{GrnAlg}(X) \rightarrow \text{Mod}(X) \) which maps a sheaf of graded algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules \( \mathcal{F} \) by the canonical isomorphism \( \mathcal{F} \rightarrow \mathbb{T}^1(\mathcal{F}) \).

Proof. The definition of a sheaf of graded \( \mathcal{O}_X \)-algebras \( \mathcal{I} \) includes the provision of a sheaf of modules \( \mathcal{I}_1 \), and any morphism of sheaves of graded algebras must induce a morphism of sheaves of modules between these degree 1 components, so the functor \( (-)_1 \) is well defined. It is not difficult to check \( \mathcal{F} \rightarrow \mathbb{T}^1(\mathcal{F}) \) is natural in \( \mathcal{F} \), and therefore defines a natural transformation \( 1 \rightarrow (-)_1 \mathbb{T}(-) \).

We have to show that if \( \mathcal{F} \) is a sheaf of graded \( \mathcal{O}_X \)-algebras and \( \phi : \mathcal{F} \rightarrow \mathcal{I}_1 \) a morphism of sheaves of modules, then there exists a unique morphism of sheaves of graded algebras \( \Phi : \mathbb{T}(\mathcal{F}) \rightarrow \mathcal{I} \) such that \( \Phi_1 \) makes the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{I}_1 \\
\mathbb{T}(\mathcal{F}) & \xrightarrow{\Phi_1} & \mathcal{I}_1
\end{array}
\]

By Proposition 2 the composite \( \mathcal{F} \rightarrow \mathcal{I}_1 \rightarrow \mathcal{I} \) induces a morphism of sheaves of algebras \( \Phi : \mathbb{T}(\mathcal{F}) \rightarrow \mathcal{I} \) unique with the property that \( \mathcal{F} \rightarrow \mathbb{T}(\mathcal{F}) \rightarrow \mathcal{I} \) is \( \mathcal{F} \rightarrow \mathcal{I} \). It is straightforward to check that \( \Phi \) is a morphism of sheaves of graded algebras, and \( \Phi_1 \) makes (1) commute. Uniqueness is easily checked, which proves that \( \mathbb{T}(-) \) is left adjoint to \( (-)_1 \). \( \square \)
Proposition 4. If $U \subseteq X$ is open then the following diagram commutes up to a canonical natural equivalence

$$\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{\tau(-)} & n\text{Alg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{\tau(-)} & n\text{Alg}(U)
\end{array}$$

For a sheaf of modules $\mathcal{F}$ on $X$ the natural isomorphism $\tau(\mathcal{F}|_U) \rightarrow \mathcal{F}(\mathcal{F}|_U)$ has the action $a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_n$.

Proof. Let $n\text{Alg}(X)$ denote the category of presheaves of $O_X$-algebras. Then associating a sheaf of modules $\mathcal{F}$ with the presheaf $P(U) = \mathcal{F}(\mathcal{F}(U))$ defines a functor $\text{Mod}(X) \rightarrow n\text{Alg}(X)$. Clearly $\tau(-)$ is the composite of this functor with sheafification $n\text{Alg}(X) \rightarrow n\text{Alg}(X)$. So by (SOA, Lemma 3) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

$$\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{f} & n\text{Alg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{f} & n\text{Alg}(U)
\end{array}$$

In fact it is easy to see that this diagram commutes in the strictest sense, that is, the two legs of the diagram are the same functor. \hfill \Box

Proposition 5. Let $f : (X, O_X) \rightarrow (Y, O_Y)$ be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence

$$\begin{array}{ccc}
n\text{Alg}(X) & \xrightarrow{f_*} & n\text{Alg}(Y) \\
\downarrow & \tau(-) & \downarrow \tau(-) \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}$$

For a sheaf of modules $\mathcal{F}$ on $X$ the natural isomorphism $f_*\mathcal{F}|_U \rightarrow \mathcal{F}(f_*\mathcal{F}|_U)$ has the action $a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_n$.

Proof. Using (SOA, Lemma 12) we reduce immediately to showing the following diagram commutes up to a canonical natural equivalence

$$\begin{array}{ccc}
n\text{Alg}(X) & \xrightarrow{f_*} & n\text{Alg}(Y) \\
\downarrow & & \downarrow \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}$$

where the vertical functors are the “presheaf” tensor algebra functors given in Proposition 4. Let $V \subseteq X$ be open. In our construction of the $O_Y(V)$-module $\mathcal{F}(f^{-1}V)^{\otimes n}$ for $n \geq 2$ we see that we actually produce the same underlying group as when we define $\mathcal{F}(f^{-1}V)^{\otimes n}$ over $O_X(f^{-1}V)$. Together with the ring isomorphism $f^*_\mathcal{F}$ this gives an isomorphism of abelian groups $\rho_V : T_{O_Y(V)}(\mathcal{F}(f^{-1}V)) \rightarrow T_{O_X(f^{-1}V)}(\mathcal{F}(f^{-1}V))$

$$T_{O_Y(V)}(\mathcal{F}(f^{-1}V)) = O_Y(V) \oplus \mathcal{F}(f^{-1}V) \oplus \mathcal{F}(f^{-1}V)^{\otimes 2} \oplus \cdots$$

$$\downarrow$$

$$T_{O_X(f^{-1}V)}(\mathcal{F}(f^{-1}V)) = O_X(f^{-1}V) \oplus \mathcal{F}(f^{-1}V) \oplus \mathcal{F}(f^{-1}V)^{\otimes 2} \oplus \cdots$$

It is not hard to see this is an isomorphism of $O_Y(V)$-algebras natural in $V$ and also in $\mathcal{F}$, which completes the proof. \hfill \Box
Proposition 6. Let $X = \text{Spec} A$ be an affine scheme and $M$ an $A$-module. Then there is a canonical isomorphism $\theta : T(M) \rightarrow T(M^\sim)$ of sheaves of $O_X$-algebras which is natural in $M$. We have

$$\theta_U(a_1 \otimes \cdots \otimes a_n/s_1 \cdots s_n) = a_1/s_1 \otimes \cdots \otimes a_n/s_n$$

where $U \subseteq X$ is open, $a_i \in M$ and $s_i \in A$ with $U \subseteq D(s_1 \cdots s_n)$.

Proof. Consider the following diagram consisting of adjoint pairs of functors (see Ex 5.3, Proposition 2, (SOA, Proposition 5) and (TES, Proposition 7))

\[
\begin{array}{ccc}
\text{AnAlg} & \overset{\Gamma}{\longrightarrow} & \text{nAlg}(X) \\
T & \downarrow & T(X) \\
\text{AMod} & \overset{\Gamma}{\longrightarrow} & \text{Mod}(X)
\end{array}
\]

The two composites $\text{nAlg}(X) \rightarrow \text{AMod}$ are equal, so $T$ and $\sim T$ are both left adjoints for the same functor. Therefore they must be canonically naturally equivalent, which is what we wanted to show. The isomorphism $\theta : T(M) \rightarrow T(M^\sim)$ is unique with the property that $\theta_X(m/1) = m/1$ for every $m \in M$ (one should think carefully about what the notation $m/1$ means in both cases). This is an isomorphism of sheaves of algebras, so it is now easy to see $\theta$ has the desired effect on the special sections in the statement of the Proposition.

Corollary 7. Let $X$ be a scheme and $\mathcal{F}$ a sheaf of modules on $X$. If $\mathcal{F}$ is quasi-coherent then so is $T(\mathcal{F})$.

Proof. For $x \in X$ let $U$ be an open affine neighborhood of $x$ and $f : U \rightarrow \text{Spec} O_X(U)$ the canonical isomorphism. Then $f_*\mathcal{F}|_U \cong \mathcal{F}(U)^\sim$ and combining Proposition 6, Proposition 5 and Proposition 4 we see that

$$f_*(T(\mathcal{F})|_U) \cong f_*(T(\mathcal{F}|_U))$$

$$\cong T(f_*\mathcal{F}|_U)$$

$$\cong T(\mathcal{F}(U))$$

$$\cong T(\mathcal{F}(U))^\sim$$

This is an isomorphism of sheaves of algebras, which is more than enough to show that $T(\mathcal{F})$ is quasi-coherent.

Proposition 8. Let $X$ be a scheme and $\mathcal{F}$ a sheaf of modules on $X$. If $\mathcal{F}$ is quasi-coherent and $U \subseteq X$ is affine then there is a canonical isomorphism of graded $O_X(U)$-algebras natural in $\mathcal{F}$ and the affine open set $U$

$$\tau : T(\mathcal{F}(U)) \rightarrow T(\mathcal{F})(U)$$

$$f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n$$

Proof. We make $T(\mathcal{F})(U)$ into a graded $O_X(U)$-algebra as in (SOA, Proposition 40). Using (SOA, Proposition 4) (c) and Corollary 7 we get an isomorphism of $O_X(U)$-algebras $\tau$, and it is not hard to check it has the desired action on $f_1 \otimes \cdots \otimes f_n$, and is therefore a morphism of graded algebras. Note that $\tau$ is actually the sheafification morphism $P \rightarrow T(\mathcal{F})$ evaluated at $U$, from which naturality in $\mathcal{F}$ and inclusions of affine open sets $V \subseteq U$ is obvious.

Lemma 9. Let $(X, O_X)$ be a ringed space. Then $T(O_X)$ is a sheaf of commutative graded $O_X$-algebras.

Proof. It suffices to show that the presheaf $P(U) = T(O_X(U))$ is commutative, which follows immediately from (TES, Lemma 8).
3 Sheaves of Symmetric Algebras

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules, and for an open set $U$ let $H(U)$ be the commutative $\mathcal{O}_X(U)$-algebra given by the symmetric algebra $S(\mathcal{F}(U))$. That is, $S(\mathcal{F}(U))$ is the commutative graded $\mathcal{O}_X(U)$-algebra obtained as a quotient of $T(\mathcal{F}(U))$ by the two-sided ideal $I$ generated by the elements of the form $x \otimes y - y \otimes x$. For an inclusion $V \subseteq U$ the morphism of rings $T(\mathcal{F}(U)) \to T(\mathcal{F}(V))$ defined earlier induces a morphism of rings $S(\mathcal{F}(U)) \to S(\mathcal{F}(V))$ fitting into a commutative diagram

$$
\begin{array}{ccc}
T(\mathcal{F}(U)) & \to & S(\mathcal{F}(U)) \\
\downarrow & & \downarrow \\
T(\mathcal{F}(V)) & \to & S(\mathcal{F}(V))
\end{array}
$$

This makes $H$ into a presheaf of commutative $\mathcal{O}_X$-algebras, and if $P$ is the presheaf of algebras $P(U) = T(\mathcal{F}(U))$ then the canonical projections give a morphism of presheaves of algebras $P \to H$. Let $\mathcal{S}(\mathcal{F})$ denote the sheaf of commutative $\mathcal{O}_X$-algebras obtained by sheafifying $H$. Sheafifying $P \to H$ gives a canonical morphism of sheaves of algebras $T(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$, which is an epimorphism of sheaves of modules. The morphism of presheaves of modules $\mathcal{F} \to H$ given pointwise by the canonical injection $\mathcal{F}(U) \to \mathcal{S}(\mathcal{F}(U))$ composes with $H \to \mathcal{S}(\mathcal{F})$ to give a monomorphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{S}(\mathcal{F})$. The morphisms we have just defined fit into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} & \to & T(\mathcal{F}) \\
\downarrow & & \downarrow \\
\mathcal{S}(\mathcal{F}) & \to & \mathcal{S}(\mathcal{F})
\end{array}
$$

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of modules, whose associated presheaves of symmetric algebras are $H, G$ respectively, then we define a morphism of presheaves of $\mathcal{O}_X$-algebras $\phi' : H \to G$ by $\phi'_U = S(\phi_U)$. This sheafifies to give a morphism of sheaves of $\mathcal{O}_X$-algebras $\mathcal{S}(\phi) : \mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{G})$. This defines a functor

$$
\mathcal{S}(-) : \text{Mod}(X) \to \text{Alg}(X)
$$

Note that the following diagrams commute

$$
\begin{array}{ccc}
S(\mathcal{F}) & \xrightarrow{\mathcal{S}(\phi)} & S(\mathcal{G}) \\
\downarrow & & \downarrow \\
T(\mathcal{F}) & \xrightarrow{T(\phi)} & T(\mathcal{G})
\end{array}
$$

For $d \geq 0$ let $H_d$ denote the sub-presheaf of $\mathcal{O}_X$-modules of $H$ given by $H_d(U) = S^d(\mathcal{F}(U))$, which is the submodule of $S(\mathcal{F}(U))$ given by the image of $\mathcal{F}(U) \otimes_{d \geq 0} H_d \to H$ is an isomorphism (coproduct of presheaves of modules) and $H_d H_e \subseteq H_{d+e}, 1 \in H_0(X)$. Let $S^d(\mathcal{F})$ denote the subalgebra of $S(\mathcal{F})$ given by the image of $\mathcal{A}(H_d) \to \mathcal{A}H = S(\mathcal{F})$. Then $S(\mathcal{F})$ together with the submodules $S^d(\mathcal{F})$ is a sheaf of graded $\mathcal{O}_X$-algebras. Note that $S^1(\mathcal{F})$ is the image of the monomorphism $\mathcal{F} \to S(\mathcal{F})$ and $S^0(\mathcal{F})$ is the image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \to S(\mathcal{F})$ (this latter morphism is also a monomorphism of sheaves of modules, so $S^0(\mathcal{F}) \cong \mathcal{O}_X$ and $S^1(\mathcal{F}) \cong \mathcal{F}$ as sheaves of modules).

It is clear that if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of modules, then $S(\phi)$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras, so we also have a functor

$$
\mathcal{S}(-) : \text{Mod}(X) \to \text{GrAlg}(X)
$$
As usual, given \( r \in \mathcal{O}_X(U) \) we write \( r \) for the corresponding element of \( S^0(\mathcal{F})(U) \). Similarly if \( a \in \mathcal{F}(U) \) we write \( a \) for the corresponding element of \( S^1(\mathcal{F})(U) \). For \( n > 1 \) and sections \( a_1, \ldots, a_n \in \mathcal{F}(U) \) the product \( a_1 \cdots a_n \) in the ring \( S(\mathcal{F}(U)) \) is the coset of the tensor \( a_1 \otimes \cdots \otimes a_n \).

The image of this product in \( S^0(\mathcal{F})(U) \) via \( H \rightarrow S(\mathcal{F}) \) is just the product of the \( a_i \) considered as sections of \( S^1(\mathcal{F})(U) \), so we can write this section as \( a_1 \cdots a_n \) with no ambiguity. In this notation, for a morphism of sheaves of modules \( \phi : \mathcal{F} \rightarrow \mathcal{I} \) we have

\[
S(\phi)|_U (r + a_{11}a_{22} + \cdots + a_{hh}) = r + \phi_U(a_{11}) + \phi_U(a_{21})\phi_U(a_{22}) + \cdots + \phi_U(a_{hh})
\]

**Proposition 10.** If \( q \in S(\mathcal{F})(U) \) then for every \( x \in U \) there is an open neighborhood \( x \in V \subseteq U \) such that \( q|_V = q_1 + \cdots + q_n \) where each \( q_k \) has the form

\[
q_k = r + a_{11} + a_{21}a_{22} + \cdots + a_{hh}
\]

where \( r \in \mathcal{O}_X(V) \) and \( a_{ij} \in \mathcal{F}(V) \).

**Proof.** This follows immediately from the fact that \( S(\mathcal{F}) \) is the sheafification of the presheaf \( H \) defined above. \( \square \)

**Proposition 11.** The functor \( S(-) : \mathfrak{M}(X) \rightarrow \mathfrak{A}(X) \) is left adjoint to the forgetful functor \( \mathfrak{A}(X) \rightarrow \mathfrak{M}(X) \). The unit of the adjunction is given for a sheaf of modules \( \mathcal{F} \) by the canonical morphism \( \mathcal{F} \rightarrow S(\mathcal{F}) \).

**Proof.** Let \( \mathcal{F} \) be a sheaf of commutative \( \mathcal{O}_X \)-algebras and \( \phi : \mathcal{F} \rightarrow \mathcal{I} \) a morphism of sheaves of modules. We have to show there exists a unique morphism of sheaves of algebras \( \Phi : S(\mathcal{F}) \rightarrow \mathcal{I} \) making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{I} \\
\downarrow & & \downarrow \\
S(\mathcal{F}) & \xrightarrow{\Phi} & \mathcal{I}
\end{array}
\]

One argues as for \( T(-) \), using the properties of the symmetric algebra given in our TES notes. \( \square \)

**Proposition 12.** The functor \( S(-) : \mathfrak{M}(X) \rightarrow \mathfrak{G}(X) \) is left adjoint to the functor \( (-)_1 : \mathfrak{G}(X) \rightarrow \mathfrak{M}(X) \) which maps a sheaf of commutative graded algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules \( \mathcal{F} \) by the canonical isomorphism \( \mathcal{F} \rightarrow S^1(\mathcal{F}) \).

**Proof.** The canonical morphism \( \mathcal{F} \rightarrow S^1(\mathcal{F}) \) is natural in \( \mathcal{F} \), and we have to show that if \( \mathcal{F} \) is a sheaf of commutative graded \( \mathcal{O}_X \)-algebras and \( \phi : \mathcal{F} \rightarrow \mathcal{I}_1 \) a morphism of sheaves of modules, then there exists a unique morphism of sheaves of graded algebras \( \Phi : S(\mathcal{F}) \rightarrow \mathcal{I} \) such that \( \Phi_1 \) makes the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{I}_1 \\
\downarrow & & \downarrow \\
S^1(\mathcal{F}) & \xrightarrow{\Phi_1} & \mathcal{I}
\end{array}
\] \hspace{1cm} (2)

By Proposition 11 the composite \( \mathcal{F} \rightarrow \mathcal{I}_1 \rightarrow \mathcal{I} \) induces a morphism of sheaves of algebras \( \Phi : S(\mathcal{F}) \rightarrow \mathcal{I} \) unique with the property that \( \mathcal{F} \rightarrow S(\mathcal{F}) \rightarrow \mathcal{I} \) is \( \mathcal{F} \rightarrow \mathcal{I} \). It is easy to check that \( \Phi \) is a morphism of sheaves of graded algebras, and \( \Phi_1 \) makes (2) commute. Uniqueness is clear, which proves that \( S(-) \) is left adjoint to \( (-)_1 \). \( \square \)
Proposition 13. If $U \subseteq X$ is open then the following diagram commutes up to canonical natural equivalence

\[
\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{S(-)} & \text{Alg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{S(-)} & \text{Alg}(U)
\end{array}
\]

For a sheaf of modules $\mathcal{F}$ on $X$ the natural isomorphism $S(\mathcal{F}|_U) \rightarrow S(\mathcal{F})|_U$ has the action $a_1 \cdots a_n \mapsto a_1 \cdot \cdots \cdot a_n$.

Proof. Let $\text{Alg}(X)$ denote the category of presheaves of commutative $\mathcal{O}_X$-algebras. Associating a sheaf of modules $\mathcal{F}$ with the presheaf $H(U) = S(\mathcal{F}(U))$ defines a functor $\text{Mod}(X) \rightarrow \text{Alg}(X)$. Clearly $S(-)$ is the composite of this functor with sheafification $\text{Alg}(X) \rightarrow \text{Alg}(X)$. So by (MRS, Lemma 24) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

\[
\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{f^*} & \text{Alg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{f^*} & \text{Alg}(U)
\end{array}
\]

But it is not hard to check that this diagram actually commutes. \qed

Proposition 14. Let $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence

\[
\begin{array}{ccc}
\text{Alg}(X) & \xrightarrow{f_*} & \text{Alg}(Y) \\
\downarrow & & \downarrow \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}
\]

For a sheaf of modules $\mathcal{F}$ on $X$ the natural isomorphism $f_*S(\mathcal{F}) \rightarrow S(f_*\mathcal{F})$ has the action $a_1 \cdots a_n \mapsto a_1 \cdot \cdots \cdot a_n$.

Proof. Using (SOA, Lemma 12) we reduce immediately to showing the following diagram commutes up to a canonical natural equivalence

\[
\begin{array}{ccc}
\text{Alg}(X) & \xrightarrow{f_*} & \text{Alg}(Y) \\
\downarrow & & \downarrow \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}
\]

where the vertical functors are the “presheaf” symmetric algebra functors given in Proposition 13. In Proposition 5 we defined an isomorphism of $\mathcal{O}_Y(V)$-algebras $T_{\mathcal{O}_Y(V)}(\mathcal{F}(f^{-1}V)) \rightarrow T_{\mathcal{O}_X(f^{-1}V)}(\mathcal{F}(f^{-1}V))$ natural in $V$ and $\mathcal{F}$ which induces the necessary isomorphism of symmetric algebras. \qed

Proposition 15. Let $X = \text{Spec}A$ be an affine scheme and $M$ an $A$-module. Then there is a canonical isomorphism $\theta : S(M)^- \rightarrow S(M)$ of sheaves of $\mathcal{O}_X$-algebras which is natural in $M$. We have

\[
\theta_U(a_1 \cdots a_n/s_1 \cdots s_n) = a_1/s_1 \cdots a_n/s_n
\]

where $U \subseteq X$ is open, $a_i \in M$ and $s_i \in A$ with $U \subseteq D(s_1 \cdots s_n)$. 8
Proof. Consider the following diagram consisting of adjoint pairs of functors (see (Ex 5.3), (SOA, Proposition 5), Proposition 11 and (TES, Proposition 33))

\[
\begin{array}{ccc}
\mathcal{A} \text{Alg} & \xrightarrow{\Gamma} & \mathcal{A} \text{Alg}(X) \\
\downarrow S & & \downarrow S(-) \\
\mathcal{A} \text{Mod} & \xrightarrow{\Gamma} & \mathcal{A} \text{Mod}(X)
\end{array}
\]

The two composites \( \mathcal{A} \text{Alg}(X) \to \mathcal{A} \text{Mod} \) are equal, so \( \tilde{S} \) and \( S(-) \) are both left adjoints for the same functor. Therefore they must be canonically naturally equivalent, which is what we wanted to show. The isomorphism \( \theta : S(M) \to S\tilde{M} \) is unique with the property that \( \theta_X(m/1) = m/1 \) for every \( m \in M \). This is an isomorphism of sheaves of algebras, so it is easy to check \( \theta \) has the desired effect on the sections in the statement of the Proposition.

Corollary 16. Let \( X \) be a scheme and \( \mathcal{F} \) a sheaf of modules on \( X \). If \( \mathcal{F} \) is quasi-coherent then \( \tilde{S}(\mathcal{F}) \) is also quasi-coherent.

Proof. For \( x \in X \) let \( U \) be an open affine neighborhood of \( x \) and \( f : U \to \text{Spec} \mathcal{O}_X(U) \) the canonical isomorphism. Then \( f_* \mathcal{F}|_U \cong \mathcal{F}(U) \) and combining Proposition 15, Proposition 14 and Proposition 13 we see that

\[
f_*(\tilde{S}(\mathcal{F})|_U) \cong f_*(\tilde{S}(\mathcal{F}|_U))
\]

This is an isomorphism of sheaves of algebras, which shows that \( \tilde{S}(\mathcal{F}) \) is quasi-coherent.

Corollary 18. Let \( X \) be a scheme and \( \mathcal{F} \) a quasi-coherent sheaf of modules on \( X \). Then \( \tilde{S}(\mathcal{F}) \) is locally generated by \( S^1(\mathcal{F}) \) as an \( \mathcal{O}_X \)-algebra.

Proof. Using Proposition 17 we reduce immediately to showing that for a commutative ring \( A \) and \( A \)-module \( M \) the graded \( A \)-algebra \( S(M) \) is generated by \( S^1(M) \) as an \( A \)-algebra, which is obvious.

4 Sheaves of Exterior Algebras

Throughout this section \((X, \mathcal{O}_X)\) is a ringed space. Let \( \mathcal{F} \) be a sheaf of \( \mathcal{O}_X \)-modules, and for any open set \( U \) let \( H(U) \) be the (noncommutative) \( \mathcal{O}_X(U) \)-algebra given by the exterior algebra
That is, $\Lambda \mathcal{F}(U)$ is the graded $\mathcal{O}_X(U)$-algebra obtained as a quotient of $T(\mathcal{F}(U))$ by the two-sided ideal $I$ generated by the elements of the form $x \otimes x$. For an inclusion $V \subseteq U$ the morphism of rings $T(\mathcal{F}(U)) \to T(\mathcal{F}(V))$ defined earlier induces a morphism of rings $\Lambda \mathcal{F}(U) \to \Lambda \mathcal{F}(V)$ fitting into a commutative diagram

\[
\begin{array}{ccc}
T(\mathcal{F}(U)) & \longrightarrow & \Lambda \mathcal{F}(U) \\
\downarrow & & \downarrow \\
T(\mathcal{F}(V)) & \longrightarrow & \Lambda \mathcal{F}(V)
\end{array}
\]

This makes $H$ into a presheaf of $\mathcal{O}_X$-algebras, and if $P$ is the presheaf of algebras $P(U) = T(\mathcal{F}(U))$ then the canonical projections give a morphism of presheaves of algebras $P \to H$. Let $\Lambda \mathcal{F}$ denote the sheaf of $\mathcal{O}_X$-algebras obtained by sheafifying $H$. Sheafifying $P \to H$ gives a canonical morphism of sheaves of algebras $T(\mathcal{F}) \to \Lambda \mathcal{F}$, which is an epimorphism of sheaves of modules. The morphism of presheaves of modules $\mathcal{F} \to H$ given pointwise by the canonical injection $\mathcal{F}(U) \to \Lambda \mathcal{F}(U)$ composes with $H \to \Lambda \mathcal{F}$ to give a monomorphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \Lambda \mathcal{F}$. The morphisms we have just defined fit into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & T(\mathcal{F}) \\
\downarrow & & \downarrow \\
\Lambda \mathcal{F} & \longrightarrow & \Lambda \mathcal{F}
\end{array}
\]

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of modules, whose associated presheaves of exterior algebras are $H, G$ respectively, then we define a morphism of presheaves of algebras $\phi : \mathcal{F} \to \mathcal{G}$ which is the submodule of $\Lambda \mathcal{F}$ given by the image of the canonical morphism of sheaves of modules $\mathcal{F} \to \Lambda \mathcal{F}$, so $\phi(U)$ is an isomorphism. This defines a functor $\Lambda(-) : \mathcal{Mod}(X) \to \mathcal{sAlg}(X)$.

Note that the following diagrams commute

\[
\begin{array}{ccc}
\Lambda \mathcal{F} & \stackrel{\Lambda \phi}{\longrightarrow} & \Lambda \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{F} & \stackrel{\phi}{\longrightarrow} & \mathcal{G} \\
\Lambda \mathcal{F} & \longrightarrow & \Lambda \mathcal{G}
\end{array}
\]

For $d \geq 0$ let $H_d$ denote the sub-presheaf of $\mathcal{O}_X$-modules of $H$ given by $H_d(U) = \Lambda^d(\mathcal{F}(U))$, which is the submodule of $\Lambda(\mathcal{F}(U))$ given by the image of $\mathcal{F}(U) \otimes^d \to T(\mathcal{F}(U)) \to \Lambda(\mathcal{F}(U))$. In particular there are isomorphism of presheaves of modules $H_0 \cong \mathcal{O}_X$ and $H_1 \cong \mathcal{F}$. By construction the induced morphism $\bigoplus_{d \geq 0} H_d \to H$ is an isomorphism (coproduct of presheaves of modules) and $H_{d+e} \subseteq H_d \otimes H_e$, $1 \in H_0(X)$. Let $\Lambda^d \mathcal{F}$ denote the submodule of $\Lambda \mathcal{F}$ given by the image of $a H_d \longrightarrow a H = \Lambda \mathcal{F}$. Then $\Lambda \mathcal{F}$ together with the submodules $\Lambda^d \mathcal{F}$ is a sheaf of super $\mathcal{O}_X$-algebras. Note that $\Lambda^0 \mathcal{F}$ is the image of the monomorphism $\mathcal{F} \to \Lambda \mathcal{F}$ and $\Lambda^0 \mathcal{F}$ is the image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \to \Lambda \mathcal{F}$ (this latter morphism is also a monomorphism of sheaves of modules, so $\Lambda^0 \mathcal{F} \cong \mathcal{O}_X$ and $\Lambda^1 \mathcal{F} \cong \mathcal{F}$ as sheaves of modules).

It is clear that if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of modules, then $\Lambda \phi$ is a morphism of sheaves of super $\mathcal{O}_X$-algebras, so we also have a functor $\Lambda(-) : \mathcal{Mod}(X) \to \mathcal{sAlg}(X)$.

As usual, given $r \in \mathcal{O}_X(U)$ we write $r$ for the corresponding element of $\Gamma(U, \Lambda^0 \mathcal{F})$. Similarly if $a \in \mathcal{F}(U)$ we write $a$ for the corresponding element of $\Gamma(U, \Lambda^1 \mathcal{F})$. For $n > 1$ and $a_1, \ldots, a_n \in \mathcal{F}(U)$ we write $a_1 \wedge \cdots \wedge a_n$ for the element of $\Gamma(U, \Lambda^n \mathcal{F})$ which is the image of $a_1 \wedge \cdots \wedge a_n \in \Lambda^n(\mathcal{F}(U))$. 

\[
\Lambda(-) : \mathcal{Mod}(X) \to \mathcal{sAlg}(X)
\]
under $H \to \bigwedge F$. This is just the product of the individual $a_i$ considered as elements of $\Gamma(U, \bigwedge^1 F)$. In this notation, for a morphism of sheaves of modules $\phi : F \to G$ we have

$$(\wedge \phi)_U(r + a_{11} + a_{21} \wedge a_{22} + \cdots + a_{hh} \wedge \cdots \wedge a_{hh}) = r + \phi_U(a_{11}) + \phi_U(a_{21}) \wedge \phi_U(a_{22}) + \cdots + \phi_U(a_{hh}) \wedge \cdots \wedge \phi_U(a_{hh})$$

For $d \geq 0$ let $\wedge^d : \bigwedge F \to \bigwedge^d G$ be the morphism of sheaves of modules induced on the graded submodules by $\wedge \phi$, which is unique making the following diagram of sheaves of modules commute

\[
\begin{array}{ccc}
\bigwedge F & \xrightarrow{\wedge \phi} & \bigwedge G \\
\downarrow & & \downarrow \\
\bigwedge^d F & \xrightarrow{\wedge^d \phi} & \bigwedge^d G
\end{array}
\]

This defines a functor $\bigwedge^d(-) : \text{Mod}(X) \to \text{Mod}(X)$.

**Proposition 19.** If $q \in \Gamma(U, \bigwedge F)$ then for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $q|_V = q_1 + \cdots + q_n$ where each $q_k$ has the form

$$q_k = r + a_{11} + a_{21} \wedge a_{22} + \cdots + a_{hh} \wedge \cdots \wedge a_{hh}$$

where $r \in \mathcal{O}_X(V)$ and $a_{ij} \in F(V)$.

**Definition 1.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. For $n \geq 1$ we say a multilinear morphism (MRS,Definition 9) $f : \mathcal{F} \times \cdots \times \mathcal{F} \to G$ from the $n$-fold product into a sheaf of abelian groups $G$ is alternating if for every open $U \subseteq X$ the map $f_U$ is alternating. That is, for every open $U \subseteq X$ we have $f_U(m_1, \ldots, m_n) = 0$ whenever $m_i = m_j$ for $i \neq j$. The canonical morphism of sheaves of sets

$$\gamma : \mathcal{F} \times \cdots \times \mathcal{F} \to \bigwedge^n \mathcal{F}$$

$$(a_1, \ldots, a_n) \mapsto a_1 \wedge \cdots \wedge a_n$$

is clearly an alternating multilinear form. In fact this is the universal alternating multilinear form, as we will see in a moment. If $n = 1$ then an alternating multilinear map is just a morphism of sheaves of abelian groups $\mathcal{F} \to G$ and an alternating multilinear form is a morphism of sheaves of modules.

**Proposition 20.** Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, G$ sheaves of modules on $X$ and suppose $f : \mathcal{F} \times \cdots \times \mathcal{F} \to G$ is an alternating multilinear form out of the $n$-fold product for $n \geq 1$. Then there is a unique morphism of sheaves of modules $\theta : \bigwedge^n \mathcal{F} \to G$ making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} \times \cdots \times \mathcal{F} & \xrightarrow{\gamma} & \bigwedge^n \mathcal{F} \\
\downarrow & \searrow \phi & \downarrow \\
G & & G
\end{array}
\]

**Proof.** For each open $U \subseteq X$ we have an alternating multilinear form $f_U : \mathcal{F}(U)^n \to G(U)$, which by (TES,Lemma 18) corresponds to a morphism of $\mathcal{O}_X(U)$-modules $\phi_U : \bigwedge^n \mathcal{F}(U) \to G(U)$. Together these define a morphism of presheaves of $\mathcal{O}_X(U)$-modules $\phi : H_d \to G$. The induced morphism $\theta : \bigwedge^n \mathcal{F} \to G$ is the one we require. \hfill \square

**Proposition 21.** The functor $\bigwedge(-) : \text{Mod}(X) \to \text{sAlg}(X)$ is left adjoint to the functor $(-)_1 : \text{sAlg}(X) \to \text{Mod}(X)$ which maps a sheaf of super algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules $\mathcal{F}$ by the canonical isomorphism $\mathcal{F} \to \bigwedge^1 \mathcal{F}$.  

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Proof. The canonical morphism \( \eta : \mathcal{F} \to \Lambda^1 \mathcal{F} \) is natural in \( \mathcal{F} \), and we have to show that if \( \mathcal{X} \) is a sheaf of super \( \mathcal{O}_X \)-algebras and \( \phi : \mathcal{F} \to \mathcal{X} \) a morphism of sheaves of modules, then there exists a unique morphism of sheaves of super algebras \( \Phi : \Lambda \mathcal{F} \to \mathcal{X} \) such that \( \Phi \) makes the following diagram commute

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow{\eta} & & \downarrow{\Phi} \\
\Lambda^1 \mathcal{F} & & \\
\end{array}
\]

Let \( \Phi_0 \) be the canonical morphism \( \Lambda^0 \mathcal{F} \to \mathcal{X}_0 \) and let \( \Phi_1 \) be the composite \( \phi \eta^{-1} \). For \( n > 1 \) we define a multilinear form

\[
f : \mathcal{F} \times \cdots \times \mathcal{F} \to \mathcal{X}_n
\]

\[
f(a_1, \ldots, a_n) = \phi v(a_1) \cdots \phi v(a_n)
\]

which is alternating since \( \mathcal{X} \) is a super \( \mathcal{O}_X \)-algebra. By Proposition 20 there is an induced morphism \( \Phi_n : \Lambda^n \mathcal{F} \to \mathcal{X}_n \). The morphisms \( \Phi_n \) for \( n \geq 0 \) induce a morphism of sheaves of graded \( \mathcal{O}_X \)-modules \( \Phi : \Lambda \mathcal{F} \to \mathcal{X} \) out of the coproduct. In fact with a little work one checks that \( \Phi \) is a morphism of super \( \mathcal{O}_X \)-algebras, while \( \Phi_1 \) trivially makes (3) commute. Uniqueness is clear, which proves that \( \Lambda(-) \) is left adjoint to \((-)_1 \).

**Proposition 22.** If \( U \subseteq X \) is open then the following diagram commutes up to canonical natural equivalence

\[
\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{\Lambda(-)} & \text{sAlg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{\Lambda(-)} & \text{sAlg}(U)
\end{array}
\]

For a sheaf of modules \( \mathcal{F} \) on \( X \) the natural isomorphism \( \Lambda(\mathcal{F}|_U) \cong (\Lambda \mathcal{F})|_U \) has the action \( a_1 \wedge \cdots \wedge a_n \mapsto a_1 \wedge \cdots \wedge a_n \). In particular there is an isomorphism of sheaves of modules \( \Lambda^d(\mathcal{F}|_U) \cong (\Lambda^d \mathcal{F})|_U \) natural in \( \mathcal{F} \).

**Proof.** Let \( \text{GrnAlg}(X) \) denote the category of presheaves of graded \( \mathcal{O}_X \)-algebras. Associating a sheaf of modules \( \mathcal{F} \) with the presheaf of graded \( \mathcal{O}_X \)-algebras \( H(U) = \Lambda(\mathcal{F}(U)) \) defines a functor \( \text{Mod}(X) \to \text{GrnAlg}(X) \). Clearly \( \Lambda(-) \) is the composite of this functor with sheafification \( \text{GrnAlg}(X) \to \text{Alg}(X) \). So by (SOA, Lemma 38) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

\[
\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{f} & \text{GrnAlg}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{f} & \text{GrnAlg}(U)
\end{array}
\]

But it is not hard to see that this diagram actually commutes. \( \square \)

**Corollary 23.** If \( U \subseteq X \) is open and \( \mathcal{F} \) is a sheaf of modules on \( X \) then for any \( d \geq 0 \) there is a canonical isomorphism \( \Lambda^d(\mathcal{F}|_U) \cong (\Lambda^d \mathcal{F})|_U \) of sheaves of modules on \( U \) natural in \( \mathcal{F} \).

**Proposition 24.** Let \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence

\[
\begin{array}{ccc}
\text{sAlg}(X) & \xrightarrow{f_\ast} & \text{sAlg}(Y) \\
\downarrow{\Lambda(-)} & & \downarrow{\Lambda(-)} \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}
\]
For a sheaf of modules $\mathcal{F}$ on $X$ the natural isomorphism $f_*(\bigwedge \mathcal{F}) \to \bigwedge (f_* \mathcal{F})$ has the action $a_1 \wedge \cdots \wedge a_n \mapsto a_1 \wedge \cdots \wedge a_n$.

**Proof.** Using (SOA, Lemma 44) we reduce immediately to showing that the following diagram commutes up to a canonical natural equivalence

$$
\begin{array}{ccc}
\text{GrnAlg}(X) & \xrightarrow{f_*} & \text{GrnAlg}(Y) \\
\downarrow & & \downarrow \\
\text{Mod}(X) & \xrightarrow{f_*} & \text{Mod}(Y)
\end{array}
$$

where the vertical functors are the “presheaf” exterior algebra functors given in Proposition 22. In Proposition 5 we defined an isomorphism of $\mathcal{O}_Y(V)$-algebras $T_{\mathcal{O}_Y(V)}(\mathcal{F}(f^{-1}V)) \to T_{\mathcal{O}_X(f^{-1}V)}(\mathcal{F}(f^{-1}V))$ natural in $V$ and $\mathcal{F}$ which induces the necessary isomorphism of exterior algebras.

**Lemma 25.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a sheaf of modules on $X$. For $y \in X$ there is a canonical isomorphism of graded $\mathcal{O}_{X,y}$-algebras natural in $\mathcal{F}$

$$
\tau : (\bigwedge ^d \mathcal{F})_y \to \bigwedge ^d \mathcal{F}_y
$$

$$
(U, a_1 \wedge \cdots \wedge a_n) \mapsto (U, a_1) \wedge \cdots \wedge (U, a_n)
$$

In particular for $d \geq 0$ there is a canonical isomorphism of $\mathcal{O}_{X,y}$-modules $(\bigwedge ^d \mathcal{F})_y \cong \bigwedge ^d \mathcal{F}_y$ natural in $\mathcal{F}$.

**Proof.** Let $H$ be the presheaf $H(U) = \bigwedge (\mathcal{F}(U))$. Then $H_y$ becomes a graded $\mathcal{O}_{X,y}$-algebra with the canonical grading, and in fact $H_y$ is a super $\mathcal{O}_{X,y}$-algebra (TES, Definition 4). Suppose $T$ is a super $\mathcal{O}_{X,y}$-algebra and that $\varphi : \mathcal{F}_y \to T_1$ is a morphism of $\mathcal{O}_{X,y}$-modules. For each open neighborhood $U$ of $y$ the morphism of $\mathcal{O}_X(U)$-modules $\mathcal{F}(U) \to \mathcal{F}_y \to T_1$ induces a morphism of super $\mathcal{O}_X(U)$-algebras $\bigwedge (\mathcal{F}(U)) \to T$ (TES, Proposition 7). Taking the direct limit we obtain a well-defined morphism of super $\mathcal{O}_{X,y}$-algebras $\tau : H_y \to T$ defined for $a \in \mathcal{F}(U)$ by $(U, a) \mapsto \varphi(U, a)$. If $\eta : \mathcal{F}_y \cong (H_y)_1$ is the canonical isomorphism of $\mathcal{O}_{X,y}$-modules then $\tau$ is trivially the unique morphism of super algebras making the following diagram commute

$$
\begin{array}{ccc}
\mathcal{F}_y & \xrightarrow{\varphi} & T_1 \\
\downarrow & \ & \downarrow \\
(H_y)_1 & \xrightarrow{\eta} & T
\end{array}
$$

But this is the universal property that identifies the super algebra $\bigwedge \mathcal{O}_{X,y} \mathcal{F}_y$. Therefore we have a canonical isomorphism of graded $\mathcal{O}_{X,y}$-algebras

$$
\tau : H_y \to \bigwedge \mathcal{F}_y
$$

$$
\tau(U, a_1 \wedge \cdots \wedge a_n) = (U, a_1) \wedge \cdots \wedge (U, a_n)
$$

composing with the canonical isomorphism $H_y \cong (\bigwedge \mathcal{F})_y$ we have the desired isomorphism. Naturality in $\mathcal{F}$ is easily checked.

**Proposition 26.** Let $X = \text{Spec} A$ be an affine scheme and $M$ an $A$-module. Then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras natural in $M$

$$
\theta : (\bigwedge M)^{\sim} \to \bigwedge (M^{\sim})
$$

$$
(a_1 \wedge \cdots \wedge a_n)/(s_1 \cdots s_n) \mapsto a_1/s_1 \wedge \cdots \wedge a_n/s_n
$$
Proof. See (SOA, Definition 13) for the definition of the sheaf of graded \( O_X \)-algebras \((\land M)^{\sim}\). If \( A = 0 \) then this is trivial, so assume \( A \) is nonzero. For \( p \in \text{Spec}A \) we have an isomorphism of \( O_{X,p} \)-algebras, using (TES, Corollary 16) and Lemma 25

\[
\theta_p : (\land A M)^{\sim}_p \cong (\land A M)_p \cong \land_{A_p} M_p \cong \land_{O_{X,p}} (M^\sim)_p \cong (\land M^\sim)_p
\]

It is not hard to check that \( \mathrm{germ}_p \theta_U(s) = \theta_p(\mathrm{germ}_p s) \) defines the necessary isomorphism of sheaves of graded algebras. Naturality in \( M \) is also easily checked. \( \square \)

**Corollary 27.** Let \( X \) be a scheme and \( \mathcal{F} \) a sheaf of modules on \( X \). If \( \mathcal{F} \) is quasi-coherent then so are \( \land \mathcal{F}, \land^d \mathcal{F} \) for every \( d \geq 0 \). If \( X \) is noetherian and \( \mathcal{F} \) coherent then \( \land^d \mathcal{F} \) is coherent for \( d \geq 0 \).

**Proof.** For \( x \in X \) let \( U \) be an affine open neighborhood of \( x \) and \( f : U \to \text{Spec}O_X(U) \) the canonical isomorphism. Then \( f_* (\mathcal{F})_{|U} \cong (\mathcal{F})_{|U}^{\sim} \) and combining Proposition 26, Proposition 24 and Proposition 22 we see that there is an isomorphism of sheaves of graded algebras

\[
f_* ((\land \mathcal{F})_{|U}) \cong f_* ((\land (\mathcal{F})_{|U}))
\]

\[
\cong (\land (f_* \mathcal{F})_{|U})
\]

\[
\cong (\land (\mathcal{F}(U)^{\sim}))
\]

\[
\cong (\land (\mathcal{F}(U)))^{\sim}
\]

This shows that \( \land \mathcal{F} \) is quasi-coherent. Since \( \land \mathcal{F} = \bigoplus_{d \geq 0} \land^d \mathcal{F} \) it follows from (MOS, Lemma 1) that \( \land^d \mathcal{F} \) is a quasi-coherent sheaf of modules for \( d \geq 0 \). Since (4) is an isomorphism of sheaves of graded algebras, we deduce an isomorphism of sheaves of modules \( f_* ((\land^d \mathcal{F})_{|U}) \cong (\land^d \mathcal{F}(U))^{\sim} \) for \( d \geq 0 \). If \( X \) is noetherian and \( \mathcal{F} \) coherent then \( \mathcal{F}(U) \) is a finitely generated \( O_X(U) \)-module, so \( \land^d \mathcal{F}(U) \) is finitely generated (TES, Lemma 11) and consequently \( \land^d \mathcal{F} \) is coherent. \( \square \)

**Corollary 28.** Let \( X \) be a nonempty scheme and \( \mathcal{F} \) a locally free sheaf of finite rank \( n \geq 1 \). Then for \( 0 \leq d \leq n \) the sheaf \( \land^d \mathcal{F} \) is locally free of rank \( \binom{n}{d} \). If \( d > n \) then \( \land^d \mathcal{F} = 0 \).

**Proof.** If \( x \in X \) then we can find an affine open neighborhood \( U \) of \( x \) with \( \mathcal{F}_{|U} \) free of rank \( n \). Then \( \mathcal{F}(U) \) is a free \( O_X(U) \)-module of rank \( n \) (MOS, Lemma 6) and therefore \( \land^d \mathcal{F}(U) \) is a free module of rank \( \binom{n}{d} \) if \( d \leq n \) and is zero otherwise (TES, Proposition 21). The result now follows from the isomorphism \( f_* ((\land^d \mathcal{F})_{|U}) \cong (\land^d \mathcal{F}(U))^{\sim} \) of Corollary 27. \( \square \)

**Proposition 29.** Let \( f : (X, O_X) \to (Y, O_Y) \) be a morphism of ringed spaces and \( \mathcal{F} \) a sheaf of modules on \( Y \). Then there is a canonical isomorphism of sheaves of graded \( O_X \)-algebras natural in \( \mathcal{F} \)

\[
\xi : f^*(\land \mathcal{F}) \to \land (f^* \mathcal{F})
\]

\[
[V, a_1 \land \cdots \land a_n] \otimes (b_1 \cdots b_n) \mapsto ([V, a_1] \otimes b_1) \land \cdots \land ([V, a_n] \otimes b_n)
\]

In particular for \( d \geq 0 \) there is a canonical isomorphism of \( O_X \)-modules \( f^*(\land^d \mathcal{F}) \cong \land^d f^* \mathcal{F} \) natural in \( \mathcal{F} \).

**Proof.** See (SOA, Section 3.1) for the definition of the inverse image of a sheaf of graded algebras. For \( x \in X \) we use Lemma 25, (SOA, Lemma 45) and (TES, Proposition 15) to obtain an
One checks that \( \text{germ}_x \xi_U(s) = \xi_x(\text{germ}_x s) \) gives a well-defined isomorphism of sheaves of graded \( \mathcal{O}_X \)-algebras. Naturality in \( F \) is easily checked.

**Lemma 30.** Let \((X, \mathcal{O}_X)\) be a ringed space and \( \psi : \mathcal{F} \to \mathcal{G} \) an epimorphism of sheaves of \( \mathcal{O}_X \)-modules. Then \( \bigwedge^d \psi : \bigwedge^d \mathcal{F} \to \bigwedge^d \mathcal{G} \) is an epimorphism for \( d \geq 1 \).

**Proof.** By Lemma 25 we reduce to showing that if \( R \) is a commutative ring, \( \psi : M \to N \) an epimorphism of \( R \)-modules then \( \bigwedge^d M \to \bigwedge^d N \) is an epimorphism, which is trivial.

**Remark 1.** Let \((X, \mathcal{O}_X)\) be a ringed space, \( \mathcal{F} \) a sheaf of \( \mathcal{O}_X \)-modules. Then for \( a, b \geq 1 \) the product in the sheaf of graded \( \mathcal{O}_X \)-algebras \( \bigwedge \mathcal{F} \) induces a morphism of sheaves of \( \mathcal{O}_X \)-modules

\[
\bigwedge^a \mathcal{F} \otimes \bigwedge^b \mathcal{F} \to \bigwedge^{a+b} \mathcal{F}
\]

\( a \otimes b \mapsto ab \quad (5) \)

**Proposition 31.** Let \((X, \mathcal{O}_X)\) be a nontrivial ringed space and suppose we have an exact sequence of locally free sheaves

\[
0 \to \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{H} \to 0
\]

where \( \mathcal{G}, \mathcal{H} \) are locally free of finite ranks \( a, b \) respectively. Then there is a canonical isomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \bigwedge^a \mathcal{F} \cong \bigwedge^a \mathcal{G} \otimes \bigwedge^b \mathcal{H} \).

**Proof.** See (MRS,Remark 13) for the definition of a nontrivial ringed space. In the cases where one of \( a, b \) are zero there is a trivial isomorphism, so assume \( a, b \geq 1 \). Tensoring \( \bigwedge^a \mathcal{F} \to \bigwedge^a \mathcal{G} \) with \( \bigwedge^b \mathcal{F} \) and then composing with (5) gives a morphism of sheaves of \( \mathcal{O}_X \)-modules

\[
\theta : \bigwedge^a \mathcal{G} \otimes \bigwedge^b \mathcal{F} \to \bigwedge^{a+b} \mathcal{F}
\]

Using Lemma 25 to reduce to the case of rings and exterior powers of modules, one checks that \( \theta \) is an epimorphism of sheaves of modules (see (TES,Section 3.3.1) for the case of rings). Tensoring the epimorphism \( \bigwedge^a \mathcal{F} \to \bigwedge^a \mathcal{G} \) with \( \bigwedge^b \mathcal{F} \) gives an epimorphism of sheaves of \( \mathcal{O}_X \)-modules

\[
\rho : \bigwedge^a \mathcal{G} \otimes \bigwedge^b \mathcal{F} \to \bigwedge^a \mathcal{G} \otimes \bigwedge^b \mathcal{H}
\]

We claim that \( \text{Ker}(\theta)_x = \text{Ker}(\rho)_x \) for every \( x \in X \), which follows from Lemma 25 and (TES,Proposition 32). Since \( \theta, \rho \) are epimorphisms with the same kernel, it follows that there is a unique isomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \bigwedge^{a+b} \mathcal{F} \cong \bigwedge^a \mathcal{G} \otimes \bigwedge^b \mathcal{H} \).
Proposition 32. Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{F}\) a sheaf of modules on \(X\). Then for \(n \geq 1\) there is a canonical morphism of sheaves of modules natural in \(\mathcal{F}\)

\[
\beta: \bigwedge^n (\mathcal{F}^\vee) \rightarrow \left( \bigwedge^n \mathcal{F} \right)^\vee
\]

\[
\beta_U(\nu_1 \wedge \cdots \wedge \nu_n)^V(a_1 \wedge \cdots \wedge a_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} (\nu_i)^V(a_{\sigma(i)})
\]

Proof. For an open subset \(U \subseteq X\) and \(\nu_1, \ldots, \nu_n \in \Gamma(U, \mathcal{F}^\vee)\) we define a morphism of sheaves of sets

\[
f: \mathcal{F}|_U \times \cdots \times \mathcal{F}|_U \rightarrow \mathcal{O}_X|_U
\]

\[
f_V(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} (\nu_i)^V(a_{\sigma(i)}) = \det((\nu_i)^V(a_j))
\]

Using the standard properties of the determinant, one checks that \(f\) is an alternating multilinear form. By Proposition 20 there is a corresponding morphism of sheaves of modules \(\bigwedge^n (\mathcal{F}|_U) \rightarrow \mathcal{O}_X|_U\). Let \(\theta_{\nu_1, \ldots, \nu_n}\) denote the composite of this morphism with the canonical isomorphism \(\bigwedge^n (\mathcal{F}|_U) \cong (\bigwedge^n \mathcal{F})|_U\) of Proposition 22. One checks that the following map is also an alternating multilinear form

\[
\tau: \mathcal{F}^\vee \times \cdots \times \mathcal{F}^\vee \rightarrow \left( \bigwedge^n \mathcal{F} \right)^\vee
\]

\[
\tau_U(\nu_1, \ldots, \nu_n) = \theta_{\nu_1, \ldots, \nu_n}
\]

which by Proposition 20 must correspond to a morphism of sheaves of modules \(\beta\) with the required property.

Corollary 33. Let \(X\) be a nonempty scheme, \(\mathcal{L}\) a locally free sheaf of finite rank. Then for \(n \geq 1\) the canonical morphism of sheaves of modules

\[
\beta: \bigwedge^n (\mathcal{L}^\vee) \rightarrow \left( \bigwedge^n \mathcal{L} \right)^\vee
\]

of Proposition 32 is an isomorphism.

Proof. Let \(\mathcal{L}\) be a locally free sheaf of rank \(r \geq 0\). If \(r = 0\) then \(\mathcal{L} = 0\) and \(\beta\) is trivially an isomorphism. If \(r \geq 1\) but \(n > r\) then the domain and codomain of \(\beta\) are both zero, so once again \(\beta\) is an isomorphism. So we may assume \(1 \leq r \leq n\). For \(x \in X\) the \(\mathcal{O}_{X,x}\)-module \(\mathcal{L}_x\) is free of rank \(r\). Using Lemma 25, Corollary 28 and (MRS,Corollary 92) we have a commutative diagram

\[
\begin{array}{ccc}
\{ \bigwedge^n (\mathcal{L}^\vee) \}_x & \xrightarrow{\beta_x} & \{ (\bigwedge^n \mathcal{L} )^\vee \}_x \\
\downarrow & & \downarrow \\
\bigwedge^n_{\mathcal{O}_{X,x}} (\mathcal{L}_x)^\vee & \xrightarrow{} & (\bigwedge^n_{\mathcal{O}_{X,x}} \mathcal{L}_x)^\vee
\end{array}
\]

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where the bottom morphism is the isomorphism of \((TES, Corollary 31)\). Therefore \(\beta_x\) is an isomorphism for every \(x \in X\), which implies that \(\beta\) is an isomorphism and completes the proof. \(\Box\)

**Proposition 34.** Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{F}, \mathcal{G}\) sheaves of modules on \(X\). Then there is a canonical isomorphism of sheaves of super \(\mathcal{O}_X\)-algebras

\[
\delta : \bigwedge \mathcal{F} \otimes \bigwedge \mathcal{G} \to \bigwedge (\mathcal{F} \oplus \mathcal{G})
\]

\[(f_1 \wedge \cdots \wedge f_m) \otimes (g_1 \wedge \cdots \wedge g_n) \mapsto (f_1, 0) \wedge \cdots \wedge (f_m, 0) \wedge (0, g_1) \wedge \cdots \wedge (0, g_n)\]

In particular for \(d \geq 0\) there is a canonical isomorphism of sheaves of modules

\[
\bigoplus_{m+n=d} \bigwedge^m \mathcal{F} \otimes \bigwedge^n \mathcal{G} \to \bigwedge^d (\mathcal{F} \oplus \mathcal{G})
\]

**Proof.** By the tensor product \(\bigwedge \mathcal{F} \otimes \bigwedge \mathcal{G}\) we mean the super \(\mathcal{O}_X\)-algebra of \((SOA, Proposition 52)\). By Proposition 21 the functor \(\bigwedge(-) : \text{Mod}(X) \to \text{sAlg}(X)\) has a right adjoint and therefore preserves all colimits. It follows that the morphisms \(\bigwedge \mathcal{F} \to \bigwedge (\mathcal{F} \oplus \mathcal{G})\) and \(\bigwedge \mathcal{G} \to \bigwedge (\mathcal{F} \oplus \mathcal{G})\) are a coproduct in the category \(\text{sAlg}(X)\). But the super \(\mathcal{O}_X\)-algebra \(\bigwedge \mathcal{F} \otimes \bigwedge \mathcal{G}\) is also a coproduct by \((SOA, Proposition 52), so there is a canonical isomorphism of super \(\mathcal{O}_X\)-algebras \(\bigwedge \mathcal{F} \otimes \bigwedge \mathcal{G} \to \bigwedge (\mathcal{F} \oplus \mathcal{G})\), as claimed. The second statement follows immediately from \((MRS, Lemma 101)\). \(\Box\)

**Corollary 35.** Let \(X\) be a scheme and \(\mathcal{L}\) an invertible sheaf of modules on \(X\). Then for \(d \geq 1\) there is a canonical isomorphism of sheaves of modules

\[
\delta : \bigwedge^d \mathcal{L} \to \mathcal{L}^\otimes d
\]

**Proof.** Here \(\mathcal{L}^d\) denotes the coproduct of \(d\) copies of \(\mathcal{L}\). We can assume \(X\) is nonempty, so we have canonical isomorphisms \(\bigwedge^0 \mathcal{L} \cong \mathcal{O}_X, \bigwedge^1 \mathcal{L} \cong \mathcal{L}\) and \(\bigwedge^n \mathcal{L} = 0\) for \(n > 1\) by Corollary 28. For \(d \geq 1\) the sheaf \(\mathcal{L}^d\) is locally free of rank \(d\), so again by Corollary 28 we have \(\bigwedge^n \mathcal{L}^d = 0\) for \(n > d\). We construct the isomorphism \(\delta\) recursively. For \(d = 1\) we take the canonical isomorphism \(\delta_1 : \bigwedge^1 \mathcal{L} \cong \mathcal{L}\). Suppose that the isomorphism \(\delta_i\) has been constructed for \(1 \leq i < d\). Using Proposition 34 and \(\delta_{d-1}\) we have a canonical isomorphism

\[
\bigwedge^d \mathcal{L} \cong \bigwedge^d (\mathcal{L} \oplus \mathcal{L}^{d-1}) \cong (\bigwedge^1 \mathcal{L} \otimes \bigwedge^{d-1} \mathcal{L}) \oplus (\bigwedge^0 \mathcal{L} \otimes \bigwedge^d \mathcal{L}^{d-1})
\]

as required. \(\Box\)

## 5 Sheaves of Polynomial Algebras

Let \(\mathcal{F}\) be a sheaf of modules on \(X\), and let \(n \geq 1\) be an integer. We define a presheaf \(P\) on \(X\) as follows: For an open subset \(U \subseteq X\) we define

\[
P(U) = \mathcal{F}(U)[x_1, \ldots, x_n]
\]

That is, \(P(U)\) is the graded \(\mathcal{O}_X(U)\)-module of all polynomials in \(n\) variables with coefficients from \(\mathcal{F}(U)\). The homogenous polynomials of degree \(m\) give the \(m\)th graded piece. For open subsets \(V \subseteq U\) the morphism of modules \(\mathcal{F}(U) \to \mathcal{F}(V)\) induces \(P(U) \to P(V)\), and with these morphisms it is clear that \(P\) is a presheaf of graded modules on \(X\). Restriction acts by restriction on the coefficients:

\[
(f + f_1 x_1 + \cdots + f_n x_n + \cdots)|_V = f|_V + (f_1|_V)x_1 + \cdots + (f_n|_V)x_n + \cdots
\]
If \( \phi : \mathcal{F} \to \mathcal{G} \) is a morphism of sheaves of modules, and \( Q \) is the presheaf of polynomial modules for \( \mathcal{G} \), then we get a morphism of presheaves of graded modules

\[
\varphi : \mathcal{F} \to Q
\]

\[
\varphi_U : \mathcal{F}(U)[x_1, \ldots, x_n] \to \mathcal{G}(U)[x_1, \ldots, x_n]
\]

Therefore there is a morphism of presheaves of modules \( \mathcal{F} \to P \) given pointwise by the canonical injection \( \mathcal{F}(U) \to \mathcal{F}(U)[x_1, \ldots, x_n] \).

**Definition 2.** Let \( (X, \mathcal{O}_X) \) be a ringed space and \( \mathcal{F} \) a sheaf of modules on \( X \). For \( n \geq 1 \) the polynomial module with coefficients in \( \mathcal{F} \), denoted \( \mathcal{F}[x_1, \ldots, x_n] \) is the sheaf of graded modules on \( X \) given by the sheafification of the presheaf \( P(U) = \mathcal{F}(U)[x_1, \ldots, x_n] \) above. If \( \phi : \mathcal{F} \to \mathcal{G} \) is a morphism of sheaves of modules, there is an induced morphism of sheaves of graded modules \( \mathcal{F}[x_1, \ldots, x_n] \to \mathcal{G}[x_1, \ldots, x_n] \) given by the sheafification of \( \varphi \). This defines a functor

\[
(-)[x_1, \ldots, x_n] : \mathcal{Mod}(X) \to \mathcal{GMod}(X)
\]

The composite \( \mathcal{F} \to P \to \mathcal{F}[x_1, \ldots, x_n] \) gives a monomorphism of sheaves of modules \( \eta : \mathcal{F} \to \mathcal{F}[x_1, \ldots, x_n] \) which is clearly natural in \( \mathcal{F} \). The image of this morphism is clearly \( \mathcal{F}[x_1, \ldots, x_n]_0 \).

If \( \mathcal{F} \) is a sheaf of algebras on \( X \) then \( P(U) \) becomes a \( \mathcal{O}_X(U) \)-algebra via the ring morphism

\[
\mathcal{O}_X(U) \to \mathcal{F}(U) \to \mathcal{F}(U)[x_1, \ldots, x_n]
\]

and in this way \( P \) is a presheaf of graded algebras on \( X \). Therefore \( \mathcal{F}[x_1, \ldots, x_n] \) is a sheaf of graded \( \mathcal{O}_X \)-algebras with the same grading as above. For every open set \( U \subseteq X \) we write \( x_i \) for the polynomial in \( P(U) \) whose only nonzero coefficient is the identity \( 1 \in \mathcal{F}(U) \) on the monomial \( x_i \).

So we have a section \( x_i \in \mathcal{F}[x_1, \ldots, x_n](U) \) homogenous of degree 1. We use a similar notation for any monomial in \( x_1, \ldots, x_n \). If \( \phi : \mathcal{F} \to \mathcal{G} \) is a morphism of sheaves of algebras, then \( \mathcal{F}[x_1, \ldots, x_n] \to \mathcal{G}[x_1, \ldots, x_n] \) is clearly a morphism of sheaves of graded algebras, so we get functors

\[
(-)[x_1, \ldots, x_n] : \mathcal{Alg}(X) \to \mathcal{GAlg}(X)
\]

\[
(-)[x_1, \ldots, x_n] : \mathcal{Alg}(X) \to \mathcal{GAlg}(X)
\]

In particular for \( n \geq 1 \) we have a commutative sheaf of graded \( \mathcal{O}_X \)-algebras \( \mathcal{O}_X[x_1, \ldots, x_n] \) with \( \mathcal{O}_X[x_1, \ldots, x_{n+1}] = \mathcal{O}_X \).

**Proposition 36.** Let \( (X, \mathcal{O}_X) \) be a ringed space and \( \mathcal{F} \) a sheaf of commutative algebras on \( X \). For \( n \geq 1 \) there is a bijection natural in \( \mathcal{F} \)

\[
\beta : \text{Hom}_{\mathcal{Alg}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathcal{F}) \cong \mathcal{F}(X)^n
\]

\[
\phi \mapsto (\phi_X(x_1), \ldots, \phi_X(x_n))
\]

**Proof.** It is clear that \( \beta \) is a morphism of abelian groups. Using the fact that sheafification is a left adjoint, we have a bijection

\[
\text{Hom}_{\mathcal{Alg}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathcal{F}) \cong \text{Hom}_{\mathcal{Alg}(X)}(P, \mathcal{F})
\]

where \( P \) sheafifies to give \( \mathcal{O}_X[x_1, \ldots, x_n] \). Since \( P(U) = \mathcal{O}_X(U)[x_1, \ldots, x_n] \) it is not hard to check that \( \text{Hom}(P, \mathcal{F}) \cong \mathcal{F}(X)^n \) under the map \( \psi \mapsto (\psi_X(x_1), \ldots, \psi_X(x_n)) \). Together with (6) this gives the desired isomorphism \( \beta \). Naturality in \( \mathcal{F} \) is easily checked. If we are given a tuple \( (a_1, \ldots, a_n) \) with \( a_i \in \mathcal{F}(X) \) then the corresponding morphism \( \mathcal{O}_X[x_1, \ldots, x_n] \to \mathcal{F} \) is induced by \( P \to \mathcal{F} \) defined pointwise by the morphism of \( \mathcal{O}_X(U) \)-algebras \( \mathcal{O}_X(U)[x_1, \ldots, x_n] \to \mathcal{F}(U) \) corresponding to \( (a_1|U, \ldots, a_n|U) \).
Proposition 37. Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{F}\) a sheaf of graded commutative algebras on \(X\). Then for \(n \geq 1\) a morphism of sheaves of algebras \(\phi : \mathcal{O}_X[x_1, \ldots, x_n] \rightarrow \mathcal{F}\) is a morphism of sheaves of graded algebras if and only if \(\phi(x_i) \in \mathcal{F}_i(X)\) for \(1 \leq i \leq n\).

Proof. The condition is clearly necessary. To see it is sufficient, we can reduce to showing that the corresponding morphism \(\phi' : P \rightarrow \mathcal{F}\) of presheaves of algebras preserves grade, which is obvious.

Corollary 38. Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{F}\) a sheaf of graded commutative algebras on \(X\). For \(n \geq 1\) there is a bijection natural in \(\mathcal{F}\)

\[
\beta : \text{Hom}_{\mathcal{O}_{\text{alg}}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathcal{F}) \cong \mathcal{F}_1^n
\]

\[
\phi \mapsto (\phi_X(x_1), \ldots, \phi_X(x_n))
\]

Proof. This is straightforward, using the technique of Proposition 4.

Proposition 39. If \(U \subseteq X\) is open and \(n \geq 1\) then the following diagram commutes up to a canonical natural equivalence

\[
\begin{array}{ccc}
\text{Mod}(X) & \xrightarrow{(-)|_{x_1,..,x_n}} & \text{Mod}(X) \\
\downarrow & & \downarrow \\
\text{Mod}(U) & \xrightarrow{(-)|_{x_1,..,x_n}} & \text{Mod}(U)
\end{array}
\]

For a sheaf of modules \(\mathcal{F}\) on \(X\) the natural isomorphism \(\mathcal{F}|_U[x_1, \ldots, x_n] \rightarrow \mathcal{F}|_U[x_1, \ldots, x_n]\) has the action \(ax_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto ax_1^{\alpha_1} \cdots x_n^{\alpha_n}\).

Proof. This is straightforward, using the technique of Proposition 5.

Lemma 41. Let \(A\) be a commutative ring, \(p\) a prime ideal of \(A\) and \(M\) an \(A\)-module. For \(n \geq 1\) there is a canonical isomorphism of \(A_p\)-modules \(\phi_p : M[x_1, \ldots, x_n]_p \rightarrow M_p[x_1, \ldots, x_n]\) defined by \(\phi_p(f/s)(a) = f(a)/s\).

Lemma 42. Let \((X, \mathcal{O}_X)\) be a ringed space and \(\mathcal{F}\) a sheaf of modules on \(X\). For \(n \geq 1\) and \(y \in X\) there is a canonical isomorphism of \(\mathcal{O}_{\mathcal{X},y}\)-modules natural in \(\mathcal{F}\)

\[
\gamma : \mathcal{F}[x_1, \ldots, x_n]_y \rightarrow \mathcal{F}[x_1, \ldots, x_n]_y
\]

\[
(U, ax_1^{\alpha_1} \cdots x_n^{\alpha_n}) \mapsto (U, a)x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

Proof. Let \(P\) be the presheaf \(P(U) = \mathcal{F}(U)[x_1, \ldots, x_n]\). It is not hard to check that the following is a well-defined isomorphism of \(\mathcal{O}_{\mathcal{X},y}\)-modules

\[
\tau : P_y \rightarrow \mathcal{F}_y[x_1, \ldots, x_n]
\]

\[
\tau(U, f)(a) = (U, f(a))
\]

We define \(\gamma\) to be the composite of the canonical isomorphism \(\mathcal{F}[x_1, \ldots, x_n]_y \cong P_y\) with \(\tau\). Naturality in \(\mathcal{F}\) is easily checked.
Proposition 43. Let $X = \text{Spec} \mathcal{A}$ be an affine scheme and $M$ an $A$-module. Then for $n \geq 1$ there is a canonical isomorphism of sheaves of modules natural in $M$

$$\psi : \tilde{M}[x_1, \ldots, x_n] \to M[x_1, \ldots, x_n]^\sim$$

$$(m/s)x_1^{a_1} \cdots x_n^{a_n} \mapsto mx_1^{a_1} \cdots x_n^{a_n}/s$$

Proof. For open $U \subseteq X$ and $p \in U$ and a polynomial $f \in \tilde{M}(U)[x_1, \ldots, x_n]$ let $f(p)$ denote the polynomial $\alpha \mapsto f(\alpha)(p)$ of $M_p[x_1, \ldots, x_n]$. Let $P$ be the presheaf $P(U) = \tilde{M}(U)[x_1, \ldots, x_n]$ which sheafifies to give $\tilde{M}[x_1, \ldots, x_n]$. We have a morphism of presheaves of modules

$$\psi' : P \to M[x_1, \ldots, x_n]^\sim$$

$$\psi'_p(f)(p) = \phi_p^{-1}(f(p))$$

where $\phi_p$ is the isomorphism defined in Lemma 41. To see that $\psi'_p(f)$ is a well-defined section, take the nonzero coefficients of $f$ and find an open neighborhood of $p$ small enough so all these coefficients are of the form $m/s$. Then on that neighborhood $\psi'_p(f)$ will be of the form $g/t$ for a polynomial $g \in M[x_1, \ldots, x_n]$. For any prime $p$ there is a commutative diagram

$$\begin{array}{ccc}
P_p & \xrightarrow{\psi'_p} & M[x_1, \ldots, x_n]^\sim_p \\
\downarrow & & \downarrow \\
\tilde{M}_p[x_1, \ldots, x_n] & \to & M_p[x_1, \ldots, x_n] \to M[x_1, \ldots, x_n]_p
\end{array}$$

Therefore $\psi'_p$ is an isomorphism, and hence so is the morphism of sheaves of modules $\psi : \tilde{M}[x_1, \ldots, x_n] \to M[x_1, \ldots, x_n]^\sim$ induced by $\psi'_p$. Naturality in $M$ is easily checked. \hfill \square

Corollary 44. Let $X$ be a scheme and $\mathcal{F}$ a sheaf of modules on $X$. If $\mathcal{F}$ is quasi-coherent, then the same is true of $\mathcal{F}[x_1, \ldots, x_n]$ for any $n \geq 1$.

Proof. For $x \in X$ let $U$ be an affine open neighborhood of $x$ and $f : U \to \text{Spec} \mathcal{O}_X(U)$ the canonical isomorphism. Then $f_*\mathcal{F}|_U \cong \mathcal{F}(U)^\sim$ and combining Proposition 39, Proposition 40 and Proposition 43 we see that

$$f_*(\mathcal{F}[x_1, \ldots, x_n]|_U) \cong f_*(\mathcal{F}|_U[x_1, \ldots, x_n])$$

$$\cong (f_*\mathcal{F}|_U)[x_1, \ldots, x_n]$$

$$\cong \mathcal{F}(U)[x_1, \ldots, x_n]$$

$$\cong \mathcal{F}(U)[x_1, \ldots, x_n]^\sim$$

This shows that $\mathcal{F}[x_1, \ldots, x_n]$ is a quasi-coherent sheaf of modules, as required. \hfill \square

Proposition 45. Let $X$ be a scheme and $\mathcal{F}$ quasi-coherent sheaf of modules on $X$. For $n \geq 1$ and affine open $U \subseteq X$ there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-modules natural in $\mathcal{F}$ and the affine open set $U$

$$\tau : \mathcal{F}(U)[x_1, \ldots, x_n] \cong \mathcal{F}[x_1, \ldots, x_n](U)$$

$$ax_1^{a_1} \cdots x_n^{a_n} \mapsto ax_1^{a_1} \cdots x_n^{a_n}$$

If $\mathcal{F}$ is a sheaf of algebras, this is an isomorphism of graded $\mathcal{O}_X(U)$-algebras.

Proof. We make $\mathcal{F}[x_1, \ldots, x_n](U)$ into a graded $\mathcal{O}_X(U)$-module as in (SOA, Proposition 40). Using Proposition (H,5.1(d)) and Corollary 44 we get an isomorphism $\mathcal{O}_X(U)$-modules $\tau$, which is easily seen to preserve grade. If $\mathcal{F}$ is a sheaf of algebras, then $\tau$ is clearly a morphism of sheaves of algebras. Note that $\tau$ is actually the sheafification morphism $P \to \mathcal{F}[x_1, \ldots, x_n]$ evaluated at $U$, from which it follows that $\tau$ is natural in $\mathcal{F}$ and inclusions of open affines $V \subseteq U$. \hfill \square
Corollary 46. Let $X$ be a scheme and $\mathcal{F}$ a quasi-coherent sheaf of commutative algebras on $X$. Then for $n \geq 1$, $\mathcal{F}[x_1, \ldots, x_n]$ is locally finitely generated by $\mathcal{F}[x_1, \ldots, x_n]_1$ as an $\mathcal{F}[x_1, \ldots, x_n]_0$-algebra.

Proof. This follows immediately from Proposition 45.

Lemma 47. Let $X$ be a nonempty scheme and $\mathcal{F}$ a free $\mathcal{O}_X$-module of rank $n \geq 1$. For any basis $f_1, \ldots, f_n \in \mathcal{F}(X)$ there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras

$$\beta : \mathcal{O}_X[x_1, \ldots, x_n] \to \mathcal{S}(\mathcal{F})$$

$$x_1 \mapsto f_1$$

Proof. By a “basis” we mean a coproduct $\{\phi_i : \mathcal{O}_X \to \mathcal{F}\}_{1 \leq i \leq n}$, but since morphisms $\mathcal{O}_X \to \mathcal{F}$ correspond to global sections of $\mathcal{F}$, there is no harm in calling the elements $\langle \phi_i \rangle_X(1)$ a basis. We know such a morphism of sheaves of graded $\mathcal{O}_X$-algebras $\beta$ exists by Corollary 38. Since $\mathcal{F}$ is quasi-coherent, we reduce by (MOS, Lemma 2) to showing that $\beta_U$ is a bijection for every affine open $U \subseteq X$, and this follows at once from (TES, Lemma 34), Proposition 45, and Proposition 17.

Lemma 48. If $X$ is a scheme then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras

$$\beta : \mathcal{O}_X[x] \to \mathcal{T}(\mathcal{O}_X)$$

$$x \mapsto 1 \in \mathcal{T}^1(\mathcal{O}_X)(X)$$

Proof. First of all, we know from Lemma 9 that $\mathcal{T}(\mathcal{O}_X)$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. So such a morphism of sheaves of graded $\mathcal{O}_X$-algebras $\beta$ exists by Corollary 38. We reduce by (MOS, Lemma 2) to showing that $\beta_U$ is a bijection for every affine open $U \subseteq X$, and this follows at once from (TES, Lemma 8), Proposition 45, and Proposition 8.

## 6 Sheaves of Ideal Products

Suppose for every $n \geq 0$ we have a sheaf of ideals $\mathcal{J}_n$ on $X$ satisfying the following conditions

$$\mathcal{J}_0 = \mathcal{O}_X$$

$$\mathcal{J}_n \subseteq \mathcal{J}_m \quad m \leq n$$

$$\mathcal{J}_n \mathcal{J}_m \subseteq \mathcal{J}_{m+n} \quad m, n \geq 0$$

By $\mathcal{J}_n \mathcal{J}_m \subseteq \mathcal{J}_{n+m}$ we mean that the ideal product $\mathcal{J}_n \mathcal{J}_m$, which is a sheaf of ideals on $X$, is contained in $\mathcal{J}_{m+n}$. This is equivalent to having $\mathcal{J}_n(U) \mathcal{J}_m(U) \subseteq \mathcal{J}_{m+n}(U)$ as ideals in $\mathcal{O}_X(U)$ for every open subset $U \subseteq X$. Let $P$ be the following presheaf of modules on $\mathcal{O}_X$ (coproduct of presheaves of modules)

$$P = \bigoplus_{n \geq 0} \mathcal{J}_n = \mathcal{O}_X \oplus \mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \cdots$$

For open $U \subseteq X$ define a product on the $\mathcal{O}_X(U)$-module $P(U)$ by

$$(xy)_i = \sum_{d+e=i} x_d y_e$$

It is easy to check this is a commutative graded $\mathcal{O}_X(U)$-algebra with identity $(1, 0, \ldots)$ and graded piece $\mathcal{J}_n(U)$ in degree $n \geq 0$. As usual we identify elements of $\mathcal{J}_n(U)$ with sequences in $P(U)$ with only one nonzero entry. In that case multiplication in $P(U)$ is just multiplication in $\mathcal{O}_X(U)$ where you have to keep track of the grade. With component-wise restriction it is clear that $P$ is a presheaf of $\mathcal{O}_X(U)$-algebras. Therefore the sheafification $\mathcal{P}$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras, where the submodule $\mathcal{P}_n$ for $n \geq 0$ is the image of the canonical morphism
\( J_n \rightarrow P \rightarrow \mathcal{P} \). In fact these morphisms \( J_n \rightarrow \mathcal{P} \) are the canonical coproduct of sheaves of modules

\[
\mathcal{P} = \bigoplus_{n \geq 0} J_n
\]

As usual, given \( a \in J_n(U) \) for \( n \geq 0 \) and some open set \( U \subseteq U \) we simply write \( a \) for the corresponding element of \( \mathcal{P}(U) \) which is the image under \( P \rightarrow \mathcal{P} \) of the sequence in \( P(U) \) with a single nonzero entry given by \( a \) in the \( n \)th place. Equivalently this is the image of \( a \) under \( J_n \rightarrow \mathcal{P} \).

**Definition 3.** Let \((X, \mathcal{O}_X)\) be a ringed space, \( \{J_n\}_{n \geq 0} \) a collection of sheaves of ideals satisfying (7), (8), (9). Then the coproduct of sheaves of modules \( \bigoplus_{n \geq 0} J_n \) becomes a sheaf of commutative graded \( \mathcal{O}_X \)-algebras in a canonical way. In particular if \( J \) is a sheaf of ideals then we can set \( J_n = J^n \) for \( n \geq 1 \) (the \( n \)-fold product), and in this case we write \( B(J) \) for \( \bigoplus_{n \geq 0} J_n \) and \( B^n(J) \) for the submodule of degree \( n \) for \( n \geq 0 \). Note that \( B^0(J) = \mathcal{O}_X \).

If \( J \subseteq \mathcal{K} \) then there is a canonical morphism of graded \( \mathcal{O}_X \)-algebras \( B(J) \rightarrow B(\mathcal{K}) \) which is the sheafification of the morphism given component-wise by the inclusion \( J \rightarrow \mathcal{K} \). If \( \mathcal{K} = J \) this is the identity, and if \( J \subseteq \mathcal{K} \subseteq \mathcal{L} \) then the composite \( B(J) \rightarrow B(\mathcal{K}) \rightarrow B(\mathcal{L}) \) is just \( B(J) \rightarrow B(\mathcal{L}) \).

**Proposition 49.** If \( J \) is a sheaf of ideals and \( q \in B(J)(U) \) then for every \( x \in U \) there is an open neighborhood \( x \in V \subseteq U \) such that \( q|_V = q_1 + \cdots + q_s \) where each \( q_k \) has the form

\[
q_k = r + a_{11} + a_{21}x_{22} + \cdots + a_{h1} \cdots a_{hh}
\]

where \( r \in \mathcal{O}_X(V) \) and \( a_{ij} \in J(V) \). In the sum above, an \( n \)-fold product is given grade \( n \) for \( n \geq 1 \).

**Proposition 50.** If \( U \subseteq X \) is open and \( J \) is a sheaf of ideals on \( X \) then there is a canonical isomorphism of sheaves of graded \( \mathcal{O}_X \)-algebras natural in \( J \)

\[
\mathcal{B}(J|_U) \rightarrow \mathcal{B}(J)|_U \quad \quad a_1 \cdots a_n \mapsto a_1 \cdots a_n
\]

**Proposition 51.** Let \( f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \) be an isomorphism of ringed spaces with inverse \( h \) and let \( J \) be a sheaf of ideals on \( X \). Then there is a canonical isomorphism of sheaves of \( \mathcal{O}_Y \)-algebras

\[
\rho : f_\ast B(J) \rightarrow B(J \cdot \mathcal{O}_Y) \quad \quad a_1 \cdots a_n \mapsto h_U\#(a_1) \cdots h_U\#(a_n)
\]

where \( U \subseteq X \) is open and \( a_i \in J(U) \).

**Proof.** Here \( J \cdot \mathcal{O}_Y \) denotes the sheaf of ideals on \( Y \) corresponding to \( J \) under (MRS, Lemma 49). Let \( P \) be the presheaf of algebras on \( X \) sheafifying to give \( B(J) \) and let \( Q \) sheafify to give \( B^0(J \cdot \mathcal{O}_Y) \). Then we have isomorphisms of presheaves of \( \mathcal{O}_Y \)-modules (using (MRS, Proposition 52))

\[
f_\ast P = f_\ast \mathcal{O}_X \oplus f_\ast J \oplus f_\ast (J^2) \oplus \cdots \cong \mathcal{O}_Y \oplus (J \cdot \mathcal{O}_Y) \oplus (J^2 \cdot \mathcal{O}_Y) \oplus \cdots \cong \mathcal{O}_Y \oplus (J \cdot \mathcal{O}_Y) \oplus (J \cdot \mathcal{O}_Y)^2 \oplus \cdots = Q
\]

In fact this is an isomorphism of presheaves of algebras, and together with the canonical isomorphism of sheaves of algebras \( a(f_\ast P) \cong f_\ast(aP) \) this gives our isomorphism \( \rho \) of sheaves of algebras. \( \square \)

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If \( A \) is a ring with ideal \( \mathfrak{a} \) then we can define a commutative graded \( A \)-algebra
\[
B(\mathfrak{a}) = \bigoplus_{n \geq 0} \mathfrak{a}^n = A \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots
\]
with the product \((xy)_i = \sum_{d+e=i} x_d y_e\). If \( \mathfrak{b} \subseteq \mathfrak{a} \) are ideals then the inclusions give a morphism of graded \( A \)-algebras \( B(\mathfrak{b}) \rightarrow B(\mathfrak{a}) \). If \( A \) is an integral domain then clearly so is \( B(\mathfrak{a}) \).

**Proposition 52.** Let \( X = \text{Spec } A \) be an affine scheme and \( \mathfrak{a} \subseteq A \) an ideal. Then there is a canonical isomorphism of sheaves of \( \mathcal{O}_X \)-algebras
\[
\psi : \mathcal{B}(\mathfrak{a}) \rightarrow B(\mathfrak{a})^\sim
\]
\[
a_1/s_1 \cdots a_n/s_n \mapsto a_1 \cdots a_n/s_1 \cdots s_n
\]

**Proof.** Since the functor \( \sim : \text{AMod} \rightarrow \text{Mod}(X) \) preserves all colimits, there is an isomorphism of sheaves of modules
\[
\mathcal{B}(\mathfrak{a}) = \mathcal{O}_X \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots = \mathcal{O}_X \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots \\
\cong B(\mathfrak{a})^\sim
\]
To check it is a morphism of sheaves of algebras we can reduce to the case of sections of the special form given in the statement, which is easy. \(\square\)

**Proposition 53.** Let \( X \) be a scheme and \( \mathcal{I} \) a sheaf of ideals on \( X \). If \( \mathcal{I} \) is quasi-coherent then so is \( \mathcal{B}(\mathcal{I}) \).

**Proof.** This follows immediately from (MOS, Corollary 12) and (MOS, Proposition 25). \(\square\)

**Proposition 54.** Let \( X \) be a scheme and \( \mathcal{I} \) a sheaf of ideals on \( X \). If \( \mathcal{I} \) is quasi-coherent and \( U \subseteq X \) is affine then there is a canonical isomorphism of graded \( \mathcal{O}_X(U) \)-algebras natural in \( U \) and the affine open subset \( U \)
\[
\tau : B(\mathcal{I}(U)) \rightarrow \mathcal{B}(\mathcal{I})(U)
\]
\[
a_1 \cdots a_n \mapsto a_1 \cdots a_n
\]

**Proof.** We make \( \mathcal{B}(\mathcal{I})(U) \) into a graded \( \mathcal{O}_X(U) \)-algebra as in (SOA, Proposition 40). Let \( P \) be the presheaf \( P(U) = B(\mathcal{I}(U)) \) of \( \mathcal{O}_X \)-algebras, which sheafifies to give \( \mathcal{B}(\mathcal{I}) \) by (MOS, Proposition 13), and let \( \tau \) be the canonical morphism \( P \rightarrow \mathcal{B}(\mathcal{I}) \) evaluated at \( U \). This is a morphism of graded \( \mathcal{O}_X(U) \)-algebras, and it follows from (MOS, Lemma 6) that \( \tau \) is an isomorphism. \(\square\)

**Corollary 55.** Let \( X \) be a scheme and \( \mathcal{I} \) a quasi-coherent sheaf of ideals on \( X \). Then \( \mathcal{B}(\mathcal{I}) \) is locally generated by \( \mathcal{B}(\mathcal{I})_1 \) as a \( \mathcal{B}(\mathcal{I}) \)-algebra (locally finitely generated if \( \mathcal{I} \) is coherent).

**Proof.** Using Proposition 54 we reduce immediately to showing that for a commutative ring \( A \) and ideal \( \mathfrak{a} \) the graded \( A \)-algebra \( B(\mathfrak{a}) \) is generated by \( B(\mathfrak{a})_1 \) as an \( A \)-algebra, with \( B(\mathfrak{a}) \) finitely generated by \( B(\mathfrak{a})_1 \) if \( \mathfrak{a} \) is finitely generated. This is easy enough to check. \(\square\)

**Proposition 56.** If \( A \) is a ring then there is a canonical isomorphism of graded \( A \)-algebras
\[
\beta : A[x] \rightarrow B(A)
\]
\[
x \mapsto (0,1,0,\ldots)
\]

**Corollary 57.** If \( X \) is a scheme then there is a canonical isomorphism of sheaves of graded \( \mathcal{O}_X \)-algebras
\[
\beta : \mathcal{O}_X[x] \rightarrow \mathcal{B}(\mathcal{O}_X)
\]
\[
\hat{x} \mapsto 1 \in \mathcal{B}^1(\mathcal{O}_X)(X)
\]
Proof. We know from Proposition 53 that $\mathcal{B}(\mathcal{O}_X)$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. So such a morphism of sheaves of graded $\mathcal{O}_X$-algebras $\beta$ exists by Corollary 38. We reduce by (MOS,Lemma 2) to showing that $\beta_U$ is a bijection for every affine open $U \subseteq X$, and this follows from Proposition 56, Proposition 45 and Proposition 54. \qed
In summary we have the following diagram of functors. Pairs of functors going in opposite directions are adjoint pairs, with the left adjoint on the left.