Special Sheaves of Algebras

Daniel Murfet

October 5, 2006

Contents

1 Introduction

In this note "ring" means a not necessarily commutative ring. If A is a commutative ring then an A-algebra is a ring morphism $A \longrightarrow B$ whose image is contained in the center of B. We allow noncommutative sheaves of rings, but if we say $(X, \mathcal{O}_X)$ is a ringed space then we mean $\mathcal{O}_X$ is a sheaf of commutative rings. Throughout this note $(X, \mathcal{O}_X)$ is a ringed space. Associated to this ringed space are the following categories:

$$\mathfrak{Mod}(X), \mathfrak{GrMod}(X), \mathfrak{Alg}(X), \mathfrak{nAlg}(X), \mathfrak{GrAlg}(X), \mathfrak{GrnAlg}(X)$$

We show that the forgetful functors $\mathfrak{Alg}(X) \longrightarrow \mathfrak{Mod}(X)$ and $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{Mod}(X)$ have left adjoints. If A is a nonzero commutative ring, the forgetful functors $A\mathbf{Alg} \longrightarrow A\mathbf{Mod}$ and $A\mathbf{nAlg} \longrightarrow A\mathbf{Mod}$ have left adjoints given by the symmetric algebra and tensor algebra constructions respectively.

2 Sheaves of Tensor Algebras

Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules, and for an open set U let $P(U)$ be the $\mathcal{O}_X(U)$-algebra given by the tensor algebra $T(\mathscr{F}(U))$. That is,

$$P(U) = \mathcal{O}_X(U) \oplus \mathscr{F}(U) \oplus \mathscr{F}(U)^{\otimes 2} \oplus \cdots$$

For an inclusion $V \subseteq U$ let $\rho : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ and $\eta : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ be the morphisms of abelian groups given by restriction. For $n \geq 2$ we define a multilinear map

$$\begin{gathered}\mathscr{F}(U) \times \cdots \times \mathscr{F}(U) \longrightarrow \mathscr{F}(V) \otimes \cdots \otimes \mathscr{F}(V) \\ (m_1, \ldots, m_n) \mapsto m_1|_V \otimes \cdots \otimes m_n|_V\end{gathered}$$

Let $\eta^{\otimes n}$ denote the induced morphism of abelian groups $\mathscr{F}(U)^{\otimes n} \longrightarrow \mathscr{F}(V)^{\otimes n}$. Then $\rho \oplus \eta \oplus \eta^{\otimes 2} \oplus \cdots$ gives a morphism of abelian groups $P(U) \longrightarrow P(V)$ compatible with the module structures and the ring morphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$. It is then readily seen that P is a presheaf of $\mathcal{O}_X$-algebras, and we let $\mathbb{T}(\mathscr{F})$ be the sheaf of $\mathcal{O}_X$-algebras given by the sheafification. The morphism of presheaves of modules $\mathscr{F} \longrightarrow P$ given pointwise by the canonical injection $\mathscr{F}(U) \longrightarrow T(\mathscr{F}(U))$ composes with $P \longrightarrow \mathbb{T}(\mathscr{F})$ to give a monomorphism of sheaves of $\mathcal{O}_X$-modules $\mathscr{F} \longrightarrow \mathbb{T}(\mathscr{F})$.

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, whose associated presheaves of tensor algebras are P, Q respectively, then we define a morphism of presheaves of $\mathcal{O}_X$-algebras $\phi' : P \longrightarrow Q$ by $\phi'_U = T(\phi_U)$. That is,

$$\phi'_U = 1 \oplus \phi_U \oplus \phi_U^{\otimes 2} \oplus \phi_U^{\otimes 3} \oplus \cdots$$

Let $\mathbb{T}(\phi) : \mathbb{T}(\mathscr{F}) \longrightarrow \mathbb{T}(\mathscr{G})$ denote the morphism of sheaves of $\mathcal{O}_X$-algebras given by the sheafification of ϕ'. This defines a functor

$$\mathbb{T}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{nAlg}(X)$$

Note that the following diagram of sheaves of modules commutes

$$\begin{tikzcd}
\mathbb{T}(\mathscr{F}) \arrow[r, "\mathbb{T}(\phi)"] & \mathbb{T}(\mathscr{G}) \\
\mathscr{F} \arrow[u] \arrow[r, "\phi"'] & \mathscr{G} \arrow[u]
\end{tikzcd}$$

For $d \geq 0$ let P_d denote the sub-presheaf of $\mathcal{O}_X$-modules of P given by $P_d(U) = T^d(\mathscr{F}(U))$, which is the submodule of $T(\mathscr{F}(U))$ given by the isomorphic copy of $\mathscr{F}(U)^{\otimes d}$. In particular there are isomorphisms of presheaves of modules $P_0 \cong \mathcal{O}_X$ and $P_1 \cong \mathscr{F}$. By construction the induced morphism $\bigoplus_{d \geq 0} P_d \longrightarrow P$ is an isomorphism (coproduct of presheaves of modules) and $P_d P_e \subseteq P_{d+e}, 1 \in P_0(X)$. Let $\mathbb{T}^d(\mathscr{F})$ denote the submodule of $\mathbb{T}(\mathscr{F})$ given by the image of $\mathbf{a}P_d \longrightarrow \mathbf{a}P = \mathbb{T}(\mathscr{F})$. Then $\mathbb{T}(\mathscr{F})$ together with the submodules $\mathbb{T}^d(\mathscr{F})$ is a sheaf of graded $\mathcal{O}_X$-algebras. Note that $\mathbb{T}^1(\mathscr{F})$ is the image of the monomorphism $\mathscr{F} \longrightarrow \mathbb{T}(\mathscr{F})$ and $\mathbb{T}^0(\mathscr{F})$ is the image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \longrightarrow \mathbb{T}(\mathscr{F})$ (this latter morphism is also a monomorphism of sheaves of modules, so $\mathbb{T}^0(\mathscr{F}) \cong \mathcal{O}_X$ and $\mathbb{T}^1(\mathscr{F}) \cong \mathscr{F}$ as sheaves of modules). More generally for $d \geq 0$ there is a canonical monomorphism of sheaves of modules $\mathscr{F}^{\otimes d} \longrightarrow \mathbb{T}(\mathscr{F})$ whose image is $\mathbb{T}^d(\mathscr{F})$ so we have (in the category $\mathfrak{Mod}(X)$)

$$\mathbb{T}(\mathscr{F}) \cong \bigoplus_{d \geq 0} \mathbb{T}^d(\mathscr{F}) \cong \bigoplus_{d \geq 0} \mathscr{F}^{\otimes d}$$

and the product in $\mathbb{T}(\mathscr{F})$ is described by commutativity of the following diagram for $d, e \geq 0$

$$\begin{tikzcd}
\mathscr{F}^{\otimes d} \otimes \mathscr{F}^{\otimes e} \arrow[r] \arrow[d] & \mathbb{T}(\mathscr{F}) \otimes \mathbb{T}(\mathscr{F}) \arrow[d] \\
\mathscr{F}^{\otimes (d+e)} \arrow[r] & \mathbb{T}(\mathscr{F})
\end{tikzcd}$$

It is clear that if $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, then $\mathbb{T}(\phi)$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras, so we also have a functor

$$\mathbb{T}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrnAlg}(X)$$

As usual, given $r \in \mathcal{O}_X(U)$ we also write r for the corresponding element of $\mathbb{T}^0(\mathscr{F})(U)$. Similarly if $a \in \mathscr{F}(U)$ we write a for the corresponding element of $\mathbb{T}^1(\mathscr{F})(U)$. For $n > 1$ and $a_1, \ldots, a_n \in \mathscr{F}(U)$ we write $a_1 \dot{\otimes} \cdots \dot{\otimes} a_n$ for the element of $\mathbb{T}^n(\mathscr{F})(U)$. In this notation, for a morphism of sheaves of modules $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ we have

$$\begin{aligned}
\mathbb{T}(\phi)_U(r + a_{11} + a_{21} \dot{\otimes} a_{22} + \cdots + a_{h1} \dot{\otimes} \cdots \dot{\otimes} a_{hh}) = r + \phi_U(a_{11}) + \phi_U(a_{21}) \dot{\otimes} \phi_U(a_{22}) \\
+ \cdots + \phi_U(a_{h1}) \dot{\otimes} \cdots \dot{\otimes} \phi_U(a_{hh})
\end{aligned}$$

Proposition 1. *If $q \in \mathbb{T}(\mathscr{F})(U)$ then for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $q|_V = q_1 + \cdots + q_s$ where each q_k has the form*

$$q_k = r + a_{11} + a_{21} \dot{\otimes} a_{22} + \cdots + a_{h1} \dot{\otimes} \cdots \dot{\otimes} a_{hh}$$

where $r \in \mathcal{O}_X(V)$ and $a_{ij} \in \mathscr{F}(V)$.

Proof. This follows immediately from the fact that $\mathbb{T}(\mathscr{F})$ is the sheafification of the presheaf P defined above. □

Proposition 2. *The functor $\mathbb{T}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{nAlg}(X)$ is left adjoint to the forgetful functor $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{Mod}(X)$. The unit of the adjunction is given for a sheaf of modules $\mathscr{F}$ by the canonical morphism $\mathscr{F} \longrightarrow \mathbb{T}(\mathscr{F})$.*

Proof. Let $\mathscr{S}$ be a sheaf of $\mathcal{O}_X$-algebras and $\phi : \mathscr{F} \longrightarrow \mathscr{S}$ a morphism of sheaves of modules. We have to show there exists a unique morphism of sheaves of algebras $\Phi : \mathbb{T}(\mathscr{F}) \longrightarrow \mathscr{S}$ making the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d] & \mathscr{S} \\ \mathbb{T}(\mathscr{F}) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

We use the results of our notes on Tensor, Exterior and Symmetric algebras (TES) in what follows. For every open set U there is a unique morphism of $\mathcal{O}_X(U)$-algebras $\Phi'_U : T(\mathscr{F}(U)) \longrightarrow \mathscr{S}(U)$ making the following diagram commute

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathscr{S}(U) \\ T(\mathscr{F}(U)) \arrow[ur, "\Phi'_U"'] & \end{tikzcd}$$

This defines a morphism of presheaves of $\mathcal{O}_X$-algebras $\Phi' : P \longrightarrow \mathscr{S}$, which induces a morphism of sheaves of $\mathcal{O}_X$-algebras $\Phi : \mathbb{T}(\mathscr{F}) \longrightarrow \mathscr{S}$ with the required property. □

Proposition 3. *The functor $\mathbb{T}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrnAlg}(X)$ is left adjoint to the functor $(-)_1 : \mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{Mod}(X)$ which maps a sheaf of graded algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules $\mathscr{F}$ by the canonical isomorphism $\mathscr{F} \longrightarrow \mathbb{T}^1(\mathscr{F})$.*

Proof. The definition of a sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{S}$ includes the provision of a sheaf of modules $\mathscr{S}_1$, and any morphism of sheaves of graded algebras must induce a morphism of sheaves of modules between these degree 1 components, so the functor $(-)_1$ is well defined. It is not difficult to check $\mathscr{F} \longrightarrow \mathbb{T}^1(\mathscr{F})$ is natural in $\mathscr{F}$, and therefore defines a natural transformation $1 \longrightarrow (-)_1\mathbb{T}(-)$.

We have to show that if $\mathscr{S}$ is a sheaf of graded $\mathcal{O}_X$-algebras and $\phi : \mathscr{F} \longrightarrow \mathscr{S}_1$ a morphism of sheaves of modules, then there exists a unique morphism of sheaves of graded algebras $\Phi : \mathbb{T}(\mathscr{F}) \longrightarrow \mathscr{S}$ such that Φ_1 makes the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d] & \mathscr{S}_1 \\ \mathbb{T}^1(\mathscr{F}) \arrow[ur, "\Phi_1"'] & \end{tikzcd} \tag{1}$$

By Proposition 2 the composite $\mathscr{F} \longrightarrow \mathscr{S}_1 \longrightarrow \mathscr{S}$ induces a morphism of sheaves of algebras $\Phi : \mathbb{T}(\mathscr{F}) \longrightarrow \mathscr{S}$ unique with the property that $\mathscr{F} \longrightarrow \mathbb{T}(\mathscr{F}) \longrightarrow \mathscr{S}$ is $\mathscr{F} \longrightarrow \mathscr{S}$. It is straightforward to check that Φ is a morphism of sheaves of *graded* algebras, and Φ_1 makes (1) commute. Uniqueness is easily checked, which proves that $\mathbb{T}(-)$ is left adjoint to $(-)_1$. □

Proposition 4. *If $U \subseteq X$ is open then the following diagram commutes up to a canonical natural equivalence*

$$\begin{tikzcd} \mathfrak{Mod}(X) \arrow[r, "\mathbb{T}(-)"] \arrow[d] & \mathbf{n}\mathfrak{Alg}(X) \arrow[d] \\ \mathfrak{Mod}(U) \arrow[r, "\mathbb{T}(-)"'] & \mathbf{n}\mathfrak{Alg}(U) \end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $\mathbb{T}(\mathscr{F}|_U) \longrightarrow \mathbb{T}(\mathscr{F})|_U$ has the action $a_1 \dot\otimes \cdots \dot\otimes a_n \mapsto a_1 \dot\otimes \cdots \dot\otimes a_n$.

Proof. Let $nAlg(X)$ denote the category of presheaves of $\mathcal{O}_X$-algebras. Then associating a sheaf of modules $\mathscr{F}$ with the presheaf $P(U) = T(\mathscr{F}(U))$ defines a functor $\mathfrak{Mod}(X) \longrightarrow nAlg(X)$. Clearly $\mathbb{T}(-)$ is the composite of this functor with sheafification $nAlg(X) \longrightarrow \mathbf{n}\mathfrak{Alg}(X)$. So by (SOA,Lemma 3) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

$$\begin{tikzcd} \mathfrak{Mod}(X) \arrow[r] \arrow[d] & nAlg(X) \arrow[d] \\ \mathfrak{Mod}(U) \arrow[r] & nAlg(U) \end{tikzcd}$$

In fact it is easy to see that this diagram commutes in the strictest sense, that is, the two legs of the diagram are the same functor. ☐

Proposition 5. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd} \mathbf{n}\mathfrak{Alg}(X) \arrow[r, Rightarrow, "f_*"] & \mathbf{n}\mathfrak{Alg}(Y) \\ \mathfrak{Mod}(X) \arrow[u, "\mathbb{T}(-)"] \arrow[r, Rightarrow, "f_*"'] & \mathfrak{Mod}(Y) \arrow[u, "\mathbb{T}(-)"'] \end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $f_*\mathbb{T}(\mathscr{F}) \longrightarrow \mathbb{T}(f_*\mathscr{F})$ has the action $a_1 \dot\otimes \cdots \dot\otimes a_n \mapsto a_1 \dot\otimes \cdots \dot\otimes a_n$.

Proof. Using (SOA,Lemma 12) we reduce immediately to showing the following diagram commutes up to a canonical natural equivalence

$$\begin{tikzcd} nAlg(X) \arrow[r, Rightarrow, "f_*"] & nAlg(Y) \\ \mathfrak{Mod}(X) \arrow[u] \arrow[r, Rightarrow, "f_*"'] & \mathfrak{Mod}(Y) \arrow[u] \end{tikzcd}$$

where the vertical functors are the "presheaf" tensor algebra functors given in Proposition 4. Let $V \subseteq X$ be open. In our construction of the $\mathcal{O}_Y(V)$-module $\mathscr{F}(f^{-1}V)^{\otimes n}$ for $n \geq 2$ we see that we actually produce the same underlying group as when we define $\mathscr{F}(f^{-1}V)^{\otimes n}$ over $\mathcal{O}_X(f^{-1}V)$. Together with the ring isomorphism $f_V^\#$ this gives an isomorphism of abelian groups $\rho_V : T_{\mathcal{O}_Y(V)}(\mathscr{F}(f^{-1}V)) \longrightarrow T_{\mathcal{O}_X(f^{-1}V)}(\mathscr{F}(f^{-1}V))$

$$T_{\mathcal{O}_Y(V)}(\mathscr{F}(f^{-1}V)) = \mathcal{O}_Y(V) \oplus \mathscr{F}(f^{-1}V) \oplus \mathscr{F}(f^{-1}V)^{\otimes 2} \oplus \cdots$$
$$\Downarrow$$
$$T_{\mathcal{O}_X(f^{-1}V)}(\mathscr{F}(f^{-1}V)) = \mathcal{O}_X(f^{-1}V) \oplus \mathscr{F}(f^{-1}V) \oplus \mathscr{F}(f^{-1}V)^{\otimes 2} \oplus \cdots$$

It is not hard to see this is an isomorphism of $\mathcal{O}_Y(V)$-algebras natural in V and also in $\mathscr{F}$, which completes the proof. ☐

Proposition 6. *Let $X = SpecA$ be an affine scheme and M an A-module. Then there is a canonical isomorphism $\theta : T(M)^{\sim} \longrightarrow \mathbb{T}(M^{\sim})$ of sheaves of $\mathcal{O}_X$-algebras which is natural in M. We have*

$$\theta_U(a_1 \otimes \cdots \otimes a_n \dot{/} s_1 \cdots s_n) = a_1 \dot{/} s_1 \dot{\otimes} \cdots \dot{\otimes} a_n \dot{/} s_n$$

where $U \subseteq X$ is open, $a_i \in M$ and $s_i \in A$ with $U \subseteq D(s_1 \cdots s_n)$.

Proof. Consider the following diagram consisting of adjoint pairs of functors (see Ex 5.3, Proposition 2, (SOA,Proposition 5) and (TES,Proposition 7))

$$\begin{tikzcd}
A\mathbf{nAlg} \arrow[r, shift left, "\sim"] \arrow[d, shift left] & \mathfrak{nAlg}(X) \arrow[l, shift left, "\Gamma"] \arrow[d, shift left] \\
A\mathbf{Mod} \arrow[u, shift left, "T"] \arrow[r, shift left, "\sim"] & \mathfrak{Mod}(X) \arrow[l, shift left, "\Gamma"] \arrow[u, shift left, "\mathbb{T}"]
\end{tikzcd}$$

The two composites $\mathfrak{nAlg}(X) \longrightarrow A\mathbf{Mod}$ are equal, so $\widetilde{-}T$ and $\mathbb{T}\widetilde{-}$ are both left adjoints for the same functor. Therefore they must be canonically naturally equivalent, which is what we wanted to show. The isomorphism $\theta : T(M)^{\sim} \longrightarrow \mathbb{T}(\widetilde{M})$ is unique with the property that $\theta_X(m\dot{/}1) = m\dot{/}1$ for every $m \in M$ (one should think carefully about what the notation $m\dot{/}1$ means in both cases). This is an isomorphism of sheaves of algebras, so it is now easy to see θ has the desired effect on the special sections in the statement of the Proposition. $\square$

Corollary 7. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent then so is $\mathbb{T}(\mathscr{F})$.*

Proof. For $x \in X$ let U be an open affine neighborhood of x and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism. Then $f_*\mathscr{F}|_U \cong \mathscr{F}(U)^{\sim}$and combining Proposition 6, Proposition 5 and Proposition 4 we see that

$$\begin{aligned}
f_*(\mathbb{T}(\mathscr{F})|_U) &\cong f_*(\mathbb{T}(\mathscr{F}|_U)) \\
&\cong \mathbb{T}(f_*\mathscr{F}|_U) \\
&\cong \mathbb{T}(\widetilde{\mathscr{F}(U)}) \\
&\cong T(\mathscr{F}(U))^{\sim}
\end{aligned}$$

This is an isomorphism of sheaves of algebras, which is more than enough to show that $\mathbb{T}(\mathscr{F})$ is quasi-coherent. $\square$

Proposition 8. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent and $U \subseteq X$ is affine then there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-algebras natural in $\mathscr{F}$ and the affine open set U*

$$\begin{gathered}
\tau : T(\mathscr{F}(U)) \longrightarrow \mathbb{T}(\mathscr{F})(U) \\
f_1 \otimes \cdots \otimes f_n \mapsto f_1 \dot{\otimes} \cdots \dot{\otimes} f_n
\end{gathered}$$

Proof. We make $\mathbb{T}(\mathscr{F})(U)$ into a graded $\mathcal{O}_X(U)$-algebra as in (SOA,Proposition 40). Using (SOA,Proposition 4) (c) and Corollary 7 we get an isomorphism of $\mathcal{O}_X(U)$-algebras τ, and it is not hard to check it has the desired action on $f_1 \otimes \cdots \otimes f_n$, and is therefore a morphism of graded algebras. Note that τ is actually the sheafification morphism $P \longrightarrow \mathbb{T}(\mathscr{F})$ evaluated at U, from which naturality in $\mathscr{F}$ and inclusions of affine open sets $V \subseteq U$ is obvious. $\square$

Lemma 9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then $\mathbb{T}(\mathcal{O}_X)$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras.*

Proof. It suffices to show that the presheaf $P(U) = T(\mathcal{O}_X(U))$ is commutative, which follows immediately from (TES,Lemma 8) $\square$

3 Sheaves of Symmetric Algebras

Throughout this section $(X, \mathcal{O}_X)$ is a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules, and for an open set U let $H(U)$ be the commutative $\mathcal{O}_X(U)$-algebra given by the symmetric algebra $S(\mathscr{F}(U))$. That is, $S(\mathscr{F}(U))$ is the commutative graded $\mathcal{O}_X(U)$-algebra obtained as a quotient of $T(\mathscr{F}(U))$ by the two-sided ideal I generated by the elements of the form $x \otimes y - y \otimes x$. For an inclusion $V \subseteq U$ the morphism of rings $T(\mathscr{F}(U)) \longrightarrow T(\mathscr{F}(V))$ defined earlier induces a morphism of rings $S(\mathscr{F}(U)) \longrightarrow S(\mathscr{F}(V))$ fitting into a commutative diagram

$$\begin{tikzcd} T(\mathscr{F}(U)) \arrow[r] \arrow[d] & S(\mathscr{F}(U)) \arrow[d] \\ T(\mathscr{F}(V)) \arrow[r] & S(\mathscr{F}(V)) \end{tikzcd}$$

This makes H into a presheaf of commutative $\mathcal{O}_X$-algebras, and if P is the presheaf of algebras $P(U) = T(\mathscr{F}(U))$ then the canonical projections give a morphism of presheaves of algebras $P \longrightarrow H$. Let $\mathbb{S}(\mathscr{F})$ denote the sheaf of commutative $\mathcal{O}_X$-algebras obtained by sheafifying H. Sheafifying $P \longrightarrow H$ gives a canonical morphism of sheaves of algebras $\mathbb{T}(\mathscr{F}) \longrightarrow \mathbb{S}(\mathscr{F})$, which is an epimorphism of sheaves of modules. The morphism of presheaves of modules $\mathscr{F} \longrightarrow H$ given pointwise by the canonical injection $\mathscr{F}(U) \longrightarrow S(\mathscr{F}(U))$ composes with $H \longrightarrow \mathbb{S}(\mathscr{F})$ to give a monomorphism of sheaves of $\mathcal{O}_X$-modules $\mathscr{F} \longrightarrow \mathbb{S}(\mathscr{F})$. The morphisms we have just defined fit into a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathbb{T}(\mathscr{F}) \arrow[d] \\ & \mathbb{S}(\mathscr{F}) \end{tikzcd}$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, whose associated presheaves of symmetric algebras are H, G respectively, then we define a morphism of presheaves of $\mathcal{O}_X$-algebras $\phi' : H \longrightarrow G$ by $\phi'_U = S(\phi_U)$. This sheafifies to give a morphism of sheaves of $\mathcal{O}_X$-algebras $\mathbb{S}(\phi) : \mathbb{S}(\mathscr{F}) \longrightarrow \mathbb{S}(\mathscr{G})$. This defines a functor

$$\mathbb{S}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Alg}(X)$$

Note that the following diagrams commute

$$\begin{tikzcd} \mathbb{S}(\mathscr{F}) \arrow[r, "\mathbb{S}(\phi)"] & \mathbb{S}(\mathscr{G}) \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"'] & \mathscr{G} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{T}(\mathscr{F}) \arrow[r, "\mathbb{T}(\phi)"] \arrow[d] & \mathbb{T}(\mathscr{G}) \arrow[d] \\ \mathbb{S}(\mathscr{F}) \arrow[r, "\mathbb{S}(\phi)"'] & \mathbb{S}(\mathscr{G}) \end{tikzcd}$$

For $d \geq 0$ let H_d denote the sub-presheaf of $\mathcal{O}_X$-modules of H given by $H_d(U) = S^d(\mathscr{F}(U))$, which is the submodule of $S(\mathscr{F}(U))$ given by the image of $\mathscr{F}(U)^{\otimes d} \longrightarrow T(\mathscr{F}(U)) \longrightarrow S(\mathscr{F}(U))$. In particular there are isomorphisms of presheaves of modules $H_0 \cong \mathcal{O}_X$ and $H_1 \cong \mathscr{F}$. By construction the induced morphism $\bigoplus_{d \geq 0} H_d \longrightarrow H$ is an isomorphism (coproduct of presheaves of modules) and $H_d H_e \subseteq H_{d+e}, 1 \in H_0(X)$. Let $\mathbb{S}^d(\mathscr{F})$ denote the submodule of $\mathbb{S}(\mathscr{F})$ given by the image of $\mathbf{a}H_d \longrightarrow \mathbf{a}H = \mathbb{S}(\mathscr{F})$. Then $\mathbb{S}(\mathscr{F})$ together with the submodules $\mathbb{S}^d(\mathscr{F})$ is a sheaf of graded $\mathcal{O}_X$-algebras. Note that $\mathbb{S}^1(\mathscr{F})$ is the image of the monomorphism $\mathscr{F} \longrightarrow \mathbb{S}(\mathscr{F})$ and $\mathbb{S}^0(\mathscr{F})$ is the image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \longrightarrow \mathbb{S}(\mathscr{F})$ (this latter morphism is also a monomorphism of sheaves of modules, so $\mathbb{S}^0(\mathscr{F}) \cong \mathcal{O}_X$ and $\mathbb{S}^1(\mathscr{F}) \cong \mathscr{F}$ as sheaves of modules).

It is clear that if $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, then $\mathbb{S}(\phi)$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras, so we also have a functor

$$\mathbb{S}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrAlg}(X)$$

As usual, given $r \in \mathcal{O}_X(U)$ we write r for the corresponding element of $\mathbb{S}^0(\mathscr{F})(U)$. Similarly if $a \in \mathscr{F}(U)$ we write a for the corresponding element of $\mathbb{S}^1(\mathscr{F})(U)$. For $n > 1$ and sections $a_1, \ldots, a_n \in \mathscr{F}(U)$ the product $a_1 \cdots a_n$ in the ring $S(\mathscr{F}(U))$ is the coset of the tensor $a_1 \otimes \cdots \otimes a_n$. The image of this product in $\mathbb{S}^n(\mathscr{F})(U)$ via $H \longrightarrow \mathbb{S}(\mathscr{F})$ is just the product of the a_i considered as sections of $\mathbb{S}^1(\mathscr{F})(U)$, so we can write this section as $a_1 \cdots a_n$ with no ambiguity. In this notation, for a morphism of sheaves of modules $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ we have

$$\begin{aligned}\mathbb{S}(\phi)_U(r + a_{11} + a_{21}a_{22} + \cdots + a_{h1} \cdots a_{hh}) &= r + \phi_U(a_{11}) + \phi_U(a_{21})\phi_U(a_{22}) \\ &\quad + \cdots + \phi_U(a_{h1}) \cdots \phi_U(a_{hh})\end{aligned}$$

Proposition 10. *If $q \in \mathbb{S}(\mathscr{F})(U)$ then for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $q|_V = q_1 + \cdots + q_s$ where each q_k has the form*

$$q_k = r + a_{11} + a_{21}a_{22} + \cdots + a_{h1} \cdots a_{hh}$$

where $r \in \mathcal{O}_X(V)$ and $a_{ij} \in \mathscr{F}(V)$.

Proof. This follows immediately from the fact that $\mathbb{S}(\mathscr{F})$ is the sheafification of the presheaf H defined above. □

Proposition 11. *The functor $\mathbb{S}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Alg}(X)$ is left adjoint to the forgetful functor $\mathfrak{Alg}(X) \longrightarrow \mathfrak{Mod}(X)$. The unit of the adjunction is given for a sheaf of modules $\mathscr{F}$ by the canonical morphism $\mathscr{F} \longrightarrow \mathbb{S}(\mathscr{F})$.*

Proof. Let $\mathscr{S}$ be a sheaf of commutative $\mathcal{O}_X$-algebras and $\phi : \mathscr{F} \longrightarrow \mathscr{S}$ a morphism of sheaves of modules. We have to show there exists a unique morphism of sheaves of algebras $\Phi : \mathbb{S}(\mathscr{F}) \longrightarrow \mathscr{S}$ making the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d] & \mathscr{S} \\ \mathbb{S}(\mathscr{F}) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

One argues as for $\mathbb{T}(-)$, using the properties of the symmetric algebra given in our TES notes. □

Proposition 12. *The functor $\mathbb{S}(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrAlg}(X)$ is left adjoint to the functor $(-)_1 : \mathfrak{GrAlg}(X) \longrightarrow \mathfrak{Mod}(X)$ which maps a sheaf of commutative graded algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules $\mathscr{F}$ by the canonical isomorphism $\mathscr{F} \longrightarrow \mathbb{S}^1(\mathscr{F})$.*

Proof. The canonical morphism $\mathscr{F} \longrightarrow \mathbb{S}^1(\mathscr{F})$ is natural in $\mathscr{F}$, and we have to show that if $\mathscr{S}$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras and $\phi : \mathscr{F} \longrightarrow \mathscr{S}_1$ a morphism of sheaves of modules, then there exists a unique morphism of sheaves of graded algebras $\Phi : \mathbb{S}(\mathscr{F}) \longrightarrow \mathscr{S}$ such that Φ_1 makes the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d] & \mathscr{S}_1 \\ \mathbb{S}^1(\mathscr{F}) \arrow[ur, "\Phi_1"'] & \end{tikzcd} \tag{2}$$

By Proposition 11 the composite $\mathscr{F} \longrightarrow \mathscr{S}_1 \longrightarrow \mathscr{S}$ induces a morphism of sheaves of algebras $\Phi : \mathbb{S}(\mathscr{F}) \longrightarrow \mathscr{S}$ unique with the property that $\mathscr{F} \longrightarrow \mathbb{S}(\mathscr{F}) \longrightarrow \mathscr{S}$ is $\mathscr{F} \longrightarrow \mathscr{S}$. It is easy to check that Φ is a morphism of sheaves of *graded* algebras, and Φ_1 makes (2) commute. Uniqueness is clear, which proves that $\mathbb{S}(-)$ is left adjoint to $(-)_1$. □

Proposition 13. *If $U \subseteq X$ is open then the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd}
\mathfrak{Mod}(X) \arrow[r, "\mathbb{S}(-)"] \arrow[d] & \mathfrak{Alg}(X) \arrow[d] \\
\mathfrak{Mod}(U) \arrow[r, "\mathbb{S}(-)"'] & \mathfrak{Alg}(U)
\end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $\mathbb{S}(\mathscr{F}|_U) \longrightarrow \mathbb{S}(\mathscr{F})|_U$ has the action $a_1 \cdots a_n \mapsto a_1 \cdots a_n$.

Proof. Let $Alg(X)$ denote the category of presheaves of commutative $\mathcal{O}_X$-algebras. Associating a sheaf of modules $\mathscr{F}$ with the presheaf $H(U) = S(\mathscr{F}(U))$ defines a functor $\mathfrak{Mod}(X) \longrightarrow Alg(X)$. Clearly $\mathbb{S}(-)$ is the composite of this functor with sheafification $Alg(X) \longrightarrow \mathfrak{Alg}(X)$. So by (MRS,Lemma 24) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

$$\begin{tikzcd}
\mathfrak{Mod}(X) \arrow[r] \arrow[d] & Alg(X) \arrow[d] \\
\mathfrak{Mod}(U) \arrow[r] & Alg(U)
\end{tikzcd}$$

But it is not hard to check that this diagram actually commutes. □

Proposition 14. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd}
\mathfrak{Alg}(X) \arrow[r, Rightarrow, "f_*"] & \mathfrak{Alg}(Y) \\
\mathfrak{Mod}(X) \arrow[r, Rightarrow, "f_*"'] \arrow[u, "\mathbb{S}(-)"] & \mathfrak{Mod}(Y) \arrow[u, "\mathbb{S}(-)"']
\end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $f_*\mathbb{S}(\mathscr{F}) \longrightarrow \mathbb{S}(f_*\mathscr{F})$ has the action $a_1 \cdots a_n \mapsto a_1 \cdots a_n$.

Proof. Using (SOA,Lemma 12) we reduce immediately to showing the following diagram commutes up to a canonical natural equivalence

$$\begin{tikzcd}
Alg(X) \arrow[r, Rightarrow, "f_*"] & Alg(Y) \\
\mathfrak{Mod}(X) \arrow[r, Rightarrow, "f_*"'] \arrow[u] & \mathfrak{Mod}(Y) \arrow[u]
\end{tikzcd}$$

where the vertical functors are the "presheaf" symmetric algebra functors given in Proposition 13. In Proposition 5 we defined an isomorphism of $\mathcal{O}_Y(V)$-algebras $T_{\mathcal{O}_Y(V)}(\mathscr{F}(f^{-1}V)) \longrightarrow T_{\mathcal{O}_X(f^{-1}V)}(\mathscr{F}(f^{-1}V))$ natural in V and $\mathscr{F}$ which induces the necessary isomorphism of symmetric algebras. □

Proposition 15. *Let $X = SpecA$ be an affine scheme and M an A-module. Then there is a canonical isomorphism $\theta : S(M)^{\sim} \longrightarrow \mathbb{S}(\widetilde{M})$ of sheaves of $\mathcal{O}_X$-algebras which is natural in M. We have*

$$\theta_U(a_1 \cdots a_n \dot{/} s_1 \cdots s_n) = a_1 \dot{/} s_1 \cdots a_n \dot{/} s_n$$

where $U \subseteq X$ is open, $a_i \in M$ and $s_i \in A$ with $U \subseteq D(s_1 \cdots s_n)$.

Proof. Consider the following diagram consisting of adjoint pairs of functors (see (Ex 5.3), (SOA,Proposition 5), Proposition 11 and (TES,Proposition 33))

$$\begin{tikzcd}
A\mathbf{Alg} \arrow[r, bend left, "\sim"] \arrow[d, bend left] & \mathfrak{Alg}(X) \arrow[l, bend left, "\Gamma"] \arrow[d, bend left] \\
A\mathbf{Mod} \arrow[u, bend left, "S"] \arrow[r, bend left, "\sim"] & \mathfrak{Mod}(X) \arrow[l, bend left, "\Gamma"] \arrow[u, bend left, "\mathbb{S}(-)"]
\end{tikzcd}$$

The two composites $\mathfrak{Alg}(X) \longrightarrow A\mathbf{Mod}$ are equal, so $\widetilde{-}S$ and $\mathbb{S}(-)\widetilde{-}$ are both left adjoints for the same functor. Therefore they must be canonically naturally equivalent, which is what we wanted to show. The isomorphism $\theta : S(M)^{\sim} \longrightarrow \mathbb{S}(\widetilde{M})$ is unique with the property that $\theta_X(\dot{m/1}) = \dot{m}/1$ for every $m \in M$. This is an isomorphism of sheaves of algebras, so it is easy to check θ has the desired effect on the sections in the statement of the Proposition. □

Corollary 16. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent then so is $\mathbb{S}(\mathscr{F})$.*

Proof. For $x \in X$ let U be an open affine neighborhood of x and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism. Then $f_*\mathscr{F}|_U \cong \mathscr{F}(U)^{\sim}$and combining Proposition 15, Proposition 14 and Proposition 13 we see that

$$\begin{aligned}
f_*(\mathbb{S}(\mathscr{F})|_U) &\cong f_*(\mathbb{S}(\mathscr{F}|_U)) \\
&\cong \mathbb{S}((f_*\mathscr{F}|_U)) \\
&\cong \mathbb{S}(\widetilde{\mathscr{F}(U)}) \\
&\cong S(\mathscr{F}(U))^{\sim}
\end{aligned}$$

This is an isomorphism of sheaves of algebras, which shows that $\mathbb{S}(\mathscr{F})$ is quasi-coherent. □

Proposition 17. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent and $U \subseteq X$ is affine then there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-algebras natural in $\mathscr{F}$ and the affine open set U*

$$\begin{aligned}
\tau : S(\mathscr{F}(U)) &\longrightarrow \mathbb{S}(\mathscr{F})(U) \\
f_1 \cdots f_n &\mapsto f_1 \cdots f_n
\end{aligned}$$

Proof. We make $\mathbb{S}(\mathscr{F})(U)$ into a graded $\mathcal{O}_X(U)$-algebra as in (SOA,Proposition 40). Using (SOA,Proposition 4) (c) and Corollary 16 we get an isomorphism $\mathcal{O}_X(U)$-algebras τ with the required properties. Note that τ is actually the sheafification morphism $H \longrightarrow \mathbb{S}(\mathscr{F})$ evaluated at U, which makes it clear that τ is natural in $\mathscr{F}$ and the inclusion of affine open sets $V \subseteq U$. □

Corollary 18. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. Then $\mathbb{S}(\mathscr{F})$ is locally generated by $\mathbb{S}^1(\mathscr{F})$ as an $\mathbb{S}^0(\mathscr{F})$-algebra.*

Proof. Using Proposition 17 we reduce immediately to showing that for a commutative ring A and A-module M the graded A-algebra $S(M)$ is generated by $S^1(M)$ as an A-algebra, which is obvious. □

4 Sheaves of Exterior Algebras

Througout this section $(X, \mathcal{O}_X)$ is a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules, and for any open set U let $H(U)$ be the (noncommutative) $\mathcal{O}_X(U)$-algebra given by the exterior algebra

$\bigwedge \mathscr{F}(U)$. That is, $\bigwedge \mathscr{F}(U)$ is the graded $\mathcal{O}_X(U)$-algebra obtained as a quotient of $T(\mathscr{F}(U))$ by the two-sided ideal I generated by the elements of the form $x \otimes x$. For an inclusion $V \subseteq U$ the morphism of rings $T(\mathscr{F}(U)) \longrightarrow T(\mathscr{F}(V))$ defined earlier induces a morphism of rings $\bigwedge \mathscr{F}(U) \longrightarrow \bigwedge \mathscr{F}(V)$ fitting into a commutative diagram

$$\begin{tikzcd} T(\mathscr{F}(U)) \arrow[r] \arrow[d] & \bigwedge \mathscr{F}(U) \arrow[d] \\ T(\mathscr{F}(V)) \arrow[r] & \bigwedge \mathscr{F}(V) \end{tikzcd}$$

This makes H into a presheaf of $\mathcal{O}_X$-algebras, and if P is the presheaf of algebras $P(U) = T(\mathscr{F}(U))$ then the canonical projections give a morphism of presheaves of algebras $P \longrightarrow H$. Let $\bigwedge \mathscr{F}$ denote the sheaf of $\mathcal{O}_X$-algebras obtained by sheafifying H. Sheafifying $P \longrightarrow H$ gives a canonical morphism of sheaves of algebras $\mathbb{T}(\mathscr{F}) \longrightarrow \bigwedge \mathscr{F}$, which is an epimorphism of sheaves of modules. The morphism of presheaves of modules $\mathscr{F} \longrightarrow H$ given pointwise by the canonical injection $\mathscr{F}(U) \longrightarrow \bigwedge \mathscr{F}(U)$ composes with $H \longrightarrow \bigwedge \mathscr{F}$ to give a monomorphism of sheaves of $\mathcal{O}_X$-modules $\mathscr{F} \longrightarrow \bigwedge \mathscr{F}$. The morphisms we have just defined fit into a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathbb{T}(\mathscr{F}) \arrow[d] \\ & \bigwedge \mathscr{F} \end{tikzcd}$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, whose associated presheaves of exterior algebras are H, G respectively, then we define a morphism of presheaves of $\mathcal{O}_X$-algebras $\phi' : H \longrightarrow G$ by $\phi'_U = \wedge \phi_U$. This sheafifies to give a morphism of sheaves of $\mathcal{O}_X$-algebras $\bigwedge \mathscr{F} \longrightarrow \bigwedge \mathscr{G}$. This defines a functor

$$\bigwedge(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{nAlg}(X)$$

Note that the following diagrams commute

$$\begin{tikzcd} \bigwedge \mathscr{F} \arrow[r, "\bigwedge \phi"] & \bigwedge \mathscr{G} \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"'] & \mathscr{G} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{T}(\mathscr{F}) \arrow[r, "\mathbb{T}(\phi)"] \arrow[d] & \mathbb{T}(\mathscr{G}) \arrow[d] \\ \bigwedge \mathscr{F} \arrow[r, "\bigwedge \phi"'] & \bigwedge \mathscr{G} \end{tikzcd}$$

For $d \geq 0$ let H_d denote the sub-presheaf of $\mathcal{O}_X$-modules of H given by $H_d(U) = \bigwedge^d(\mathscr{F}(U))$, which is the submodule of $\bigwedge(\mathscr{F}(U))$ given by the image of $\mathscr{F}(U)^{\otimes d} \longrightarrow T(\mathscr{F}(U)) \longrightarrow \bigwedge(\mathscr{F}(U))$. In particular there are isomorphism of presheaves of modules $H_0 \cong \mathcal{O}_X$ and $H_1 \cong \mathscr{F}$. By construction the induced morphism $\bigoplus_{d \geq 0} H_d \longrightarrow H$ is an isomorphism (coproduct of presheaves of modules) and $H_d H_e \subseteq H_{d+e}, 1 \in H_0(X)$. Let $\bigwedge^d \mathscr{F}$ denote the submodule of $\bigwedge \mathscr{F}$ given by the image of $\mathbf{a}H_d \longrightarrow \mathbf{a}H = \bigwedge \mathscr{F}$. Then $\bigwedge \mathscr{F}$ together with the submodules $\bigwedge^d \mathscr{F}$ is a sheaf of super $\mathcal{O}_X$-algebras. Note that $\bigwedge^1 \mathscr{F}$ is the image of the monomorphism $\mathscr{F} \longrightarrow \bigwedge \mathscr{F}$ and $\bigwedge^0 \mathscr{F}$ is the image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \longrightarrow \bigwedge \mathscr{F}$ (this latter morphism is also a monomorphism of sheaves of modules, so $\bigwedge^0 \mathscr{F} \cong \mathcal{O}_X$ and $\bigwedge^1 \mathscr{F} \cong \mathscr{F}$ as sheaves of modules).

It is clear that if $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, then $\bigwedge \phi$ is a morphism of sheaves of super $\mathcal{O}_X$-algebras, so we also have a functor

$$\bigwedge(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{sAlg}(X)$$

As usual, given $r \in \mathcal{O}_X(U)$ we write r for the corresponding element of $\Gamma(U, \bigwedge^0 \mathscr{F})$. Similarly if $a \in \mathscr{F}(U)$ we write a for the corresponding element of $\Gamma(U, \bigwedge^1 \mathscr{F})$. For $n > 1$ and $a_1, \ldots, a_n \in \mathscr{F}(U)$ we write $a_1 \dot{\wedge} \cdots \dot{\wedge} a_n$ for the element of $\Gamma(U, \bigwedge^n \mathscr{F})$ which is the image of $a_1 \wedge \cdots \wedge a_n \in \bigwedge^n(\mathscr{F}(U))$

under $H \longrightarrow \bigwedge \mathscr{F}$. This is just the product of the individual a_i considered as elements of $\Gamma(U, \bigwedge^1 \mathscr{F})$. In this notation, for a morphism of sheaves of modules $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ we have

$$\begin{aligned}(\wedge\phi)_U(r + a_{11} + a_{21} \dot{\wedge} a_{22} + \cdots + a_{h1} \dot{\wedge} \cdots \dot{\wedge} a_{hh}) = r + \phi_U(a_{11}) + \phi_U(a_{21}) \dot{\wedge} \phi_U(a_{22})\\ + \cdots + \phi_U(a_{h1}) \dot{\wedge} \cdots \dot{\wedge} \phi_U(a_{hh})\end{aligned}$$

For $d \geq 0$ let $\wedge^d \phi : \bigwedge^d \mathscr{F} \longrightarrow \bigwedge^d \mathscr{G}$ be the morphism of sheaves of modules induced on the graded submodules by $\wedge\phi$, which is unique making the following diagram of sheaves of modules commute

$$\begin{tikzcd}
\bigwedge \mathscr{F} \arrow[r, "\wedge \phi"] & \bigwedge \mathscr{G} \\
\bigwedge^d \mathscr{F} \arrow[u] \arrow[r, "\wedge^d \phi"'] & \bigwedge^d \mathscr{G} \arrow[u]
\end{tikzcd}$$

This defines a functor $\bigwedge^d(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$.

Proposition 19. *If $q \in \Gamma(U, \bigwedge \mathscr{F})$ then for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $q|_V = q_1 + \cdots + q_s$ where each q_k has the form*

$$q_k = r + a_{11} + a_{21} \dot{\wedge} a_{22} + \cdots + a_{h1} \dot{\wedge} \cdots \dot{\wedge} a_{hh}$$

where $r \in \mathcal{O}_X(V)$ and $a_{ij} \in \mathscr{F}(V)$.

Definition 1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. For $n \geq 1$ we say a multilinear morphism (MRS,Definition 9) $f : \mathscr{F} \times \cdots \times \mathscr{F} \longrightarrow \mathscr{G}$ from the n-fold product into a sheaf of abelian groups $\mathscr{G}$ is *alternating* if for every open $U \subseteq X$ the map f_U is alternating. That is, for every open $U \subseteq X$ we have $f_U(m_1, \ldots, m_n) = 0$ whenever $m_i = m_j$ for $i \neq j$. The canonical morphism of sheaves of sets

$$\gamma : \mathscr{F} \times \cdots \times \mathscr{F} \longrightarrow \bigwedge^n \mathscr{F}$$
$$(a_1, \ldots, a_n) \mapsto a_1 \dot{\wedge} \cdots \dot{\wedge} a_n$$

is clearly an alternating multilinear form. In fact this is the universal alternating multilinear form, as we will see in a moment. If $n = 1$ then an alternating multilinear map is just a morphism of sheaves of abelian groups $\mathscr{F} \longrightarrow \mathscr{G}$ and an alternating multilinear form is a morphism of sheaves of modules.

Proposition 20. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}, \mathscr{G}$ sheaves of modules on X and suppose $f : \mathscr{F} \times \cdots \times \mathscr{F} \longrightarrow \mathscr{G}$ is an alternating multilinear form out of the n-fold product for $n \geq 1$. Then there is a* unique *morphism of sheaves of modules $\theta : \bigwedge^n \mathscr{F} \longrightarrow \mathscr{G}$ making the following diagram commute*

$$\begin{tikzcd}
\mathscr{F} \times \cdots \times \mathscr{F} \arrow[r, "\gamma"] \arrow[d, "f"'] & \bigwedge^n \mathscr{F} \arrow[dl, "\theta"] \\
\mathscr{G} &
\end{tikzcd}$$

Proof. For each open $U \subseteq X$ we have an alternating multilinear form $f_U : \mathscr{F}(U)^n \longrightarrow \mathscr{G}(U)$, which by (TES,Lemma 18) corresponds to a morphism of $\mathcal{O}_X(U)$-modules $\phi_U : \wedge^n \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$. Together these define a morphism of presheaves of $\mathcal{O}_X(U)$-modules $\phi : H_d \longrightarrow \mathscr{G}$. The induced morphism $\theta : \bigwedge^n \mathscr{F} \longrightarrow \mathscr{G}$ is the one we require. $\square$

Proposition 21. *The functor $\bigwedge(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{sAlg}(X)$ is left adjoint to the functor $(-)_1 : \mathfrak{sAlg}(X) \longrightarrow \mathfrak{Mod}(X)$ which maps a sheaf of super algebras to its degree 1 component. The unit of the adjunction is given for a sheaf of modules $\mathscr{F}$ by the canonical isomorphism $\mathscr{F} \longrightarrow \bigwedge^1 \mathscr{F}$.*

Proof. The canonical morphism $\eta : \mathscr{F} \longrightarrow \bigwedge^1 \mathscr{F}$ is natural in $\mathscr{F}$, and we have to show that if $\mathscr{S}$ is a sheaf of super $\mathcal{O}_X$-algebras and $\phi : \mathscr{F} \longrightarrow \mathscr{S}_1$ a morphism of sheaves of modules, then there exists a unique morphism of sheaves of super algebras $\Phi : \bigwedge \mathscr{F} \longrightarrow \mathscr{S}$ such that Φ_1 makes the following diagram commute

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d, "\eta"'] & \mathscr{S}_1 \\ \bigwedge^1 \mathscr{F} \arrow[ur, "\Phi_1"'] & \end{tikzcd} \tag{3}$$

Let Φ_0 be the canonical morphism $\bigwedge^0 \mathscr{F} \longrightarrow \mathscr{S}_0$ and let Φ_1 be the composite $\phi\eta^{-1}$. For $n > 1$ we define a multilinear form

$$f : \mathscr{F} \times \cdots \times \mathscr{F} \longrightarrow \mathscr{S}_n$$
$$f_U(a_1, \ldots, a_n) = \phi_U(a_1) \cdots \phi_U(a_n)$$

which is alternating since $\mathscr{S}$ is a super $\mathcal{O}_X$-algebra. By Proposition 20 there is an induced morphism $\Phi_n : \bigwedge^n \mathscr{F} \longrightarrow \mathscr{S}_n$. The morphisms Φ_n for $n \geq 0$ induce a morphism of sheaves of graded $\mathcal{O}_X$-modules $\Phi : \bigwedge \mathscr{F} \longrightarrow \mathscr{S}$ out of the coproduct. In fact with a little work one checks that Φ is a morphism of super $\mathcal{O}_X$-algebras, while Φ_1 trivially makes (3) commute. Uniqueness is clear, which proves that $\bigwedge(-)$ is left adjoint to $(-)_1$. $\square$

Proposition 22. *If $U \subseteq X$ is open then the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd} \mathfrak{Mod}(X) \arrow[r, "\bigwedge(-)"] \arrow[d] & \mathfrak{sAlg}(X) \arrow[d] \\ \mathfrak{Mod}(U) \arrow[r, "\bigwedge(-)"'] & \mathfrak{sAlg}(U) \end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $\bigwedge(\mathscr{F}|_U) \cong (\bigwedge \mathscr{F})|_U$ has the action $a_1 \dot\wedge \cdots \dot\wedge a_n \mapsto a_1 \dot\wedge \cdots \dot\wedge a_n$. In particular there is an isomorphism of sheaves of modules $\bigwedge^d(\mathscr{F}|_U) \cong (\bigwedge^d \mathscr{F})|_U$ natural in $\mathscr{F}$.

Proof. Let $GrnAlg(X)$ denote the category of presheaves of graded $\mathcal{O}_X$-algebras. Associating a sheaf of modules $\mathscr{F}$ with the presheaf of graded $\mathcal{O}_X$-algebras $H(U) = \bigwedge(\mathscr{F}(U))$ defines a functor $\mathfrak{Mod}(X) \longrightarrow GrnAlg(X)$. Clearly $\bigwedge(-)$ is the composite of this functor with sheafification $GrnAlg(X) \longrightarrow \mathfrak{GrnAlg}(X)$. So by (SOA,Lemma 38) it suffices to show that the following diagram of functors commutes up to a canonical natural equivalence

$$\begin{tikzcd} \mathfrak{Mod}(X) \arrow[r] \arrow[d] & GrnAlg(X) \arrow[d] \\ \mathfrak{Mod}(U) \arrow[r] & GrnAlg(U) \end{tikzcd}$$

But it is not hard to see that this diagram actually commutes. $\square$

Corollary 23. *If $U \subseteq X$ is open and $\mathscr{F}$ is a sheaf of modules on X then for any $d \geq 0$ there is a canonical isomorphism $\bigwedge^d(\mathscr{F}|_U) \cong \left(\bigwedge^d \mathscr{F}\right)|_U$ of sheaves of modules on U natural in $\mathscr{F}$.*

Proposition 24. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd} \mathfrak{sAlg}(X) \arrow[r, Rightarrow, "f_*"] & \mathfrak{sAlg}(Y) \\ \mathfrak{Mod}(X) \arrow[r, Rightarrow, "f_*"'] \arrow[u, "\bigwedge(-)"] & \mathfrak{Mod}(Y) \arrow[u, "\bigwedge(-)"'] \end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $f_*(\bigwedge \mathscr{F}) \longrightarrow \bigwedge(f_*\mathscr{F})$ has the action $a_1 \dot\wedge \cdots \dot\wedge a_n \mapsto a_1 \dot\wedge \cdots \dot\wedge a_n$.

Proof. Using (SOA,Lemma 44) we reduce immediately to showing that the following diagram commutes up to a canonical natural equivalence

$$\begin{tikzcd}
GrnAlg(X) \arrow[r, Rightarrow, "f_*"] & GrnAlg(Y) \\
\mathfrak{Mod}(X) \arrow[u] \arrow[r, Rightarrow, "f_*"'] & \mathfrak{Mod}(Y) \arrow[u]
\end{tikzcd}$$

where the vertical functors are the "presheaf" exterior algebra functors given in Proposition 22. In Proposition 5 we defined an isomorphism of $\mathcal{O}_Y(V)$-algebras $T_{\mathcal{O}_Y(V)}(\mathscr{F}(f^{-1}V)) \longrightarrow T_{\mathcal{O}_X(f^{-1}V)}(\mathscr{F}(f^{-1}V))$ natural in V and $\mathscr{F}$ which induces the necessary isomorphism of exterior algebras. $\square$

Lemma 25. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. For $y \in X$ there is a canonical isomorphism of graded $\mathcal{O}_{X,y}$-algebras natural in $\mathscr{F}$*

$$\tau : (\bigwedge \mathscr{F})_y \longrightarrow \bigwedge_{\mathcal{O}_{X,y}} \mathscr{F}_y$$
$$(U, a_1 \dot\wedge \cdots \dot\wedge a_n) \mapsto (U, a_1) \wedge \cdots \wedge (U, a_n)$$

In particular for $d \geq 0$ there is a canonical isomorphism of $\mathcal{O}_{X,y}$-modules $(\bigwedge^d \mathscr{F})_y \cong \bigwedge^d_{\mathcal{O}_{X,y}} \mathscr{F}_y$ natural in $\mathscr{F}$.

Proof. Let H be the presheaf $H(U) = \bigwedge(\mathscr{F}(U))$. Then H_y becomes a graded $\mathcal{O}_{X,y}$-algebra with the canonical grading, and in fact H_y is a super $\mathcal{O}_{X,y}$-algebra (TES,Definition 4). Suppose T is a super $\mathcal{O}_{X,y}$-algebra and that $\varphi : \mathscr{F}_y \longrightarrow T_1$ is a morphism of $\mathcal{O}_{X,y}$-modules. For each open neighborhood U of y the morphism of $\mathcal{O}_X(U)$-modules $\mathscr{F}(U) \longrightarrow \mathscr{F}_y \longrightarrow T_1$ induces a morphism of super $\mathcal{O}_X(U)$-algebras $\bigwedge(\mathscr{F}(U)) \longrightarrow T$ (TES,Proposition 7). Taking the direct limit we obtain a well-defined morphism of super $\mathcal{O}_{X,y}$-algebras $\tau : H_y \longrightarrow T$ defined for $a \in \mathscr{F}(U)$ by $(U, a) \mapsto \varphi(U, a)$. If $\eta : \mathscr{F}_y \cong (H_y)_1$ is the canonical isomorphism of $\mathcal{O}_{X,y}$-modules then τ is trivially the unique morphism of super algebras making the following diagram commute

$$\begin{tikzcd}
\mathscr{F}_y \arrow[r, "\varphi"] \arrow[d, Rightarrow] & T_1 \\
(H_y)_1 \arrow[ur, "\tau_1"'] &
\end{tikzcd}$$

But this is the universal property that identifies the super algebra $\bigwedge_{\mathcal{O}_{X,y}} \mathscr{F}_y$. Therefore we have a canonical isomorphism of graded $\mathcal{O}_{X,y}$-algebras

$$\tau : H_y \longrightarrow \bigwedge_{\mathcal{O}_{X,y}} \mathscr{F}_y$$
$$\tau(U, a_1 \wedge \cdots \wedge a_n) = (U, a_1) \wedge \cdots \wedge (U, a_n)$$

composing with the canonical isomorphism $H_y \cong (\bigwedge \mathscr{F})_y$ we have the desired isomorphism. Naturality in $\mathscr{F}$ is easily checked. $\square$

Proposition 26. *Let $X = SpecA$ be an affine scheme and M an A-module. Then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras natural in M*

$$\theta : (\bigwedge M)^\sim \longrightarrow \bigwedge(M^\sim)$$
$$(a_1 \wedge \cdots \wedge a_n)\dot/(s_1 \cdots s_n) \mapsto a_1/s_1 \dot\wedge \cdots \dot\wedge a_n/s_n$$

Proof. See (SOA,Definition 13) for the definition of the sheaf of graded $\mathcal{O}_X$-algebras $(\bigwedge M)^{\sim}$. If $A = 0$ then this is trivial, so assume A is nonzero. For $\mathfrak{p} \in SpecA$ we have an isomorphism of $\mathcal{O}_{X,\mathfrak{p}}$-algebras, using (TES,Corollary 16) and Lemma 25

$$\theta_{\mathfrak{p}} : (\wedge_A M)^{\sim}{}_{\mathfrak{p}} \cong (\wedge_A M)_{\mathfrak{p}} \cong \wedge_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \cong \wedge_{\mathcal{O}_{X,\mathfrak{p}}} (M^{\sim})_{\mathfrak{p}} \cong (\wedge M^{\sim})_{\mathfrak{p}}$$

It is not hard to check that $germ_{\mathfrak{p}} \theta_U(s) = \theta_{\mathfrak{p}}(germ_{\mathfrak{p}} s)$ defines the necessary isomorphism of sheaves of graded algebras. Naturality in M is also easily checked. $\square$

Corollary 27. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent then so are $\bigwedge \mathscr{F}, \bigwedge^d \mathscr{F}$ for every $d \geq 0$. If X is noetherian and $\mathscr{F}$ coherent then $\bigwedge^d \mathscr{F}$ is coherent for $d \geq 0$.*

Proof. For $x \in U$ let U be an affine open neighborhood of x and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism. Then $f_* \mathscr{F}|_U \cong \mathscr{F}(U)^{\sim}$ and combining Proposition 26, Proposition 24 and Proposition 22 we see that there is an isomorphism of sheaves of graded algebras

$$\begin{aligned} f_*((\bigwedge \mathscr{F})|_U) &\cong f_*(\bigwedge(\mathscr{F}|_U)) \\ &\cong \bigwedge(f_* \mathscr{F}|_U) \\ &\cong \bigwedge(\mathscr{F}(U)^{\sim}) \\ &\cong (\bigwedge \mathscr{F}(U))^{\sim} \end{aligned} \tag{4}$$

This shows that $\bigwedge \mathscr{F}$ is quasi-coherent. Since $\bigwedge \mathscr{F} = \bigoplus_{d \geq 0} \bigwedge^d \mathscr{F}$ it follows from (MOS,Lemma 1) that $\bigwedge^d \mathscr{F}$ is a quasi-coherent sheaf of modules for $d \geq 0$. Since (4) is an isomorphism of sheaves of graded algebras, we deduce an isomorphism of sheaves of modules $f_*((\bigwedge^d \mathscr{F})|_U) \cong (\bigwedge^d \mathscr{F}(U))^{\sim}$ for $d \geq 0$. If X is noetherian and $\mathscr{F}$ coherent then $\mathscr{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module, so $\bigwedge^d \mathscr{F}(U)$ is finitely generated (TES,Lemma 11) and consequently $\bigwedge^d \mathscr{F}$ is coherent. $\square$

Corollary 28. *Let X be a nonempty scheme and $\mathscr{F}$ a locally free sheaf of finite rank $n \geq 1$. Then for $0 \leq d \leq n$ the sheaf $\bigwedge^d \mathscr{F}$ is locally free of rank $\binom{n}{d}$. If $d > n$ then $\bigwedge^d \mathscr{F} = 0$.*

Proof. If $x \in X$ then we can find an affine open neighborhood U of x with $\mathscr{F}|_U$ free of rank n. Then $\mathscr{F}(U)$ is a free $\mathcal{O}_X(U)$-module of rank n (MOS,Lemma 6) and therefore $\bigwedge^d \mathscr{F}(U)$ is a free module of rank $\binom{n}{d}$ if $d \leq n$ and is zero otherwise (TES,Proposition 21). The result now follows from the isomorphism $f_*((\bigwedge^d \mathscr{F})|_U) \cong (\bigwedge^d \mathscr{F}(U))^{\sim}$ of Corollary 27. $\square$

Proposition 29. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of modules on Y. Then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras natural in $\mathscr{F}$*

$$\xi : f^*(\bigwedge \mathscr{F}) \longrightarrow \bigwedge(f^* \mathscr{F})$$
$$[V, a_1 \dot{\wedge} \cdots \dot{\wedge} a_n] \dot{\otimes} (b_1 \cdots b_n) \mapsto ([V, a_1] \dot{\otimes} b_1) \dot{\wedge} \cdots \dot{\wedge} ([V, a_n] \dot{\otimes} b_n)$$

In particular for $d \geq 0$ there is a canonical isomorphism of $\mathcal{O}_X$-modules $f^*(\bigwedge^d \mathscr{F}) \cong \bigwedge^d f^* \mathscr{F}$ natural in $\mathscr{F}$.

Proof. See (SOA,Section 3.1) for the definition of the inverse image of a sheaf of graded algebras. For $x \in X$ we use Lemma 25, (SOA,Lemma 45) and (TES,Proposition 15) to obtain an isomorphism of graded $\mathcal{O}_{X,x}$-algebras

$$\begin{aligned}\xi_x : f^*(\bigwedge \mathscr{F})_x &\cong (\bigwedge \mathscr{F})_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}\\ &\cong (\bigwedge_{\mathcal{O}_{Y,f(x)}} \mathscr{F}_{f(x)}) \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}\\ &\cong \bigwedge_{\mathcal{O}_{X,x}} (\mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x})\\ &\cong \bigwedge_{\mathcal{O}_{X,x}} (f^*\mathscr{F})_x \cong \Big(\bigwedge (f^*\mathscr{F})\Big)_x\end{aligned}$$

One checks that $germ_x \xi_U(s) = \xi_x(germ_x s)$ gives a well-defined isomorphism of sheaves of graded $\mathcal{O}_X$-algebras. Naturality in $\mathscr{F}$ is easily checked. $\square$

Lemma 30. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ an epimorphism of sheaves of $\mathcal{O}_X$-modules. Then $\bigwedge^d \psi : \bigwedge^d \mathscr{F} \longrightarrow \bigwedge^d \mathscr{G}$ is an epimorphism for $d \geq 1$.*

Proof. By Lemma 25 we reduce to showing that if R is a commutative ring, $\psi : M \longrightarrow N$ an epimorphism of R-modules then $\wedge^d M \longrightarrow \wedge^d N$ is an epimorphism, which is trivial. $\square$

Remark 1. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. Then for $a, b \geq 1$ the product in the sheaf of graded $\mathcal{O}_X$-algebras $\bigwedge \mathscr{F}$ induces a morphism of sheaves of $\mathcal{O}_X$-modules

$$\begin{gathered}\bigwedge^a \mathscr{F} \otimes \bigwedge^b \mathscr{F} \longrightarrow \bigwedge^{a+b} \mathscr{F}\\ a \dot{\otimes} b \mapsto ab\end{gathered} \tag{5}$$

Proposition 31. *Let $(X, \mathcal{O}_X)$ be a nontrivial ringed space and suppose we have an exact sequence of locally free sheaves*

$$0 \longrightarrow \mathscr{G} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

where $\mathscr{G}, \mathscr{H}$ are locally free of finite ranks a, b respectively. Then there is a canonical isomorphism of sheaves of $\mathcal{O}_X$-modules $\bigwedge^{a+b} \mathscr{F} \cong \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{H}$.

Proof. See (MRS,Remark 13) for the definition of a *nontrivial* ringed space. In the cases where one of a, b are zero there is a trivial isomorphism, so assume $a, b \geq 1$. Tensoring $\bigwedge^a \mathscr{G} \longrightarrow \bigwedge^a \mathscr{F}$ with $\bigwedge^b \mathscr{F}$ and then composing with (5) gives a morphism of sheaves of $\mathcal{O}_X$-modules

$$\theta : \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{F} \longrightarrow \bigwedge^{a+b} \mathscr{F}$$

Using Lemma 25 to reduce to the case of rings and exterior powers of modules, one checks that θ is an epimorphism of sheaves of modules (see (TES,Section 3.3.1) for the case of rings). Tensoring the epimorphism $\bigwedge^b \mathscr{F} \longrightarrow \bigwedge^b \mathscr{H}$ with $\bigwedge^a \mathscr{G}$ gives an epimorphism of sheaves of $\mathcal{O}_X$-modules

$$\rho : \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{F} \longrightarrow \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{H}$$

We claim that $Ker(\theta) = Ker(\rho)$. It suffices to show that $Ker(\theta)_x = Ker(\rho)_x$ for every $x \in X$, which follows from Lemma 25 and (TES,Proposition 32). Since θ, ρ are epimorphisms with the same kernel, it follows that there is a unique isomorphism of sheaves of $\mathcal{O}_X$-modules $\bigwedge^{a+b} \mathscr{F} \cong$

$\bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{H}$ fitting into the following commutative diagram

$$\begin{tikzcd} & \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{H} \\ \bigwedge^a \mathscr{G} \otimes \bigwedge^b \mathscr{F} \arrow[ur, "\rho"] \arrow[dr, "\theta"'] & \\ & \bigwedge^{a+b} \mathscr{F} \arrow[uu, Rightarrow] \end{tikzcd}$$

which completes the proof. □

Proposition 32. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. Then for $n \geq 1$ there is a canonical morphism of sheaves of modules natural in $\mathscr{F}$*

$$\beta : \bigwedge^n (\mathscr{F}^\vee) \longrightarrow (\bigwedge^n \mathscr{F})^\vee$$

$$\beta_U(\nu_1 \dot\wedge \cdots \dot\wedge \nu_n)_V(a_1 \dot\wedge \cdots \dot\wedge a_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n (\nu_i)_V(a_{\sigma(i)})$$

Proof. For an open subset $U \subseteq X$ and $\nu_1, \ldots, \nu_n \in \Gamma(U, \mathscr{F}^\vee)$ we define a morphism of sheaves of sets

$$f : \mathscr{F}|_U \times \cdots \times \mathscr{F}|_U \longrightarrow \mathcal{O}_X|_U$$

$$f_V(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n (\nu_i)_V(a_{\sigma(i)}) = det((v_i)_V(a_j))$$

Using the standard properties of the determinant, one checks that f is an alternating multilinear form. By Proposition 20 there is a corresponding morphism of sheaves of modules $\bigwedge^n(\mathscr{F}|_U) \longrightarrow \mathcal{O}_X|_U$. Let $\theta_{\nu_1,\ldots,\nu_n}$ denote the composite of this morphism with the canonical isomorphism $\bigwedge^n(\mathscr{F}|_U) \cong (\bigwedge^n \mathscr{F})|_U$ of Proposition 22. One checks that the following map is also an alternating multilinear form

$$\tau : \mathscr{F}^\vee \times \cdots \times \mathscr{F}^\vee \longrightarrow (\bigwedge^n \mathscr{F})^\vee$$

$$\tau_U(\nu_1, \ldots, \nu_n) = \theta_{\nu_1,\ldots,\nu_n}$$

which by Proposition 20 must correspond to a morphism of sheaves of modules β with the required property. □

Corollary 33. *Let X be a nonempty scheme, $\mathscr{L}$ a locally free sheaf of finite rank. Then for $n \geq 1$ the canonical morphism of sheaves of modules*

$$\beta : \bigwedge^n (\mathscr{L}^\vee) \longrightarrow (\bigwedge^n \mathscr{L})^\vee$$

of Proposition 32 is an isomorphism.

Proof. Let $\mathscr{L}$ be a locally free sheaf of rank $r \geq 0$. If $r = 0$ then $\mathscr{L} = 0$ and β is trivially an isomorphism. If $r \geq 1$ but $n > r$ then the domain and codomain of β are both zero, so once again β is an isomorphism. So we may assume $1 \leq r \leq n$. For $x \in X$ the $\mathcal{O}_{X,x}$-module $\mathscr{L}_x$ is free of rank r. Using Lemma 25, Corollary 28 and (MRS,Corollary 92) we have a commutative diagram

$$\begin{tikzcd} \{\bigwedge^n(\mathscr{L}^\vee)\}_x \arrow[r, "\beta_x"] \arrow[d, Rightarrow] & \{(\bigwedge^n \mathscr{L})^\vee\}_x \arrow[d, Rightarrow] \\ \bigwedge^n_{\mathcal{O}_{X,x}}(\mathscr{L}_x)^\vee \arrow[r, Rightarrow] & (\bigwedge^n_{\mathcal{O}_{X,x}} \mathscr{L}_x)^\vee \end{tikzcd}$$

where the bottom morphism is the isomorphism of (TES,Corollary 31). Therefore β_x is an isomorphism for every $x \in X$, which implies that β is an isomorphism and completes the proof. $\square$

Proposition 34. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ sheaves of modules on X. Then there is a canonical isomorphism of sheaves of super $\mathcal{O}_X$-algebras*

$$\delta : \bigwedge \mathscr{F} \otimes \bigwedge \mathscr{G} \longrightarrow \bigwedge (\mathscr{F} \oplus \mathscr{G})$$
$$(f_1 \dot{\wedge} \cdots \dot{\wedge} f_m) \dot{\otimes} (g_1 \dot{\wedge} \cdots \dot{\wedge} g_n) \mapsto (f_1, 0) \dot{\wedge} \cdots \dot{\wedge} (f_m, 0) \dot{\wedge} (0, g_1) \dot{\wedge} \cdots \dot{\wedge} (0, g_n)$$

In particular for $d \geq 0$ there is a canonical isomorphism of sheaves of modules

$$\bigoplus_{m+n=d} \bigwedge^m \mathscr{F} \otimes \bigwedge^n \mathscr{G} \longrightarrow \bigwedge^d (\mathscr{F} \oplus \mathscr{G})$$

Proof. By the tensor product $\bigwedge \mathscr{F} \otimes \bigwedge \mathscr{G}$ we mean the super $\mathcal{O}_X$-algebra of (SOA,Proposition 52). By Proposition 21 the functor $\bigwedge(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{sAlg}(X)$ has a right adjoint and therefore preserves all colimits. It follows that the morphisms $\bigwedge \mathscr{F} \longrightarrow \bigwedge(\mathscr{F} \oplus \mathscr{G})$ and $\bigwedge \mathscr{G} \longrightarrow \bigwedge(\mathscr{F} \oplus \mathscr{G})$ are a coproduct in the category $\mathfrak{sAlg}(X)$. But the super $\mathcal{O}_X$-algebra $\bigwedge \mathscr{F} \otimes \bigwedge \mathscr{G}$ is also a coproduct by (SOA,Proposition 52), so there is a canonical isomorphism of super $\mathcal{O}_X$-algebras $\bigwedge \mathscr{F} \otimes \bigwedge \mathscr{G} \longrightarrow \bigwedge(\mathscr{F} \oplus \mathscr{G})$, as claimed. The second statement follows immediately from (MRS,Lemma 101). $\square$

Corollary 35. *Let X be a scheme and $\mathscr{L}$ an invertible sheaf of modules on X. Then for $d \geq 1$ there is a canonical isomorphism of sheaves of modules*

$$\delta : \bigwedge^d \mathscr{L}^d \longrightarrow \mathscr{L}^{\otimes d}$$

Proof. Here $\mathscr{L}^d$ denotes the coproduct of d copies of $\mathscr{L}$. We can assume X is nonempty, so we have canonical isomorphisms $\bigwedge^0 \mathscr{L} \cong \mathcal{O}_X, \bigwedge^1 \mathscr{L} \cong \mathscr{L}$ and $\bigwedge^n \mathscr{L} = 0$ for $n > 1$ by Corollary 28. For $d \geq 1$ the sheaf $\mathscr{L}^d$ is locally free of rank d, so again by Corollary 28 we have $\bigwedge^n \mathscr{L}^d = 0$ for $n > d$. We construct the isomorphism δ recursively. For $d = 1$ we take the canonical isomorphism $\delta_1 : \bigwedge^1 \mathscr{L} \cong \mathscr{L}$. Suppose that the isomorphism δ_i has been constructed for $1 \leq i < d$. Using Proposition 34 and δ_{d-1} we have a canonical isomorphism

$$\begin{aligned} \bigwedge^d \mathscr{L}^d \cong \bigwedge^d (\mathscr{L} \oplus \mathscr{L}^{d-1}) &\cong \Big(\bigwedge^1 \mathscr{L} \otimes \bigwedge^{d-1} \mathscr{L}^{d-1}\Big) \oplus \Big(\bigwedge^0 \mathscr{L} \otimes \bigwedge^d \mathscr{L}^{d-1}\Big) \\ &\cong \mathscr{L} \otimes \mathscr{L}^{\otimes(d-1)} = \mathscr{L}^{\otimes d} \end{aligned}$$

as required. $\square$

5 Sheaves of Polynomial Algebras

Let $\mathscr{F}$ be a sheaf of modules on X, and let $n \geq 1$ be an integer. We define a presheaf P on X as follows: For an open subset $U \subseteq X$ we define

$$P(U) = \mathscr{F}(U)[x_1, \ldots, x_n]$$

That is, $P(U)$ is the graded $\mathcal{O}_X(U)$-module of all polynomials in n variables with coefficients from $\mathscr{F}(U)$. The homogenous polynomials of degree m give the mth graded piece. For open subsets $V \subseteq U$ the morphism of modules $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ induces $P(U) \longrightarrow P(V)$, and with these morphisms it is clear that P is a presheaf of graded modules on X. Restriction acts by restriction on the coefficients:

$$(f + f_1 x_1 + \cdots + f_n x_n + \cdots)|_V = f|_V + (f_1|_V)x_1 + \cdots + (f_n|_V)x_n + \cdots$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, and Q is the presheaf of polynomial modules for $\mathscr{G}$, then we get a morphism of presheaves of graded modules

$$\varphi : P \longrightarrow Q$$
$$\varphi_U : \mathscr{F}(U)[x_1, \ldots, x_n] \longrightarrow \mathscr{G}(U)[x_1, \ldots, x_n]$$
$$f x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \phi_U(f) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

There is a morphism of presheaves of modules $\mathscr{F} \longrightarrow P$ given pointwise by the canonical injection $\mathscr{F}(U) \longrightarrow \mathscr{F}(U)[x_1, \ldots, x_n]$.

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. For $n \geq 1$ the *polynomial module with coefficients in* $\mathscr{F}$, denoted $\mathscr{F}[x_1, \ldots, x_n]$ is the sheaf of graded modules on X given by the sheafification of the presheaf $P(U) = \mathscr{F}(U)[x_1, \ldots, x_n]$ above. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules, there is an induced morphism of sheaves of graded modules $\mathscr{F}[x_1, \ldots, x_n] \longrightarrow \mathscr{G}[x_1, \ldots, x_n]$ given by the sheafification of φ above. This defines a functor

$$(-)[x_1, \ldots, x_n] : \mathfrak{Mod}(X) \longrightarrow \mathfrak{GrMod}(X)$$

The composite $\mathscr{F} \longrightarrow P \longrightarrow \mathscr{F}[x_1, \ldots, x_n]$ gives a monomorphism of sheaves of modules $\eta : \mathscr{F} \longrightarrow \mathscr{F}[x_1, \ldots, x_n]$ which is clearly natural in $\mathscr{F}$. The image of this morphism is clearly $\mathscr{F}[x_1, \ldots, x_n]_0$.

If $\mathscr{F}$ is a sheaf of algebras on X then $P(U)$ becomes a $\mathcal{O}_X(U)$-algebra via the ring morphism

$$\mathcal{O}_X(U) \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{F}(U)[x_1, \ldots, x_n]$$

and in this way P is a presheaf of graded algebras on X. Therefore $\mathscr{F}[x_1, \ldots, x_n]$ is a sheaf of graded $\mathcal{O}_X$-algebras with the same grading as above. For every open set $U \subseteq X$ we write x_i for the polynomial in $P(U)$ whose only nonzero coefficient is the identity $1 \in \mathscr{F}(U)$ on the monomial x_i. So we have a section $\dot{x_i} \in \mathscr{F}[x_1, \ldots, x_n](U)$ homogenous of degree 1. We use a similar notation for any monomial in $x_1, \ldots, x_n$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of algebras, then $\mathscr{F}[x_1, \ldots, x_n] \longrightarrow \mathscr{G}[x_1, \ldots, x_n]$ is clearly a morphism of sheaves of graded algebras, so we get functors

$$(-)[x_1, \ldots, x_n] : \mathfrak{nAlg}(X) \longrightarrow \mathfrak{GrnAlg}(X)$$
$$(-)[x_1, \ldots, x_n] : \mathfrak{Alg}(X) \longrightarrow \mathfrak{GrAlg}(X)$$

In particular for $n \geq 1$ we have a commutative sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{O}_X[x_1, \ldots, x_n]$ with $\mathcal{O}_X[x_1, \ldots, x_n]_0 = \mathcal{O}_X$.

Proposition 36. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of commutative algebras on X. For $n \geq 1$ there is a bijection natural in $\mathscr{F}$*

$$\beta : Hom_{\mathfrak{Alg}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathscr{F}) \cong \mathscr{F}(X)^n$$
$$\phi \mapsto (\phi_X(\dot{x_1}), \ldots, \phi_X(\dot{x_n}))$$

Proof. It is clear that β is a morphism of abelian groups. Using the fact that sheafification is a left adjoint, we have a bijection

$$Hom_{\mathfrak{Alg}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathscr{F}) \cong Hom_{Alg(X)}(P, \mathscr{F}) \tag{6}$$

where P sheafifies to give $\mathcal{O}_X[x_1, \ldots, x_n]$. Since $P(U) = \mathcal{O}_X(U)[x_1, \ldots, x_n]$ it is not hard to check that $Hom(P, \mathscr{F}) \cong \mathscr{F}(X)^n$ under the map $\psi \mapsto (\psi_X(x_1), \ldots, \psi_X(x_n))$. Together with (6) this gives the desired isomorphism β. Naturality in $\mathscr{F}$ is easily checked. If we are given a tuple $(a_1, \ldots, a_n)$ with $a_i \in \mathscr{F}(X)$ then the corresponding morphism $\mathcal{O}_X[x_1, \ldots, x_n] \longrightarrow \mathscr{F}$ is induced by $P \longrightarrow \mathscr{F}$ defined pointwise by the morphism of $\mathcal{O}_X(U)$-algebras $\mathcal{O}_X(U)[x_1, \ldots, x_n] \longrightarrow \mathscr{F}(U)$ corresponding to $(a_1|_U, \ldots, a_n|_U)$. $\square$

Proposition 37. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of graded commutative algebras on X. Then for $n \geq 1$ a morphism of sheaves of algebras $\phi : \mathcal{O}_X[x_1, \ldots, x_n] \longrightarrow \mathscr{F}$ is a morphism of sheaves of graded algebras if and only if $\phi_X(\dot{x_i}) \in \mathscr{F}_1(X)$ for $1 \leq i \leq n$.*

Proof. The condition is clearly necessary. To see it is sufficient, we can reduce to showing that the corresponding morphism $\phi' : P \longrightarrow \mathscr{F}$ of presheaves of algebras preserves grade, which is obvious. $\square$

Corollary 38. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of graded commutative algebras on X. For $n \geq 1$ there is a bijection natural in $\mathscr{F}$*

$$\beta : Hom_{\mathfrak{GrAlg}(X)}(\mathcal{O}_X[x_1, \ldots, x_n], \mathscr{F}) \cong \mathscr{F}_1(X)^n$$
$$\phi \mapsto (\phi_X(\dot{x_1}), \ldots, \phi_X(\dot{x_n}))$$

Proposition 39. *If $U \subseteq X$ is open and $n \geq 1$ then the following diagram commutes up to a canonical natural equivalence*

$$\begin{tikzcd}
\mathfrak{Mod}(X) \arrow[r, "(-)[x_1,\ldots,x_n]"] \arrow[d] & \mathfrak{Mod}(X) \arrow[d] \\
\mathfrak{Mod}(U) \arrow[r, "(-)[x_1,\ldots,x_n]"'] & \mathfrak{Mod}(U)
\end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $\mathscr{F}|_U[x_1, \ldots, x_n] \longrightarrow \mathscr{F}[x_1, \ldots, x_n]|_U$ has the action $a x_1^{\alpha_1} \dot{\cdots} x_n^{\alpha_n} \mapsto a x_1^{\alpha_1} \dot{\cdots} x_n^{\alpha_n}$.

Proof. This is straightforward, using the technique of Proposition 4. $\square$

Proposition 40. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then for $n \geq 1$ the following diagram commutes up to canonical natural equivalence*

$$\begin{tikzcd}
\mathfrak{Mod}(X) \arrow[r, Rightarrow, "f_*"] & \mathfrak{Mod}(Y) \\
\mathfrak{Mod}(X) \arrow[r, Rightarrow, "f_*"'] \arrow[u, "(-)[x_1,\ldots,x_n]"] & \mathfrak{Mod}(Y) \arrow[u, "(-)[x_1,\ldots,x_n]"']
\end{tikzcd}$$

For a sheaf of modules $\mathscr{F}$ on X the natural isomorphism $f_*(\mathscr{F}[x_1, \ldots, x_n]) \cong (f_*\mathscr{F})[x_1, \ldots, x_n]$ has the action $a x_1^{\alpha_1} \dot{\cdots} x_n^{\alpha_n} \mapsto a x_1^{\alpha_1} \dot{\cdots} x_n^{\alpha_n}$.

Proof. Once again, straightforward using the technique of Proposition 5. $\square$

Lemma 41. *Let A be a commutative ring, $\mathfrak{p}$ a prime ideal of A and M an A-module. For $n \geq 1$ there is a canonical isomorphism of $A_{\mathfrak{p}}$-modules $\phi_{\mathfrak{p}} : M[x_1, \ldots, x_n]_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}[x_1, \ldots, x_n]$ defined by $\phi_{\mathfrak{p}}(f/s)(\alpha) = f(\alpha)/s$.*

Lemma 42. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules on X. For $n \geq 1$ and $y \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,y}$-modules natural in $\mathscr{F}$*

$$\gamma : \mathscr{F}[x_1, \ldots, x_n]_y \longrightarrow \mathscr{F}_y[x_1, \ldots, x_n]$$
$$(U, a x_1^{\alpha_1} \dot{\cdots} x_n^{\alpha_n}) \mapsto (U, a) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

Proof. Let P be the presheaf $P(U) = \mathscr{F}(U)[x_1, \ldots, x_n]$. It is not hard to check that the following is a well-defined isomorphism of $\mathcal{O}_{X,y}$-modules

$$\tau : P_y \longrightarrow \mathscr{F}_y[x_1, \ldots, x_n]$$
$$\tau(U, f)(\alpha) = (U, f(\alpha))$$

We define γ to be the composite of the canonical isomorphism $\mathscr{F}[x_1, \ldots, x_n]_y \cong P_y$ with τ. Naturality in $\mathscr{F}$ is easily checked. $\square$

Proposition 43. *Let $X = SpecA$ be an affine scheme and M an A-module. Then for $n \geq 1$ there is a canonical isomorphism of sheaves of modules natural in M*

$$\psi : \widetilde{M}[x_1, \ldots, x_n] \longrightarrow M[x_1, \ldots, x_n]^{\sim}$$
$$(\dot{m/s})x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \dot{mx_1^{\alpha_1} \cdots x_n^{\alpha_n}/s}$$

Proof. For open $U \subseteq X$ and $\mathfrak{p} \in U$ and a polynomial $f \in \widetilde{M}(U)[x_1, \ldots, x_n]$ let $f(\mathfrak{p})$ denote the polynomial $\alpha \mapsto f(\alpha)(\mathfrak{p})$ of $M_{\mathfrak{p}}[x_1, \ldots, x_n]$. Let P be the presheaf $P(U) = \widetilde{M}(U)[x_1, \ldots, x_n]$ which sheafifies to give $\widetilde{M}[x_1, \ldots, x_n]$. We have a morphism of presheaves of modules

$$\psi' : P \longrightarrow M[x_1, \ldots, x_n]^{\sim}$$
$$\psi'_U(f)(\mathfrak{p}) = \phi_{\mathfrak{p}}^{-1}(f(\mathfrak{p}))$$

where $\phi_{\mathfrak{p}}$ is the isomorphism defined in Lemma 41. To see that $\psi'_U(f)$ is a well-defined section, take the nonzero coefficients of f and find an open neighborhood of $\mathfrak{p}$ small enough so all these coefficients are of the form $\dot{m/s}$. Then on that neighborhood $\psi'_U(f)$ will be of the form $\dot{g/t}$ for a polynomial $g \in M[x_1, \ldots, x_n]$. For any prime $\mathfrak{p}$ there is a commutative diagram

$$\begin{tikzcd}
P_{\mathfrak{p}} \arrow[rr, "\psi'_{\mathfrak{p}}"] \arrow[d, Rightarrow] & & M[x_1, \ldots, x_n]^{\sim}_{\mathfrak{p}} \arrow[d, Rightarrow] \\
\widetilde{M}_{\mathfrak{p}}[x_1, \ldots, x_n] \arrow[r, Rightarrow] & M_{\mathfrak{p}}[x_1, \ldots, x_n] \arrow[r, Rightarrow] & M[x_1, \ldots, x_n]_{\mathfrak{p}}
\end{tikzcd}$$

Therefore $\psi'_{\mathfrak{p}}$ is an isomorphism, and hence so is the morphism of sheaves of modules $\psi : \widetilde{M}[x_1, \ldots, x_n] \longrightarrow M[x_1, \ldots, x_n]^{\sim}$ induced by ψ'. Naturality in M is easily checked. $\square$

Corollary 44. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent, then the same is true of $\mathscr{F}[x_1, \ldots, x_n]$ for any $n \geq 1$.*

Proof. For $x \in X$ let U be an affine open neighborhood of x and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism. Then $f_*\mathscr{F}|_U \cong \mathscr{F}(U)^{\sim}$and combining Proposition 39, Proposition 40 and Proposition 43 we see that

$$\begin{aligned}
f_*(\mathscr{F}[x_1, \ldots, x_n]|_U) &\cong f_*(\mathscr{F}|_U[x_1, \ldots, x_n]) \\
&\cong (f_*\mathscr{F}|_U)[x_1, \ldots, x_n] \\
&\cong \widetilde{\mathscr{F}(U)}[x_1, \ldots, x_n] \\
&\cong \mathscr{F}(U)[x_1, \ldots, x_n]^{\sim}
\end{aligned}$$

This shows that $\mathscr{F}[x_1, \ldots, x_n]$ is a quasi-coherent sheaf of modules, as required. $\square$

Proposition 45. *Let X be a scheme and $\mathscr{F}$ quasi-coherent sheaf of modules on X. For $n \geq 1$ and affine open $U \subseteq X$ there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-modules natural in $\mathscr{F}$ and the affine open set U*

$$\tau : \mathscr{F}(U)[x_1, \ldots, x_n] \cong \mathscr{F}[x_1, \ldots, x_n](U)$$
$$ax_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \dot{ax_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

If $\mathscr{F}$ is a sheaf of algebras, this is an isomorphism of graded $\mathcal{O}_X(U)$-algebras.

Proof. We make $\mathscr{F}[x_1, \ldots, x_n](U)$ into a graded $\mathcal{O}_X(U)$-module as in (SOA,Proposition 40). Using Proposition (H,5.1(d)) and Corollary 44 we get an isomorphism $\mathcal{O}_X(U)$-modules τ, which is easily seen to preserve grade. If $\mathscr{F}$ is a sheaf of algebras, then τ is clearly a morphism of sheaves of algebras. Note that τ is actually the sheafification morphism $P \longrightarrow \mathscr{F}[x_1, \ldots, x_n]$ evaluated at U, from which it follows that τ is natural in $\mathscr{F}$ and inclusions of open affines $V \subseteq U$. $\square$

Corollary 46. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of commutative algebras on X. Then for $n \geq 1$, $\mathscr{F}[x_1, \ldots, x_n]$ is locally finitely generated by $\mathscr{F}[x_1, \ldots, x_n]_1$ as an $\mathscr{F}[x_1, \ldots, x_n]_0$-algebra.*

Proof. This follows immediately from Proposition 45. $\square$

Lemma 47. *Let X be a nonempty scheme and $\mathscr{F}$ a free $\mathcal{O}_X$-module of rank $n \geq 1$. For any basis $f_1, \ldots, f_n \in \mathscr{F}(X)$ there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras*

$$\begin{aligned} \beta : \mathcal{O}_X[x_1, \ldots, x_n] &\longrightarrow \mathbb{S}(\mathscr{F}) \\ \dot{x}_i &\mapsto f_i \end{aligned}$$

Proof. By a "basis" we mean a coproduct $\{\phi_i : \mathcal{O}_X \longrightarrow \mathscr{F}\}_{1 \leq i \leq n}$, but since morphisms $\mathcal{O}_X \longrightarrow \mathscr{F}$ correspond to global sections of $\mathscr{F}$, there is no harm in calling the elements $(\phi_i)_X(1)$ a basis. We know such a morphism of sheaves of graded $\mathcal{O}_X$-algebras β exists by Corollary 38. Since $\mathscr{F}$ is quasi-coherent, we reduce by (MOS,Lemma 2) to showing that β_U is a bijection for every affine open $U \subseteq X$, and this follows at once from (TES,Lemma 34), Proposition 45, and Proposition 17. $\square$

Lemma 48. *If X is a scheme then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras*

$$\begin{aligned} \beta : \mathcal{O}_X[x] &\longrightarrow \mathbb{T}(\mathcal{O}_X) \\ \dot{x} &\mapsto 1 \in \mathbb{T}^1(\mathcal{O}_X)(X) \end{aligned}$$

Proof. First of all, we know from Lemma 9 that $\mathbb{T}(\mathcal{O}_X)$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. So such a morphism of sheaves of graded $\mathcal{O}_X$-algebras β exists by Corollary 38. We reduce by (MOS,Lemma 2) to showing that β_U is a bijection for every affine open $U \subseteq X$, and this follows at once from (TES,Lemma 8), Proposition 45, and Proposition 8. $\square$

6 Sheaves of Ideal Products

Suppose for every $n \geq 0$ we have a sheaf of ideals $\mathscr{J}_n$ on X satisfying the following conditions

$$\mathscr{J}_0 = \mathcal{O}_X \tag{7}$$

$$\mathscr{J}_n \subseteq \mathscr{J}_m \qquad m \leq n \tag{8}$$

$$\mathscr{J}_n \mathscr{J}_m \subseteq \mathscr{J}_{m+n} \qquad m, n \geq 0 \tag{9}$$

By $\mathscr{J}_n \mathscr{J}_m \subseteq \mathscr{J}_{n+m}$ we mean that the ideal product $\mathscr{J}_n \mathscr{J}_m$, which is a sheaf of ideals on X, is contained in $\mathscr{J}_{m+n}$. This is equivalent to having $\mathscr{J}_n(U)\mathscr{J}_m(U) \subseteq \mathscr{J}_{m+n}(U)$ as ideals in $\mathcal{O}_X(U)$ for every open subset $U \subseteq X$. Let P be the following presheaf of modules on $\mathcal{O}_X$ (coproduct of presheaves of modules)

$$P = \bigoplus_{n \geq 0} \mathscr{J}_n = \mathcal{O}_X \oplus \mathscr{J}_1 \oplus \mathscr{J}_2 \oplus \cdots$$

For open $U \subseteq X$ define a product on the $\mathcal{O}_X(U)$-module $P(U)$ by

$$(xy)_i = \sum_{d+e=i} x_d y_e$$

It is easy to check this is a commutative graded $\mathcal{O}_X(U)$-algebra with identity $(1, 0, \ldots)$ and graded piece $\mathscr{J}_n(U)$ in degree $n \geq 0$. As usual we identify elements of $\mathscr{J}_n(U)$ with sequences in $P(U)$ with only one nonzero entry. In that case multiplication in $P(U)$ is just multiplication in $\mathcal{O}_X(U)$ where you have to keep track of the grade. With component-wise restriction it is clear that P is a presheaf of $\mathcal{O}_X(U)$-algebras. Therefore the sheafification $\mathscr{P}$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras, where the submodule $\mathscr{P}_n$ for $n \geq 0$ is the image of the canonical morphism

$\mathscr{J}_n \longrightarrow P \longrightarrow \mathscr{P}$. In fact these morphisms $\mathscr{J}_n \longrightarrow \mathscr{P}$ are the canonical coproduct of sheaves of modules

$$\mathscr{P} = \bigoplus_{n \geq 0} \mathscr{J}_n$$

As usual, given $a \in \mathscr{J}_n(U)$ for $n \geq 0$ and some open set $U \subseteq U$ we simply write a for the corresponding element of $\mathscr{P}(U)$ which is the image under $P \longrightarrow \mathscr{P}$ of the sequence in $P(U)$ with a single nonzero entry given by a in the nth place. Equivalently this is the image of a under $\mathscr{J}_n \longrightarrow \mathscr{P}$.

Definition 3. Let $(X, \mathcal{O}_X)$ be a ringed space, $\{\mathscr{J}_n\}_{n\geq 0}$ a collection of sheaves of ideals satisfying (7), (8), (9). Then the coproduct of sheaves of modules $\bigoplus_{n\geq 0} \mathscr{J}_n$ becomes a sheaf of commutative graded $\mathcal{O}_X$-algebras in a canonical way. In particular if $\mathscr{J}$ is a sheaf of ideals then we can set $\mathscr{J}_n = \mathscr{J}^n$ for $n \geq 1$ (the n-fold product), and in this case we write $\mathbb{B}(\mathscr{J})$ for $\bigoplus_{n\geq 0} \mathscr{J}_n$ and $\mathbb{B}^n(\mathscr{J})$ for the submodule of degree n for $n \geq 0$. Note that $\mathbb{B}^0(\mathscr{J}) = \mathcal{O}_X$.

If $\mathscr{J} \subseteq \mathscr{K}$ then there is a canonical morphism of graded $\mathcal{O}_X$-algebras $\mathbb{B}(\mathscr{J}) \longrightarrow \mathbb{B}(\mathscr{K})$ which is the sheafification of the morphism given component-wise by the inclusion $\mathscr{J} \longrightarrow \mathscr{K}$. If $\mathscr{K} = \mathscr{J}$ this is the identity, and if $\mathscr{J} \subseteq \mathscr{K} \subseteq \mathscr{L}$ then the composite $\mathbb{B}(\mathscr{J}) \longrightarrow \mathbb{B}(\mathscr{K}) \longrightarrow \mathbb{B}(\mathscr{L})$ is just $\mathbb{B}(\mathscr{J}) \longrightarrow \mathbb{B}(\mathscr{L})$.

Proposition 49. *If $\mathscr{J}$ is a sheaf of ideals and $q \in \mathbb{B}(\mathscr{J})(U)$ then for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $q|_V = q_1 + \cdots + q_s$ where each q_k has the form*

$$q_k = r + a_{11} + a_{21}a_{22} + \cdots + a_{h1} \cdots a_{hh}$$

where $r \in \mathcal{O}_X(V)$ and $a_{ij} \in \mathscr{J}(V)$. In the sum above, an n-fold product is given grade n for $n \geq 1$.

Proposition 50. *If $U \subseteq X$ is open and $\mathscr{J}$ is a sheaf of ideals on X then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras natural in $\mathscr{J}$*

$$\begin{gathered} \mathbb{B}(\mathscr{J}|_U) \longrightarrow \mathbb{B}(\mathscr{J})|_U \\ a_1 \cdots a_n \mapsto a_1 \cdots a_n \end{gathered}$$

Proposition 51. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces with inverse h and let $\mathscr{J}$ be a sheaf of ideals on X. Then there is a canonical isomorphism of sheaves of $\mathcal{O}_Y$-algebras*

$$\begin{gathered} \rho : f_*\mathbb{B}(\mathscr{J}) \longrightarrow \mathbb{B}(\mathscr{J} \cdot \mathcal{O}_Y) \\ a_1 \cdots a_n \mapsto h^\#_U(a_1) \cdots h^\#_U(a_n) \end{gathered}$$

where $U \subseteq X$ is open and $a_i \in \mathscr{J}(U)$.

Proof. Here $\mathscr{J} \cdot \mathcal{O}_Y$ denotes the sheaf of ideals on Y corresponding to $\mathscr{J}$ under (MRS,Lemma 49). Let P be the presheaf of algebras on X sheafifying to give $\mathbb{B}(\mathscr{J})$ and let Q sheafify to give $\mathbb{B}(\mathscr{J} \cdot \mathcal{O}_Y)$. Then we have isomorphisms of presheaves of $\mathcal{O}_Y$-modules (using (MRS,Proposition 52))

$$\begin{aligned} f_*P &= f_*\mathcal{O}_X \oplus f_*\mathscr{J} \oplus f_*(\mathscr{J}^2) \oplus \cdots \\ &\cong \mathcal{O}_Y \oplus (\mathscr{J} \cdot \mathcal{O}_Y) \oplus (\mathscr{J}^2 \cdot \mathcal{O}_Y) \oplus \cdots \\ &\cong \mathcal{O}_Y \oplus (\mathscr{J} \cdot \mathcal{O}_Y) \oplus (\mathscr{J} \cdot \mathcal{O}_Y)^2 \oplus \cdots \\ &= Q \end{aligned}$$

In fact this is an isomorphism of presheaves of algebras, and together with the canonical isomorphism of sheaves of algebras $\mathbf{a}(f_*P) \cong f_*(\mathbf{a}P)$ this gives our isomorphism ρ of sheaves of algebras. $\square$

If A is a ring with ideal $\mathfrak{a}$ then we can define a commutative graded A-algebra

$$B(\mathfrak{a}) = \bigoplus_{n \geq 0} \mathfrak{a}^n = A \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots$$

with the product $(xy)_i = \sum_{d+e=i} x_d y_e$. If $\mathfrak{b} \subseteq \mathfrak{a}$ are ideals then the inclusions give a morphism of graded A-algebras $B(\mathfrak{b}) \longrightarrow B(\mathfrak{a})$. If A is an integral domain then clearly so is $B(\mathfrak{a})$.

Proposition 52. *Let $X = SpecA$ be an affine scheme and $\mathfrak{a} \subseteq A$ an ideal. Then there is a canonical isomorphism of sheaves of $\mathcal{O}_X$-algebras*

$$\begin{aligned} \psi : \mathbb{B}(\widetilde{\mathfrak{a}}) &\longrightarrow B(\mathfrak{a})^\sim \\ \dot{a_1/s_1} \cdots \dot{a_n/s_n} &\mapsto a_1 \cdots a_n / s_1 \cdots s_n \end{aligned}$$

Proof. Since the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ preserves all colimits, there is an isomorphism of sheaves of modules

$$\begin{aligned} \mathbb{B}(\widetilde{\mathfrak{a}}) &= \mathcal{O}_X \oplus \widetilde{\mathfrak{a}} \oplus \widetilde{\mathfrak{a}}^2 \oplus \cdots \\ &= \mathcal{O}_X \oplus \widetilde{\mathfrak{a}} \oplus \widetilde{\mathfrak{a}^2} \oplus \cdots \\ &\cong B(\mathfrak{a})^\sim \end{aligned}$$

To check it is a morphism of sheaves of algebras we can reduce to the case of sections of the special form given in the statement, which is easy. □

Proposition 53. *Let X be a scheme and $\mathscr{J}$ a sheaf of ideals on X. If $\mathscr{J}$ is quasi-coherent then so is $\mathbb{B}(\mathscr{J})$.*

Proof. This follows immediately from (MOS,Corollary 12) and (MOS,Proposition 25). □

Proposition 54. *Let X be a scheme and $\mathscr{J}$ a sheaf of ideals on X. If $\mathscr{J}$ is quasi-coherent and $U \subseteq X$ is affine then there is a canonical isomorphism of graded $\mathcal{O}_X(U)$-algebras natural in $\mathscr{J}$ and the affine open subset U*

$$\begin{aligned} \tau : B(\mathscr{J}(U)) &\longrightarrow \mathbb{B}(\mathscr{J})(U) \\ a_1 \cdots a_n &\mapsto \dot{a_1 \cdots a_n} \end{aligned}$$

Proof. We make $\mathbb{B}(\mathscr{J})(U)$ into a graded $\mathcal{O}_X(U)$-algebra as in (SOA,Proposition 40). Let P be the presheaf $P(U) = B(\mathscr{J}(U))$ of $\mathcal{O}_X$-algebras, which sheafifies to give $\mathbb{B}(\mathscr{J})$ by (MOS,Proposition 13), and let τ be the canonical morphism $P \longrightarrow \mathbb{B}(\mathscr{J})$ evaluated at U. This is a morphism of graded $\mathcal{O}_X(U)$-algebras, and it follows from (MOS,Lemma 6) that τ is an isomorphism. □

Corollary 55. *Let X be a scheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X. Then $\mathbb{B}(\mathscr{J})$ is locally generated by $\mathbb{B}(\mathscr{J})_1$ as a $\mathbb{B}(\mathscr{J})_0$-algebra (locally finitely generated if $\mathscr{J}$ is coherent).*

Proof. Using Proposition 54 we reduce immediately to showing that for a commutative ring A and ideal $\mathfrak{a}$ the graded A-algebra $B(\mathfrak{a})$ is generated by $B(\mathfrak{a})_1$ as an A-algebra, with $B(\mathfrak{a})$ finitely generated by $B(\mathfrak{a})_1$ if $\mathfrak{a}$ is finitely generated. This is easy enough to check. □

Proposition 56. *If A is a ring then there is a canonical isomorphism of graded A-algebras*

$$\begin{aligned} \beta : A[x] &\longrightarrow B(A) \\ x &\mapsto (0, 1, 0, \ldots) \end{aligned}$$

Corollary 57. *If X is a scheme then there is a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras*

$$\begin{aligned} \beta : \mathcal{O}_X[x] &\longrightarrow \mathbb{B}(\mathcal{O}_X) \\ \dot{x} &\mapsto 1 \in \mathbb{B}^1(\mathcal{O}_X)(X) \end{aligned}$$

Proof. We know from Proposition 53 that $\mathbb{B}(\mathcal{O}_X)$ is a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. So such a morphism of sheaves of graded $\mathcal{O}_X$-algebras β exists by Corollary 38. We reduce by (MOS,Lemma 2) to showing that β_U is a bijection for every affine open $U \subseteq X$, and this follows from Proposition 56, Proposition 45 and Proposition 54. $\square$

In summary we have the following diagram of functors. Pairs of functors going in opposite directions are adjoint pairs, with the left adjoint on the left.