Sheaves of Groups and Rings

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May 16, 2006

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1 Sheaves of Sets

Definition 1. Let X be a topological space and $U \subseteq X$ an open subset. An *open cover* of U is a set $\mathcal{S}$ of open subsets of U, with the property that the union set of $\mathcal{S}$ is U. Note that the empty set $\mathcal{S} = \emptyset$ is an open cover of $\emptyset$, as is the set $\mathcal{S} = \{\emptyset\}$. If we want to exclude the trivial case where $\mathcal{S}$ is empty, we refer to a *nonempty open cover*.

Let $\mathcal{S}$ be an open cover of U and P a presheaf of sets on X. A *matching family* for the cover $\mathcal{S}$ and presheaf P is a function $f : \mathcal{S} \longrightarrow \coprod_{U \in \mathcal{S}} P(U)$ with $f(U) \in P(U)$ for each $U \in \mathcal{S}$, with the property that $f(U)|_{U \cap V} = f(V)|_{U \cap V}$ for every pair $U, V \in \mathcal{S}$. Observe that if $\mathcal{S}$ is empty then there is *precisely* one matching family for $\mathcal{S}$ and P, given by the empty function $f : \emptyset \longrightarrow \emptyset$. Also notice that a matching family *cannot* contain two sections over the same open set. An *amalgamation* of such a matching family is an element $s \in P(U)$ with $s|_U = f(U)$ for every $U \in \mathcal{S}$.

Definition 2. Let X be a topological space and $U \subseteq X$ an open subset. An *indexed open cover* of U is an index set I (possibly empty) together with a function $c : I \longrightarrow \{V \,|\, V \subseteq U \text{ is open}\}$ with the property that the union set of $Im(c)$ is U. Note that the empty set I together with the unique function $c : \emptyset \longrightarrow \{V \,|\, V \subseteq U \text{ is open}\}$ is an open cover of $\emptyset$. Any open cover $\mathcal{S}$ becomes an indexed open cover in the obvious way.

Let (I, c) be an indexed open cover of U and P a presheaf of sets on X. An *indexed matching family* for the cover (I, c) and presheaf P is a function $f : I \longrightarrow \coprod_{i \in I} P(c(i))$ with $f(i) \in P(U_i)$ for each $i \in I$, with the property that $f(i)|_{c(i) \cap c(j)} = f(j)|_{c(i) \cap c(j)}$ for every pair $i, j \in I$. Observe that if I is empty then there is *precisely* one matching family for (I, c) and P, given by the empty function $f : \emptyset \longrightarrow \emptyset$. Notice that *indexed* matching families may contain multiple sections over the same open set. An *amalgamation* of such a matching family is an element $s \in P(U)$ with $s|_{c(i)} = f(i)$ for every $i \in I$.

Definition 3. Let X be a topological space, and let $\mathcal{O}(X)$ denote the category obtained from the partially ordered set of open sets of X. A *presheaf of sets* on X is a contravariant functor $P : \mathcal{O}(X) \longrightarrow$ **Sets**, and a *morphism* of presheaves of sets is a natural transformation. This defines the category $P(X)$ of presheaves of sets on X.

A presheaf P is a *sheaf* if for any open set U and open cover $\mathcal{S}$ of U there is a *unique* amalgamation of every matching family for the cover $\mathcal{S}$ and presheaf P. A *morphism* of sheaves is a morphism of presheaves, and this defines the category $Sh(X)$ of sheaves of sets on X.

Remark 1. Let X, P be as in Definition 3. Then P is a sheaf if and only if for every open set U and *indexed* open cover (I, c) of U there is a unique amalgamation of every *indexed* matching family for the cover (I, c) and presheaf P.

Remark 2. In practice, the following characterisation is more convenient. A presheaf P of sets on X is a sheaf if and only if

(i) $P(\emptyset)$ is a singleton.

(ii) For every nonempty open set U and open cover $\{V_i\}_{i\in I}$ by nonempty open sets V_i together with matching sections $s_i \in P(V_i)$, then there is a unique amalgamation $s \in P(V)$.

Remark 3. Since $P(X)$ is a functor category and **Sets** is complete and cocomplete, the category $P(X)$ is complete and cocomplete. Limits and colimits are computed pointwise.

Remark 4. Throughout our notes, when we say "pointwise" we mean "for every open set" (the notation comes from category theory, where we view open sets as objects of the category $\mathcal{O}(X)$). In particular, "pointwise" does *not* mean "for the stalk at every point".

Remark 5. Let X be a topological space and P a presheaf of sets on X. If $U, V \subseteq X$ are open and $s \in P(U), t \in P(V)$ then the set $\{x \in U \cap V \mid germ_x t = germ_x s\}$ is clearly open.

Definition 4. Let X be a topological space and P a presheaf of sets on X. For each $x \in X$ let P_x be the usual stalk and define $\Lambda_P = \coprod_{x\in X} P_x$ (this is the usual disjoint union). Given an open set $U \subseteq X$ we say a function $t : U \longrightarrow \Lambda_P$ is *regular* if satisfies the following two conditions

(i) $t(x) \in P_x$ for every $x \in U$.

(ii) For each $x \in U$ there is an open neighborhood $x \in V \subseteq U$ and $s \in P(V)$ such that $t(y) = germ_y s$ for each $y \in V$.

This says that t is *locally* an element of P. Let $\mathbf{a}P(V)$ denote the set of all regular functions $V \longrightarrow \Lambda_P$. The restriction of a regular function is regular, and $\mathbf{a}P(V)$ is a sheaf of sets. Let $\phi : P \longrightarrow Q$ be a morphism of presheaves of sets. Then we define a morphism of sheaves of sets

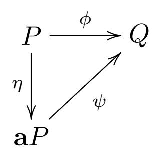
$$\mathbf{a}(\phi) : \mathbf{a}P \longrightarrow \mathbf{a}Q$$
$$\mathbf{a}(\phi)_U(t)(x) = \phi_x(t(x))$$

This defines the functor $\mathbf{a}(-) : P(X) \longrightarrow Sh(X)$, called the *sheafification functor*. For any presheaf P there is a canonical morphism of presheaves of sets natural in P

$$\eta : P \longrightarrow \mathbf{a}P$$
$$\eta_U(s)(x) = germ_x s$$

Throughout our notes, we denote the section $\eta_U(s) \in \Gamma(U, \mathbf{a}P)$ by $\dot{s}$.

Proposition 1. *Let $\phi : P \longrightarrow Q$ be a morphism of presheaves of sets on a topological space X, where Q is a sheaf. Then there is a* unique *morphism of presheaves of sets $\psi : \mathbf{a}P \longrightarrow Q$ making the following diagram commute*



Proof. Let an open set $U \subseteq X$ and $t \in \mathbf{a}P(U)$ be given. We can find a nonempty open cover $\{W_i\}_{i\in I}$ of U together with $t_i \in P(W_i)$ such that $t|_{W_i} = \dot{t}_i$ for each $i \in I$. Then the sections $\phi_{W_i}(t_i)$ form a matching family for Q, with a unique amalgamation $t' \in Q(U)$. If we define $\phi_U(t) = t'$ then t' is the unique element of $Q(U)$ with $germ_x t' = \phi_x(t(x))$ for every $x \in U$ (so t' does not depend on the choice of cover) and this defines a morphism of sheaves of sets ψ with the necessary property. □

Corollary 2. *Let X be a topological space. Then we have an adjoint pair*

$$Sh(X) \underset{\mathbf{i}}{\overset{\mathbf{a}}{\leftrightarrows}} P(X) \qquad \mathbf{a} \dashv \mathbf{i} \tag{1}$$

where $\mathbf{i} : Sh(X) \longrightarrow P(X)$ is the inclusion. The functor $\mathbf{a}$ preserves finite limits and monomorphisms, so $Sh(X)$ is a giraud subcategory of $P(X)$.

Proof. The adjunction is an immediate consequence of Proposition 1. You need to see our topos notes to check that $\mathbf{a}$ preserves finite limits, which shows that $Sh(X)$ is a giraud subcategory of $P(X)$. □

Corollary 3. *A morphism in $Sh(X)$ is a monomorphism in $Sh(X)$ iff. it is a monomorphism in $P(X)$, so iff. it is a pointwise injective map. A morphism in $Sh(X)$ is an isomorphism iff. it is an isomorphism in $P(X)$, so iff. it is pointwise bijective.*

Remark 6. If D is a diagram of sheaves and morphisms then the limit in $P(X)$ is a sheaf and is a limit in $Sh(X)$. So limits in $Sh(X)$ are computed pointwise. Let L be the colimit for D in $P(X)$. Then $\mathbf{a}L$ together with the morphisms $D_i \longrightarrow L \longrightarrow \mathbf{a}L$ are a colimit for D in $Sh(X)$.

1.1 Direct and Inverse Image

Definition 5. Let $f : X \longrightarrow Y$ be a continuous map of spaces and P a sheaf of sets on X. Then define a sheaf of sets f_*P on Y by $(f_*P)(V) = P(f^{-1}V)$ with the obvious restriction maps. If $\phi : P \longrightarrow P'$ is a morphism of sheaves of sets then define a morphism of sheaves of sets

$$f_*\phi : f_*P \longrightarrow f_*P'$$
$$(f_*\phi)_V = \phi_{f^{-1}V}$$

This defines the functor $f_*(-) : Sh(X) \longrightarrow Sh(Y)$, called the *direct image functor*

Definition 6. Let $f : X \longrightarrow Y$ be a continuous map of spaces and Q a sheaf of sets on Y. Define a presheaf of sets Q_X on X by $Q_X(U) = \varinjlim_{V \supseteq f(U)} Q(V)$ and let $f^{-1}Q$ be the sheaf of sets $\mathbf{a}Q_X$. If $\phi : Q \longrightarrow Q'$ is a morphism of sheaves of sets, define a morphism of presheaves of sets

$$\phi_X : Q_X \longrightarrow Q'_X$$
$$(\phi_X)_U(V, t) = (V, \phi_V(t))$$

and let $f^{-1}\phi : f^{-1}Q \longrightarrow f^{-1}Q'$ be $\mathbf{a}\phi_X$. This defines the functor $f^{-1}(-) : Sh(Y) \longrightarrow Sh(X)$, called the *inverse image functor*. Given open sets $U \subseteq X$ and $V \supseteq f(U)$ together with $s \in Q(V)$ we denote the image of the equivalence class of (V, s) in $\Gamma(U, f^{-1}Q)$ by $[V, s]$. In this notation, $(f^{-1}\phi)_U([V, s]) = [V, \phi_V(s)]$.

Proposition 4. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and let Q be a sheaf of sets on Y. Then for $x \in X$ there is a canonical isomorphism natural in Q*

$$\lambda : (f^{-1}Q)_x \longrightarrow Q_{f(x)}$$
$$germ_x[V, s] \mapsto germ_{f(x)}s$$

Proof. Let Q be a sheaf of sets on X. Then $(U, s) \mapsto s(x)$ defines an isomorphism $(f^{-1}Q)_x \longrightarrow (Q_X)_x$. We define

$$\phi : (Q_X)_x \longrightarrow Q_{f(x)}$$
$$(V, (W, t)) \mapsto (W, t)$$

It is not hard to check that this is a well-defined isomorphism. Composed with the isomorphism $(f^{-1}Q)_x \cong (Q_X)_x$ this gives the desired natural isomorphism. □

Proposition 5. *Let $f : X \longrightarrow Y$ be a continuous map of spaces. Then we have an adjoint pair*

$$Sh(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh(Y) \qquad f^{-1} \dashv f_* \tag{2}$$

For sheaves P, Q on X, Y respectively the unit and counit are defined by

$$\eta : Q \longrightarrow f_* f^{-1} Q \qquad\qquad \varepsilon : f^{-1} f_* P \to P$$
$$\eta_V(s) = [V, s] \qquad\qquad \varepsilon_U([V, s]) = s|_U$$

Proof. We already know that η is the unit of the adjunction. Given a morphism of sheaves $\phi : Q \longrightarrow f_* P$ the morphism $\psi : f^{-1}Q \longrightarrow P$ corresponding to ϕ under the adjunction is defined as follows. The morphism ψ is induced by the following morphism of presheaves

$$\psi' : Q_X \longrightarrow P$$
$$\psi'_U(V, s) = \phi_V(s)|_U$$

In particular, we have $\psi_U([V, s]) = \phi_V(s)|_U$. Taking $\phi = 1$ gives the definition of the counit. □

Lemma 6. *Let X be a topological space and $U \subseteq X$ an open subset with inclusion $i : U \longrightarrow X$. For any sheaf of sets P on X there is a canonical isomorphism of sheaves of sets natural in P*

$$\alpha : P|_U \longrightarrow i^{-1} P$$
$$\alpha_V(s) = [V, s]$$

That is, there is a canonical natural equivalence $(-)|_U \cong i^{-1}$.

Proof. See our Section 2.5 notes p.29. □

Definition 7. Let X be a topological space and $U \subseteq X$ an open subset with inclusion $j : U \longrightarrow X$. Let P be a sheaf of sets on U and let P_E be the following presheaf on X

$$P_E(V) = \begin{cases} P(V) & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

with the obvious restriction maps, where 0 denotes the singleton $\{\emptyset\}$. We denote the sheaf of sets $\mathbf{a}P_E$ by $j_!P$ and call it the *extension by zero* of P to X. Let $\phi : P \longrightarrow P'$ be a morphism of sheaves of sets, and define a morphism of presheaves

$$\phi_E : P_E \longrightarrow P'_E$$
$$(\phi_E)_V = \begin{cases} \phi_V & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Let $j_!\phi : j_!P \longrightarrow j_!P'$ be the morphism $\mathbf{a}\phi_E$. This defines the functor $j_!(-) : Sh(U) \longrightarrow Sh(X)$.

Proposition 7. *Let X be a topological space and $U \subseteq X$ an open subset with inclusion $j : U \longrightarrow X$. Then we have an adjoint pair*

$$Sh(X) \underset{j_!}{\overset{(-)|_U}{\rightleftarrows}} Sh(U) \qquad j_! \dashv (-)|_U \tag{3}$$

Proof. Let P be a sheaf of sets on X and F the presheaf $(P|_U)_E$ on X. There is an obvious morphism of presheaves $F \longrightarrow P$ which is the identity for $V \subseteq U$ and zero otherwise. Denote by $\varepsilon : j_!(P|_U) \longrightarrow P$ the induced morphism of sheaves of sets, which is natural in P. If Q is a sheaf of sets on U and $b : j_!Q \longrightarrow P$ a morphism of sheaves of sets, there is a unique morphism of sheaves of sets $c : Q \longrightarrow P|_U$ making the following diagram commute

$$\begin{array}{ccc} j_!Q & \xrightarrow{b} & P \\ {\scriptstyle j_!c}\downarrow & \nearrow{\scriptstyle \varepsilon} & \\ j_!(P|_U) & & \end{array}$$

Therefore ε is the counit of an adjunction $j_!(-) \dashv (-)|_U$, as required. Let Q be a sheaf of sets on U. The unit of this adjunction is the canonical isomorphism $\eta : Q \longrightarrow (j_!Q)|_U$ defined by $\eta_V(s) = \dot{s}$. ☐

Corollary 8. *With the notation of Proposition 7 we have the following facts*

(i) For any sheaf of sets Q on U there is a canonical bijection $Q_x \longrightarrow (j_!Q)_x$ natural in Q for every $x \in U$. For $x \notin U$ we have $(j_!Q)_x = 0$.

(ii) For any sheaf of sets P on X the counit $\varepsilon : j_!(P|_U) \longrightarrow P$ is a monomorphism.

Proof. (i) is a trivial consequence of the fact that the unit $\eta : Q \longrightarrow (j_!Q)|_U$ is an isomorphism. As for (ii), we need only show that ε_x is injective for every $x \in X$. For $x \notin U$ this is trivial since $j_!(P|_U)_x = 0$, and for $x \in U$ the map ε_x is an isomorphism. ☐

2 Abelian groups

Definition 8. Let X be a topological space and $\mathcal{C}$ the small category of open sets of X. In our Algebra in a Category notes we defined *presheaves of abelian groups* and *sheaves of abelian groups*. Let $Ab(X)$ denote the complete grothendieck abelian category of all presheaves of abelian groups on X (ALCAT,Corollary 3) and $\mathfrak{Ab}(X)$ the complete grothendieck abelian category of all sheaves of abelian groups on X (ALCAT,Corollary 4).

Let P be a presheaf of abelian groups on X and $\mathbf{a}P$ the sheafification of P considered as a sheaf of sets. For $x \in X$ the stalk P_x is an abelian group in a canonical way, and by adding sections pointwise $\mathbf{a}P$ becomes a sheaf of abelian groups and $\eta : P \longrightarrow \mathbf{a}P$ a morphism of presheaves of abelian groups. If $\phi : P \longrightarrow Q$ is a morphism of presheaves of abelian groups, then $\mathbf{a}\phi : \mathbf{a}P \longrightarrow \mathbf{a}Q$ is a morphism of sheaves of abelian groups, and this defines an additive functor

$$\mathbf{a} : Ab(X) \longrightarrow \mathfrak{Ab}(X)$$

The natural transformation $\eta : 1 \longrightarrow \mathbf{ia}$ is the unit of an adjunction

$$\mathfrak{Ab}(X) \underset{\mathbf{a}}{\overset{\mathbf{i}}{\rightleftarrows}} Ab(X) \qquad \mathbf{a} \dashv \mathbf{i} \tag{4}$$

where $\mathbf{i}$ is the inclusion.

The subobjects $Sub\mathscr{F}$ of a sheaf $\mathscr{F}$ form a set (here $Sub\mathscr{F}$ consists of equivalence classes of monics with codomain $\mathscr{F}$ under the subobject equivalence relation). A morphism $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a monomorphism in $\mathfrak{Ab}(X)$ iff. ϕ_U is injective for all open U. A *subsheaf* is a monomorphism $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ in $\mathfrak{Ab}(X)$ such that ϕ_U is the inclusion of a subset for all open U. Every equivalence class of subobjects contains precisely one subsheaf. For some more details see our Hartshorne Section 2.1 notes.

Lemma 9. *Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves. Then the subsheaf $Im\phi$ is defined by the following condition: given an open set U and $s \in \mathscr{G}(U)$ we have $s \in (Im\phi)(U)$ if and only if for every point $x \in U$ there is an open set V with $x \in V \subseteq U$ and $s|_V \in Im(\phi_V)$.*

Lemma 10. *Let $\phi : G \longrightarrow \mathscr{F}$ be a monomorphism of presheaves with $G(U) \subseteq \mathscr{F}(U)$ for all U, where $\mathscr{F}$ is a sheaf. Then the induced morphism of sheaves $\mathbf{a}G \longrightarrow \mathscr{F}$ is a monomorphism whose image I is defined by the following condition: given an open set U and $s \in \mathscr{F}(U)$ we have $s \in I(U)$ if and only if for every point $x \in U$ there is an open set V with $x \in V \subseteq U$ and $s|_V \in G(V)$.*

As usual in a grothendieck abelian category, we can define the union of any set $\{\mathscr{G}_i\}_{i\in I}$ of subobjects of a sheaf $\mathscr{F}$. This is denoted by $\sum_i \mathscr{G}_i$. For the empty set this is just the zero sheaf, so let I be nonempty. The subsheaf $\sum_i \mathscr{G}_i$ is the image of the induced morphism $\bigoplus_i \mathscr{G}_i \longrightarrow \mathscr{F}$ (coproduct in $\mathfrak{Ab}(X)$). We can describe the subsheaf $\sum_i \mathscr{G}_i$ explicitly as follows.

Lemma 11. *Let $\{\mathscr{G}_i\}_{i\in I}$ be a nonempty set of subsheaves of a sheaf $\mathscr{F}$. Then the subsheaf $\sum_i \mathscr{G}_i$ is defined by the following condition: given an open set U and $s \in \mathscr{F}(U)$ we have $s \in (\sum_i \mathscr{G}_i)(U)$ if and only if for every point $x \in U$ there exists an open neighborhood V with $x \in V \subseteq U$ and elements $a_{i_1}, \ldots, a_{i_n}$ with $a_{i_k} \in \mathscr{G}_{i_k}(V)$ such that $s|_V = a_{i_1} + \cdots + a_{i_n}$.*

Proof. The easiest way to see this is to take the sum in $Ab(X)$, then since the reflection functor $\mathbf{a}$ is exact you just have to sheafify the image of the morphism $P \longrightarrow \mathscr{F}$ out of the presheaf coproduct P. ☐

We say the sum is *direct* if the induced morphism $\bigoplus_i \mathscr{G}_i \longrightarrow \mathscr{F}$ is a monomorphism. If $U \subseteq X$ is an open subset, restricting sheaves and their morphisms defines an additive functor $-|_U : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(U)$. If $U = X$ this is the identity, and for $W \subseteq U$ it is clear that $\mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(U) \to \mathfrak{Ab}(W)$ is the functor associated to the inclusion $W \subseteq X$.

Since $\mathfrak{Ab}(X)$ is a grothendieck abelian category it follows from Mitchell III 1.2 that if $\mathscr{F}_i$ are a nonempty direct family of subsheaves of a sheaf of abelian groups $\mathscr{F}$ then the induced morphism $\varinjlim \mathscr{F}_i \longrightarrow \mathscr{F}$ is in fact a monomorphism, equal as a subobject to the categorical union $\sum_i \mathscr{F}_i$. In the case where X is a noetherian space we can give a nice characterisation of this submodule.

Lemma 12. *Let $\{\alpha_i : \mathscr{F}_i \longrightarrow \mathscr{F}\}_{i\in I}$ be a nonempty family of subsheaves of a sheaf of abelian groups $\mathscr{F}$. Then the intersection is the following subsheaf*

$$\left(\bigcap_i \mathscr{F}_i\right)(U) = \bigcap_i \mathscr{F}_i(U)$$

If X is a noetherian topological space and the $\mathscr{F}_i$ are a direct family of subsheaves, then the union is the following subsheaf

$$\left(\sum_i \mathscr{F}_i\right)(U) = \bigcup_i \mathscr{F}_i(U)$$

Proof. A subsheaf of $\mathscr{G}$ of $\mathscr{F}$ is determined by the subsets $\mathscr{G}(U) \subseteq \mathscr{F}(U)$, although not all such assignments of subsets arise from a subsheaf. Clearly for any open set U, $\cap_i \mathscr{F}_i(U)$ is a subgroup of $\mathscr{F}(U)$. These subgroups close under restriction and the presheaf of abelian groups $\bigcap_i \mathscr{F}_i$ defined in this way is clearly a sheaf with the property of the intersection.

Now suppose that X is noetherian and that the $\mathscr{F}_i$ are a direct family of subsheaves (that is, for every pair $i, j \in I$ there is k with $\mathscr{F}_i \leq \mathscr{F}_k, \mathscr{F}_j \leq \mathscr{F}_k$) then for every open U, the $\mathscr{F}_i(U)$ form a direct family of subgroups of $\mathscr{F}(U)$. Therefore the union $\cup_i \mathscr{F}_i(U)$ is a subgroup of $\mathscr{F}(U)$. These subgroups close under restriction and to give a presheaf $\sum_i \mathscr{F}_i$ of abelian groups. By (H, Ex II.2.13(a)) every open subset of X is quasi-compact. So in showing that $\sum_i \mathscr{F}_i$ is a sheaf we can restrict to considering a finite family of matching sections $s_k \in \Gamma(\sum_i \mathscr{F}_i, U_k)$ where $1 \leq k \leq n$ and the U_k are an open cover of a nonempty open set U. Suppose that $s_i \in \mathscr{F}_{i_k}(U_k)$ for $1 \leq k \leq n$ and let $\ell \in I$ be such that $\mathscr{F}_{i_k} \leq \mathscr{F}_\ell$ for $1 \leq k \leq n$. Then the s_k are a matching family for $\mathscr{F}_\ell$ which can be amalgamated to give $s \in \mathscr{F}_\ell(U) \subseteq \Gamma(\sum_i \mathscr{F}_i, U)$ which shows that $\sum_i \mathscr{F}_i$ is a sheaf. It clearly has the property of the union. ☐

Definition 9. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups on X and $\{s_i\}_{i\in I}$ a nonempty set of sections $s_i \in \mathscr{F}(U_i)$ over open subsets $U_i \subseteq X$. The *subsheaf of* $\mathscr{F}$ *generated by the set* $\{s_i\}$ is the intersection of all subsheaves of $\mathscr{F}$ containing all the s_i. This is the smallest subsheaf containing the s_i, in the sense that it precedes any other such subsheaf. We say $\mathscr{F}$ is *finitely generated* if there is a nonempty finite set of sections $\{s_1, \ldots, s_n\}$ generating $\mathscr{F}$.

Lemma 13. *Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Then $\mathscr{F}$ is the direct limit of all its finitely generated subsheaves.*

Proof. Let $\{\mathscr{F}_\alpha\}_{\alpha\in\Lambda}$ be the set of all finitely generated subsheaves. This is certainly a direct family of subsheaves of $\mathscr{F}$. To show that the inclusion $\mathscr{F}_\alpha \longrightarrow \mathscr{F}$ are a direct limit, it suffices to show that $\mathscr{F} = \sum_i \mathscr{F}_i$. But any section $s \in \mathscr{F}(U)$ generates a finitely generated subsheaf, so this is trivial. □

Lemma 14. *Let $\psi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X and suppose $\mathscr{H}$ is a subsheaf of $\mathscr{G}$. Then the inverse image $\psi^{-1}\mathscr{H}$ is the subsheaf defined by*

$$(\psi^{-1}\mathscr{H})(U) = \psi_U^{-1}(\mathscr{H}(U))$$

Proof. This clearly defines a subsheaf of $\mathscr{F}$. The morphism $\psi^{-1}\mathscr{H} \longrightarrow \mathscr{H}$ defined by restricting ψ to give $\psi_U^{-1}\mathscr{H}(U) \longrightarrow \mathscr{H}(U)$ clearly fits into a commutative diagram

$$\begin{array}{ccc} \psi^{-1}\mathscr{H} & \longrightarrow & \mathscr{H} \\ \downarrow & & \downarrow \\ \mathscr{F} & \underset{\psi}{\longrightarrow} & \mathscr{G} \end{array}$$

It is not difficult to check that this is a pullback, as required. □

2.1 Stalks

Throughout this section X denotes a fixed topological space. All sheaves of abelian groups will be over X.

Lemma 15. *Suppose we have a sequence of sheaves of abelian groups*

$$\mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}''$$

This sequence is exact if and only if $\mathscr{F}'_x \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{F}''_x$ is an exact sequence of abelian groups for every $x \in X$.

Proposition 16. *Let D be a diagram of sheaves of abelian groups. A cocone $\{H, \rho_i : D_i \longrightarrow H\}_{i\in D}$ in $\mathfrak{Ab}(X)$ is a colimit if and only if $\{H_x, \rho_{i,x} : D_{i,x} \longrightarrow H_x\}_{i\in D}$ is a colimit in $\mathbf{Ab}$ for every $x \in X$.*

Proof. The categories $\mathfrak{Ab}(X), \mathbf{Ab}$ are complete, so let $\mu_i : D_i \longrightarrow Y$ be a colimit for D in $\mathfrak{Ab}(X)$. There is an induced morphism $\eta : Y \longrightarrow H$ with $\eta\mu_i = \rho_i$, and the morphisms ρ_i are a colimit iff. η is an isomorphism. But this is iff. η_x is an isomorphism for all $x \in X$, which is iff. the morphisms $\rho_{i,x}$ are a colimit for all $x \in X$. This completes the proof. □

For any point $x \in X$ we have the additive stalk functor

$$(-)_x : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$$

which is exact. Given an abelian group M we define a sheaf of abelian groups by

$$\Gamma(U, Sky_x(M)) = \begin{cases} M & x \in U \\ 0 & \text{otherwise} \end{cases}$$

One defines Sky_x on morphisms in the obvious way, so that we have an additive functor

$$Sky_x(-) : \mathbf{Ab} \longrightarrow \mathfrak{Ab}(X)$$

For a sheaf of abelian groups $\mathscr{F}$ and abelian group M we have natural morphisms

$$\eta : \mathscr{F} \longrightarrow Sky_x(\mathscr{F}_x), \quad \eta_U(m) = (U, m)$$
$$\varepsilon : (Sky_x M)_x \longrightarrow M, \quad \varepsilon(U, m) = m$$

In fact these are the unit and counit of an adjunction $(-)_x \dashv Sky_x$. It is clear that ε is a natural equivalence, from which we deduce that Sky_x is fully faithful (AC,Proposition 21). Let Z be the closure of the point $x \in X$, and define for a sheaf of abelian groups $\mathscr{F}$

$$\Gamma_Z(\mathscr{F}) = \{s \in \Gamma(X, \mathscr{F}) \,|\, s|_{X \setminus Z} = 0\}$$

For a morphism of sheaves of abelian groups $\mathscr{F} \longrightarrow \mathscr{G}$ we have a morphism of abelian groups $\Gamma_Z(\mathscr{F}) \longrightarrow \Gamma_Z(\mathscr{G})$ and this defines an additive functor

$$\Gamma_Z(-) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$$

Clearly $\Gamma_Z(Sky_x M) = M$ for any abelian group M, so the identity is a trivial natural transformation $\eta : M \longrightarrow \Gamma_Z(Sky_x M)$. Given a sheaf of abelian groups $\mathscr{G}$, an abelian group M and a morphism of abelian groups $\alpha : M \longrightarrow \Gamma_Z(\mathscr{G})$ we define a morphism of sheaves of abelian groups

$$\phi : Sky_x M \longrightarrow \mathscr{G}$$
$$\phi_U(m) = \alpha(m)|_U$$

for open U containing x. The condition on elements of $\Gamma_Z(\mathscr{G})$ makes this a morphism of sheaves of abelian groups, and it is certainly unique such that $\Gamma_Z(\phi) = \alpha$. In other words, η is the unit of an adjunction $Sky_x(-) \dashv \Gamma_Z(-)$, and we have a triple of adjoints

$$(-)_x \dashv Sky_x(-) \dashv \Gamma_Z(-)$$

which in particular means that $Sky_x(-)$ is exact and preserves all limits and colimits.

2.2 Sheaf Hom

Let X be a topological space and $\mathscr{F}, \mathscr{G}$ sheaves of abelian groups on X. Define the following presheaf of abelian groups

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})(U) = Hom_{\mathfrak{Ab}(U)}(\mathscr{F}|_U, \mathscr{G}|_U)$$
$$(\phi|_V)_W = \phi_W$$

We showed in (H,Ex.1.15) that this is a sheaf of abelian groups. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of abelian groups, we define a morphism of sheaves of abelian groups

$$\mathscr{H}om(\phi, \mathscr{G}) : \mathscr{H}om(\mathscr{F}', \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})$$
$$\mathscr{H}om(\phi, \mathscr{G})_U(\psi) = \psi\phi|_U$$

this defines the additive contravariant functor $\mathscr{H}om(-, \mathscr{G}) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of abelian groups, we define a morphism of sheaves of abelian groups

$$\mathscr{H}om(\mathscr{F}, \phi) : \mathscr{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}')$$
$$\mathscr{H}om(\mathscr{F}, \phi)_U(\psi) = \phi|_U\psi$$

this defines the additive covariant functor $\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$.

2.3 Tensor products

Let X be a topological space and $\mathscr{F},\mathscr{G}$ sheaves of abelian groups on X. Define the following presheaf of abelian groups

$$Z(U) = \mathscr{F}(U) \otimes_{\mathbb{Z}} \mathscr{G}(U)$$
$$(a \otimes b)|_V = a|_V \otimes b|_V$$

Let $\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G}$ be the associated sheaf of abelian groups. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of abelian groups and Z' sheafifies to give $\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{G}$ then $\phi' : Z \longrightarrow Z'$ defined by $\phi'_U = \phi_U \otimes 1$ is a morphism of presheaves of abelian groups, which sheafifies to give a morphism of sheaves of abelian groups

$$\phi \otimes_{\mathbb{Z}} \mathscr{G} : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} \longrightarrow \mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{G}$$
$$a \dot{\otimes} b \mapsto \phi(a) \dot{\otimes} b$$

This defines the additive covariant functor $- \otimes_{\mathbb{Z}} \mathscr{G} : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of abelian groups and Z' sheafifies to give $\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G}'$ then $\phi' : Z \longrightarrow Z'$ defined by $\phi'_U = 1 \otimes \phi_U$ is a morphism of presheaves of abelian groups, which sheafifies to give a morphism of sheaves of abelian groups

$$\mathscr{F} \otimes_{\mathbb{Z}} \phi : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} \longrightarrow \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G}'$$
$$a \dot{\otimes} b \mapsto a \dot{\otimes} \phi(b)$$

This defines the additive covariant functor $\mathscr{F} \otimes_{\mathbb{Z}} - : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ and $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ are morphisms of sheaves of abelian groups then

$$(\phi \otimes_{\mathbb{Z}} \mathscr{G}')(\mathscr{F} \otimes_{\mathbb{Z}} \psi) = (\mathscr{F}' \otimes_{\mathbb{Z}} \psi)(\phi \otimes_{\mathbb{Z}} \mathscr{G})$$

and we denote this morphism by $\phi \otimes_{\mathbb{Z}} \psi$. This defines a covariant functor

$$- \otimes_{\mathbb{Z}} - : \mathfrak{Ab}(X) \times \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$$
$$(\phi \otimes_{\mathbb{Z}} \psi)_U(a \dot{\otimes} b) = \phi_U(a) \dot{\otimes} \psi_U(b)$$

Lemma 17. *For any sheaves of abelian groups $\mathscr{F},\mathscr{G}$ there is an isomorphism of sheaves of abelian groups natural in $\mathscr{F},\mathscr{G}$*

$$\tau : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathbb{Z}} \mathscr{F}$$
$$a \dot{\otimes} b \mapsto b \dot{\otimes} a$$

called the twist

Proof. For a sheaf of abelian groups $\mathscr{G}$ let Z sheafify to give $\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G}$ and Z' sheafify to give $\mathscr{G} \otimes_{\mathbb{Z}} \mathscr{F}$. Define the isomorphism of presheaves of abelian groups $\tau' : Z \longrightarrow Z'$ to be the twist $\mathscr{F}(U) \otimes_{\mathbb{Z}} \mathscr{G}(U) \longrightarrow \mathscr{G}(U) \otimes_{\mathbb{Z}} \mathscr{F}(U)$ for every open set $U \subseteq X$. Clearly the sheafification of τ' is the required natural isomorphism τ. ☐

Proposition 18. *For sheaves of abelian groups $\mathscr{F},\mathscr{E},\mathscr{G}$ there is an isomorphism of abelian groups natural in all three variables*

$$\zeta : Hom_{\mathfrak{Ab}(X)}(\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{E}, \mathscr{G}) \longrightarrow Hom_{\mathfrak{Ab}(X)}(\mathscr{F}, \mathscr{H}om(\mathscr{E},\mathscr{G}))$$

Proof. Let $\phi : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{E} \longrightarrow \mathscr{G}$, $U \subseteq X$ and $s \in \mathscr{F}(U)$ be given. We define $\zeta(\phi)_U(s) : \mathscr{E}|_U \longrightarrow \mathscr{G}|_U$ by

$$\zeta(\phi)_U(s)_V(e) = \phi_V(s|_V \dot{\otimes} e)$$

It is not difficult to check that $\zeta(\phi)_U(s)$ and $\zeta(\phi) : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E},\mathscr{G})$ are morphisms of sheaves of abelian groups. So our map ζ is a well-defined morphism of abelian groups. Naturality in all three variables is not difficult to check. It is clear that ζ is injective. To see that it is surjective, let $\alpha : \mathscr{F} \longrightarrow \mathscr{H}om(\mathscr{E}, \mathscr{G})$ be given. For $U \subseteq X$ define a bilinear map

$$\mathscr{F}(U) \times \mathscr{E}(U) \longrightarrow \mathscr{G}(U)$$
$$(s, e) \mapsto \alpha_U(s)_U(e)$$

This induces a morphism of presheaves of abelian groups $P \longrightarrow \mathscr{G}$ where P sheafifies to give $\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{E}$. So finally we induce a morphism of sheaves of abelian groups

$$\phi : \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{E} \longrightarrow \mathscr{G}$$
$$\phi_U(s \dot{\otimes} e) = \alpha_U(s)_U(e)$$

It is easy to check that $\zeta(\phi) = \alpha$, which completes the proof that ζ is an isomorphism. □

Corollary 19. *For any sheaf of abelian groups $\mathscr{E}$ there is an adjunction of covariant additive functors*

$$\mathfrak{Ab}(X) \underset{\mathscr{H}om(\mathscr{E},-)}{\overset{-\otimes_{\mathbb{Z}}\mathscr{E}}{\rightleftarrows}} \mathfrak{Ab}(X) \qquad - \otimes_{\mathbb{Z}} \mathscr{E} \dashv \mathscr{H}om(\mathscr{E}, -)$$

In particular $\mathscr{E} \otimes_{\mathbb{Z}} -$ and $- \otimes_{\mathbb{Z}} \mathscr{E}$ preserve all colimits.

Proof. The adjunction follows from Proposition 18, and since $- \otimes_{\mathbb{Z}} \mathscr{E}$ is naturally equivalent to $\mathscr{E} \otimes_{\mathbb{Z}} -$, if one has a right adjoint so does the other. □

Proposition 20. *Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups. Then for $x \in X$ there is an isomorphism of abelian groups natural in $\mathscr{F}, \mathscr{G}$*

$$\alpha : (\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G})_x \longrightarrow \mathscr{F}_x \otimes_{\mathbb{Z}} \mathscr{G}_x$$
$$(U, a \dot{\otimes} b) \mapsto (U, a) \otimes (U, b)$$

Proof. Let Z be the presheaf of abelian groups $Z(U) = \mathscr{F}(U) \otimes_{\mathbb{Z}} \mathscr{G}(U)$. Define

$$\varepsilon : \mathscr{F}_x \times \mathscr{G}_x \longrightarrow Z_x$$
$$((U, s), (V, t)) \mapsto (U \cap V, s|_{U \cap V} \otimes t|_{U \cap V})$$

It is easily checked that ε is bilinear, and we claim that ε is actually a tensor product. Suppose M is an abelian group and $\phi : \mathscr{F}_x \times \mathscr{G}_x \longrightarrow M$ is bilinear. For $x \in U$ the map $\theta'_U : \mathscr{F}(U) \times \mathscr{G}(U) \longrightarrow M$ defined by $(s, t) \mapsto \phi((U, s), (U, t))$ is bilinear and hence induces $\theta_U : Z(U) \longrightarrow M$. Since θ commutes with restriction, we get a morphism of abelian groups $\psi : Z_x \longrightarrow M$ given by

$$\psi((U, s \otimes t)) = \phi((U, s), (U, t))$$

Clearly $\psi\varepsilon = \phi$ and ψ is unique, so ε is a tensor product and there is an isomorphism of abelian groups $\alpha' : Z_x \longrightarrow \mathscr{F}_x \otimes_{\mathbb{Z}} \mathscr{G}_x$ defined by $(U, s \otimes t) \mapsto (U, s) \otimes (U, t)$. Combined with the canonical isomorphism of abelian groups $(\mathscr{F} \otimes_{\mathbb{Z}} \mathscr{G})_x \cong Z_x$ this gives the desired isomorphism α. Naturality is easily checked. □

2.4 Inverse and Direct Image

Definition 10. Let $f : X \longrightarrow Y$ be a continuous map of spaces and $\mathscr{F}$ a sheaf of abelian groups on X. Then the sheaf of sets $f_*\mathscr{F}$ of Definition 5 is a sheaf of abelian groups in the obvious way. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of abelian groups, then so is $f_*\phi$. This defines an additive functor $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$, which we still call the *direct image functor*.

Definition 11. Let $f : X \longrightarrow Y$ be a continuous map of spaces and $\mathscr{G}$ a sheaf of abelian groups on Y. Then the sheaf of sets $f^{-1}\mathscr{G}$ of Definition 6 is a sheaf of abelian groups in the obvious way. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of abelian groups, then so is $f^{-1}\phi$. This defines an additive functor $f^{-1} : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$, which we still call the *inverse image functor*.

Proposition 21. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and let $\mathscr{G}$ be a sheaf of abelian groups on Y. Then for $x \in X$ there is a canonical isomorphism of abelian groups natural in $\mathscr{G}$*

$$\lambda : (f^{-1}\mathscr{G})_x \longrightarrow \mathscr{G}_{f(x)}$$
$$germ_x[V, s] \mapsto germ_{f(x)} s$$

Proof. The map λ of Proposition 4 is clearly a morphism of abelian groups, and gives the desired natural isomorphism. □

Lemma 22. *Let $f : X \longrightarrow Y$ be a continuous map of spaces. Then the functor $f^{-1} : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$ is exact.*

Proof. This follows immediately from Lemma 15 and Proposition 21. □

Proposition 23. *Let $f : X \longrightarrow Y$ be a continuous map of spaces. Then we have an adjoint pair*

$$\mathfrak{Ab}(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathfrak{Ab}(Y) \qquad f^{-1} \dashv f_* \tag{5}$$

For sheaves $\mathscr{F}, \mathscr{G}$ of abelian groups on X, Y respectively the unit and counit are defined by

$$\eta : \mathscr{G} \longrightarrow f_* f^{-1}\mathscr{G} \qquad\qquad \varepsilon : f^{-1} f_* \mathscr{F} \to \mathscr{F}$$
$$\eta_V(s) = [V, s] \qquad\qquad \varepsilon_U([V, s]) = s|_U$$

Proof. Given a sheaf of abelian groups $\mathscr{G}$ on Y the morphism of sheaves of sets $\eta : \mathscr{G} \longrightarrow f_* f^{-1}\mathscr{G}$ defined in Proposition 5 is a morphism of sheaves of abelian groups natural in $\mathscr{G}$. Let $\phi : \mathscr{G} \longrightarrow f_*\mathscr{F}$ be a morphism of sheaves of abelian groups. The morphism $\psi : f^{-1}\mathscr{G} \longrightarrow \mathscr{F}$ defined in Proposition 5 by $\psi_U([V, s]) = \phi_V(s)|_U$ is a morphism of sheaves of abelian groups, and this defines the required adjunction with the counit given above. □

Lemma 24. *Let X be a topological space and $U \subseteq X$ an open subset with inclusion $i : U \longrightarrow X$. Then for any sheaf of abelian groups $\mathscr{F}$ on X there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\alpha : \mathscr{F}|_U \longrightarrow i^{-1}\mathscr{F}$$
$$\alpha_V(s) = [V, s]$$

That is, there is a canonical natural equivalence $(-)|_U \cong i^{-1}$ of additive functors $\mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(U)$.

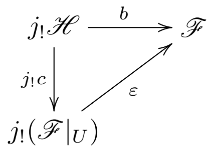
Proof. The isomorphism of sheaves of sets α defined in Lemma 6 is easily checked to be a morphism of sheaves of abelian groups. □

Definition 12. Let X be a topological space and $U \subseteq X$ an open subset with inclusion $j : U \longrightarrow X$. If $\mathscr{F}$ is a sheaf of abelian groups on U then the sheaf of sets $j_!\mathscr{F}$ of Definition 7 is a sheaf of abelian groups in the obvious way. If $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ is a morphism of sheaves of abelian groups, then so is $j_!\phi : j_!\mathscr{F} \longrightarrow j_!\mathscr{F}'$. This defines an additive functor $j_!(-) : \mathfrak{Ab}(U) \longrightarrow \mathfrak{Ab}(X)$.

Proposition 25. *Let X be a topological space and $U \subseteq X$ an open subset with inclusion $j : U \longrightarrow X$. Then we have an adjoint pair*

$$\mathfrak{Ab}(X) \underset{j_!}{\overset{(-)|_U}{\rightleftarrows}} \mathfrak{Ab}(U) \qquad j_! \dashv (-)|_U \tag{6}$$

Proof. Let $\mathscr{G}$ be a sheaf of abelian groups on X. The morphism $\varepsilon : j_!(\mathscr{G}|_U) \longrightarrow \mathscr{G}$ of Proposition 7 is easily seen to be a morphism of sheaves of abelian groups natural in $\mathscr{G}$. If $\mathscr{H}$ is a sheaf of abelian groups on U and $b : j_!\mathscr{H} \longrightarrow \mathscr{F}$ a morphism of sheaves of abelian groups, there is a unique morphism of sheaves of abelian groups $c : \mathscr{H} \longrightarrow \mathscr{F}|_U$ making the following diagram commute



Therefore ε is the counit of an adjunction $j_!(-) \dashv (-)|_U$. For a sheaf of abelian groups $\mathscr{H}$ on U the unit of this adjunction is the canonical isomorphism $\eta : \mathscr{H} \longrightarrow (j_!\mathscr{H})|_U$. □

Remark 7. With the notation of Proposition 25 the unit gives a canonical isomorphism of abelian groups $\mathscr{H}_x \longrightarrow (j_!\mathscr{H})_x$ for $x \in U$, for any sheaf of abelian groups $\mathscr{H}$ on U. If $x \notin U$ then we have $(j_!\mathscr{H})_x = 0$. By the same argument as Corollary 8 one checks that the counit $\varepsilon : j_!(\mathscr{F}|_U) \longrightarrow \mathscr{F}$ is a monomorphism for any sheaf of abelian groups $\mathscr{F}$ on X.

Lemma 26. *Let X be a topological space with open subset $U \subseteq X$ and set $Z = X \setminus U$. If $\mathscr{F}$ is a sheaf of abelian groups on X then there is an exact sequence*

$$0 \longrightarrow j_!(\mathscr{F}|_U) \longrightarrow \mathscr{F} \longrightarrow i_*(i^{-1}\mathscr{F}) \longrightarrow 0 \tag{7}$$

where $j : U \longrightarrow X, i : Z \longrightarrow X$ are the inclusions.

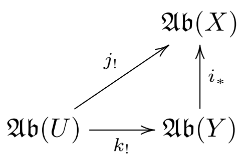
Proof. We know that the canonical morphism $\varepsilon : j_!(\mathscr{F}|_U) \longrightarrow \mathscr{F}$ is a monomorphism with ε_x an isomorphism for $x \in U$ and zero otherwise. It is not hard to see that the unit $\eta : \mathscr{F} \longrightarrow i_*(i^{-1}\mathscr{F})$ is an epimorphism with η_x an isomorphism for $x \notin U$ and zero otherwise. Exactness of (7) is now easily checked by looking at exactness on stalks. □

Lemma 27. *Let X be a topological space with open subset $U \subseteq X$ and set $Z = X \setminus U$. If $\mathscr{F}$ is a sheaf of abelian groups on X with $\mathscr{F}|_U = 0$ then there is a canonical isomorphism $\mathscr{F} \cong i_*(i^{-1}\mathscr{F})$ where $i : Z \longrightarrow X$ is the inclusion.*

Lemma 28. *The functor $j_! : \mathfrak{Ab}(U) \longrightarrow \mathfrak{Ab}(X)$ is exact.*

Proof. This is a trivial consequence of Lemma 15. □

Lemma 29. *Let X be a topological space, U an open subset, Y a closed subset and assume $U \subseteq Y$. Let $j : U \longrightarrow X, i : Y \longrightarrow X$ and $k : U \longrightarrow Y$ be the inclusions. Then for any sheaf of abelian groups $\mathscr{F}$ on U there is a canonical isomorphism $j_!\mathscr{F} \cong i_*k_!\mathscr{F}$ natural in $\mathscr{F}$. That is, the following diagram of functors commutes up to canonical natural equivalence*



Proof. Let $\mathscr{F}$ be a sheaf of abelian groups on U. Let $\eta : \mathscr{F} \longrightarrow (k_!\mathscr{F})|_U$ be the canonical morphism, which we showed above is an isomorphism. Since $(k_!\mathscr{F})|_U = (i_*k_!\mathscr{F})|_U$ we can use the adjunction $j_! \dashv (-)|_U$ to produce a morphism $\tau : j_!\mathscr{F} \longrightarrow i_*k_!\mathscr{F}$. We show that this is an isomorphism of sheaves of abelian groups.

Just to be clear, let P be the presheaf on X sheafifying to give $j_!\mathscr{F}$. Define a morphism of presheaves $P \longrightarrow i_*k_!\mathscr{F}$ on open $V \subseteq U$ to be $\eta_V : \mathscr{F}(V) \longrightarrow \Gamma(V, k_!\mathscr{F})$. This induces the morphism τ out of the sheafification. We show that τ is an isomorphism by showing that τ_x is an isomorphism of abelian groups for all $x \in X$. There are three cases:

- ($x \notin Y$). Then $(j_!\mathscr{F})_x = (i_* k_! \mathscr{F})_x = 0$ so τ_x is trivially an isomorphism.
- ($x \in Y \setminus U$). Then $(j_!\mathscr{F})_x = 0$ and $(i_* k_! \mathscr{F})_x \cong (k_!\mathscr{F})_x = 0$ so τ_x is trivially an isomorphism.
- ($x \in U$). Then $(j_!\mathscr{F})_x \cong \mathscr{F}_x$ and $(i_* k_! \mathscr{F})_x \cong (k_!\mathscr{F})_x \cong \mathscr{F}_x$ and it is not hard to see that τ_x corresponds to the identity under these isomorphisms, and therefore τ_x is an isomorphism.

It is not hard to check that τ is natural in $\mathscr{F}$, completing the proof. □

In a sense to be made precise in the next lemma, the group $\Gamma(V, j_!\mathscr{F})$ consists of all sections in $\Gamma(U \cap V, \mathscr{F})$ which can be extended by zero to all of V. That is, those sections that "fade out" towards the edge of $U \cap V$ inside V.

Lemma 30. *Let X be a topological space and U an open subset with inclusion $j : U \longrightarrow X$. For any sheaf of abelian groups $\mathscr{F}$ on U there is a canonical monomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\psi : j_!\mathscr{F} \longrightarrow j_*\mathscr{F}$$

whose image is the subsheaf $\mathscr{G}$ defined by

$$\Gamma(V, \mathscr{G}) = \{ s \in \Gamma(V \cap U, \mathscr{F}) \,|\, \textit{for every } x \in V \setminus U \textit{ there is an open set } x \in W \subseteq V \textit{ such that } s|_{W \cap U} = 0 \}$$

Proof. Let $\mathscr{F}_E$ be the presheaf of abelian groups on X defined by $\Gamma(V, \mathscr{F}_E) = \Gamma(V, \mathscr{F})$ for $V \subseteq U$ and zero otherwise. This is a subpresheaf of $j_*\mathscr{F}$ and the induced morphism of sheaves $\psi : j_!\mathscr{F} \longrightarrow j_*\mathscr{F}$ has an image $\mathscr{G}$ defined by the condition of Lemma 10. Examining this condition one checks that $\Gamma(V, \mathscr{G}) \subseteq \Gamma(V, j_*\mathscr{F})$ consists of the sections satisfying the given condition. □

Definition 13. Let X be a topological space and $Z \subseteq X$ a closed subset with inclusion $i : Z \longrightarrow X$ and open complement U. If $\mathscr{G}$ is a sheaf of abelian groups on X then we define a presheaf of abelian groups $i^?\mathscr{G}$ on Z by

$$\Gamma(V, i^?\mathscr{G}) = \{ s \in \mathscr{G}(V \cup U) \,|\, s|_U = 0 \}$$

That is, the sections with support in Z. Together with the obvious restriction maps, this defines a *sheaf* of abelian groups on Z. If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of abelian groups, then we define a morphism of sheaves of abelian groups

$$i^?\phi : i^?\mathscr{G} \longrightarrow i^?\mathscr{G}'$$
$$(i^?\phi)_V(s) = \phi_{V \cup U}(s)$$

This defines an additive functor $i^?(-) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Z)$.

Proposition 31. *Let X be a topological space and $Z \subseteq X$ a closed subset with inclusion $i : Z \longrightarrow X$. Then we have an adjoint pair*

$$\mathfrak{Ab}(X) \underset{i_*}{\overset{i^?}{\rightleftarrows}} \mathfrak{Ab}(Z) \qquad i_* \dashv i^? \tag{8}$$

Proof. Let $\mathscr{F}$ be a sheaf of abelian groups on Z. It is clear that $i^? i_* \mathscr{F} = \mathscr{F}$ and we let $\eta : \mathscr{F} \longrightarrow i^? i_* \mathscr{F}$ be the identity. If $\mathscr{G}$ is a sheaf of abelian groups on X and $\varphi : \mathscr{F} \longrightarrow i^?\mathscr{G}$ a morphism of sheaves of abelian groups, then we define a morphism of sheaves of abelian groups

$$\psi : i_*\mathscr{F} \longrightarrow \mathscr{G}$$
$$\psi_T(s) = \varphi_{T \cap Z}(s)|_T$$

which is unique making the following diagram commute

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\varphi} & i^{?}\mathscr{G} \\
\downarrow & \nearrow_{\psi} & \\
i^{?}i_*\mathscr{F} & &
\end{array}
$$

which shows that η is the unit of an adjunction $i_* \dashv i^{?}$. For a sheaf of abelian groups $\mathscr{G}$ on X the counit $\varepsilon : i_*i^{?}\mathscr{G} \longrightarrow \mathscr{G}$ is defined by $\varepsilon_T(s) = s|_T$. $\square$

In summary, we have the following result

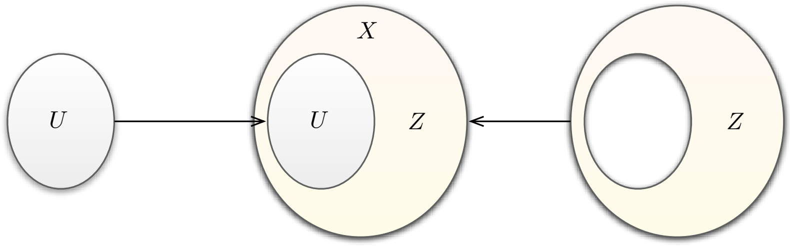
Theorem 32. *Let X be a topological space with open subset U and closed complement Z. Let $j : U \longrightarrow X$ and $i : Z \longrightarrow X$ be the inclusions. Then we have six additive functors*

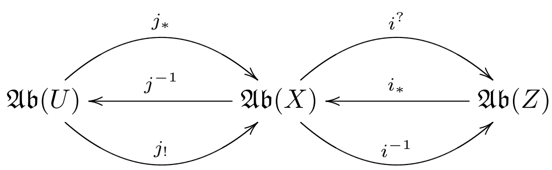
$$
\begin{aligned}
j_! &: \mathfrak{Ab}(U) \longrightarrow \mathfrak{Ab}(X) & \qquad i^{-1} &: \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Z) \\
j^{-1} &: \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(U) & \qquad i_* &: \mathfrak{Ab}(Z) \longrightarrow \mathfrak{Ab}(X) \\
j_* &: \mathfrak{Ab}(U) \longrightarrow \mathfrak{Ab}(X) & \qquad i^{?} &: \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Z)
\end{aligned}
$$

and two triples of adjunctions

$$
j_! \dashv j^{-1} \dashv j_* \qquad\qquad i^{-1} \dashv i_* \dashv i^{?}
$$

And the functors $j_!, j^{-1}$ and i_*, i^{-1} are exact.





More generally, we can define the functors $j_!$ and $i^{?}$ for open and closed *embeddings* respectively.

Definition 14. Let $f : X \longrightarrow Y$ be a continuous map of spaces. We say f is an *open embedding* if f induces a homeomorphism of X with an open subset of Y. We say that f is a *closed embedding* if f induces a homeomorphism of X with a closed subset of Y. We say that f is an *embedding* if f induces a homeomorphism of X with a subspace of Y.

Definition 15. Let $f : X \longrightarrow Y$ be an open embedding. If $\mathscr{G}$ is a sheaf of abelian groups on Y, then we write $\mathscr{G}|_f$ (or more commonly just $\mathscr{G}|_X$) to denote the sheaf of abelian groups on X defined by $\Gamma(U, \mathscr{G}|_X) = \Gamma(f(U), \mathscr{G})$. This defines an additive functor $(-)|_X : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$ which is canonically naturally equivalent to $f^{-1} : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$.

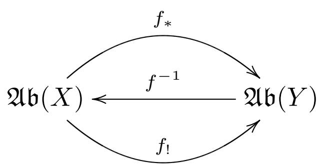
Definition 16. Let $f : X \longrightarrow Y$ be an open embedding and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathscr{F}_E$ be the following presheaf of abelian groups on Y

$$\mathscr{F}_E(V) = \begin{cases} \mathscr{F}(f^{-1}V) & V \subseteq f(X) \\ 0 & \text{otherwise} \end{cases}$$

with the obvious restriction maps. We denote the sheaf of abelian groups $\mathbf{a}\mathscr{F}_E$ by $f_!\mathscr{F}$. Let $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ be a morphism of sheaves of abelian groups, and define a morphism of presheaves

$$\phi_E : \mathscr{F}_E \longrightarrow \mathscr{F}'_E$$
$$(\phi_E)_V = \begin{cases} \phi_{f^{-1}V} & V \subseteq f(X) \\ 0 & \text{otherwise} \end{cases}$$

Let $f_!\phi : f_!\mathscr{F} \longrightarrow f_!\mathscr{F}'$ be the morphism $\mathbf{a}\phi_E$. This defines an additive functor $f_!(-) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$. If $\mathscr{G}$ is a sheaf of abelian groups on Y we define a morphism of presheaves of abelian groups $(\mathscr{G}|_X)_E \longrightarrow \mathscr{G}$ to be the identity on open sets contained in $f(X)$, and zero otherwise. Denote by $\varepsilon : f_!(\mathscr{G}|_X) \longrightarrow \mathscr{G}$ the induced morphism of sheaves of abelian groups, which is natural in $\mathscr{G}$. As before, one checks that ε is the counit of an adjunction $f_! \dashv (-)|_X$. So once again we have a triple of adjoints



$$f_! \dashv f^{-1} \dashv f_*$$

The functors $f_!$ and f^{-1} are exact.

Lemma 33. *Let $f : X \longrightarrow Y$ be an open embedding. Then the functor $(-)|_X : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$ preserves injectives.*

Proof. This follows immediately from the fact that $(-)|_X$ has an exact left adjoint (AC,Proposition 25). ☐

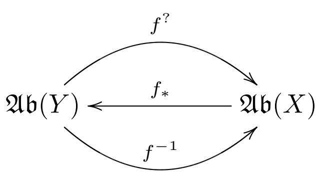
Definition 17. Let $f : X \longrightarrow Y$ be a closed embedding with closed image Z and set $U = Y \setminus Z$. Let $\mathscr{G}$ be a sheaf of abelian groups on Y and define a sheaf of abelian groups on X by

$$\Gamma(V, f^?\mathscr{G}) = \{ s \in \mathscr{G}(f(V) \cup U) \,|\, s|_U = 0 \}$$

If $\phi : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of sheaves of abelian groups, then we define a morphism of sheaves of abelian groups

$$f^?\phi : f^?\mathscr{G} \longrightarrow f^?\mathscr{G}'$$
$$(f^?\phi)_V(s) = \phi_{f(V)\cup U}(s)$$

This defines an additive functor $f^?(-) : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$. Let $\mathscr{F}$ be a sheaf of abelian groups on X. It is clear that $f^?f_*\mathscr{F} = \mathscr{F}$, and we let $\eta : \mathscr{F} \longrightarrow f^?f_*\mathscr{F}$ be the identity. As before, this is the unit of an adjunction $f_* \dashv f^?$. So once again we have a triple of adjoints



$$f^{-1} \dashv f_* \dashv f^?$$

The functors f_* and f^{-1} are exact.

Definition 18. Let X be a topological space and $Z \subseteq X$ a closed subset with open complement U. If $\mathscr{F}$ is a sheaf of abelian groups on X we define a subsheaf $\mathscr{H}^0_Z(\mathscr{F})$ by

$$\begin{aligned}\Gamma(V, \mathscr{H}^0_Z(\mathscr{F})) &= \{s \in \mathscr{F}(V) \,|\, Supp(s) \subseteq Z \cap V\} \\ &= \{s \in \mathscr{F}(V) \,|\, s|_{U \cap V} = 0\}\end{aligned}$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of abelian groups, then there is an induced morphism of sheaves of abelian groups $\mathscr{H}^0_Z(\phi) : \mathscr{H}^0_Z(\mathscr{F}) \longrightarrow \mathscr{H}^0_Z(\mathscr{G})$, and this defines an additive functor $\mathscr{H}^0_Z(-) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$.

In the next result, we show that the functor $f^?$ consists of taking the subsheaf with supports in the image of f, and then applying the usual inverse image functor.

Lemma 34. *Let $f : X \longrightarrow Y$ be a closed embedding and $\mathscr{F}$ a sheaf of abelian groups on Y. Then there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\begin{aligned}\lambda : f^?(\mathscr{F}) &\longrightarrow f^{-1}\mathscr{H}^0_Z(\mathscr{F}) \\ \lambda_V(s) &= [f(V) \cup U, s]\end{aligned}$$

where Z is the closed image of f and $U = Y \setminus Z$.

Proof. By definition we have $\Gamma(V, f^?(\mathscr{F})) = \Gamma(f(V) \cup U, \mathscr{H}^0_Z(\mathscr{F}))$ for any open set $V \subseteq X$. Let Q be the presheaf of abelian groups on X which sheafifies to give $f^{-1}\mathscr{H}^0_Z(\mathscr{F})$. Then $\tau_V(s) = (f(V) \cup U, s)$ defines a morphism of presheaves of abelian groups $\tau : f^?(\mathscr{F}) \longrightarrow Q$. We claim that τ is an isomorphism.

If $\tau_V(s) = 0$ then $s|_T = 0$ for some open set $f(V) \subseteq T \subseteq f(V) \cup U$. Then U, T is an open cover of $f(V) \cup U$ and s restricts to zero on both these open sets, so $s = 0$. To see that τ_V is surjective, let an open set $T \supseteq f(V)$ and $s \in \Gamma(T, \mathscr{H}^0_Z(\mathscr{F}))$ be given. We may as well assume $T \cap f(X) = f(V)$. Then U, T is an open cover of $f(V) \cup U$ and the sections $0, s$ agree on the overlap, so there is a unique section $t \in \Gamma(f(V) \cup U, \mathscr{H}^0_Z(\mathscr{F}))$ with $t|_T = s$. Then clearly $\tau_V(t) = s$ and the proof is complete. Since Q is a sheaf, the canonical morphism $Q \longrightarrow f^{-1}\mathscr{H}^0_Z(\mathscr{F})$ is an isomorphism, and we let λ be the composite $f^?(\mathscr{F}) \longrightarrow Q \cong f^{-1}\mathscr{H}^0_Z(\mathscr{F})$, which is clearly natural in $\mathscr{F}$. □

Lemma 35. *Let $f : X \longrightarrow Y$ be a closed embedding, set $Z = f(X), U = Y \setminus Z$ and let $\mathscr{F}$ be a sheaf of abelian groups on Y with $\mathscr{F}|_U = 0$. Then there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\lambda : f^?(\mathscr{F}) \longrightarrow f^{-1}(\mathscr{F})$$

In particular for $x \in X$ there is a canonical isomorphism of abelian groups natural in $\mathscr{F}$

$$\begin{aligned}\delta : f^?(\mathscr{F})_x &\longrightarrow \mathscr{F}_{f(x)} \\ germ_x s &\mapsto germ_{f(x)} s\end{aligned}$$

Proof. If $\mathscr{F}|_U = 0$ then $\mathscr{H}^0_Z(\mathscr{F}) = \mathscr{F}$ so Lemma 34 gives an isomorphism of sheaves of abelian groups $f^?(\mathscr{F}) \longrightarrow f^{-1}\mathscr{F}$. The isomorphism on stalks follows immediately from Proposition 21. □