Sheaves of Algebras

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1 Introduction

In this note "ring" means a not necessarily commutative ring. If A is a commutative ring then an A-*algebra* is a ring morphism $A \longrightarrow B$ whose image is contained in the center of B. A morphism of A-algebras is a ring morphism compatible with these morphisms (TES,Definition 1). We denote the category of A-algebras by $A\mathbf{nAlg}$ and the full subcategory of commutative A-algebras by $A\mathbf{Alg}$. We only consider algebras over commutative rings.

By (MOS,Corollary 26) we can generalise the definition of quasi-coherent modules to modules over a sheaf of rings, with no risk of ambiguity.

Definition 1. Let X be a topological space and $\mathscr{S}$ a sheaf of rings. We say a sheaf $\mathscr{F}$ of $\mathscr{S}$-modules is *quasi-coherent* if every point $x \in X$ has an open neighborhood U such that $\mathscr{F}|_U$ can be written as the cokernel of free objects in the abelian category $\mathfrak{Mod}(\mathscr{S}|_U)$. This property is stable under isomorphism. The full replete subcategory of $\mathfrak{Mod}(\mathscr{S})$ consisting of the quasi-coherent sheaves of modules is denoted $\mathfrak{Qco}(\mathscr{S})$.

Definition 2. Let X be a topological space and $\mathscr{S}$ a sheaf of graded rings. We say a sheaf $\mathscr{F}$ of graded $\mathscr{S}$-modules is *quasi-coherent* if it is quasi-coherent as a sheaf of $\mathscr{S}$-modules. The full replete subcategory of $\mathfrak{GrMod}(\mathscr{S})$ consisting of the quasi-coherent sheaves of graded modules is denoted $\mathfrak{QcoGrMod}(\mathscr{S})$.

2 Sheaves of Algebras

Definition 3. Let $(X, \mathcal{O}_X)$ be a ringed space. A *presheaf of $\mathcal{O}_X$-algebras* is a presheaf of $\mathscr{F}$ of abelian groups on X, such that for each open set $U \subseteq X$ the group $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-algebra, and for each inclusion of open sets $V \subseteq U$ the restriction morphism $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is a morphism of rings, and for $r \in \mathcal{O}_X(U)$ and $m \in \mathscr{F}(U)$ we have $(r \cdot m)|_V = r|_V \cdot m|_V$. A *morphism* of presheaves of $\mathcal{O}_X$-algebras is a morphism of $\mathcal{O}_X$-modules which is also a morphism of presheaves of rings. This defines the category $nAlg(X)$ of presheaves of $\mathcal{O}_X$-algebras. Let $\mathfrak{nAlg}(X)$ denote the full subcategory of all sheaves of $\mathcal{O}_X$-algebras. Note that a sheaf of $\mathcal{O}_X$-algebras is a sheaf of $\mathcal{O}_X$-modules and also a sheaf of rings

We say a presheaf of $\mathcal{O}_X$-algebras $\mathscr{F}$ is *commutative* if $\mathscr{F}(U)$ is a commutative ring for every open set U, and we denote the full subcategory of commutative presheaves of $\mathcal{O}_X$-algebras by $Alg(X)$. Similarly the full subcategory of sheaves of $\mathcal{O}_X$-algebras is denoted by $\mathfrak{Alg}(X)$. We say a presheaf of commutative $\mathcal{O}_X$-algebras $\mathscr{F}$ is a *domain*, or is a *presheaf of $\mathcal{O}_X$-domains*, if for every nonempty open set U, $\mathscr{F}(U)$ is an integral domain.

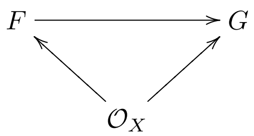
If X is a scheme then we say that a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-algebras is *quasi-coherent* or *coherent* if it has these properties when considered as a sheaf of $\mathcal{O}_X$-modules. Let $\mathfrak{QconAlg}(X)$ denote the full subcategory of $\mathfrak{nAlg}(X)$ consisting of the quasi-coherent sheaves of $\mathcal{O}_X$-algebras, and similarly let $\mathfrak{QcoAlg}(X)$ denote the full subcategory of $\mathfrak{Alg}(X)$ consisting of the quasi-coherent sheaves of commutative $\mathcal{O}_X$-algebras.

Definition 4. Let X be a scheme and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. We say that $\mathscr{F}$ is *locally finitely generated as an $\mathcal{O}_X$-algebra* if for every open affine subset $U \subseteq X$, $\mathscr{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-algebra. We say $\mathscr{F}$ is *locally an integral domain* if $\mathscr{F}(U)$ is an integral domain for every nonempty open affine subset $U \subseteq X$.

There are obvious forgetful functors $\mathfrak{Alg}(X) \longrightarrow \mathfrak{Mod}(X)$ and $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{Mod}(X)$, and we will discuss the adjoints of these functors later. If F is a presheaf of $\mathcal{O}_X$-algebras then for every point $x \in X$ the stalk F_x is an $\mathcal{O}_{X,x}$-algebra, which is commutative if F is. This gives functors $\mathfrak{Alg}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Alg}$ and $\mathfrak{nAlg}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{nAlg}$.

Definition 5. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. An *$\mathcal{O}_X$-subalgebra* of $\mathscr{F}$ is a monomorphism $\phi : \mathscr{G} \longrightarrow \mathscr{F}$ in the category $\mathfrak{Alg}(X)$ with the property that ϕ_U is the inclusion of a subset for every open $U \subseteq X$. In that case $\mathscr{G}(U)$ is an $\mathcal{O}_X(U)$-subalgebra of $\mathscr{F}(U)$ and we can identify $\mathscr{G}_x$ with an $\mathcal{O}_{X,x}$-subalgebra of $\mathscr{F}_x$ for every $x \in X$. Every subobject of $\mathscr{F}$ in the category $\mathfrak{Alg}(X)$ is equivalent as a subobject to a unique $\mathcal{O}_X$-subalgebra.

If $\mathscr{F}$ is a presheaf of $\mathcal{O}_X$-algebras then there is a canonical morphism of presheaves of rings $\mathcal{O}_X \longrightarrow \mathscr{F}$ given pointwise by the canonical algebra morphism $\mathcal{O}_X(U) \longrightarrow \mathscr{F}(U)$. In fact this is a morphism of presheaves of $\mathcal{O}_X$-algebras. Giving the structure of a presheaf of $\mathcal{O}_X$-algebras to a presheaf of rings F is equivalent to giving a morphism of presheaves of rings $\mathcal{O}_X \longrightarrow F$ in the obvious way. If F, G are presheaves of rings with morphisms $\mathcal{O}_X \longrightarrow F, \mathcal{O}_X \longrightarrow G$ then a morphism of presheaves of rings $F \longrightarrow G$ is a morphism of presheaves of $\mathcal{O}_X$-algebras if and only if the following diagram commutes



In other words:

Lemma 1. *Let $Rng(X)$ and $\mathfrak{Rng}(X)$ denote the categories of presheaves and sheaves of* commutative *rings on X respectively. Let $\mathcal{O}_X$ be a sheaf of rings. Then mapping a presheaf (sheaf) of commutative $\mathcal{O}_X$-algebras $\mathscr{F}$ to the canonical morphism $\mathcal{O}_X \longrightarrow \mathscr{F}$ defines isomorphisms of categories*

$$\mathcal{O}_X / Rng(X) \cong Alg(X)$$
$$\mathcal{O}_X / \mathfrak{Rng}(X) \cong \mathfrak{Alg}(X)$$

Let F be a presheaf of $\mathcal{O}_X$-algebras. The sheafification $\mathscr{F}$ has a canonical structure as a sheaf of $\mathcal{O}_X$-modules, and using the $\mathcal{O}_{X,x}$-algebra structure on F_x it is clear that $\mathscr{F}$ becomes a sheaf of $\mathcal{O}_X$-algebras. If F is commutative then so is $\mathscr{F}$. The canonical morphism $F \longrightarrow \mathscr{F}$ is a morphism of presheaves of $\mathcal{O}_X$-algebras.

If $\phi : F \longrightarrow G$ is a morphism of presheaves of $\mathcal{O}_X$-algebras then the morphism $\mathbf{a}\phi : \mathscr{F} \longrightarrow \mathscr{G}$ between the sheafifications of presheaves of modules is a morphism of sheaves of algebras, so sheafification gives a functor $\mathbf{a} : nAlg(X) \longrightarrow \mathfrak{nAlg}(X)$. This functor is left adjoint to the inclusion $\mathbf{i} : \mathfrak{Alg}(X) \longrightarrow nAlg(X)$. Let $\eta : 1 \longrightarrow \mathbf{ia}$ be the natural transformation that is pointwise the canonical morphism $\eta_F : F \longrightarrow \mathscr{F}$ above. Then if $\mathscr{H}$ is any sheaf $\mathcal{O}_X$-algebras and $F \longrightarrow \mathscr{H}$ a morphism of presheaves of algebras, there is a unique morphism of sheaves of algebras $\mathscr{F} \longrightarrow \mathscr{H}$ making the following diagram commute

$$\begin{array}{ccc} F & \longrightarrow & \mathscr{H} \\ \downarrow & \nearrow & \\ \mathscr{F} & & \end{array}$$

Similarly sheafification gives a functor $\mathbf{a} : Alg(X) \longrightarrow \mathfrak{Alg}(X)$ which is left adjoint to the inclusion $\mathfrak{Alg}(X) \longrightarrow Alg(X)$ with the unit being the same morphisms $F \longrightarrow \mathscr{F}$ defined above.

If $U \subseteq X$ is an open subset then restriction defines functors $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{nAlg}(U)$ and $\mathfrak{Alg}(X) \longrightarrow \mathfrak{Alg}(U)$. Obviously restricting to $U = X$ is the identity, and if $V \subseteq U \subseteq X$ then the composite of the restrictions is restriction from X to V.

Lemma 2. *Let $\phi : F \longrightarrow G$ be a morphism in $nAlg(X)$ or $\mathfrak{nAlg}(X)$. Then*

- *ϕ is a monomorphism $\Leftarrow$ ϕ_U is injective for all $U \subseteq X$.*
- *ϕ is an isomorphism $\Leftrightarrow$ ϕ_U is bijective for all $U \subseteq X$.*

Let $\phi : F \longrightarrow G$ be a morphism in $Alg(X)$. Then

- *ϕ is a monomorphism $\Leftrightarrow$ ϕ_U is injective for all $U \subseteq X$.*
- *ϕ is an epimorphism $\Leftrightarrow$ ϕ_U is a ring epimorphism for all $U \subseteq X$.*
- *ϕ is an isomorphism $\Leftrightarrow$ ϕ_U is bijective for all $U \subseteq X$.*

Let $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism in $\mathfrak{Alg}(X)$. Then

- *ϕ is a monomorphism $\Leftrightarrow$ ϕ_U is injective for all $U \subseteq X$ $\Leftrightarrow$ ϕ_x is injective for all $x \in X$.*
- *ϕ is an isomorphism $\Leftrightarrow$ ϕ_U is bijective for all $U \subseteq X$ $\Leftrightarrow$ ϕ_x is bijective for all $x \in X$.*

Proof. The statements for $nAlg(X)$ and $\mathfrak{nAlg}(X)$ are trivial. One could probably also show injective $\Leftrightarrow$ monic, using a sheafified version of the free algebra $\mathbb{Z}\langle x \rangle$, but I haven't checked this.

Since the category of commutative rings is complete and cocomplete, the same is true of $Rng(X)$ and $\mathfrak{Rng}X$. It follows from Lemma 1 and our notes on Coslice categories that a morphism in $Alg(X)$ is a monomorphism, epimorphism or isomorphism iff. it has this property as a morphism of $Rng(X)$. In which case the claims follow from our "Morphism cheat sheet" in Section 2.1. The claims for $\mathfrak{Alg}(X)$ follow in the same way. $\square$

Lemma 3. *Let $U \subseteq X$ be an open subset. Then the following diagrams of functors commute up to canonical natural equivalence*

$$\begin{array}{ccc} nAlg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{nAlg}(X) \\ \downarrow{\scriptstyle |_U} & & \downarrow{\scriptstyle |_U} \\ nAlg(U) & \xrightarrow[\mathbf{a}]{} & \mathfrak{nAlg}(U) \end{array} \qquad \begin{array}{ccc} Alg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{Alg}(X) \\ \downarrow{\scriptstyle |_U} & & \downarrow{\scriptstyle |_U} \\ Alg(U) & \xrightarrow[\mathbf{a}]{} & \mathfrak{Alg}(U) \end{array}$$

Proof. The analogous result for presheaves of sets, groups or rings is easily checked (see *p*.7 of our Section 2.1 notes). Let F be a presheaf of algebras on X, and let $\phi_x : (F|_U)_x \longrightarrow F_x$ be the canonical isomorphism of rings (compatible with the ring isomorphism $(\mathcal{O}_X|_U)_x \cong \mathcal{O}_{X,x}$) for $x \in U$. Then we define

$$\eta : \mathbf{a}(F|_U) \longrightarrow (\mathbf{a}F)|_U$$
$$\eta_W(s)(x) = \phi_x(s(x))$$

This is a morphism of sheaves of $\mathcal{O}_X$-algebras, which is clearly an isomorphism. Naturality in F is also easily checked. The same argument applies to the second diagram. □

If $X = SpecA$ is an affine scheme then there are additive functors $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ and $\Gamma : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$ which form an adjoint pair. We want to extend this adjunction to algebras. Let A be a commutative ring and B an A-algebra (so B may be noncommutative). If S is a multiplicatively closed subset of A then the $S^{-1}A$-module $S^{-1}B$ can be given a ring structure via $b/s \cdot c/t = bc/st$. This makes $S^{-1}B$ into a (noncommutative) $S^{-1}A$-algebra.

Definition 6. Let A be a commutative ring and set $X = SpecA$. Let B be an A-algebra (B may be noncommutative). For a prime ideal $\mathfrak{p} \in SpecA$ the module $B_\mathfrak{p}$ is canonically an $A_\mathfrak{p}$-algebra, and we make the sheaf of $\mathcal{O}_X$-modules $\widetilde{B}$ into a sheaf of $\mathcal{O}_X$-algebras using this pointwise structure. If $\phi : B \longrightarrow C$ is a morphism of A-algebras then the morphism $\widetilde{B} \longrightarrow \widetilde{C}$ of sheaves of modules is clearly a morphism of sheaves of algebras. If B is commutative then clearly so is $\widetilde{B}$, so we have defined functors $\widetilde{-} : A\mathbf{nAlg} \longrightarrow \mathfrak{nAlg}(X)$ and $\widetilde{-} : A\mathbf{Alg} \longrightarrow \mathfrak{Alg}(X)$. If $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-algebras then $\Gamma(\mathscr{F})$ is an A-algebra via the ring morphism $A \cong \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathscr{F})$, so taking global sections defines functors $\Gamma : \mathfrak{nAlg}(X) \longrightarrow A\mathbf{nAlg}$ and $\Gamma : \mathfrak{Alg}(X) \longrightarrow A\mathbf{Alg}$.

Proposition 4. *Let A be a commutative ring, B an A-algebra and $\widetilde{B}$ the sheaf of algebras on $X = SpecA$ corresponding to B. Then*

(a) For each $\mathfrak{p} \in X$ there is an isomorphism of rings $\widetilde{B}_\mathfrak{p} \cong B_\mathfrak{p}$ compatible with the ring isomorphism $\mathcal{O}_{X,\mathfrak{p}} \cong A_\mathfrak{p}$.

(b) For any $f \in A$ there is an isomorphism of rings $\widetilde{B}(D(f)) \cong B_f$ compatible with the ring isomorphism $\mathcal{O}_X(D(f)) \cong A_f$.

(c) In particular $\Gamma(X, \widetilde{B}) \cong B$ as A-algebras.

Proof. The isomorphisms in (5.1) give the necessary ring isomorphisms in the current context. □

Proposition 5. *Let A be a commutative ring and set $X = SpecA$. Then we have two pairs of adjoint functors of the form $\widetilde{\ } \dashv \Gamma$*

$$A\mathbf{nAlg} \underset{\Gamma}{\overset{\sim}{\rightleftarrows}} \mathfrak{nAlg}(X) \qquad A\mathbf{Alg} \underset{\Gamma}{\overset{\sim}{\rightleftarrows}} \mathfrak{Alg}(X)$$

In both cases the unit $B \longrightarrow \Gamma(\widetilde{B})$ and counit $\widetilde{\Gamma(\mathscr{F})} \longrightarrow \mathscr{F}$ are as for modules, so $\mathscr{F}$ is a quasi-coherent sheaf of algebras iff. the counit is an isomorphism.

Proof. For an A-algebra B let $\eta : B \longrightarrow \Gamma(\widetilde{B})$ be the morphism of A-modules given in our solution to Ex 5.3 (where we established the adjunction for modules). This is clearly a morphism of A-algebras natural in B. Suppose we are given a morphism of A-algebras $\phi : B \longrightarrow \Gamma(\mathscr{F})$ for a sheaf of algebras $\mathscr{F}$. Let $\psi : \widetilde{B} \longrightarrow \mathscr{F}$ be the morphism of sheaves of modules defined in Ex 5.3, where we showed that this is the unique morphism of sheaves of modules making the following diagram commute

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \Gamma(\mathscr{F}) \\ {\scriptstyle\eta}\downarrow & \nearrow_{\psi_X} & \\ \Gamma(\widetilde{B}) & & \end{array}$$

So to complete the proof (for both noncommutative and commutative sheaves of algebras) it suffices to show that ψ is a morphism of sheaves of algebras. This is easy to check, given the explicit definition in Ex 5.3. $\square$

Lemma 6. *Let A be a commutative ring and set $X = SpecA$. The functors $\widetilde{-} : A\mathbf{nAlg} \longrightarrow \mathfrak{nAlg}(X)$ and $\widetilde{-} : A\mathbf{Alg} \longrightarrow \mathfrak{Alg}(X)$ are fully faithful and preserve epimorphisms and all colimits.*

Proof. We showed in (5.2) that the functor $A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is fully faithful, from which it follows immediately that the two functors above are fully faithful. The other properties follow directly from Proposition 5. $\square$

Lemma 7. *Let A be a commutative ring and set $X = SpecA$. The functors $\widetilde{-}$ give equivalences*

$$A\mathbf{nAlg} \cong \mathfrak{QconAlg}(X) \quad A\mathbf{Alg} \cong \mathfrak{QcoAlg}(X)$$

2.1 Direct and inverse image

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then there is a pair of adjoint functors

$$\mathfrak{Mod}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathfrak{Mod}(Y) \qquad f^* \dashv f_*$$

We want to define direct and inverse image for sheaves of algebras. Let $\mathscr{F}$ be a sheaf of algebras on X, and let $\mathcal{O}_X \longrightarrow \mathscr{F}$ be the canonical morphism of sheaves of rings (remember that $\mathscr{F}$ is allowed to be noncommutative). Then the composite $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathscr{F}$ makes $f_*\mathscr{F}$ into a sheaf of $\mathcal{O}_Y$-algebras, so this defines the functors $f_* : \mathfrak{nAlg}(X) \longrightarrow \mathfrak{nAlg}(Y)$ and $f_* : \mathfrak{Alg}(X) \longrightarrow \mathfrak{Alg}(Y)$.

If $\mathscr{S}$ is a sheaf of rings on Y then we can define a sheaf of rings $f^{-1}\mathscr{S}$ as in Section 2.1, since the definition there does not depend on $\mathscr{S}$ being commutative. So we get a functor from sheaves of rings on Y to sheaves of rings on X.

Now let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-algebras. The morphism of sheaves of rings $\mathcal{O}_Y \longrightarrow \mathscr{G}$ gives a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \longrightarrow f^{-1}\mathscr{G}$ and together with $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ (the adjoint partner of $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$) we see that the rings $f^{-1}\mathscr{G}(U)$ and $\mathcal{O}_X(U)$ are $f^{-1}\mathcal{O}_Y(U)$-algebras for every open subset $U \subseteq X$. Therefore the tensor product $f^{-1}\mathscr{G}(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)$ is canonically a $\mathcal{O}_X(U)$-algebra. See our Tensor, Exterior, Symmetric algebra notes for the construction of this algebra. We already know that $P(U) = f^{-1}\mathscr{G}(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)$ is a presheaf of $\mathcal{O}_X$-modules, so it is clearly a presheaf of $\mathcal{O}_X$-algebras. Therefore the sheaf of $\mathcal{O}_X$-modules $f^*\mathscr{G}$ is canonically a sheaf of $\mathcal{O}_X$-algebras. If $\mathscr{G}$ is commutative then so is $f^*\mathscr{G}$. If $\phi : \mathscr{G} \longrightarrow \mathscr{H}$ is a morphism of sheaves of $\mathcal{O}_Y$-algebras then $f^*\phi : f^*\mathscr{G} \longrightarrow f^*\mathscr{H}$, which we know to be a morphism of modules, is easily seen to be a morphism of sheaves of algebras. So we have the desired functors $f^* : \mathfrak{nAlg}(Y) \longrightarrow \mathfrak{nAlg}(X)$ and $f^* : \mathfrak{Alg}(Y) \longrightarrow \mathfrak{Alg}(X)$. By construction the following two diagrams of functors commute:

$$\begin{array}{ccc} \mathfrak{nAlg}(X) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathfrak{nAlg}(Y) \\ \downarrow & & \downarrow \\ \mathfrak{Mod}(X) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathfrak{Mod}(Y) \end{array} \qquad \begin{array}{ccc} \mathfrak{Alg}(X) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathfrak{Alg}(Y) \\ \downarrow & & \downarrow \\ \mathfrak{Mod}(X) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathfrak{Mod}(Y) \end{array}$$

Proposition 8. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then for a sheaf of algebras $\mathscr{G}$ on Y the canonical morphism of modules $\eta : \mathscr{G} \longrightarrow f_*f^*\mathscr{G}$ is a morphism of algebras,* *and these morphisms are the unit of an adjunction $f^* \dashv f_*$ (for sheaves of commutative algebras also). That is, we have two adjoint pairs of functors*

$$\mathfrak{Alg}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathfrak{Alg}(Y) \qquad f^* \dashv f_*$$

$$\mathfrak{nAlg}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathfrak{nAlg}(Y) \qquad f^* \dashv f_*$$

Proof. Using our explicit definition in Section 2.5 it is easy to check that η is a morphism of sheaves of $\mathcal{O}_Y$-algebras, and these morphisms therefore give a natural transformation $1 \longrightarrow f_* f^*$. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-algebras and $\phi : \mathscr{G} \longrightarrow f_*\mathscr{F}$ a morphism of sheaves of algebras. Then there is a unique morphism of sheaves of modules $\psi : f^*\mathscr{G} \longrightarrow \mathscr{F}$ making the following diagram commute

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\phi} & f_*\mathscr{F} \\ {\scriptstyle\eta}\downarrow & \nearrow {\scriptstyle f_*\psi} & \\ f_*f^*\mathscr{G} & & \end{array}$$

To complete the proof, it suffices to check that ψ is a morphism of sheaves of algebras. Using our explicit definition in Section 2.5 this is easily checked. ☐

Proposition 9. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then $g_* f_* = (gf)_*$ for functors between the categories of sheaves of algebras, and sheaves of commutative algebras.*

Proof. We know that $(gf)_* = g_* f_*$ as functors between the categories of sheaves of modules, and it is not hard to see the equality holds in our current context as well. ☐

Corollary 10. *If $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is an isomorphism of ringed spaces then we have isomorphisms of categories $f_* : \mathfrak{nAlg}(X) \longrightarrow \mathfrak{nAlg}(Y)$ and $f_* : \mathfrak{Alg}(X) \longrightarrow \mathfrak{Alg}(Y)$.*

Lemma 11. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of $\mathcal{O}_Y$-algebras. Then for $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-algebras*

$$\tau : (f^*\mathscr{F})_x \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$
$$germ_x([V, s] \dot{\otimes} b) \mapsto germ_{f(x)} s \otimes germ_x b$$

Proof. It is easily checked that the isomorphism of (MRS, Proposition 20) is an isomorphism of rings. ☐

Lemma 12. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagrams of functors commute up to canonical natural equivalence*

$$\begin{array}{ccc} Alg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{Alg}(X) \\ {\scriptstyle f_*}\Downarrow & & \Downarrow{\scriptstyle f_*} \\ Alg(Y) & \xrightarrow[\mathbf{a}]{} & \mathfrak{Alg}(Y) \end{array} \qquad \begin{array}{ccc} nAlg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{nAlg}(X) \\ {\scriptstyle f_*}\Downarrow & & \Downarrow{\scriptstyle f_*} \\ nAlg(Y) & \xrightarrow[\mathbf{a}]{} & \mathfrak{nAlg}(Y) \end{array}$$

Proof. The isomorphism given in (MRS, Lemma 25) gives the necessary isomorphism of sheaves of algebras. ☐

Theorem 13. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ be morphisms of schemes and consider the following diagrams of functors*

$$\begin{array}{ccc} \mathfrak{nAlg}(X) & \xleftarrow{f^*} & \mathfrak{nAlg}(Y) \\ & {}_{(gf)^*}\nwarrow \quad \nearrow_{g^*} & \\ & \mathfrak{nAlg}(Z) & \end{array} \qquad \begin{array}{ccc} \mathfrak{Alg}(X) & \xleftarrow{f^*} & \mathfrak{Alg}(Y) \\ & {}_{(gf)^*}\nwarrow \quad \nearrow_{g^*} & \\ & \mathfrak{Alg}(Z) & \end{array}$$

We claim that both diagrams commute up to a canonical natural equivalence.

Proof. Given a sheaf of (commutative) $\mathcal{O}_Z$-algebras $\mathscr{F}$ we need only show the isomorphism of sheaves of modules $\alpha : (gf)^*\mathscr{F} \longrightarrow f^*g^*\mathscr{F}$ given in our Inverse and Direct Image notes is a morphism of sheaves of rings. It therefore suffices to show the morphism $\alpha' : Q \longrightarrow Q'$ given there is a morphism of presheaves of rings, and this comes down to showing $\kappa : g^{-1}\mathscr{F} \longrightarrow g^*\mathscr{F}$ is a morphism of sheaves of rings, which is trivial. $\square$

Lemma 14. *Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ an open subset with inclusion $i : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$. Then we have the following functors*

$$\mathfrak{nAlg}(U) \underset{-|_U}{\overset{i_*}{\underset{i^*}{\leftleftarrows\!\!\!\rightarrow}}} \mathfrak{nAlg}(X) \qquad \mathfrak{Alg}(U) \underset{-|_U}{\overset{i_*}{\underset{i^*}{\leftleftarrows\!\!\!\rightarrow}}} \mathfrak{Alg}(X)$$

We claim that in both cases there is a canonical natural equivalence $i^* \cong -|_U$.

Proof. In our Section 2.5 notes we give a natural isomorphism of sheaves of modules $\gamma : \mathscr{F}|_U \longrightarrow i^*\mathscr{F}$ and it is easy to check this is an isomorphism of sheaves of algebras. $\square$

Proposition 15. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets with $f(U) \subseteq V$. Let $g : U \longrightarrow V$ be morphism induced by f. Then the following diagram on the left commutes up to canonical natural equivalence, and if $U = f^{-1}V$ then the diagram on the right commutes*

$$\begin{array}{ccc} \mathfrak{nAlg}(X) & \xleftarrow{f^*} & \mathfrak{nAlg}(Y) \\ {\scriptstyle |_U}\downarrow & & \downarrow{\scriptstyle |_V} \\ \mathfrak{nAlg}(U) & \xleftarrow[g^*]{} & \mathfrak{nAlg}(V) \end{array} \qquad \begin{array}{ccc} \mathfrak{nAlg}(X) & \xrightarrow{f_*} & \mathfrak{nAlg}(Y) \\ {\scriptstyle |_U}\downarrow & & \downarrow{\scriptstyle |_V} \\ \mathfrak{nAlg}(U) & \xrightarrow[g_*]{} & \mathfrak{nAlg}(V) \end{array}$$

The same is true with $\mathfrak{nAlg}(X)$ replaced by $\mathfrak{Alg}(X)$.

Proof. If $U = f^{-1}V$ then it is easy to see that $(-|_V)f_* = g_*(-|_U)$. If $i : U \longrightarrow X$ and $j : V \longrightarrow Y$ are the inclusions then using Lemma 14, Theorem 13 and the fact that $jg = fi$ we have a natural equivalence

$$\begin{aligned} (-|_U)f^* &\cong i^*f^* \cong (fi)^* = (jg)^* \\ &\cong g^*j^* \cong g^*(-|_V) \end{aligned}$$

as required. $\square$

Proposition 16. *Let $\varphi : A \longrightarrow B$ be a morphism of commutative rings, $X = SpecA, Y = SpecB$ and $f : Y \longrightarrow X$ the corresponding morphism of schemes. Then the following diagrams commute up to canonical natural equivalence*

$$\begin{array}{ccc} \mathfrak{nAlg}(Y) & \xrightarrow{f_*} & \mathfrak{nAlg}(X) \\ {\scriptstyle \sim}\uparrow & & \uparrow{\scriptstyle \sim} \\ B\mathbf{nAlg} & \xrightarrow[{}_A-]{} & A\mathbf{nAlg} \end{array} \qquad \begin{array}{ccc} \mathfrak{nAlg}(Y) & \xleftarrow{f^*} & \mathfrak{nAlg}(X) \\ {\scriptstyle \sim}\uparrow & & \uparrow{\scriptstyle \sim} \\ B\mathbf{nAlg} & \xleftarrow[-\otimes_A B]{} & A\mathbf{nAlg} \end{array}$$

The same is true with $\mathfrak{nAlg}(-)$ *replaced by* $\mathfrak{Alg}(-)$ *and* **nAlg** *replaced by* **Alg**.

Proof. Let N be a B-algebra, and $\eta : ({}_AN)^{\sim} \longrightarrow f_*(\widetilde{N})$ the natural isomorphism of sheaves of modules given in our Section 2.5 notes. It is not difficult to check this is also a morphism of sheaves of rings, which shows the first diagram commutes up to a canonical natural equivalence. The argument for the second diagram is similar. □

We say a morphism of ringed spaces $f : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is an *open immersion* if there is an open subset $U \subseteq X$ such that $f(Y) = U$ and the induced morphism $(Y, \mathcal{O}_Y) \longrightarrow (U, \mathcal{O}_X|_U)$ is an isomorphism.

Proposition 17. *Let* $f : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ *be an open immersion of ringed spaces and let* $h : (Y, \mathcal{O}_Y) \longrightarrow (U, \mathcal{O}_X|_U)$ *be the induced isomorphism with inverse* g. *We claim that there is a canonical natural equivalence* $f^* \cong g_* \circ (-)|_U$ *of functors* $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{nAlg}(Y)$. *The same is also true with* $\mathfrak{nAlg}(-)$ *replaced by* $\mathfrak{Alg}(-)$.

Proof. In our Section 2.5 notes we define a natural isomorphism $\kappa : g_*\mathscr{F}|_U \longrightarrow f^*\mathscr{F}$ of sheaves of modules, and it is not difficult to check that this is also a morphism of sheaves of rings, which completes the proof. □

Corollary 18. *Let* $f : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ *be an isomorphism of ringed spaces with inverse* h. *Then there is a canonical natural equivalence* $f^* \cong h_*$ *of functors* $\mathfrak{nAlg}(X) \longrightarrow \mathfrak{nAlg}(Y)$ *and* $\mathfrak{Alg}(X) \longrightarrow \mathfrak{Alg}(Y)$.

2.2 Modules

If X is a scheme and $\mathscr{S}$ a sheaf of $\mathcal{O}_X$-algebras, then in particular $\mathscr{S}$ is a sheaf of rings on X, so we stated in Definition 1 what it means to say that a sheaf of $\mathscr{S}$-modules is quasi-coherent. Since any sheaf of $\mathscr{S}$-modules can be considered as a sheaf of $\mathcal{O}_X$-modules, if we simply state that a sheaf of modules is "quasi-coherent" there is some possible ambiguity. The next result shows that if $\mathscr{S}$ itself is quasi-coherent, there is no possibility of confusion.

Proposition 19. *Let* X *be a scheme and* $\mathscr{S}$ *a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras. A sheaf of* $\mathscr{S}$*-modules* $\mathscr{M}$ *is quasi-coherent as a sheaf of* $\mathscr{S}$*-modules if and only if it is quasi-coherent as a sheaf of* $\mathcal{O}_X$*-modules.*

Proof. Suppose that $\mathscr{M}$ is quasi-coherent as a sheaf of $\mathscr{S}$-modules. Then by definition for every point $x \in X$ there is an open neighborhood U and an exact sequence in $\mathfrak{Mod}(\mathscr{S}|_U)$

$$\mathscr{S}|_U^I \longrightarrow \mathscr{S}|_U^J \longrightarrow \mathscr{M}|_U \longrightarrow 0$$

The coproducts in this sequence are coproducts in $\mathfrak{Ab}(X)$, and the sequence is exact as a sequence of sheaves of abelian groups, so it follows that it is exact as a sequence in $\mathfrak{Mod}(U)$. Therefore $\mathscr{M}|_U$ is a quasi-coherent sheaf of modules on U by (MOS,Proposition 25) and (H,5.7). Since x was arbitrary, this shows that $\mathscr{M}$ is a quasi-coherent $\mathcal{O}_X$-module.

Since the converse is local we can reduce to showing that if A is a commutative ring, B an A-algebra and M a B-module, then there is an exact sequence of A-modules $B^I \longrightarrow B^J \longrightarrow M \longrightarrow 0$ for some index sets I, J. The functor B**Mod** $\longrightarrow A$**Mod** induced by the ring morphism $A \longrightarrow B$ is exact and preserves coproducts, so this is trivial. □

Corollary 20. *Let* X *be a scheme and* $\mathscr{S}$ *a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras. Then* $\mathfrak{Qco}(\mathscr{S})$ *is an abelian subcategory of* $\mathfrak{Mod}(\mathscr{S})$.

Proof. Kernels, cokernels, images and finite products are computed in $\mathfrak{Mod}(\mathscr{S})$ in the same way as $\mathfrak{Mod}(X)$, so the claim follows from (AC,Lemma 39), Proposition 19 and the fact that $\mathfrak{Qco}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$. □

Lemma 21. *Let X be a scheme, $\mathscr{S}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras and $\mathscr{M}$ a quasi-coherent sheaf of $\mathscr{S}$-modules. If $U \subseteq X$ is affine and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism, then we claim the canonical isomorphism of sheaves of modules on $Spec\mathcal{O}_X(U)$*

$$\varepsilon : \mathscr{M}(U)^{\sim} \longrightarrow f_*\mathscr{M}|_U$$

is an isomorphism of sheaves of $\mathscr{S}(U)^{\sim}$-modules.

Proof. By Proposition 19, $\mathscr{M}$ is a quasi-coherent, so we certainly have an isomorphism ε. Also $\mathscr{S}(U)^{\sim}$ is a sheaf of quasi-coherent $\mathcal{O}_X$-algebras, so it makes sense to talk about modules over this sheaf of rings. Moreover, the isomorphism of sheaves of modules $\mathscr{S}(U)^{\sim} \cong f_*\mathscr{S}|_U$ is an isomorphism of sheaves of rings by Proposition 5, so $f_*\mathscr{M}|_U$ becomes a sheaf of $\mathscr{S}(U)^{\sim}$-modules via its module structure over $f_*\mathscr{S}|_U$. To check ε is a morphism of sheaves of $\mathscr{S}(U)^{\sim}$-modules it suffices to check that $\varepsilon(\dot{b/s} \cdot \dot{m/t}) = \dot{b/s} \cdot \varepsilon(\dot{m/t})$, as follows

$$germ_{\mathfrak{p}}\varepsilon(\dot{bm/st}) = \kappa_{\mathfrak{p}}(b \cdot m/st) = (D(st), \dot{1/st} \cdot (b \cdot m)|_{D(st)})$$

and

$$\begin{aligned} germ_{\mathfrak{p}}(\dot{b/s} \cdot \varepsilon(\dot{m/t})) &= germ_{\mathfrak{p}}(\dot{b/s}) \cdot germ_{\mathfrak{p}}\varepsilon(\dot{m/t}) = \kappa'_{\mathfrak{p}}(b/s) \cdot \kappa_{\mathfrak{p}}(m/t) \\ &= (D(s), \dot{1/s} \cdot b|_{D(s)}) \cdot (D(t), \dot{1/t} \cdot m|_{D(t)}) \\ &= (D(st), \dot{1/st} \cdot (b \cdot m)|_{D(st)}) \end{aligned}$$

where we have used the definition of the counit ε given in our Section 2.5 notes, and $\kappa_{\mathfrak{p}} : \mathscr{M}(U)_{\mathfrak{p}} \longrightarrow (f_*\mathscr{M}|_U)_{\mathfrak{p}}$ and $\kappa'_{\mathfrak{p}} : \mathscr{S}(U)_{\mathfrak{p}} \longrightarrow (f_*\mathscr{S}|_U)_{\mathfrak{p}}$ are the morphisms given there. ☐

Corollary 22. *Let X be a scheme and $\mathscr{S}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. If $\mathscr{M}, \mathscr{N}$ are quasi-coherent sheaves of $\mathscr{S}$-modules then $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of $\mathscr{S}$-modules.*

Proof. In light of Proposition 19, the sheaves $\mathscr{M}, \mathscr{N}$ are quasi-coherent as $\mathcal{O}_X$-modules, and it suffices to show that $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Given $x \in X$ let U be an affine open neighborhood with canonical isomorphism $f : U \longrightarrow Spec\mathcal{O}_X(U)$ such that $f_*\mathscr{M}|_U \cong \mathscr{M}(U)^{\sim}$, $f_*\mathscr{N}|_U \cong \mathscr{N}(U)^{\sim}$ and $f_*\mathscr{S}|_U \cong \mathscr{S}(U)^{\sim}$, where the latter isomorphism is of sheaves of algebras on $Spec\mathcal{O}_X(U)$ by Proposition 5. Then using (MAS,Proposition 3) and Lemma 21 we have an isomorphism of sheaves of $\mathscr{S}(U)^{\sim}$-modules (and therefore of sheaves of modules over the scheme $Spec\mathcal{O}_X(U)$)

$$\begin{aligned} f_*\left((\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})|_U\right) &\cong f_*(\mathscr{M}|_U \otimes_{\mathscr{S}|_U} \mathscr{N}|_U) \cong f_*\mathscr{M}|_U \otimes_{f_*\mathscr{S}|_U} f_*\mathscr{N}|_U && (1) \\ &\cong \mathscr{M}(U)^{\sim} \otimes_{\mathscr{S}(U)^{\sim}} \mathscr{N}(U)^{\sim} \cong (\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{N}(U))^{\sim} && (2) \end{aligned}$$

This shows that $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is quasi-coherent, and completes the proof. ☐

2.3 Ideals

Throughout this section let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{S}$ a commutative sheaf of $\mathcal{O}_X$-algebras.

Definition 7. If $\mathscr{F}, \mathscr{G}$ are submodules of $\mathscr{S}$ then there is a canonical morphism of sheaves of $\mathcal{O}_X$-modules

$$\begin{aligned} \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} &\longrightarrow \mathscr{S} \\ \dot{a \otimes b} &\mapsto ab \end{aligned}$$

Let $\mathscr{F}\mathscr{G}$ denote the submodule of $\mathscr{S}$ given by the image of this morphism. If $\mathscr{S} = \mathcal{O}_X$ then this is the ideal product defined in (MRS,Section 1.9). It follows from the next result and (MRS,Lemma 2) that $\mathscr{F}\mathscr{G}$ is the submodule of $\mathscr{S}$ given by sheafifying the presheaf $U \mapsto \mathscr{F}(U)\mathscr{G}(U)$ and is therefore the smallest submodule of $\mathscr{S}$ containing the products $\mathscr{F}(U)\mathscr{G}(U)$ for every open U. We denote by $\mathscr{F}^n$ the n-fold product for $n \geq 1$. It is clear that if $U \subseteq X$ is open then $(\mathscr{F}\mathscr{G})|_U = \mathscr{F}|_U\mathscr{F}|_U$. If $\mathscr{S} \cong \mathscr{T}$ is an isomorphism of commutative sheaves of $\mathcal{O}_X$-algebras let $\mathscr{F}', \mathscr{G}'$ denote the corresponding submodules of $\mathscr{T}$. Then $(\mathscr{F}\mathscr{G})' = \mathscr{F}'\mathscr{G}'$.

Lemma 23. *Let $\mathscr{F}, \mathscr{G}$ be submodules of $\mathscr{S}$. For open $U \subseteq X$ and $s \in \mathscr{S}(U)$ we have $s \in (\mathscr{F}\mathscr{G})(U)$ if and only if for every $x \in U$ there is an open neighborhood $x \in V \subseteq U$ such that $s|_V = s_1 + \cdots + s_n$ where each s_i is of the form $s_i = fg$ for some $f \in \mathscr{F}(V)$ and $g \in \mathscr{G}(V)$.*

Lemma 24. *We have the following properties of this product*

(i) If $\mathscr{F}, \mathscr{G}$ are submodules of $\mathscr{S}$ then $\mathscr{F}\mathscr{G} = \mathscr{G}\mathscr{F}$.

(ii) If $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are submodules of $\mathscr{S}$ then $(\mathscr{F}\mathscr{G})\mathscr{H} = \mathscr{F}(\mathscr{G}\mathscr{H})$.

(iii) If $\mathscr{F}, \mathscr{G}$ are submodules of $\mathscr{S}$ then $(\mathscr{F}\mathscr{G})_x = \mathscr{F}_x\mathscr{G}_x$.

(iv) If $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are submodules of $\mathscr{S}$ with $\mathscr{F} \subseteq \mathscr{G}$ then $\mathscr{F}\mathscr{H} \subseteq \mathscr{G}\mathscr{H}$. In particular if $\mathscr{F} \subseteq \mathscr{G}$ then $\mathscr{F}^n \subseteq \mathscr{G}^n$ for all $n \geq 1$.

Proof. (i),(ii) and (iv) follow from Lemma 23. For (iii) use the fact that the stalk functor $\mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$ is exact to see that $(\mathscr{F}\mathscr{G})_x$ is the image of the morphism $\mathscr{F}_x \otimes \mathscr{G}_x \longrightarrow \mathscr{S}_x$, which is obviously $\mathscr{F}_x\mathscr{G}_x$. □

Lemma 25. *Let $f : X \longrightarrow Y$ be an isomorphism of ringed spaces, $\mathscr{S}$ a commutative sheaf of $\mathcal{O}_X$-algebras and $\mathscr{F}, \mathscr{G}$ submodules of $\mathscr{S}$. Then $f_*(\mathscr{F}\mathscr{G}) = f_*\mathscr{F} \cdot f_*\mathscr{G}$.*

Proof. Immediate from Lemma 23. □

Lemma 26. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces, $\mathscr{S}$ a commutative sheaf of $\mathcal{O}_X$-algebras and $\mathscr{F}$ a submodule of $\mathscr{S}$. For $d > 0$ there is a canonical monomorphism of sheaves of modules on Y natural in $\mathscr{F}$*

$$\alpha : f_*(\mathscr{F})^d \longrightarrow f_*(\mathscr{F}^d)$$

If f is an isomorphism of ringed spaces, then α is also an isomorphism.

Proof. In fact for any $d > 0$, we have $f_*(\mathscr{F})^d \subseteq f_*(\mathscr{F}^d)$ as submodules of the commutative sheaf of $\mathcal{O}_Y$-algebras $f_*\mathscr{S}$, so the inclusion gives a monomorphism α. If f is an isomorphism, then it is easily checked that α is an isomorphism. □

Lemma 27. *Let A be a commutative ring, S a commutative A-algebra and M, N A-submodules of S. Then $M^\sim \cdot N^\sim = (MN)^\sim$.*

Proof. Set $X = SpecA$ and observe that since the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is exact we can identify $M^\sim, N^\sim$ and $(MN)^\sim$ with submodules of $S^\sim$ in the usual way. The fact that $M^\sim \cdot N^\sim = (MN)^\sim$ follows from commutativity of the following diagram

$$\begin{array}{ccc} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} & \longrightarrow & \widetilde{S} \\ \Downarrow & \nearrow & \\ (M \otimes_A N)^\sim & & \end{array}$$

which we can check by reducing to special sections. □

Proposition 28. *Let X be a scheme, $\mathscr{S}$ a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras and $\mathscr{F}, \mathscr{G}$ quasi-coherent submodules of $\mathscr{S}$. Then $\mathscr{F}\mathscr{G}$ is a quasi-coherent submodule of $\mathscr{S}$. If X is noetherian and $\mathscr{F}, \mathscr{G}$ coherent, then so is $\mathscr{F}\mathscr{G}$.*

Proof. Let $U \subseteq X$ be an affine open subset with canonical isomorphism $f : U \longrightarrow Spec\mathcal{O}_X(U)$. Then using Lemma 25 and Lemma 27 we have an isomorphism of sheaves of modules on $Spec\mathcal{O}_X(U)$

$$\begin{aligned} f_*(\mathscr{F}\mathscr{G})|_U &= f_*(\mathscr{F}|_U\mathscr{G}|_U) \\ &= f_*(\mathscr{F}|_U) \cdot f_*(\mathscr{G}|_U) \\ &\cong \mathscr{F}(U)^\sim \cdot \mathscr{G}(U)^\sim \\ &\cong (\mathscr{F}(U) \cdot \mathscr{G}(U))^\sim \end{aligned}$$

This shows that $\mathscr{F}\mathscr{G}$ is quasi-coherent. If X is noetherian and $\mathscr{F},\mathscr{G}$ coherent, then $\mathscr{F}(U),\mathscr{G}(U)$ are finitely generated and therefore so is $\mathscr{F}(U)\mathscr{G}(U)$, so $\mathscr{F}\mathscr{G}$ is also coherent. $\square$

Definition 8. Let $\mathscr{F},\mathscr{G}$ be submodules of $\mathscr{S}$. Then we define a submodule of $\mathscr{S}$ by

$$(\mathscr{F} :_{\mathscr{S}} \mathscr{G})(U) = \{r \in \mathscr{S}(U) \,|\, r|_V \mathscr{G}(V) \subseteq \mathscr{F}(V) \text{ for all open } V \subseteq U\}$$

It is clear that $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})\mathscr{G} \subseteq \mathscr{F}$. If $\mathscr{H}$ is another submodule of $\mathscr{S}$ then $\mathscr{H} \subseteq (\mathscr{F} :_{\mathscr{S}} \mathscr{G})$ if and only if $\mathscr{H}\mathscr{G} \subseteq \mathscr{F}$. If $U \subseteq X$ is an open subset then $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})|_U = (\mathscr{F}|_U :_{\mathscr{S}|_U} \mathscr{G}|_U)$.

Lemma 29. *Let $f : X \longrightarrow Y$ be an isomorphism of ringed spaces and $\mathscr{S}$ a commutative sheaf of $\mathcal{O}_X$-algebras. If $\mathscr{F},\mathscr{G}$ are submodules of $\mathscr{S}$ then $f_*(\mathscr{F} :_{\mathscr{S}} \mathscr{G}) = (f_*\mathscr{F} :_{f_*\mathscr{S}} f_*\mathscr{G})$.*

Lemma 30. *Let A be a commutative ring, S a commutative A-algebra and F, G A-submodules of S with G a finitely generated A-module. If $\mathfrak{p} \in SpecA$ then $(F :_S G)_\mathfrak{p} = (F_\mathfrak{p} :_{S_\mathfrak{p}} G_\mathfrak{p})$ as $A_\mathfrak{p}$-submodules of $S_\mathfrak{p}$.*

Proof. If $y \in (F :_S G)_\mathfrak{p}$ then $y = x/s$ for some $x \in (F :_S G)$ so it is not hard to see that $x/sG_\mathfrak{p} \subseteq F_\mathfrak{p}$, so $y \in (F_\mathfrak{p} :_{S_\mathfrak{p}} G_\mathfrak{p})$. For the converse, suppose that G is generated as an A-module by $g_1, \ldots, g_n$ and let $y \in (F_\mathfrak{p} :_{S_\mathfrak{p}} G_\mathfrak{p})$, say $y = b/t$. Then for every i we have $bx_i/t = a_i/s_i$ for some $a_i \in F$, and therefore $(s_i qb)x_i = qta_i \in F$ for some $q \notin \mathfrak{p}$. Taking $z = s_1 \cdots s_n$ it is not hard to see that $zqb \in (F :_S G)$ and therefore since $y = b/t = (zqb)/(zqt)$ we have $y \in (F :_S G)_\mathfrak{p}$ as required. $\square$

Proposition 31. *Let X be a scheme and $\mathscr{F},\mathscr{G}$ submodules of a commutative sheaf of $\mathcal{O}_X$-algebras $\mathscr{S}$. If $x \in X$ then we have $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})_x \subseteq (\mathscr{F}_x :_{\mathscr{S}_x} \mathscr{G}_x)$ as submodules of $\mathscr{S}_x$. If X is noetherian, $\mathscr{F},\mathscr{S}$ quasi-coherent and $\mathscr{G}$ coherent this is an equality.*

Proof. The inclusion $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})_x \subseteq (\mathscr{F}_x :_{\mathscr{S}_x} \mathscr{G}_x)$ is easily checked. Now suppose that X is noetherian, $\mathscr{F},\mathscr{H}$ quasi-coherent and $\mathscr{G}$ coherent. For the reverse inclusion we can reduce to the following situation: $X = SpecA$ for a commutative noetherian ring A, S is a commutative A-algebra, F, G are A-submodules of S with G finitely generated, $\mathfrak{p} \in SpecA$ and we have to show that the ring isomorphism $(S^\sim)_\mathfrak{p} \cong S_\mathfrak{p}$ identifies the submodules $(F^\sim :_{S^\sim} G^\sim)_\mathfrak{p}$ and $(F_\mathfrak{p} :_{S_\mathfrak{p}} G_\mathfrak{p})$.

From Lemma 30 we know that $(F_\mathfrak{p} :_{S_\mathfrak{p}} G_\mathfrak{p}) = (F :_S G)_\mathfrak{p}$. Given $a \in (F :_S G)_\mathfrak{p}$ and $s \notin \mathfrak{p}$ it is easy to see that the section $\dot{a/s} \in \Gamma(D(s), S^\sim)$ has the property that $\dot{a/s}G^\sim(V) \subseteq F^\sim(V)$ for any open $V \subseteq D(s)$, and therefore $\dot{a/s} \in \Gamma(D(s), (F^\sim :_{S^\sim} G^\sim))$ which completes the proof. $\square$

Corollary 32. *Let A be a commutative noetherian ring, S a commutative A-algebra and M, N A-submodules of S. If N is finitely generated then we have $(M^\sim :_{S^\sim} N^\sim) = (M :_S N)^\sim$ as submodules of $S^\sim$.*

Proof. It suffices to show that $(M^\sim :_{S^\sim} N^\sim)_\mathfrak{p} = (M :_S N)^\sim{}_\mathfrak{p}$ as submodules of $(S^\sim)_\mathfrak{p}$ for every point $\mathfrak{p} \in X$. But the proof of Proposition 31 shows that the isomorphism $(S^\sim)_\mathfrak{p} \cong S_\mathfrak{p}$ identifies both these ideals with $(M :_S N)_\mathfrak{p}$, so they must be equal. $\square$

Corollary 33. *Let X be a noetherian scheme and $\mathscr{F},\mathscr{G}$ quasi-coherent submodules of a commutative quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{S}$. If $\mathscr{G}$ is coherent then $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})$ is a quasi-coherent submodule of $\mathscr{S}$.*

Proof. If $U \subseteq X$ is an affine open subset and $f : U \longrightarrow Spec\mathcal{O}_X(U)$ the canonical isomorphism, then we have using Lemma 29 and Corollary 32 an isomorphism of sheaves of modules

$$\begin{aligned} f_*(\mathscr{F} :_{\mathscr{S}} \mathscr{G})|_U &= f_*(\mathscr{F}|_U :_{\mathscr{S}|_U} \mathscr{G}|_U) \\ &= (f_*\mathscr{F}|_U :_{f_*\mathscr{S}|_U} f_*\mathscr{G}|_U) \\ &\cong (\mathscr{F}(U)^\sim :_{\mathscr{S}(U)^\sim} \mathscr{G}(U)^\sim) \\ &\cong (\mathscr{F}(U) :_{\mathscr{S}(U)} \mathscr{G}(U))^\sim \end{aligned}$$

This shows that $(\mathscr{F} :_{\mathscr{S}} \mathscr{G})$ is quasi-coherent. $\square$

2.4 Generating Algebras

Lemma 34. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. The following conditions on two $\mathcal{O}_X$-subalgebras $\mathscr{G}, \mathscr{H}$ of $\mathscr{F}$ are equivalent:*

(i) *$\mathscr{G}$ precedes $\mathscr{H}$ as a subobject in the category $\mathfrak{Alg}(X)$;*

(ii) *$\mathscr{G}(U) \subseteq \mathscr{H}(U)$ for all open U;*

(iii) *$\mathscr{G}_x \subseteq \mathscr{H}_x$ as $\mathcal{O}_{X,x}$-subalgebras of $\mathcal{F}_x$ for all $x \in X$.*

Proof. Follows immediately from (MRS,Lemma 8). □

Definition 9. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. Given sections $s_i \in \mathscr{F}(U_i)$ let $G_x \subseteq \mathscr{F}_x$ be the $\mathcal{O}_{X,x}$-subalgebra generated by the set $\{germ_x s_i \,|\, x \in U_i\}$. For $U \subseteq X$ let

$$\mathscr{G}(U) = \{s \in \mathscr{F}(U) \,|\, germ_x s \in G_x \text{ for all } x \in U\}$$

Then $\mathscr{G}$ is a $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ which precedes any other $\mathcal{O}_X$-subalgebra containing the s_i. We call $\mathscr{G}$ the *$\mathcal{O}_X$-subalgebra* generated by the set $\{s_i\}$. It is easy to see that $\mathscr{G}_x = G_x$ for every $x \in X$.

Lemma 35. *Let X be a quasi-compact scheme and $\mathscr{F}$ a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. Consider the following conditions*

(i) *$\mathscr{F}$ is locally finitely generated as an $\mathcal{O}_X$-algebra;*

(ii) *There is an affine open cover $U_1, \dots, U_n$ of X such that $\mathscr{F}(U_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-algebra for $1 \le i \le n$.*

(iii) *There is a finite nonempty set of sections $s_i \in \mathscr{F}(U_i)$ such that the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by the set $\{s_i\}$ is $\mathscr{F}$.*

(iv) *For every $x \in X$, $\mathscr{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-algebra.*

We claim that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

Proof. $(i) \Rightarrow (ii)$ is trivial. $(ii) \Rightarrow (iii)$ For $1 \le i \le n$ let $s_{i1}, \dots, s_{iq_i}$ be a set of generators for $\mathscr{F}(U_i)$ as an $\mathcal{O}_X(U_i)$-algebra. Let $\mathscr{G}$ be the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by the sections $\{s_{ij}\}_{1 \le i \le n, 1 \le j \le q_i}$. We show that $\mathscr{G} = \mathscr{F}$ by proving that $\mathscr{G}_x = \mathscr{F}_x$ for every $x \in X$. Given $x \in X$, find an index with $x \in U_i$ and let $f : U_i \longrightarrow Spec\mathcal{O}_X(U_i)$ be the canonical isomorphism. By Proposition 5 there is a canonical isomorphism of sheaves of algebras $f_*\mathscr{F}|_{U_i} \cong \mathscr{F}(U_i)^{\sim}$. If x corresponds to the prime $\mathfrak{p} \in Spec\mathcal{O}_X(U_i)$ then there is an isomorphism of $\mathcal{O}_{X,x}$-algebras $\mathscr{F}_x \cong \mathscr{F}(U_i)_{\mathfrak{p}}$ where we identify $\mathcal{O}_{X,x}$ and $\mathcal{O}_X(U_i)_{\mathfrak{p}}$. This isomorphism identifies $germ_x s_{ij}$ with $s_{ij}/1$ for $1 \le j \le q_i$. Since $\mathscr{F}(U_i)$ is generated by the s_{ij} as an $\mathcal{O}_X(U_i)$-algebra, it follows that $\mathscr{F}(U_i)$ is generated by the $s_{ij}/1$ as an $\mathcal{O}_X(U_i)_{\mathfrak{p}}$-algebra, and therefore $\mathscr{F}_x$ is generated by the $germ_x s_{ij}$ as a $\mathcal{O}_{X,x}$-algebra, as required. $(iii) \Rightarrow (iv)$ is trivial. □

Lemma 36. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. If $\mathscr{M}$ is a $\mathcal{O}_X$-submodule of $\mathscr{F}$ then the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by $\mathscr{M}$ is $\mathscr{B}$, where*

$$\mathscr{B} = \sum_{n \ge 0} Im(\mathscr{M}^{\otimes n} \longrightarrow \mathscr{F}) \tag{3}$$

Proof. For $n \ge 0$ the morphism $\phi^0 : \mathcal{O}_X \longrightarrow \mathscr{F}$ is canonical and for $n = 1$ the morphism $\phi^1 : \mathscr{M} \longrightarrow \mathscr{F}$ is the inclusion. For $n \ge 2$ we define $\phi^n : \mathscr{M}^{\otimes n} \longrightarrow \mathscr{F}$ by induction, using the fact that $\mathscr{M}^{\otimes n} = \mathscr{M} \otimes \mathscr{M}^{\otimes(n-1)}$. Having defined ϕ^{n-1} consider the following $\mathcal{O}_X(U)$-bilinear map

$$\mathscr{M}(U) \times \mathscr{M}^{\otimes(n-1)}(U) \longrightarrow \mathscr{F}(U)$$
$$(m, n) \mapsto m\phi_U^{n-1}(n)$$

This induces a morphism out of the tensor product, which is natural in U and therefore induces a morphism of sheaves of $\mathcal{O}_X$-modules $\phi^n : \mathscr{M}^{\otimes n} \longrightarrow \mathscr{F}$. By the "$\mathcal{O}_X$-subalgebra generated by $\mathscr{M}$" we mean the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by the sections $m \in \mathscr{M}(U)$ for every open $U \subseteq X$, in the sense of Definition 9. In other words, the smallest $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ preceded by $\mathscr{M}$ as $\mathcal{O}_X$-submodules.

If we define $\mathscr{B}$ to be the $\mathcal{O}_X$-submodule of $\mathscr{F}$ in (3) then for every $x \in X$ we have

$$\mathscr{B}_x = \sum_{n \geq 0} Im((\mathscr{M}_x)^{\otimes n} \longrightarrow \mathscr{F}_x)$$

using (MRS,Lemma 69) and the fact that the stalk functor $(-)_x : \mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$ is exact and preserves all colimits. In other words $\mathscr{B}_x$ is the subgroup of $\mathscr{F}_x$ generated by the elements of $\mathcal{O}_{X,x}$ and arbitrary products of elements from $\mathscr{M}_x$, which is of course the $\mathcal{O}_{X,x}$-subalgebra of $\mathscr{F}_x$ generated by $\mathscr{M}_x$. If $\mathscr{G}$ is the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by $\mathscr{M}$, then it follows immediately from Definition 9 that $\mathscr{G}_x = \mathscr{B}_x$ for all $x \in X$ and therefore $\mathscr{G} = \mathscr{B}$ as $\mathcal{O}_X$-submodules of $\mathscr{F}$, which completes the proof. □

Lemma 37. *Let A be a commutative ring and set $X = SpecA$. Let B be a commutative A-algebra, M an A-submodule of B and C the A-subalgebra of B generated by M. Then the $\mathcal{O}_X$-subalgebra of $B^{\sim}$ generated by the $\mathcal{O}_X$-submodule $M^{\sim}$ is the $\mathcal{O}_X$-subalgebra $C^{\sim}$.*

Proof. The monomorphism of A-algebras $C \longrightarrow B$ gives a monomorphism of sheaves of commutative $\mathcal{O}_X$-algebras $C^{\sim} \longrightarrow B^{\sim}$, so we can identify $C^{\sim}$ with an $\mathcal{O}_X$-subalgebra of $B^{\sim}$. By assumption $C = \sum_{n \geq 0} Im(M^{\otimes n} \longrightarrow B)$, and since the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is exact and commutes with tensor products and colimits, we have

$$\widetilde{C} = \sum_{n \geq 0} Im(\widetilde{M}^{\otimes n} \longrightarrow \widetilde{B})$$

So the result follows from Lemma 36. □

2.5 Tensor Products

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{G}, \mathscr{H}$ sheaves of $\mathcal{O}_X$-algebras. For each open set $U \subseteq X$ the $\mathcal{O}_X(U)$-module $\mathscr{G}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{H}(U)$ becomes a $\mathcal{O}_X(U)$-algebra in a canonical way (TES,Lemma 12) and this defines a presheaf of $\mathcal{O}_X$-algebras. Therefore the sheafification $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ becomes a sheaf of $\mathcal{O}_X$-algebras in a canonical way, and we have canonical morphisms of sheaves of $\mathcal{O}_X$-algebras

$$\mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}, \quad a \mapsto a \dot{\otimes} 1$$
$$\mathscr{H} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}, \quad b \mapsto 1 \dot{\otimes} b$$

If R is a commutative ring, G, H commutative R-algebras and M, N modules over G, H respectively, then the R-module $M \otimes_R N$ is a module over the commutative ring $G \otimes_R H$ in a canonical way, with action $(g \otimes h) \cdot (m \otimes n) = (g \cdot m) \otimes (h \cdot n)$. There is an analog for sheaves of algebras. With the above notation, suppose that $\mathscr{G}, \mathscr{H}$ are commutative and let $\mathscr{M}, \mathscr{N}$ be sheaves of modules over $\mathscr{G}, \mathscr{H}$ respectively. Then $\mathscr{M}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{N}(U)$ is a module over $\mathscr{G}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{H}(U)$ and therefore the sheafification $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ is a sheaf of modules over the sheaf of commutative $\mathcal{O}_X$-algebras $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$, with action $(g \dot{\otimes} h) \cdot (m \dot{\otimes} n) = (g \cdot m) \dot{\otimes} (h \cdot n)$.

3 Sheaves of Graded Algebras

Definition 10. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of graded $\mathcal{O}_X$-algebras* is a sheaf of $\mathcal{O}_X$-algebras $\mathscr{B}$ together with a set of subsheaves of abelian groups $\mathscr{B}_d, d \geq 0$ making $\mathscr{B}$ into a sheaf of graded rings, such that for open U, $d \geq 0$ and $r \in \mathcal{O}_X(U), s \in \mathscr{B}_d(U)$ we have $r \cdot s \in \mathscr{B}_d(U)$. The following are equivalent definitions:

(a) A sheaf of $\mathcal{O}_X$-algebras $\mathscr{B}$ together with subsheaves of $\mathcal{O}_X$-modules $\mathscr{B}_d$ such that the morphisms $\mathscr{B}_d \longrightarrow \mathscr{B}$ induce an isomorphism $\bigoplus_{d\geq 0} \mathscr{B}_d \cong \mathscr{B}$ and $\mathscr{B}_d\mathscr{B}_e \subseteq \mathscr{B}_{d+e}$, $1 \in \mathscr{B}_0(X)$.

(b) A morphism of sheaves of graded rings $\mathcal{O}_X \longrightarrow \mathscr{B}$ where we grade $\mathcal{O}_X$ canonically.

A *morphism* of graded $\mathcal{O}_X$-algebras is a morphism of graded rings $\mathscr{B} \longrightarrow \mathscr{C}$ which is also a morphism of sheaves of $\mathcal{O}_X$-algebras. This makes the sheaves of graded $\mathcal{O}_X$-algebras into a category, denoted $\mathfrak{GrnAlg}(X)$. We denote the full subcategory of *commutative* graded $\mathcal{O}_X$-algebras by $\mathfrak{GrAlg}(X)$.

The image of the canonical morphism of sheaves of algebras $\mathcal{O}_X \longrightarrow \mathscr{B}$ is contained in the submodule $\mathscr{B}_0$. If the factorisation $\mathcal{O}_X \longrightarrow \mathscr{B}_0$ is an isomorphism then we write $\mathscr{B}_0 = \mathcal{O}_X$. If X is a scheme then we say that a sheaf of graded $\mathcal{O}_X$-algebras is *quasi-coherent* or *coherent* if it has this property as a sheaf of $\mathcal{O}_X$-algebras. Let $\mathfrak{QcoGrnAlg}(X)$ denote the full subcategory of $\mathfrak{GrnAlg}(X)$ consisting of the quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras, and similarly let $\mathfrak{QcoGrAlg}(X)$ denote the full subcategory of $\mathfrak{GrAlg}(X)$ consisting of the quasi-coherent sheaves of commutative graded $\mathcal{O}_X$-algebras.

Definition 11. Let $(X, \mathcal{O}_X)$ be a ringed space. A *presheaf of graded $\mathcal{O}_X$-algebras* is a presheaf of $\mathcal{O}_X$-algebras F together with sub-presheaves of $\mathcal{O}_X$-modules $F_d, d \geq 0$ such that the morphisms $F_d \longrightarrow F$ induce an isomorphism $\bigoplus_{d\geq 0} F_d \cong F$, and $F_dF_e \subseteq F_{d+e}$, $1 \in F_0(X)$. A *morphism* of presheaves of graded $\mathcal{O}_X$-algebras is a morphism of presheaves of $\mathcal{O}_X$-algebras which preserves grade. This defines the category $GrnAlg(X)$ of presheaves of graded $\mathcal{O}_X$-algebras.

If F is a presheaf of graded $\mathcal{O}_X$-algebras then the sheafification $\mathscr{F}$ of F is a sheaf of $\mathcal{O}_X$-algebras, and we let $\mathscr{F}_d$ be the submodule of $\mathscr{F}$ given by the image of the sheafified inclusion $\mathbf{a}F_d \longrightarrow \mathbf{a}F = \mathscr{F}$. Then the inclusions $\mathscr{F}_d \longrightarrow \mathscr{F}$ induce an isomorphism $\bigoplus_{d\geq 0} \mathscr{F}_d \cong \mathscr{F}$, and it is clear that $\mathscr{F}$ together with the $\mathscr{F}_d$ is a sheaf of graded $\mathcal{O}_X$-algebras. This defines a functor $\mathbf{a} : GrnAlg(X) \longrightarrow \mathfrak{GrnAlg}(X)$.

If $U \subseteq X$ and $\mathscr{B}$ together with subsheaves $\mathscr{B}_d$ is a sheaf of graded $\mathcal{O}_X$-algebras, then $\mathscr{B}|_U$ together with the subsheaves $\mathscr{B}_d|_U$ is a sheaf of graded $\mathcal{O}_X|_U$-algebras. Therefore restriction defines functors $\mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(U)$ and $\mathfrak{GrAlg}(X) \longrightarrow \mathfrak{GrAlg}(U)$. Clearly these functors are the identity if $U = X$, and the composite of the restriction functors for $V \subseteq U$ and $U \subseteq X$ is the functor for $V \subseteq X$.

If $\mathscr{B}$ is a sheaf of graded $\mathcal{O}_X$-algebras and $x \in X$, then $\mathscr{B}_x$ is a graded $\mathcal{O}_{X,x}$-algebra with the grading given by the images of the monomorphisms $\mathscr{B}_{d,x} \longrightarrow \mathscr{B}_x$ for $d \geq 0$. If $\phi : \mathscr{B} \longrightarrow \mathscr{C}$ is a morphism of sheaves of graded $\mathcal{O}_X$-algebras then $\phi_x : \mathscr{B}_x \longrightarrow \mathscr{C}_x$ is a morphism of graded $\mathcal{O}_{X,x}$-algebras. So taking stalks defines a functor $(-)_x : \mathfrak{GrnAlg}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{GrnAlg}$.

Lemma 38. *Let $U \subseteq X$ be an open subset. Then the following diagram of functors commutes up to canonical natural equivalence*

$$\begin{array}{ccc} GrnAlg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{GrnAlg}(X) \\ {\scriptstyle |_U}\downarrow & & \downarrow{\scriptstyle |_U} \\ GrnAlg(U) & \xrightarrow[\mathbf{a}]{} & \mathfrak{GrnAlg}(U) \end{array}$$

Proof. If F is a presheaf of graded $\mathcal{O}_X$-algebras then the isomorphism of sheaves of algebras $\mathbf{a}(F|_U) \cong (\mathbf{a}F)|_U$ clearly preserves grade. $\square$

Definition 12. Let X be a scheme and $\mathscr{F}$ a sheaf of commutative graded $\mathcal{O}_X$-algebras. We say $\mathscr{F}$ is *locally generated by $\mathscr{F}_1$ as an $\mathscr{F}_0$-algebra* if for every open affine subset $U \subseteq X$, $\mathscr{F}(U)$ is generated by $\mathscr{F}_1(U)$ as an $\mathscr{F}_0(U)$-algebra. We say $\mathscr{F}$ is *locally finitely generated by $\mathscr{F}_1$ as an $\mathscr{F}_0$-algebra* if for every open affine subset $U \subseteq X$, $\mathscr{F}(U)$ is finitely generated by $\mathscr{F}_1(U)$ as an $\mathscr{F}_0(U)$-algebra.

Proposition 39. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules. Then the component sheaves of modules $\mathscr{F}_d$ are quasi-coherent for all $d \geq 0$. If X is noetherian and $\mathscr{F}$ is coherent, then so are all the $\mathscr{F}_d$.*

Proof. This follows immediately from (MOS,Lemma 1). □

Proposition 40. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. If $U \subseteq X$ is an affine open subset then $\mathscr{F}(U)$ is a graded $\mathcal{O}_X(U)$-algebra with degree d component $\mathscr{F}_d(U)$. If $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{F}$-modules then $\mathscr{M}(U)$ is a graded $\mathscr{F}(U)$-module with degree n component $\mathscr{M}_n(U)$.*

Proof. We may assume U is nonempty. We need only show that $\mathscr{F}(U)$ is the internal direct sum of the submodules $\mathscr{F}_d(U)$ for $d \geq 0$. By assumption the morphisms $\mathscr{F}_d \longrightarrow \mathscr{F}$ are a coproduct, and since restriction $-|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ has a right adjoint the morphisms $\mathscr{F}_d|_U \longrightarrow \mathscr{F}|_U$ are also a coproduct. Since the modules $\mathscr{F}_d$ are quasi-coherent, there is a commutative diagram for every $d \geq 0$ (identifying $\mathfrak{Mod}(U)$ with $\mathfrak{Mod}(Spec\mathcal{O}_X(U))$)

$$\begin{array}{ccc} \mathscr{F}_d|_U & \longrightarrow & \mathscr{F}|_U \\ \Uparrow & & \Uparrow \\ \widetilde{\mathscr{F}_d(U)} & \longrightarrow & \widetilde{\mathscr{F}(U)} \end{array}$$

See our solution of Ex.5.3 for the definition of the vertical morphisms, and (5.4) for the proof they are isomorphisms. Therefore the morphisms on the bottom row are a coproduct, and since $\widetilde{-} : \mathcal{O}_X(U)\mathbf{Mod} \longrightarrow \mathfrak{Mod}(U)$ is fully faithful it reflects all colimits, and therefore the morphisms $\mathscr{F}_d(U) \longrightarrow \mathscr{F}(U)$ are a coproduct of $\mathcal{O}_X(U)$-modules, which proves the first claim.

Since $\mathscr{F}$ is a graded $\mathcal{O}_X$-algebra, for every open V the image of $\mathcal{O}_X(V)$ is contained in $\mathscr{F}_0(V)$, and therefore the subsheaves of abelian groups $\mathscr{M}_n$ are all $\mathcal{O}_X$-submodules. Therefore $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules with degree n component $\mathscr{M}_n$. Using the above argument, it is easy to check the second claim. □

Definition 13. Let A be a commutative ring and set $X = SpecA$. Let B be a graded A-algebra (B may be noncommutative). Then $B^{\sim}$ becomes a sheaf of $\mathcal{O}_X$-algebras in a canonical way, and together with the image of the monomorphism $B_d{}^{\sim} \longrightarrow B^{\sim}$ for $d \geq 0$, this defines a sheaf of graded $\mathcal{O}_X$-algebras. If B is commutative then $B^{\sim}$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras. This defines functors

$$\widetilde{-} : A\mathbf{GrnAlg} \longrightarrow \mathfrak{GrnAlg}(X)$$
$$\widetilde{-} : A\mathbf{GrAlg} \longrightarrow \mathfrak{GrAlg}(X)$$

If $\mathscr{F}$ is a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras then it follows from Proposition 40 that $\Gamma(\mathscr{F})$ is canonically a graded A-algebra with degree d subgroup $\Gamma(\mathscr{F}_d)$. This defines functors

$$\Gamma(-) : \mathfrak{QcoGrnAlg}(X) \longrightarrow A\mathbf{GrnAlg}$$
$$\Gamma(-) : \mathfrak{QcoGrAlg}(X) \longrightarrow A\mathbf{GrAlg}$$

Lemma 41. *Let A be a commutative ring and B a graded A-algebra. Then there is a canonical isomorphism of graded A-algebras $\Gamma(B^{\sim}) \cong B$.*

Proof. The isomorphism of Proposition 4(c) clearly preserves grade. □

3.1 Direct and inverse image

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces with the property that the functor $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ preserves coproducts. If $\mathscr{F}$ is a sheaf of graded $\mathcal{O}_X$-algebras then the $\mathcal{O}_Y$-algebra $f_*\mathscr{F}$ is a graded $\mathcal{O}_Y$-algebra with degree d subsheaf $f_*\mathscr{F}_d$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of graded $\mathcal{O}_X$-algebras then $f_*\phi : f_*\mathscr{F} \longrightarrow f_*\mathscr{G}$ is a morphism of graded $\mathcal{O}_X$-algebras, and in this way we define functors

$$f_* : \mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(Y)$$
$$f_* : \mathfrak{GrAlg}(X) \longrightarrow \mathfrak{GrAlg}(Y)$$

Now let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be any morphism of ringed spaces. If $\mathscr{F}$ is a sheaf of graded $\mathcal{O}_Y$-algebras then since the functor $f^* : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ preserves all colimits, the $\mathcal{O}_X$-algebra $f^*\mathscr{F}$ (see Section 2.1), together with the images of the monomorphisms $f^*\mathscr{F}_d \longrightarrow f^*\mathscr{F}$ of $\mathcal{O}_X$-modules, is a sheaf of graded $\mathcal{O}_X$-algebras. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of graded $\mathcal{O}_Y$-algebras then $f^*\phi : f^*\mathscr{F} \longrightarrow f^*\mathscr{G}$ is a morphism of graded $\mathcal{O}_X$-algebras, and in this way we define functors

$$f^* : \mathfrak{GrnAlg}(Y) \longrightarrow \mathfrak{GrnAlg}(X)$$
$$f^* : \mathfrak{GrAlg}(Y) \longrightarrow \mathfrak{GrAlg}(X)$$

Proposition 42. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces with the property that $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ and $g_* : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(Z)$ preserve coproducts. Then $g_* f_* = (gf)_*$ for functors between the categories of sheaves of graded algebras, and sheaves of commutative graded algebras.*

Corollary 43. *If $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is an isomorphism of ringed spaces then we have isomorphisms of categories $f_* : \mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(Y)$ and $f_* : \mathfrak{GrAlg}(X) \longrightarrow \mathfrak{GrAlg}(Y)$.*

Lemma 44. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. Then the following diagram of functors commutes up to canonical natural equivalence*

$$\begin{array}{ccc} GrnAlg(X) & \xrightarrow{\mathbf{a}} & \mathfrak{GrnAlg}(X) \\ {\scriptstyle f_*}\Downarrow & & \Downarrow{\scriptstyle f_*} \\ GrnAlg(Y) & \xrightarrow[\mathbf{a}]{} & \mathfrak{GrnAlg}(Y) \end{array}$$

Proof. The isomorphism given in Lemma 12 preserves grade, and therefore gives the desired isomorphism of sheaves of graded algebras. ☐

Lemma 45. *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of graded $\mathcal{O}_Y$-algebras. Then for $x \in X$ there is a canonical isomorphism of graded $\mathcal{O}_{X,x}$-algebras*

$$\tau : (f^*\mathscr{F})_x \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$
$$germ_x([V, s] \dot{\otimes} b) \mapsto germ_{f(x)} s \otimes germ_x b$$

Proof. It is easily checked that the isomorphism of Lemma 45 preserves grade. ☐

3.2 Modules

Proposition 46. *Let X be a scheme and $\mathscr{S}$ a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. Let $\mathscr{M}, \mathscr{N}$ be quasi-coherent sheaves of graded $\mathscr{S}$-modules. Then $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules and for open affine $U \subseteq X$ there is a canonical isomorphism of graded $\mathscr{S}(U)$-modules natural in $\mathscr{M}, \mathscr{N}$*

$$\mathscr{M}(U) \otimes_{\mathscr{S}(U)} \mathscr{N}(U) \longrightarrow (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})(U) \tag{4}$$
$$a \otimes b \mapsto a \dot{\otimes} b \tag{5}$$

Proof. We know from Corollary 22 that $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules, where the grading is defined in (MRS,Section 2.3). If $U \subseteq X$ is affine then $\mathscr{S}(U)$ is a graded $\mathcal{O}_X(U)$-algebra and $\mathscr{M}(U), \mathscr{N}(U), (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})(U)$ are graded $\mathscr{S}(U)$-modules by Proposition 40, so the claim at least makes sense. Evaluating the isomorphism (1) of Corollary 22 on global sections gives an isomorphism of $\mathscr{S}(U)$-modules of the form (4). Using our explicit description of the grading on $\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N}$ it is trivial to check that this is an isomorphism of graded $\mathscr{S}(U)$-modules. Naturality in both variables is easily checked. ☐

Proposition 47. *Let X be a scheme and $\mathscr{S}$ a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then $\mathfrak{QcoGrMod}(\mathscr{S})$ is an abelian subcategory of $\mathfrak{GrMod}(\mathscr{S})$.*

Proof. The full replete subcategory $\mathfrak{QcoGrMod}(\mathscr{S})$ is defined in Definition 2. We have shown in (LC,Corollary 10) that kernels, cokernels and coproducts are computed in $\mathfrak{GrMod}(\mathscr{S})$ as they are in $\mathfrak{Mod}(\mathscr{S})$. Therefore the claim follows from Corollary 20. $\square$

Lemma 48. *Let X be a scheme and $\mathscr{S}$ a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. If $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules then so is $\mathscr{M}\{d\}$ for any $d \in \mathbb{Z}$. Therefore we have an exact additive functor*

$$-\{d\} : \mathfrak{QcoGrMod}(\mathscr{S}) \longrightarrow \mathfrak{QcoGrMod}(\mathscr{S})$$

Moreover for an affine open subset $U \subseteq X$ we have an equality $\mathscr{M}\{d\}(U) = \mathscr{M}(U)\{d\}$ of graded $\mathscr{S}(U)$-modules natural in $\mathscr{M}$.

Proof. It follows from Proposition 19, (MOS,Lemma 1) and (MOS,Proposition 25) that $\mathscr{M}\{d\}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules (see (MRS,Definition 24) for the relevant definitions). The functor $-\{d\}$ is exact since $-\{d\} : \mathfrak{GrMod}(\mathscr{S}) \longrightarrow \mathfrak{GrMod}(\mathscr{S})$ is exact and $\mathfrak{QcoGrMod}(\mathscr{S})$ is an abelian subcategory of $\mathfrak{GrMod}(\mathscr{S})$.

If $U \subseteq X$ is affine and $\mathscr{M}(U)\{d\}$ the graded $\mathscr{S}(U)$-module defined in (GRM,Definition 10) then it is clear that $\mathscr{M}\{d\}(U) = \mathscr{M}(U)\{d\}$ as graded $\mathscr{S}$-modules. Naturality means that if $\phi : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism of quasi-coherent sheaves of graded $\mathscr{S}$-modules then $\phi\{d\}_U = \phi_U\{d\}$. $\square$

Definition 14. Let X be a scheme and $\mathscr{S}$ a commutative quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. If $\mathscr{M}$ is a quasi-coherent sheaf of graded $\mathscr{S}$-modules then we say $\mathscr{M}$ is *locally quasi-finitely generated* if for every affine open subset $U \subseteq X$ the graded $\mathscr{S}(U)$-module $\mathscr{M}(U)$ is quasi-finitely generated.

Definition 15. Let X be a scheme and $\varphi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of sheaves of graded $\mathcal{O}_X$-algebras. We say that φ is a *quasi-monomorphism*,*quasi-epimorphism* or *quasi-isomorphism* if it has this property as a morphism of sheaves of graded $\mathscr{S}$-modules (see (MRS,Definition 24)).

Proposition 49. *Let X be a scheme and $\varphi : \mathscr{S} \longrightarrow \mathscr{T}$ a morphism of quasi-coherent sheaves of graded $\mathcal{O}_X$-algebras. Then*

(i) φ is a quasi-monomorphism $\Rightarrow$ φ_U is a quasi-monomorphism for every affine open $U \subseteq X$.

(ii) φ is a quasi-epimorphism $\Rightarrow$ φ_U is a quasi-epimorphism for every affine open $U \subseteq X$.

(ii) φ is a quasi-isomorphism $\Rightarrow$ φ_U is a quasi-isomorphism for every affine open $U \subseteq X$.

Proof. Suppose $d \geq 0$ is such that $\varphi\{d\} : \mathscr{S}\{d\} \longrightarrow \mathscr{T}\{d\}$ is a monomorphism (epimorphism) of sheaves of graded $\mathscr{S}$-modules. Equivalently, $\varphi\{d\}$ is a monomorphism (epimorphism) of sheaves of $\mathcal{O}_X$-modules. By (MOS,Lemma 2) the morphism $\varphi\{d\}_U : \mathscr{S}\{d\}(U) \longrightarrow \mathscr{T}\{d\}(U)$ is injective (surjective) for every affine open subset $U \subseteq X$. Using Lemma 48 we see that $\varphi_U\{d\} : \mathscr{S}(U)\{d\} \longrightarrow \mathscr{T}(U)\{d\}$ is a monomorphism (epimorphism) from which it follows that φ_U is a quasi-monomorphism (quasi-epimorphism) of graded $\mathscr{S}(U)$-modules, as required. $\square$

3.3 Generating Graded Algebras

Proposition 50. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. Then $\mathscr{F}$ is locally generated by $\mathscr{F}_1$ as an $\mathcal{O}_X$-algebra if and only if the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by the submodule $\mathscr{F}_1$ is all of $\mathscr{F}$.*

Proof. In Definition 9 we stated what we mean by the "$\mathcal{O}_X$-subalgebra generated by $\mathscr{F}_1$", and we denote this $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ by $\mathscr{B}$. We know from Lemma 36 that

$$\mathscr{B} = \sum_{n \geq 0} Im(\mathscr{F}_1^{\otimes n} \longrightarrow \mathscr{F})$$

($\Rightarrow$) Suppose that $\mathscr{F}$ is locally generated by $\mathscr{F}_1$ as an $\mathcal{O}_X$-algebra. We show that $\mathscr{B} = \mathscr{F}$ by showing that $\mathscr{B}_x = \mathscr{F}_x$ for all $x \in X$. By construction $\mathscr{B}_x$ is the $\mathcal{O}_{X,x}$-subalgebra of $\mathscr{F}_x$ generated by the submodule $\mathscr{F}_{1,x}$. Any element of $\mathscr{F}_x$ can be written as $germ_x s$ for some open affine $U \subseteq X$ and $s \in \mathscr{F}(U)$. By assumption s can be written as a polynomial in the elements of $\mathscr{F}_1(U)$ with coefficients from $\mathcal{O}_X(U)$, and it follows immediately that $germ_x s \in \mathscr{B}_x$, as required.

($\Leftarrow$) Assume $\mathscr{B} = \mathscr{F}$ and let $U \subseteq X$ be affine. The functor $-|_U$ is exact and preserves all colimits, so we have

$$\mathscr{F}|_U = \sum_{n \geq 0} Im((\mathscr{F}_1|_U)^{\otimes n} \longrightarrow \mathscr{F}|_U)$$

If $f : U \longrightarrow Spec\mathcal{O}_X(U)$ is the canonical isomorphism, then applying f_* we have

$$f_*\mathscr{F}|_U = \sum_{n \geq 0} Im((f_*\mathscr{F}_1|_U)^{\otimes n} \longrightarrow f_*\mathscr{F}|_U) \tag{6}$$

By assumption $\mathscr{F}$ is quasi-coherent, and therefore so is $\mathscr{F}_1$. Since the subcategory of quasi-coherent modules is closed under all colimits, the sums and images in (6) are the categorical structures in $\mathfrak{Qco}(Spec\mathcal{O}_X(U))$. Applying the equivalence $\Gamma : \mathfrak{Qco}(Spec\mathcal{O}_X(U)) \longrightarrow \mathcal{O}_X(U)\mathbf{Mod}$ we obtain

$$\mathscr{F}(U) = \sum_{n \geq 0} Im(\mathscr{F}_1(U)^{\otimes n} \longrightarrow \mathscr{F}(U))$$

which says precisely that $\mathscr{F}(U)$ is generated by $\mathscr{F}_1(U)$ as an $\mathcal{O}_X(U)$-algebra, as required. □

Corollary 51. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of commutative graded $\mathcal{O}_X$-algebras. Then the following conditions are equivalent*

(i) $\mathscr{F}$ is locally generated by $\mathscr{F}_1$ as an $\mathcal{O}_X$-algebra;

(ii) The $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by $\mathscr{F}_1$ is all of $\mathscr{F}$;

(iii) There is a nonempty affine open cover $\{U_i\}_{i \in I}$ of X with the property that $\mathscr{F}(U_i)$ is generated by $\mathscr{F}_1(U_i)$ as a $\mathcal{O}_X(U_i)$-algebra for each $i \in I$.

Proof. We have already shown $(i) \Leftrightarrow (ii)$ in Proposition 50 and $(i) \Rightarrow (iii)$ is trivial, so it remains to show $(iii) \Rightarrow (ii)$. Suppose we are given such an open cover, and let $\mathscr{B}$ the $\mathcal{O}_X$-subalgebra of $\mathscr{F}$ generated by $\mathscr{F}_1$. It suffices to show that $\mathscr{B}|_{U_i} = \mathscr{F}|_{U_i}$ for each $i \in I$. If $f : U_i \longrightarrow Spec\mathcal{O}_X(U_i)$ is the canonical isomorphism, then it would suffice to show that $f_*\mathscr{B}|_{U_i} = f_*\mathscr{F}|_{U_i}$. But from Lemma 37 we know that the subalgebra of $\mathscr{F}(U_i)^{\sim}$ generated by $\mathscr{F}_1(U_i)^{\sim}$ is all of $\mathscr{F}(U_i)^{\sim}$, so using the isomorphism $f_*\mathscr{F}(U_i) \cong \mathscr{F}(U_i)^{\sim}$ we get the desired result. □

Note that we only define sheaves of algebras and sheaves of graded algebras over ringed spaces, although it is clear how to define a sheaf of graded algebras over a graded ringed space.

4 Sheaves of Super Algebras

Definition 16. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of super $\mathcal{O}_X$-algebras* is a sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{B}$ satisfying the following properties for every open subset $U \subseteq X$

(i) If $a \in \mathscr{B}_m(U), b \in \mathscr{B}_n(U)$ then $ab = (-1)^{mn}ba$.

(ii) If m is odd and $a \in \mathscr{B}_m(U)$ then $a^2 = 0$.

A *morphism* of super $\mathcal{O}_X$-algebras is a morphism of graded $\mathcal{O}_X$-algebras. This defines the category $\mathfrak{sAlg}(X)$ of sheaves of super $\mathcal{O}_X$-algebras.

Definition 17. Let $(X, \mathcal{O}_X)$ be a ringed space. A *presheaf of super $\mathcal{O}_X$-algebras* is a presheaf of graded $\mathcal{O}_X$-algebras F satisfying the following properties for every open subset $U \subseteq X$

(i) If $a \in F_m(U), b \in F_n(U)$ then $ab = (-1)^{mn}ba$.

(ii) If m is odd and $a \in F_m(U)$ then $a^2 = 0$.

A *morphism* of presheaves of super $\mathcal{O}_X$-algebras is a morphism of presheaves of graded $\mathcal{O}_X$-algebras. This defines the category $sAlg(X)$ of presheaves of super $\mathcal{O}_X$-algebras. Given a presheaf of super $\mathcal{O}_X$-algebras F, the sheafification $\mathscr{F}$ is a sheaf of graded $\mathcal{O}_X$-algebras in a canonical way (see Definition 11) and it is easy to check that $\mathscr{F}$ is a sheaf of super $\mathcal{O}_X$-algebras. This defines a functor $\mathbf{a} : sAlg(X) \longrightarrow \mathfrak{sAlg}(X)$.

Let $(X, \mathcal{O}_X)$ be a ringed space and $U \subseteq X$ an open subset. If $\mathscr{B}$ is a sheaf of super $\mathcal{O}_X$-algebras then the sheaf of graded $\mathcal{O}_X$-algebras $\mathscr{B}|_U$ is also a super $\mathcal{O}_X$-algebra. Therefore the canonical functor $\mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(U)$ restricts to give a functor $\mathfrak{sAlg}(X) \longrightarrow \mathfrak{sAlg}(U)$.

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces with the property that the functor $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ preserves coproducts. If $\mathscr{F}$ is a sheaf of super $\mathcal{O}_X$-algebras then the sheaf of graded $\mathcal{O}_X$-algebras $f_*\mathscr{F}$ is also a super $\mathcal{O}_X$-algebra. Therefore the canonical functor $f_* : \mathfrak{GrnAlg}(X) \longrightarrow \mathfrak{GrnAlg}(Y)$ restricts to give a functor $f_* : \mathfrak{sAlg}(X) \longrightarrow \mathfrak{sAlg}(Y)$. If f is an isomorphism of ringed spaces then this is an isomorphism of categories.

4.1 Tensor Products

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{G}, \mathscr{H}$ sheaves of super $\mathcal{O}_X$-algebras. We know from Section 2.5 that the usual tensor product $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ becomes a sheaf of $\mathcal{O}_X$-algebras in a canonical way. We also know that this tensor product becomes a sheaf of *graded* $\mathcal{O}_X$-modules in a canonical way. In fact the canonical grading (MRS,Lemma 101) makes $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ into a sheaf of graded $\mathcal{O}_X$-algebras and the canonical morphisms

$$\begin{aligned} u &: \mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H} \\ v &: \mathscr{H} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H} \end{aligned} \tag{7}$$

are morphisms of sheaves of graded $\mathcal{O}_X$-algebras. If we modify the product slightly, then we can make the tensor product of super $\mathcal{O}_X$-algebras into a super $\mathcal{O}_X$-algebra.

Proposition 52. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{G}, \mathscr{H}$ sheaves of super $\mathcal{O}_X$-algebras. Then the tensor product $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ is a super $\mathcal{O}_X$-algebra with multiplication defined for homogenous elements by*

$$(a \dot{\otimes} b)(a' \dot{\otimes} b') = (-1)^{deg(b)deg(a')} aa' \dot{\otimes} bb'$$

Moreover the canonical morphisms $\mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ and $\mathscr{H} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ are a coproduct in the category $\mathfrak{sAlg}(X)$.

Proof. For each $m, n \geq 0$ we have the canonical monomorphism of sheaves of modules $u_{m,n} : \mathscr{G}_m \otimes_{\mathcal{O}_X} \mathscr{H}_n \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$, which taken together are a coproduct (MRS,Lemma 101). Fix $m, n \geq 0$, an open set $U \subseteq X$, $a \in \mathscr{G}_m(U), b \in \mathscr{H}_n(U)$ and $i, j \geq 0$ and define a bilinear form(MRS,Definition 9)

$$\begin{gathered} t^{i,j}_{a,b} : \mathscr{G}_i|_U \times \mathscr{H}_j|_U \longrightarrow (\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})|_U \\ ((t^{i,j})_{a,b})_V(a', b') = (-1)^{ni} a|_V a' \dot{\otimes} b|_V b' \end{gathered}$$

By (MRS,Proposition 41) this induces a morphism of sheaves of modules $\theta^{i,j}_{a,b} : (\mathscr{G}_i \otimes_{\mathcal{O}_X} \mathscr{H}_j)|_U \longrightarrow (\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})|_U$. There is an induced morphism of sheaves of modules $\theta_{a,b} : (\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})|_U \longrightarrow (\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})|_U$ with $\theta_{a,b} \circ (u_{i,j})|_U = \theta^{i,j}_{a,b}$. This defines a bilinear form

$$\begin{gathered} \mathscr{G}_m \times \mathscr{H}_n \longrightarrow \mathscr{E}nd(\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}) \\ (a, b) \mapsto \theta_{a,b} \end{gathered}$$

which induces a morphism of sheaves of modules $\Theta_{m,n} : \mathscr{G}_m \otimes_{\mathcal{O}_X} \mathscr{H}_n \longrightarrow \mathscr{E}nd(\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})$. Together these induce a morphism out of the coproduct, which is defined for $m, n, i, j \geq 0, U \subseteq V$ and $a \in \mathscr{G}_m(U), b \in \mathscr{H}_n(U), a' \in \mathscr{G}_i(V), b' \in \mathscr{H}_j(V)$ by

$$\begin{gathered} \Theta : \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H} \longrightarrow \mathscr{E}nd(\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}) \\ \Theta_U(a \dot{\otimes} b)_V(a' \dot{\otimes} b') = (-1)^{ni} a|_V a' \dot{\otimes} b|_V b' \end{gathered}$$

For every open set $U \subseteq X$ we make the $\mathcal{O}_X(U)$-module $(\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H})(U)$ into a $\mathcal{O}_X(U)$-algebra with the product $ab = \Theta_U(a)_U(b)$ and identity $1 \dot{\otimes} 1$. This makes $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ into a sheaf of $\mathcal{O}_X$-algebras. With the canonical grading this is a sheaf of super $\mathcal{O}_X$-algebras. The morphisms of graded $\mathcal{O}_X$-modules $u : \mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ and $v : \mathscr{H} \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ of (7) are morphisms of super $\mathcal{O}_X$-algebras.

Given morphisms of super $\mathcal{O}_X$-algebras $\varphi : \mathscr{G} \longrightarrow \mathscr{T}$ and $\psi : \mathscr{H} \longrightarrow \mathscr{T}$ define a bilinear form

$$\kappa : \mathscr{G} \times \mathscr{H} \longrightarrow \mathscr{T}$$
$$\kappa_U(a,b) = \varphi_U(a)\psi_U(b)$$

which induces a morphism of sheaves of modules $\theta : \mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H} \longrightarrow \mathscr{T}$ with $\theta_U(a \dot{\otimes} b) = \varphi_U(a)\psi_U(b)$. It is not difficult to check that θ is the unique morphism of super $\mathcal{O}_X$-algebras with $\theta u = \varphi$ and $\theta v = \psi$, which proves that $\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{H}$ is a coproduct in $\mathfrak{sAlg}(X)$. $\square$