The Segre Embedding

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Throughout this note all rings are commutative, and A is a fixed ring. If S, T are graded A-algebras then the tensor product $S \otimes_A T$ becomes a graded A-algebra in a canonical way with the grading given by (TES,Lemma 13). That is, $S \otimes_A T$ is the coproduct of the morphisms of A-modules $S_d \otimes_A T_e \longrightarrow S \otimes_A T$ for $d, e \geq 0$. The canonical morphisms $p_1 : S \longrightarrow S \otimes_A T, p_2 : T \longrightarrow S \otimes_A T$ are then morphisms of graded A-algebras.

Definition 1. Let S, T be graded A-algebras. We define their *cartesian product*, denoted $S \times_A T$, to be the following graded A-algebra: as an A-module it is the sum of the images of the A-module morphisms $S_d \otimes_A T_d \longrightarrow S \otimes_A T$ for all $d \geq 0$. This is an A-subalgebra of $S \otimes_A T$ which is a graded A-algebra with grading $(S \times_A T)_d \cong S_d \otimes_A T_d$ for $d \geq 0$.

The scheme $Proj(S \times_A T)$ is covered by open subsets $D_+(f \otimes g)$ for $f \in S, g \in T$ homogenous of the same degree $d > 0$. It is not hard to check that the following are well-defined morphisms of A-algebras

$$\begin{array}{ll} \varphi_{f,g} : S_{(f)} \longrightarrow (S \times_A T)_{(f \otimes g)} & s/f^n \mapsto (s \otimes g^n)/(f \otimes g)^n \\ \psi_{f,g} : T_{(g)} \longrightarrow (S \times_A T)_{(f \otimes g)} & t/g^n \mapsto (f^n \otimes t)/(f \otimes g)^n \end{array}$$

If $h \in S, k \in T$ are homogenous of the same degree $e > 0$ then it is readily checked that the following diagram commutes (the vertical morphisms are the canonical ring morphisms)

$$\begin{array}{ccccc} S_{(f)} & \xrightarrow{\varphi_{f,g}} & (S \times_A T)_{(f \otimes g)} & \xleftarrow{\psi_{f,g}} & T_{(g)} \\ \downarrow & & \downarrow & & \downarrow \\ S_{(fh)} & \xrightarrow[\varphi_{fh,gk}]{} & (S \times_A T)_{(fh \otimes gk)} & \xleftarrow[\psi_{fh,gk}]{} & T_{(gk)} \end{array}$$

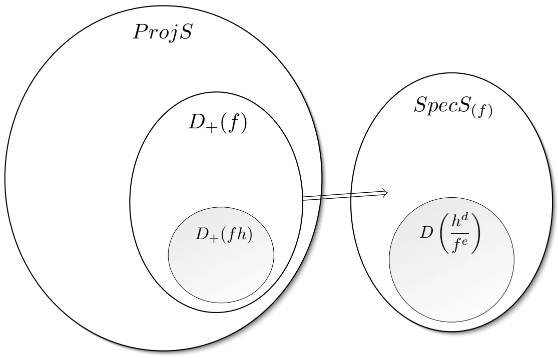
Therefore the morphisms $Spec(\varphi_{f,g})$ and $Spec(\psi_{f,g})$ glue to give morphisms of schemes over A

$$\Phi : Proj(S \times_A T) \longrightarrow ProjS, \quad \Psi : Proj(S \times_A T) \longrightarrow ProjT$$

Here Φ is the unique morphism of schemes making the left square in the following diagram commute for every pair of homogenous elements of the same positive degree $f \in S, g \in T$, and similarly for Ψ and the right square

$$\begin{array}{ccccc} ProjS & \xleftarrow{\Phi} & Proj(S \times_A T) & \xrightarrow{\Psi} & ProjT \\ \uparrow & & \uparrow & & \uparrow \\ SpecS_{(f)} & \xleftarrow[Spec(\varphi_{f,g})]{} & Spec((S \times_A T)_{(f \otimes g)}) & \xrightarrow[Spec(\psi_{f,g})]{} & SpecT_{(g)} \end{array}$$

Lemma 1. *Let S be a graded ring and $f \in S_d$ for some $d > 0$. If $h \in S_e$ for $e > 0$ then the isomorphism $D_+(f) \cong SpecS_{(f)}$ identifies the open subsets $D_+(fh)$ and $D(h^d/f^e)$.*



Proof. If $\mathfrak{p} \in ProjS$ is a homogenous prime ideal with $f \notin \mathfrak{p}$ then the image of $\mathfrak{p}$ in $SpecS_{(f)}$ is the prime ideal $\mathfrak{p}S_f \cap S_{(f)}$. It is clear that this prime belongs to $D(h^d/f^e)$ if and only if $h^d \notin \mathfrak{p}$, so if and only if $\mathfrak{p} \in D_+(fh) = D_+(f) \cap D_+(h)$. □

Lemma 2. *Let S, T be graded A-algebras and Φ, Ψ as above. If $f \in S, g \in T$ are homogenous of degree $d > 0$ then $\Phi^{-1}D_+(f) \cap \Psi^{-1}D_+(g) = D_+(f \otimes g)$.*

Proof. The inclusion $\supseteq$ is obvious, since by construction we have $\Phi(D_+(f \otimes g)) \subseteq D_+(f)$ and $\Psi(D_+(f \otimes g)) \subseteq D_+(g)$. For the reverse inclusion let $\mathfrak{p}$ be a homogenous prime of $S \times_A T$ with $\Phi(\mathfrak{p}) \in D_+(f)$ and $\Psi(\mathfrak{p}) \in D_+(g)$. There exists homogenous $h \in S, k \in T$ of the same degree $e > 0$ such that $\mathfrak{p} \in D_+(h \otimes k)$. Using Lemma 1 and the definition of the morphisms $\varphi_{h,k}, \psi_{h,k}$ we see that $h^d \otimes g^e \notin \mathfrak{p}$ and $f^e \otimes k^d \notin \mathfrak{p}$. Therefore $(f^e \otimes g^e)(h^d \otimes k^d) = f^e h^d \otimes g^e k^d \notin \mathfrak{p}$ and hence $f \otimes g \notin \mathfrak{p}$, as required. □

Proposition 3. *Let S, T be graded A-algebras, and suppose that S is generated by S_1 as an S_0-algebra and that T is generated by T_1 as a T_0-algebra. Then $Proj(S \times_A T) = ProjS \times_A ProjT$, so we have a pullback diagram*

$$\begin{array}{ccc} Proj(S \times_A T) & \xrightarrow{\Psi} & ProjT \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ ProjS & \longrightarrow & SpecA \end{array}$$

Proof. By the hypotheses on S, T the open sets of the form $D_+(f), D_+(g)$ for $f \in S_1, g \in T_1$ give open covers of $ProjS$ and $ProjT$ respectively. By the local nature of products and Lemma 2 it is enough to show that $D_+(f \otimes g) = D_+(f) \times_A D_+(g)$ or equivalently $Spec((S \times_A T)_{(f \otimes g)}) = Spec(T_{(g)}) \times_A Spec(S_{(f)})$, for every pair of homogenous elements $f \in S_1, g \in T_1$. This amounts to showing that the following diagram is a pushout of rings

$$\begin{array}{ccc} A & \longrightarrow & S_{(f)} \\ \downarrow & & \downarrow {\scriptstyle \varphi_{f,g}} \\ T_{(g)} & \xrightarrow[\psi_{f,g}]{} & (S \times_A T)_{(f \otimes g)} \end{array} \tag{1}$$

We show (1) is a pushout by showing that the ring morphism $S_{(f)} \otimes_A T_{(g)} \longrightarrow (S \times_A T)_{(f \otimes g)}$ defined by $s/f^n \otimes t/g^m \mapsto (sf^m \otimes g^n t)/(f \otimes g)^{n+m}$ is an isomorphism of rings. The proof is motivated by the technique used in (TPC,Proposition 15).

Consider the following well-defined A-bilinear map

$$S_f \times T_g \longrightarrow (S \otimes_A T)_{f \otimes g}$$
$$(s/f^n, t/g^m) \mapsto (sf^m \otimes tg^n)/(f \otimes g)^{n+m}$$

This induces a morphism of A-algebras $S_f \otimes_A T_g \longrightarrow (S \otimes_A T)_{f\otimes g}$. The canonical maps $S \longrightarrow S_f, T \longrightarrow T_g$ are morphisms of A-algebras, so we have a morphism of A-algebras $S \otimes_A T \longrightarrow S_f \otimes_A T_g$ defined by $s \otimes t \mapsto s/1 \otimes t/1$. This sends $f \otimes g$ to a unit, so there is an induced morphism of A-algebras $(S \otimes_A T)_{f\otimes g} \longrightarrow S_f \otimes_A T_g$ given by $(s \otimes t)/(f \otimes g)^n \mapsto s/f^n \otimes t/g^n$. Since we have already constructed the inverse, this is an isomorphism of A-algebras.

The rings S_f, T_g are $\mathbb{Z}$-graded, and are therefore graded A-modules. Hence $S_f \otimes_A T_g$ is a graded A-module and therefore also a $\mathbb{Z}$-graded ring. It is not hard to check that $(S \otimes_A T)_{f\otimes g} \cong S_f \otimes_A T_g$ is an isomorphism of $\mathbb{Z}$-graded rings, so it induces an isomorphism of degree zero subrings $(S \otimes_A T)_{(f\otimes g)} \cong (S_f \otimes_A T_g)_0$. The injective morphism of A-algebras $S \times_A T \longrightarrow S \otimes_A T$, which certainly does not preserve grade, nonetheless localises and restricts to give an injective ring morphism $(S \times_A T)_{(f\otimes g)} \longrightarrow (S \otimes_A T)_{(f\otimes g)} \cong (S_f \otimes_A T_g)_0$ defined by $(s \otimes t)/(f \otimes g)^n \mapsto s/f^n \otimes t/g^n$ for $s \in S_n, t \in T_n$.

Let $\alpha : S_f \otimes_{\mathbb{Z}} T_g \longrightarrow S_f \otimes_A T_g$ be the canonical morphism of groups (GRM,Section 6). The kernel of α is the abelian group P' generated by elements $(a \cdot x) \otimes y - x \otimes (a \cdot y)$ where x, y are homogenous. The morphism $S_{(f)} \otimes_{\mathbb{Z}} T_{(g)} \longrightarrow S_f \otimes_{\mathbb{Z}} T_g$ is injective since $S_{(f)}, T_{(g)}$ are direct summands of S_f, T_g respectively, and tensor products preserve colimits. Therefore the group $S_{(f)} \otimes_{\mathbb{Z}} T_{(g)}$ is isomorphic to its image in $S_f \otimes_{\mathbb{Z}} T_g$, which is mapped by α onto the image of $(S \times_A T)_{(f\otimes g)}$ in $(S_f \otimes_A T_g)_0$. So there is an isomorphism of abelian groups $(S_{(f)} \otimes_{\mathbb{Z}} T_{(g)})/P'' \cong (S \times_A T)_{(f\otimes g)}$ where $P'' = P' \cap (S_{(f)} \otimes_{\mathbb{Z}} T_{(g)})$. We can write $S_f \otimes_{\mathbb{Z}} T_g$ as the following direct sum

$$S_f \otimes_{\mathbb{Z}} T_g = \bigoplus_{p,q\in\mathbb{Z}} (S_f)_p \otimes_{\mathbb{Z}} (T_g)_q$$

Therefore it is not hard to see that P'' is generated as an abelian group by elements $(a \cdot x) \otimes y - x \otimes (a \cdot y)$ where x, y are homogenous of degree zero, that is, $x \in S_{(f)}, y \in T_{(g)}$. Hence there is an isomorphism of abelian groups

$$\begin{gathered} S_{(f)} \otimes_A T_{(g)} \cong (S_{(f)} \otimes_{\mathbb{Z}} T_{(g)})/P'' \cong (S \times_A T)_{(f\otimes g)} \\ s/f^n \otimes t/g^m \mapsto (sf^m \otimes g^n t)/(f \otimes g)^{n+m} \end{gathered}$$

This shows that (1) is a pushout, and completes the proof. □

Lemma 4. *Let S, T be graded A-algebras. If S is generated by elements $\{s_i\}_{i\in I} \subseteq S_1$ as an S_0-algebra and T is generated by $\{t_j\}_{j\in J} \subseteq T_1$ as a T_0-algebra, then $S \times_A T$ is generated by $\{s_i \otimes t_j\} \subseteq (S \times_A T)_1$ as an $(S \times_A T)_0$-algebra.*

Corollary 5. *Let A be a ring and fix integers $m, n \geq 1$. There is a canonical closed immersion $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \longrightarrow \mathbb{P}^{mn+m+n}_A$ of schemes over A, called the* Segre embedding.

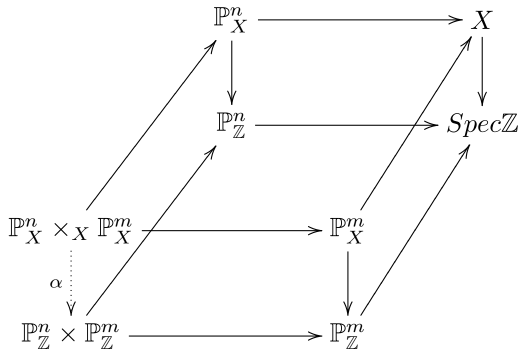
Proof. The pullback we have in mind is $Proj(A[x_0, \ldots, x_m] \times_A A[y_0, \ldots, y_n]) = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$. Consider the following morphism of graded A-algebras

$$\begin{gathered} \gamma : A[\{z_{ij}\}_{0\leq i\leq m, 0\leq j\leq n}] \longrightarrow A[x_0, \ldots, x_m] \times_A A[y_0, \ldots, y_n] \\ z_{ij} \mapsto x_i \otimes y_j \end{gathered}$$

which is surjective since the latter ring is generated as an A-algebra by the elements $x_i \otimes y_j$. Therefore the morphism of A-schemes induced by γ is the desired closed immersion. □

Proposition 6. *Let X be a scheme and fix integers $m, n \geq 1$. There is a canonical closed immersion $\mathbb{P}^m_X \times_X \mathbb{P}^n_X \longrightarrow \mathbb{P}^{mn+m+n}_X$ of schemes over X, called the* Segre embedding.

Proof. When we say "canonical" we mean that once you select specific pullbacks $\mathbb{P}^m_X$, $\mathbb{P}^n_X$, $\mathbb{P}^m_X \times_X \mathbb{P}^n_X$ and $\mathbb{P}^{mn+m+n}_X$ the definition of the closed immersion involves no arbitrary choices. Consider the following commutative diagram



where α is induced into the bottom pullback to make the diagram commute. Using standard properties of pullbacks, we see that every face of this cube is a pullback. That is,

$$\mathbb{P}^n_X \times_X \mathbb{P}^m_X = \mathbb{P}^m_X \times_{\mathbb{P}^m_{\mathbb{Z}}} (\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}})$$
$$\mathbb{P}^n_X \times_X \mathbb{P}^m_X = \mathbb{P}^n_X \times_{\mathbb{P}^n_{\mathbb{Z}}} (\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}})$$

Therefore there is a unique morphism of schemes over X, $\mathbb{P}^n_X \times_X \mathbb{P}^m_X \longrightarrow \mathbb{P}^{mn+m+n}_X$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{P}^n_X \times_X \mathbb{P}^m_X & \longrightarrow & \mathbb{P}^{mn+m+n}_X \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} & \longrightarrow & \mathbb{P}^{mn+m+n}_{\mathbb{Z}} \end{array} \tag{2}$$

where the bottom morphism is the closed immersion of Corollary 5. Once again using standard properties of pullbacks we see that (2) is a pullback, and therefore the top morphism is a closed immersion, which completes the proof. □

Corollary 7. *Let Z be a scheme and fix integers $m, n \geq 1$. There is a canonical closed immersion $\mathbb{P}^n_{\mathbb{P}^m_Z} \longrightarrow \mathbb{P}^{mn+m+n}_Z$ of schemes over Z.*

Proof. By definition we have

$$\mathbb{P}^n_{\mathbb{P}^m_Z} = \mathbb{P}^m_Z \times \mathbb{P}^n_{\mathbb{Z}} = Z \times \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$$

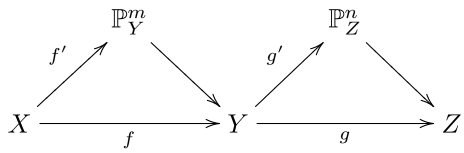
Let $\alpha : \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$ be the Segre embedding. Then the morphism

$$1_Z \times \alpha : \mathbb{P}^n_{\mathbb{P}^m_Z} = Z \times (\mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}) \longrightarrow Z \times \mathbb{P}^{mn+m+n}_{\mathbb{Z}} = \mathbb{P}^{mn+m+n}_Z$$

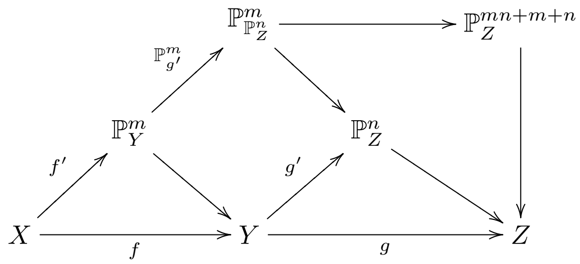
is the desired closed immersion (LocP,Proposition 1). □

Proposition 8. *The composition of projective morphisms is projective.*

Proof. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be projective morphisms, so that we have integers $m, n \geq 1$ and a commutative diagram

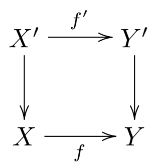


where f', g' are closed immersions. The morphism $\mathbb{P}^m_{g'} : \mathbb{P}^m_Y \longrightarrow \mathbb{P}^m_{\mathbb{P}^n_Z}$ of (TPC,Section 5) is a closed immersion, and using Corollary 7 we have a commutative diagram with the top morphism also a closed immersion



This shows that gf is projective and completes the proof. □

Lemma 9. *Projective morphisms are stable under pullback. That is, if $f : X \longrightarrow Y$ is projective and there is a pullback diagram*



then f' is projective.

Proof. This follows immediately from the construction of the morphisms $\mathbb{P}^n_f$ in (TPC,Section 5) and the fact that closed immersions are stable under pullback. □

Proposition 10. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are quasi-projective with Y noetherian, then $g \circ f$ is quasi-projective.*

Proof. Use the proof of Proposition 8, except now f', g' are immersions and we use the stability of immersions under pullback (SI,Lemma 15) and composition (SI,Lemma 16). We need Y noetherian so that $\mathbb{P}^m_Y$ is noetherian, which is the technical condition of (SI,Lemma 16). □

Proposition 11. *Let $\mathscr{P}$ be a property of morphisms of schemes such that*

(a) a closed immersion has $\mathscr{P}$.

(b) the composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$.

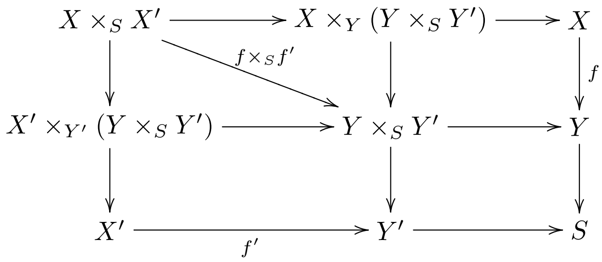
(c) $\mathscr{P}$ is stable under base extension.

Then the following holds

(d) the product of two morphisms having $\mathscr{P}$ has $\mathscr{P}$.

(e) if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$.

Proof. (d) Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be morphisms of schemes over a scheme S and form the following diagram



Using (b) and (c) it is easy to check that $f \times_S f'$ has $\mathscr{P}$. For (e) we consider X, Y as schemes over Z and f as a morphism of Z-schemes. Since g is separated over Z the graph morphism $\Gamma_f : X \longrightarrow X \times_Z Y$ is a closed immersion. Therefore Γ_f has $\mathscr{P}$ and using the definition of the graph morphism and (b), (c) we see that f also has $\mathscr{P}$. □

Corollary 12. *We have the following properties of projective morphisms*

(a) a closed immersion is projective.

(b) the composition of two projective morphisms is projective.

(c) projective morphisms are stable under base extension.

(d) the product of projective morphisms is projective.

(e) if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, and if $g \circ f$ is projective and g is separated, then f is projective.