Section 3.8 - Higher Direct Images of Sheaves

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In this note we study the higher direct image functors $R^i f_*(-)$ and the higher coinverse image functors $R^i f^!(-)$ which will play a role in our study of Serre duality. The main theorem is the proof that if $\mathscr{F}$ is quasi-coherent then so is $R^i f_*(\mathscr{F})$, which we prove first for noetherian schemes and then more generally for quasi-compact quasi-separated schemes. Most proofs are from either Hartshorne's book [1] or Kempf's paper [2], with some elaborations.

Contents

1 Definition

Definition 1. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. Then we define the *higher direct image* functors $R^i f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_* for $i \geq 0$. Since f_* is left exact there is a canonical natural equivalence $R^0 f_* \cong f_*$. For any short exact sequence of sheaves of abelian groups on X

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

there is a long exact sequence of sheaves of abelian groups on Y

$$0 \longrightarrow f_*(\mathscr{F}') \longrightarrow f_*(\mathscr{F}) \longrightarrow f_*(\mathscr{F}'') \longrightarrow R^1 f_*(\mathscr{F}') \longrightarrow \cdots$$
$$\cdots \longrightarrow R^i f_*(\mathscr{F}'') \longrightarrow R^{i+1} f_*(\mathscr{F}') \longrightarrow R^{i+1} f_*(\mathscr{F}) \longrightarrow R^{i+1} f_*(\mathscr{F}'') \longrightarrow \cdots$$

Remark 1. If the functor $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ is exact, then for $i > 0$ the higher direct image functor $R^i f_*$ is zero. In particular this is the case if f is a closed embedding (SGR,Definition 17).

Remark 2. Let X be a topological space. In (COS,Section 1.3) we defined for every $i \geq 0$ an additive functor $\mathscr{H}^i(-) : \mathfrak{Ab}(X) \longrightarrow Ab(X)$ which maps a sheaf of abelian groups $\mathscr{F}$ to the *presheaf of cohomology* defined by $\Gamma(U, \mathscr{H}^i(\mathscr{F})) = H^i(U, \mathscr{F})$. Here $H^i(U, -)$ denotes the ith right derived functor of $\Gamma(U,-) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$. By (COS,Lemma 12) the group $H^i(U,\mathscr{F})$ is canonically isomorphic to the usual cohomology group $H^i(U,\mathscr{F}|_U)$ of the sheaf $\mathscr{F}|_U$.

Proposition 1. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and $\mathscr{F}$ a sheaf of abelian groups on X. For every $i \geq 0$ there is a canonical isomorphism of sheaves of abelian groups on Y natural in $\mathscr{F}$*

$$\nu : \mathbf{a} f_* \mathscr{H}^i(\mathscr{F}) \longrightarrow R^i f_*(\mathscr{F})$$

In other words, $R^i f_*(\mathscr{F})$ is the sheafification of the presheaf

$$U \mapsto H^i(f^{-1}U, \mathscr{F})$$

Proof. For every $i \geq 0$ we have the additive functor $\mathbf{a} f_* \mathscr{H}^i(-)$ which is the composite of $\mathscr{H}^i(-)$ with the direct image for presheaves $f_* : Ab(X) \longrightarrow Ab(Y)$ and the sheafification $\mathbf{a} : Ab(Y) \longrightarrow \mathfrak{Ab}(Y)$. Note that since exactness in $Ab(X), Ab(Y)$ is computed pointwise, the functor $\mathbf{a} f_*$ is exact. Suppose we have a short exact sequence of sheaves of abelian groups on X

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

As in (COS,Section 1.3) we obtain a canonical connecting morphism $\omega^i : \mathscr{H}^i(\mathscr{F}'') \longrightarrow \mathscr{H}^{i+1}(\mathscr{F}')$ for each $i \geq 0$. The functors $\mathbf{a} f_* \mathscr{H}^i(-)$ together with the morphisms $\mathbf{a} f_* \omega^i$ define a cohomological δ-functor between $\mathfrak{Ab}(X)$ and $\mathfrak{Ab}(Y)$. As right derived functors the $H^i(U,-)$ vanish on injectives for $i > 0$, and therefore so do the functors $\mathbf{a} f_* \mathscr{H}^i(-)$. The cohomological δ-functor is therefore universal by (DF,Theorem 74).

On the other hand, the cohomological δ-functor of right derived functors $\{R^i f_*\}_{i \geq 0}$ between $\mathcal{A}$ and $\mathcal{B}$ is always universal. For any sheaf of abelian groups $\mathscr{F}$ on X there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$

$$\mathbf{a} f_* \mathscr{H}^0(\mathscr{F}) \cong \mathbf{a}(f_*\mathscr{F}) \cong f_*\mathscr{F} \cong R^0 f_*(\mathscr{F})$$

since $f_*\mathscr{F}$ is already a sheaf. This natural equivalence $\mathbf{a} f_* \mathscr{H}^0(-) \cong R^0 f_*$ induces a canonical isomorphism of cohomological δ-functors. In particular, for each $i \geq 0$ we have a canonical natural equivalence $\eta^i : \mathbf{a} f_* \mathscr{H}^i(-) \longrightarrow R^i f_*$, as required. ☐

Corollary 2. *Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Then for $i \geq 0$ there is a canonical isomorphism of sheaves of abelian groups $\mathbf{a}\mathscr{H}^i(\mathscr{F}) \longrightarrow R^i 1_*(\mathscr{F})$.*

Corollary 3. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and $V \subseteq Y$ an open subset with induced map $g : f^{-1}V \longrightarrow V$. If $\mathscr{F}$ is a sheaf of abelian groups on X then for $i \geq 0$ there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\alpha : R^i f_*(\mathscr{F})|_V \longrightarrow R^i g_*(\mathscr{F}|_{f^{-1}V})$$

Proof. Both sides are universal cohomological δ-functors between $\mathfrak{Ab}(X)$ and $\mathfrak{Ab}(V)$ in the variable $\mathscr{F}$ (DF,Definition 24), (COS,Lemma 4), (DF,Theorem 74). The canonical natural isomorphism

$$R^0 f_*(\mathscr{F})|_V \cong f_*(\mathscr{F})|_V = g_*(\mathscr{F}|_{f^{-1}V}) \cong R^0 g_*(\mathscr{F}|_{f^{-1}V})$$

induces a canonical isomorphism of the two δ-functors. In particular, for each $i \geq 0$ we have a canonical natural equivalence $\eta^i : R^i f_*(-)|_V \longrightarrow R^i g_*((-)|_{f^{-1}V})$, as required. ☐

Lemma 4. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and $g : Y \longrightarrow Z$ a closed embedding. If $\mathscr{F}$ is a sheaf of abelian groups on X then for $i \geq 0$ there is a canonical isomorphism of sheaves of abelian groups on Z natural in $\mathscr{F}$*

$$\gamma : g_* R^i f_*(\mathscr{F}) \longrightarrow R^i(gf)_*(\mathscr{F})$$

Proof. The functor $g_* : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(Z)$ is exact (SGR,Definition 17), so the result follows from (DF,Proposition 63). ☐

Remark 3. In particular we can apply Lemma 4 in the case where g is a homeomorphism, to see that taking higher direct image functors commutes with isomorphisms.

Corollary 5. *Let $f : X \longrightarrow Y$ be a continuous map of spaces and $\mathscr{F}$ a flasque sheaf of abelian groups on X. Then $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$.*

Proof. By Proposition 1 it suffices to show that $\mathscr{H}^i(\mathscr{F}) = 0$ for $i > 0$. But since $\Gamma(U, \mathscr{H}^i(\mathscr{F})) \cong H^i(U, \mathscr{F}|_U)$ (COS,Lemma 12) this follows from (COS,Proposition 5) and the fact that restrictions of flasque sheaves are flasque. □

2 Module Structure

Throughout this section let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and fix assignments of injective resolutions $\mathcal{I}, \mathcal{J}$ to the objects of $\mathfrak{Mod}(X), \mathfrak{Ab}(X)$ respectively, with respect to which all derived functors are calculated.

For clarity we denote the direct image functor for sheaves of modules by $f_*^M : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$. This is a left exact functor, and we denote its right adjoints by $R^i f_*^M(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ for $i \geq 0$. Let $U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Ab}(X)$ and $u : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Ab}(Y)$ be the forgetful functors. Then the following diagram commutes

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \xrightarrow{f_*^M} & \mathfrak{Mod}(Y) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle u} \\ \mathfrak{Ab}(X) & \xrightarrow[f_*]{} & \mathfrak{Ab}(Y) \end{array}$$

Since U, u are exact and U sends injective objects into right $f_*(-)$-acyclic objects by Corollary 5 and (COS,Lemma 3), we are in a position to apply (DF,Proposition 77) and (DF,Remark 2) to obtain a canonical natural equivalence for $i \geq 0$

$$\mu^i : u \circ R^i f_*^M(-) \longrightarrow R^i f_*(-) \circ U$$

Moreover given a short exact sequence of sheaves of modules $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ the following diagram of abelian groups commutes for $i \geq 0$

$$\begin{array}{ccc} R^i f_*^M(\mathscr{F}'') & \longrightarrow & R^{i+1} f_*^M(\mathscr{F}') \\ \Downarrow & & \Downarrow \\ R^i f_*(\mathscr{F}'') & \longrightarrow & R^{i+1} f_*(\mathscr{F}') \end{array} \qquad (1)$$

If $\mathcal{I}, \mathcal{I}'$ are two assignments of injective resolutions to the objects of $\mathfrak{Mod}(X)$ then for any sheaf of modules $\mathscr{F}$ on X the composite $R^i_{\mathcal{I}} f_*^M(\mathscr{F}) \cong R^i f_*(\mathscr{F}) \cong R^i_{\mathcal{I}'} f_*^M(\mathscr{F})$ is just the evaluation of the canonical natural equivalence $R^i_{\mathcal{I}} f_*^M(-) \cong R^i_{\mathcal{I}'} f_*^M(-)$. This means that the module structure on $R^i f_*(\mathscr{F})$ is independent of the choice of resolutions on $\mathfrak{Mod}(X)$.

Definition 2. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and fix an assignment of injective resolutions $\mathcal{J}$ to the objects of $\mathfrak{Ab}(X)$. Then for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ and $i \geq 0$ the sheaf of abelian groups $R^i f_*(\mathscr{F})$ has a canonical $\mathcal{O}_Y$-module structure. For $i = 0$ this is the structure induced by $R^0 f_*(\mathscr{F}) \cong f_*\mathscr{F}$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of $\mathcal{O}_X$-modules then $R^i f_*(\mathscr{F}) \longrightarrow R^i f_*(\mathscr{G})$ is a morphism of $\mathcal{O}_Y$-modules, so we have an additive functor $R^i f_*(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$. For a short exact sequence of sheaves of modules

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

the connecting morphism $\delta^i : R^i f_*(\mathscr{F}'') \longrightarrow R^{i+1} f_*(\mathscr{F}')$ is a morphism of $\mathcal{O}_Y$-modules for $i \geq 0$. So we have a long exact sequence of $\mathcal{O}_Y$-modules

$$0 \longrightarrow f_*(\mathscr{F}') \longrightarrow f_*(\mathscr{F}) \longrightarrow f_*(\mathscr{F}'') \longrightarrow R^1 f_*(\mathscr{F}') \longrightarrow \cdots$$

That is, the additive functors $\{R^i f_*(-)\}_{i\geq 0}$ form a universal cohomological δ-functor between $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(Y)$ (universal by Corollary 5 and (DF,Theorem 74)).

Remark 4. Let $f:(X,\mathcal{O}_X)\longrightarrow(Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathscr{F}$ be a sheaf of modules on X and fix an assignments of injective resolutions $\mathcal{J}$ to the objects of $\mathfrak{Ab}(X)$. To calculate the $\mathcal{O}_Y$-module structure on the sheaf of abelian groups $R^i f_*(\mathscr{F})$ you proceed as follows (using (DF,Remark 2)). Choose any injective resolution of $\mathscr{F}$ in $\mathfrak{Mod}(X)$

$$0\longrightarrow\mathscr{F}\longrightarrow\mathscr{I}^0\longrightarrow\mathscr{I}^1\longrightarrow\mathscr{I}^2\longrightarrow\cdots$$

and observe that this is a flasque resolution in $\mathfrak{Ab}(X)$. Suppose that the chosen injective resolution for $\mathscr{F}$ in $\mathfrak{Ab}(X)$ is

$$0\longrightarrow\mathscr{F}\longrightarrow\mathscr{J}^0\longrightarrow\mathscr{J}^1\longrightarrow\mathscr{J}^2\longrightarrow\cdots$$

Then we can lift the identity to a morphism of cochain complexes $\mathscr{I}\longrightarrow\mathscr{J}$ in $\mathfrak{Ab}(X)$ (DF,Theorem 19). Applying the functor $f_*:\mathfrak{Ab}(X)\longrightarrow\mathfrak{Ab}(Y)$ and taking cohomology at i we obtain an isomorphism of sheaves of abelian groups $R^i f_*^M(\mathscr{F})\cong R^i f_*(\mathscr{F})$, which induces the $\mathcal{O}_Y$-module structure on $R^i f_*(\mathscr{F})$.

Corollary 6. *Let $f:X\longrightarrow Y$ be a morphism of ringed spaces and $V\subseteq Y$ an open subset with induced map $g:f^{-1}V\longrightarrow V$. If $\mathscr{F}$ is a sheaf of modules on X then for $i\geq 0$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{F}$*

$$\alpha: R^i f_*(\mathscr{F})|_V\longrightarrow R^i g_*(\mathscr{F}|_{f^{-1}V})$$

Proof. Both sides are universal cohomological δ-functors between $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(V)$ in the variable $\mathscr{F}$ (DF,Definition 24) (universal by Corollary 5 and (DF,Theorem 74)). The canonical natural isomorphism

$$R^0 f_*(\mathscr{F})|_V\cong f_*(\mathscr{F})|_V=g_*(\mathscr{F}|_{f^{-1}V})\cong R^0 g_*(\mathscr{F}|_{f^{-1}V})$$

induces a canonical isomorphism of the two δ-functors. In particular, for each $i\geq 0$ we have a canonical natural equivalence $\eta^i: R^i f_*(-)|_V\longrightarrow R^i g_*((-)|_{f^{-1}V})$, as required. □

Lemma 7. *Let $f:X\longrightarrow Y$ be a morphism of ringed spaces and $g:Y\longrightarrow Z$ a closed embedding of ringed spaces. If $\mathscr{F}$ is a sheaf of modules on X then for $i\geq 0$ there is a canonical isomorphism of sheaves of modules on Z natural in $\mathscr{F}$*

$$\gamma: g_* R^i f_*(\mathscr{F})\longrightarrow R^i(gf)_*(\mathscr{F})$$

Proof. Let γ be the canonical isomorphism of sheaves of abelian groups defined in Lemma 4. We show that γ is a morphism of sheaves of modules. Choose an injective resolution $\mathscr{I}$ for $\mathscr{F}$ as an object of $\mathfrak{Mod}(X)$, and suppose that $\mathscr{J}$ is the chosen injective resolution of $\mathscr{F}$ as an object of $\mathfrak{Ab}(X)$. As in Remark 4 we lift the identity to a morphism of cochain complexes $\mathscr{I}\longrightarrow\mathscr{J}$ of sheaves of abelian groups, apply f_* and take cohomology to obtain the canonical isomorphism of sheaves of abelian groups $m: H^i(f_*\mathscr{I})\longrightarrow H^i(f_*\mathscr{J})$ which gives the latter sheaf its module structure (since the former is the cohomology of a complex of sheaves of modules).

In the same way that we induce the isomorphism γ we obtain an isomorphism of sheaves of modules $\gamma': g_* H^i(f_*\mathscr{I})\longrightarrow H^i(g_* f_*\mathscr{I})$ which fits into the following commutative diagram

$$\begin{array}{ccc} g_* H^i(f_*\mathscr{I}) & \xRightarrow{g_* m} & g_* H^i(f_*\mathscr{J}) \\ {\scriptstyle\gamma'}\Downarrow & & \Downarrow{\scriptstyle\gamma} \\ H^i(g_* f_*\mathscr{I}) & \Longrightarrow & H^i(g_* f_*\mathscr{J}) \end{array}$$

where the bottom isomorphism gives $H^i((gf)_*\mathscr{J})$ its module structure. This shows that γ is a morphism of sheaves of modules, and completes the proof. □

Proposition 8. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of modules on X. For every $i \geq 0$ there is a canonical isomorphism of sheaves of modules on Y natural in $\mathscr{F}$*

$$\nu : \mathbf{a} f_* \mathscr{H}^i(\mathscr{F}) \longrightarrow R^i f_*(\mathscr{F})$$

In other words, $R^i f_*(\mathscr{F})$ is the sheafification of the presheaf

$$U \mapsto H^i(f^{-1}U, \mathscr{F})$$

Proof. We described in (COS,Section 1.3) how the presheaf of abelian groups $\mathscr{H}^i(\mathscr{F})$ becomes a presheaf of $\mathcal{O}_X$-modules in a canonical way, and we have just described the module structure on $R^i f_*(\mathscr{F})$. The proof is the same as Proposition 1, *mutatis mutandis.* □

Recall from (CON,Definition 4) that a scheme X is *semi-separated* if it has a nonempty affine open cover with affine pairwise intersections, called a *semi-separating cover.* A semi-separated scheme is quasi-compact if and only if it has a finite semi-separating cover. If X is semi-separated with semi-separating cover $\mathfrak{U}$ then for any $V \in \mathfrak{U}$ the inclusion $V \longrightarrow X$ is affine. A similar observation applies to the elements of a *semi-separating affine basis.*

Proposition 9. *Let $f : X \longrightarrow Y$ be a morphism of semi-separated schemes with X quasi-compact, and let $\mathfrak{U}$ be a finite semi-separating cover of X. If $\mathscr{F}$ is a quasi-coherent sheaf on X, then*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots$$

is an f_*-acyclic resolution of $\mathscr{F}$ by quasi-coherent sheaves, and for $p \geq 0$ there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$

$$H^p(f_* \mathscr{C}(\mathfrak{U}, \mathscr{F})) \longrightarrow R^p f_*(\mathscr{F})$$

Proof. In (COS,Theorem 35) we showed that this resolution was quasi-coherent and $\Gamma(X, -)$-acyclic, and we seek to upgrade this acyclicity. That is, we want to show that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is right f_*-acyclic for $p \geq 0$. By Proposition 8 it would suffice to show that for $i > 0$ and $V \subseteq Y$ affine belonging to a semi-separating affine *basis* of Y

$$H^i(f^{-1}V, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_{f^{-1}V}) = 0$$

The inclusion $V \longrightarrow Y$ is affine, and therefore by pullback so is the inclusion $f^{-1}V \longrightarrow X$. For indices $i_0 < \cdots < i_p$ of the cover $\mathfrak{U}$ we write

$$u_{i_0,\ldots,i_p} : U_{i_0} \cap \cdots \cap U_{i_p} \longrightarrow X, \quad v_{i_0,\ldots,i_p} : U_{i_0} \cap \cdots \cap U_{i_p} \cap f^{-1}V \longrightarrow f^{-1}V$$

for the inclusions. Observe that both of these morphisms are affine by our assumptions, and the open set $U_{i_0} \cap \cdots \cap U_{i_p} \cap f^{-1}V$ is affine because the morphism $f^{-1}V \longrightarrow X$ is affine. We have a canonical isomorphism of sheaves of modules

$$\begin{aligned} \mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_{f^{-1}V} &\cong \bigoplus_{i_0 < \cdots < i_p} u_{i_0,\ldots,i_p *}(\mathscr{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}})|_{f^{-1}V} \\ &\cong \bigoplus_{i_0 < \cdots < i_p} v_{i_0,\ldots,i_p *}(\mathscr{F}|_{U_{i_0} \cap \cdots \cap U_{i_0} \cap f^{-1}V}) \end{aligned}$$

But for each sequence $i_0 < \cdots < i_p$ we have by (COS,Corollary 28)

$$H^i(f^{-1}V, v_{i_0,\ldots,i_p *}(\mathscr{F}|_{U_{i_0} \cap \cdots \cap U_{i_0} \cap f^{-1}V})) \cong H^i(U_{i_0} \cap \cdots \cap U_{i_p} \cap f^{-1}V, \mathscr{F}|_{U_{i_0} \cap \cdots \cap U_{i_0} \cap f^{-1}V}) = 0$$

since the higher cohomology of quasi-coherent sheaves on affine schemes vanishes. This calculation is enough to show that $H^i(f^{-1}V, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_{f^{-1}V}) = 0$ as required.

From (DTC2,Remark 14) we deduce a canonical isomorphism of sheaves of *abelian groups* $H^p(f_* \mathscr{C}(\mathfrak{U}, \mathscr{F})) \longrightarrow R^p f_*(\mathscr{F})$, which is natural with respect to morphisms of sheaves of abelian groups. □

3 Direct Image and Quasi-coherence

The following result and its corollary are central. We give two proofs: the first one given in this section only works in the case where X is noetherian, but has the advantage of being short. In Section 5 we will give a proof more generally for concentrated schemes, but this will require us to develop a little bit more technical machinery.

Proposition 10. *Let $f : X \longrightarrow Y$ be a morphism of schemes where X is noetherian and $Y = SpecA$ is affine. Then for any quasi-coherent sheaf of modules $\mathscr{F}$ on X and $i \geq 0$ there is a canonical isomorphism of sheaves of modules on Y natural in $\mathscr{F}$*

$$\beta : R^i f_*(\mathscr{F}) \longrightarrow H^i(X, \mathscr{F})^{\sim}$$

Proof. We give $R^i f_*(\mathscr{F})$ the canonical $\mathcal{O}_Y$-module structure of Definition 2. We know from (COS,Definition 5) that the abelian group $H^i(X, \mathscr{F})$ has a canonical $\Gamma(X, \mathcal{O}_X)$-module structure, and therefore also an A-module structure. Since X is noetherian the sheaf $f_*\mathscr{F}$ is quasi-coherent, and we therefore have a canonical isomorphism of sheaves of modules $f_*\mathscr{F} \cong \Gamma(X, \mathscr{F})^{\sim}$. Therefore for $i = 0$ we have a canonical isomorphism natural in $\mathscr{F}$

$$\mu^0 : R^0 f_*(\mathscr{F}) \cong f_*(\mathscr{F}) \cong \Gamma(X, \mathscr{F})^{\sim} \cong H^0(X, \mathscr{F})^{\sim}$$

Since $\tilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(Y)$ is exact, we have two cohomological δ-functors $\{R^i f_*(-)\}_{i \geq 0}$ and $\{H^i(X, -)^{\sim}\}_{i \geq 0}$ between $\mathfrak{Qco}(X)$ and $\mathfrak{Mod}(Y)$. By (COS,Corollary 22) any quasi-coherent sheaf $\mathscr{F}$ on X can be embedded in a flasque, quasi-coherent sheaf. Hence both δ-functors are effaceable for $i > 0$, and therefore universal (DF,Theorem 74). Therefore μ^0 gives rise to a canonical isomorphism μ of δ-functors. In particular, for each $i \geq 0$ we have a canonical natural equivalence $\mu^i : R^i f_*(-) \longrightarrow H^i(X, -)^{\sim}$ of additive functors $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(Y)$, as required. □

Corollary 11. *Let $f : X \longrightarrow Y$ be a morphism of schemes, with X noetherian. Then for any quasi-coherent sheaf $\mathscr{F}$ on X the sheaves $R^i f_*(\mathscr{F})$ are quasi-coherent on Y for $i \geq 0$.*

Proof. Let $y \in Y$ be given, and find an affine open neighborhood V of y together with an isomorphism $h : V \longrightarrow SpecA$. Let $g : f^{-1}V \longrightarrow V$ be the morphism induced by f. Then by Proposition 10, Lemma 7 and Corollary 6 we have an isomorphism of sheaves of modules for $i \geq 0$

$$\begin{aligned} h_*(R^i f_*(\mathscr{F})|_V) &\cong h_* R^i g_*(\mathscr{F}|_{f^{-1}V}) \\ &\cong R^i((hg)_*(\mathscr{F}|_{f^{-1}V})) \\ &\cong H^i(f^{-1}V, \mathscr{F}|_{f^{-1}V})^{\sim} \end{aligned}$$

which shows that $R^i f_*(\mathscr{F})$ is quasi-coherent. □

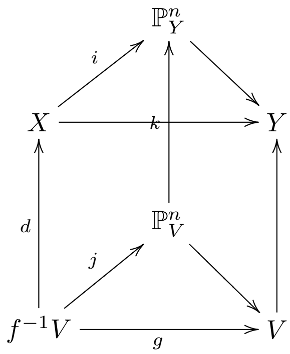
Remark 5. The proof of Proposition 10 also works under slightly different hypothesis. Let X be a quasi-compact semi-separated scheme with finite semi-separating cover $\mathfrak{U}$. Then any quasi-coherent sheaf $\mathscr{F}$ on X can be embedded in the sheaf $\mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ which by Proposition 10 is quasi-coherent and both $\Gamma(X, -)$ and f_*-acyclic. This allows us to show that both δ-functors in the proof of Proposition 10 are effaceable without the noetherian hypothesis on X, and the rest of the proof is the same.

In particular we deduce that the conclusion of Corollary 11 holds for a quasi-compact separated morphism of schemes $f : X \longrightarrow Y$. In fact we show in Section 5 that with a little more work one can weaken "separated" to "quasi-separated".

Lemma 12. *Let $f : X \longrightarrow Y$ be a morphism of schemes and $\mathscr{L}$ a very ample sheaf on X relative to Y. If $V \subseteq Y$ is open and $g : f^{-1}V \longrightarrow V$ the induced morphism of schemes, then $\mathscr{L}|_{f^{-1}V}$ is very ample relative to V.*

Proof. It is clear that $\mathscr{L}|_{f^{-1}V}$ is invertible. Suppose $i : X \longrightarrow \mathbb{P}^n_Y$ is an immersion of Y-schemes with $i^*\mathcal{O}(1) \cong \mathscr{L}$. Let $k : \mathbb{P}^n_V \longrightarrow \mathbb{P}^n_Y$ be the morphism induced by the inclusion $V \longrightarrow Y$ (see

(TPC,Section 5)). Then we induce a morphism $j : f^{-1}V \longrightarrow \mathbb{P}^n_V$ into the pullback making every face of the following diagram commute



Using standard properties of pullbacks we see that every square face is a pullback, and in particular j is an immersion of schemes over V (SI,Lemma 15). It is not difficult to see that $k^*\mathcal{O}(1) \cong \mathcal{O}(1)$ and therefore there is an isomorphism of sheaves of modules

$$j^*\mathcal{O}(1) \cong j^*k^*\mathcal{O}(1) \cong d^*i^*\mathcal{O}(1) \cong d^*\mathscr{L} \cong \mathscr{L}|_{f^{-1}V}$$

which shows that $\mathscr{L}|_{f^{-1}V}$ is very ample relative to V. ☐

Lemma 13. *Let $f : X \longrightarrow Y$ be a morphism of schemes with X noetherian and Y affine, and let $\mathscr{F}$ be a quasi-coherent sheaf of modules on X. Then $\mathscr{F}$ is generated by global sections if and only if the counit $\varepsilon : f^*f_*\mathscr{F} \longrightarrow \mathscr{F}$ is an epimorphism.*

Proof. Since X is noetherian the sheaf $f_*\mathscr{F}$ is quasi-coherent, and is therefore generated by global sections. For $x \in X$ we have a commutative diagram

$$\begin{array}{ccc} (f^*f_*\mathscr{F})_x & \xrightarrow{\varepsilon_x} & \mathscr{F}_x \\ \Downarrow & \nearrow_{\varphi} & \\ (f_*\mathscr{F})_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} & & \end{array}$$

where $\varphi((V,s)\otimes r) = r\cdot(f^{-1}V, s)$. It follows that the image of ε_x is the $\mathcal{O}_{X,x}$-submodule generated by the global sections. Therefore ε is an epimorphism if and only if $\mathscr{F}$ is generated by global sections. ☐

Theorem 14. *Let $f : X \longrightarrow Y$ be a projective morphism of noetherian schemes, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{F}$ be a coherent sheaf on X. Then*

(a) For all sufficiently large n, the natural morphism $f^*f_*(\mathscr{F}(n)) \longrightarrow \mathscr{F}(n)$ is an epimorphism.

(b) For all $i \geq 0$ the sheaf of modules $R^if_*(\mathscr{F})$ is coherent.

(c) For $i > 0$ and all sufficiently large n, we have $R^if_*(\mathscr{F}(n)) = 0$.

Proof. First we reduce to the case where Y is affine. Given a point $y \in Y$ let V be an affine open neighborhood, set $U = f^{-1}V$ and let $g : U \longrightarrow V$ the induced projective morphism of noetherian schemes (SEM,Lemma 9). By Lemma 12 the invertible sheaf $\mathscr{L}|_U$ is very ample relative to V, and it is not difficult to check that there is a commutative diagram for $n > 0$

$$\begin{array}{ccc} f^*f_*(\mathscr{F}(n))|_U & \xrightarrow{\varepsilon|_U} & \mathscr{F}(n)|_U \\ \Downarrow & & \Downarrow \\ g^*g_*(\mathscr{F}|_U(n)) & \xrightarrow[\varepsilon]{} & \mathscr{F}|_U(n) \end{array}$$

This shows that (a) is a local question, while $(b), (c)$ are local by Corollary 6. So using quasi-compactness of Y we can reduce to the case where Y is affine. Then by Lemma 13, (a) becomes the statement that for n sufficiently large $\mathscr{F}(n)$ is generated by global sections, which follows from (H, II 5.17). To prove $(b), (c)$ we can by Lemma 7 reduce slightly to the case where $Y = SpecA$ for some noetherian ring A. For $i \geq 0$ there is by Proposition 10 an isomorphism of sheaves of modules

$$R^i f_*(\mathscr{F}) \cong H^i(X, \mathscr{F})^{\sim}$$

Therefore (b) follows (COS,Theorem 43)(a), while (c) follows from (COS,Theorem 43)(b).

□

Remark 6. Let $f : X \longrightarrow Y$ be a closed immersion of noetherian schemes. In particular f is projective, so the functors $R^i f_*$ preserve quasi-coherent and coherent sheaves for $i \geq 0$.

4 Higher Coinverse Image

Definition 3. Let $f : X \longrightarrow Y$ be a closed immersion of schemes. There is an additve functor $f^! : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ which is right adjoint to f_* (MRS,Proposition 97) and we define the *higher coinverse image* functors $R^i f^! : \mathfrak{Mod}(Y) \longrightarrow \mathfrak{Mod}(X)$ to be the right derived functors of $f^!$ for $i \geq 0$. Since $f^!$ is left exact there is a canonical natural equivalence $R^0 f^! \cong f^!$. For any short exact sequence of sheaves of modules on Y

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

there is a long exact sequence of sheaves of modules on X

$$0 \longrightarrow f^!(\mathscr{F}') \longrightarrow f^!(\mathscr{F}) \longrightarrow f^!(\mathscr{F}'') \longrightarrow R^1 f^!(\mathscr{F}') \longrightarrow \cdots$$
$$\cdots \longrightarrow R^i f^!(\mathscr{F}'') \longrightarrow R^{i+1} f^!(\mathscr{F}') \longrightarrow R^{i+1} f^!(\mathscr{F}) \longrightarrow R^{i+1} f^!(\mathscr{F}'') \longrightarrow \cdots$$

Proposition 15. *Let $f : X \longrightarrow Y$ be a closed immersion of schemes and $\mathscr{F}$ a sheaf of modules on Y. Then for $i \geq 0$ there is a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$*

$$\rho^i : R^i f^!(\mathscr{F}) \longrightarrow f^{-1} \mathscr{E}xt^i_Y(f_* \mathcal{O}_X, \mathscr{F})$$

Proof. Both sides are universal cohomological δ-functors between $\mathfrak{Mod}(Y)$ and $\mathfrak{Ab}(X)$, and the case $i = 0$ is (MRS,Proposition 96), so we induce a natural equivalence in all degrees $i \geq 0$ in the usual way. □

If $f : X \longrightarrow Y$ is a closed immersion of noetherian schemes, the functor $f^!$ preserves quasi-coherent and coherent sheaves. We want to show this is also true for the higher coinverse image functors. First we prove the higher versions of (MOS,Proposition 17), (MOS,Proposition 18) and (MOS,Proposition 20).

Proposition 16. *Let $f : X \longrightarrow Y$ be a closed immersion of schemes, $V \subseteq Y$ an open subset and $g : f^{-1}V \longrightarrow V$ the induced morphism. Then for a sheaf of modules $\mathscr{F}$ on Y and $i \geq 0$ there is a canonical isomorphism of sheaves of modules on $f^{-1}V$ natural in $\mathscr{F}$*

$$\theta^i : (R^i f^! \mathscr{F})|_{f^{-1}V} \longrightarrow R^i g^!(\mathscr{F}|_V)$$

That is, there is a canonical natural equivalence $\theta^i : (-|_{f^{-1}V}) R^i f^! \longrightarrow R^i g^!(-|_V)$.

Proof. Both sides are universal cohomological δ-functors between $\mathfrak{Mod}(Y)$ and $\mathfrak{Mod}(f^{-1}V)$, and the case $i = 0$ is (MOS,Proposition 17), so we induce a natural equivalence in all degrees $i \geq 0$ in the usual way. □

Proposition 17. *Suppose there is a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle k}\Downarrow & & \Downarrow{\scriptstyle h} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

with f, g closed immersions. Then for any sheaf of modules $\mathscr{F}$ on Y and $i \geq 0$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{F}$

$$\mu^i : k_*(R^i f^! \mathscr{F}) \longrightarrow R^i g^!(h_* \mathscr{F})$$

That is, there is a canonical natural equivalence $\mu^i : k_* \circ R^i f^! \longrightarrow R^i g^! \circ h_*$.

Proof. Use the same argument given in the proof of Proposition 16, using (MOS,Proposition 18) in the case $i = 0$. □

Proposition 18. *Let $\phi : A \longrightarrow B$ be a surjective morphism of noetherian rings and $f : X \longrightarrow Y$ the corresponding closed immersion of affine schemes. For any A-module M and $i \geq 0$ there is a canonical isomorphism of sheaves of modules natural in M*

$$\zeta^i : Ext^i_A(B, M)^\sim \longrightarrow R^i f^!(\widetilde{M})$$

Proof. Both sides are cohomological δ-functors between A**Mod** and $\mathfrak{Mod}(X)$. To show they are universal, it suffices to show that for $i > 0$ they both vanish on injective A-modules. For $Ext^i_A(B, -)^\sim$ this is trivial. If I is an injective A-module, then $I^\sim$ is an injective sheaf of modules in $\mathfrak{Mod}(X)$ (COS,Remark 7), so $R^i f^!(I^\sim) = 0$ for $i > 0$. Therefore both sides are universal δ-functors. We have a natural equivalence in degree zero by (MOS,Proposition 20), and therefore a canonical natural equivalence ζ^i for $i \geq 0$ by the usual argument. □

Corollary 19. *Let $f : X \longrightarrow Y$ be a closed immersion of noetherian schemes. If $\mathscr{F}$ is a quasi-coherent sheaf of modules on Y then the sheaves $R^i f^! \mathscr{F}$ are quasi-coherent for $i \geq 0$. If $\mathscr{F}$ is coherent, then so is $R^i f^! \mathscr{F}$ for $i \geq 0$.*

Proof. Let $\mathscr{F}$ be a quasi-coherent sheaf of modules on Y, and let $x \in X$ be given. Let V be an affine open neighborhood of $f(x)$ and set $U = f^{-1}V$, which is also affine. Let $g : U \longrightarrow V, F : Spec\mathcal{O}_X(U) \longrightarrow Spec\mathcal{O}_Y(V)$ be the induced closed immersions and $k : U \longrightarrow Spec\mathcal{O}_X(U), h : V \longrightarrow Spec\mathcal{O}_Y(V)$ the canonical isomorphisms. Then using Proposition 16, Proposition 17 and Proposition 18 we have an isomorphism of sheaves of modules for $i \geq 0$

$$\begin{aligned} k_*(R^i f^! \mathscr{F})|_U &\cong k_*(R^i g^!(\mathscr{F}|_V)) \\ &\cong R^i F^! h_*(\mathscr{F}|_V) \\ &\cong R^i F^!(\mathscr{F}(V)^\sim) \\ &\cong Ext^i_{\mathcal{O}_Y(V)}(\mathcal{O}_X(U), \mathscr{F}(V))^\sim \end{aligned}$$

which shows that $R^i f^! \mathscr{F}$ is quasi-coherent. If $\mathscr{F}$ is coherent then $\mathscr{F}(V)$ is finitely generated, and hence so is the $\mathcal{O}_X(U)$-module $Ext^i_{\mathcal{O}_Y(V)}(\mathcal{O}_X(U), \mathscr{F}(V))$ (EXT,Proposition 9), which shows that $R^i f^! \mathscr{F}$ is coherent. □

Remark 7. Let $f : X \longrightarrow Y$ be a closed immersion of noetherian schemes. Then for $i \geq 0$ the functors $R^i f_*$ and $R^i f^!$ preserve quasi-coherent and coherent sheaves.

5 Direct Image and Quasi-coherence (General case)

The aim of this section is to prove that for a concentrated morphism of schemes $f : X \longrightarrow Y$ the higher direct image functors $R^i f_*(-)$ preserve quasi-coherence, thus generalising Corollary 11. We follow the elegant proof given in Kempf's paper [2]. Here is the proof in outline. First we show how to localise sheaves with respect to a global section. Given $\mathscr{F}$ and $f \in \Gamma(X, \mathcal{O}_X)$ one takes the direct limit $\mathscr{F}_{(f)}$ over the direct system

$$\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \cdots$$

where the morphisms are multiplication by f. The elementary properties of this construction take up Section 5.1. The main result is that this construction localises cohomology $H^i(X, \mathscr{F})_f \cong H^i(X, \mathscr{F}_{(f)})$. In Section 5.3 we show that localising $\mathscr{F}$ is the same as restricting to X_f and then pushing back up to the whole scheme. Finally in Section 5.4 we put all of this together and give the proof of the main theorem.

5.1 Localisations of Sheaves

A direct system of sheaves which commonly arises is successive multiplication by a section of the structure sheaf or, more generally, by a section of an invertible sheaf. Let f be a global section of an invertible sheaf $\mathscr{L}$ on a scheme X. For any sheaf of modules $\mathscr{F}$ on X, we have the direct system

$$\mathscr{F} \longrightarrow \mathscr{F} \otimes \mathscr{L} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes 2} \longrightarrow \cdots \tag{2}$$

given by multiplication by f. This direct system is natural in $\mathscr{F}$. To be clear, these morphisms are defined for sections over an open set U by

$$\begin{aligned} \mathscr{F} \longrightarrow \mathscr{F} \otimes \mathscr{L} \quad & a \mapsto a \dot{\otimes} f|_U \\ \mathscr{F} \otimes \mathscr{L}^{\otimes i} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes (i+1)} \quad & a \dot{\otimes} f_1 \dot{\otimes} \cdots \dot{\otimes} f_i \mapsto a \dot{\otimes} f_1 \dot{\otimes} \cdots \dot{\otimes} f_i \dot{\otimes} f|_U \end{aligned}$$

If we take $\mathscr{F}$ to be a sheaf of commutative algebras and $\mathscr{L} = \mathcal{O}_X$, the direct limit $\varinjlim(\mathscr{F} \otimes \mathcal{O}_X^{\otimes n})$ becomes a sheaf of commutative algebras in a canonical way. In particular $\varinjlim(\mathcal{O}_X \otimes \mathcal{O}_X^{\otimes n})$ is a sheaf of algebras, which behaves like the localisation A_f of a ring A, and for any sheaf of modules $\mathscr{F}$ the sheaf $\varinjlim(\mathscr{F} \otimes \mathcal{O}_X^{\otimes n})$ is a sheaf of modules over $\varinjlim(\mathcal{O}_X \otimes \mathcal{O}_X^{\otimes n})$ (which one should compare with M_f being a module over A_f). In the remainder of this section we check the details of these claims.

Let X be a scheme, $f \in \Gamma(X, \mathcal{O}_X)$ a global section and $\mathscr{F}$ a sheaf of commutative $\mathcal{O}_X$-algebras. We have the direct system of multiplication by f

$$\mathscr{F} \longrightarrow \mathscr{F} \otimes \mathcal{O}_X \longrightarrow \mathscr{F} \otimes \mathcal{O}_X^{\otimes 2} \longrightarrow \cdots \tag{3}$$

Each of these sheaves is a sheaf of commutative $\mathcal{O}_X$-algebras in a canonical way (SOA,Section 2.5), but multiplication by f is certainly not a morphism of $\mathcal{O}_X$-algebras so we cannot define the ring structure on $\varinjlim(\mathscr{F} \otimes \mathcal{O}_X^{\otimes n})$ in the naive way (by just taking direct limits).

The algebra structure on $\mathscr{F}$ gives a canonical morphism of sheaves of modules $\mathscr{F} \otimes \mathscr{F} \longrightarrow \mathscr{F}$ which we use to define the following morphism of sheaves of modules for $d, e \geq 0$

$$\begin{gathered} \rho^{d,e} : (\mathscr{F} \otimes \mathcal{O}_X^{\otimes d}) \otimes (\mathscr{F} \otimes \mathcal{O}_X^{\otimes e}) \cong (\mathscr{F} \otimes \mathscr{F}) \otimes \mathcal{O}_X^{\otimes (d+e)} \longrightarrow \mathscr{F} \otimes \mathcal{O}_X^{\otimes (d+e)} \\ (m \dot{\otimes} f_1 \dot{\otimes} \cdots \dot{\otimes} f_d) \dot{\otimes} (n \dot{\otimes} g_1 \dot{\otimes} \cdots \dot{\otimes} g_e) \mapsto mn \dot{\otimes} f_1 \dot{\otimes} \cdots \dot{\otimes} f_d \dot{\otimes} g_1 \dot{\otimes} \cdots \dot{\otimes} g_e \end{gathered}$$

Note that since the tensor products occur over $\mathcal{O}_X$, we can rearrange the $f_1, \ldots, f_d, g_1, \ldots, g_e$ into any order we like. This is a cruical point in what follows, which is why we can't expect the same thing to work for an arbitrary invertible sheaf $\mathscr{L}$. The following lemma simply says that if you apply the morphisms in the direct system to either term in a product (that is, stick extra factors of f on the end) and then multiply, you get the original product with the factors of f at the end.

Lemma 20. *For any $d, e \geq 0$ and $k > 0$ the following diagram commutes*

$$\begin{array}{ccc}
(\mathscr{F} \otimes \mathcal{O}_X^{\otimes d}) \otimes (\mathscr{F} \otimes \mathcal{O}_X^{\otimes e}) & \xrightarrow{\rho^{d,e}} & \mathscr{F} \otimes \mathcal{O}_X^{\otimes(d+e)} \\
\downarrow & & \downarrow \\
(\mathscr{F} \otimes \mathcal{O}_X^{\otimes(d+k)}) \otimes (\mathscr{F} \otimes \mathcal{O}_X^{\otimes e}) & \xrightarrow[\rho^{d+k,e}]{} & \mathscr{F} \otimes \mathcal{O}_X^{\otimes(d+k+e)}
\end{array}$$

The analogous diagram in the second variable also commutes. In addition the following diagram commutes

$$\begin{array}{ccc}
(\mathscr{F} \otimes \mathcal{O}_X^{\otimes d}) \otimes (\mathscr{F} \otimes \mathcal{O}_X^{\otimes e}) & \Longrightarrow & (\mathscr{F} \otimes \mathcal{O}_X^{\otimes e}) \otimes (\mathscr{F} \otimes \mathcal{O}_X^{\otimes d}) \\
{\scriptstyle \rho^{d,e}} \searrow & & \swarrow {\scriptstyle \rho^{e,d}} \\
& \mathscr{F} \otimes \mathcal{O}_X^{\otimes(d+e)} &
\end{array}$$

Proof. To check this reduce to special sections and then apply the explicit formula for $\rho^{d,e}$. There is another diagram for associativity of the product that we will need, but we leave it to the reader to write down and verify. □

With this technical preliminary out of the way, let P be the presheaf colimit of the direct system (3). That is, $P(U) = \varinjlim \Gamma(U, \mathscr{F} \otimes \mathcal{O}_X^{\otimes n})$. This is already a presheaf of $\mathcal{O}_X$-modules, and we make it into a presheaf of $\mathcal{O}_X$-algebras by defining multiplication in the following way for $d, e \geq 0$

$$(d, x) \cdot (e, y) = (d + e, \rho_U^{d,e}(x \,\dot{\otimes}\, y))$$

From Lemma 20 we infer that $P(U)$ is a commutative $\Gamma(U, \mathcal{O}_X)$-algebra and P a presheaf of commutative $\mathcal{O}_X$-algebras. It follows that the sheaf of $\mathcal{O}_X$-modules $\varinjlim(\mathscr{F} \otimes \mathcal{O}_X^{\otimes n})$ is canonically a sheaf of commutative $\mathcal{O}_X$-algebras with the following multiplication

$$(d, m \,\dot{\otimes}\, f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_d) \cdot (e, n \,\dot{\otimes}\, g_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, g_e) = (d + e, mn \,\dot{\otimes}\, f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_d \,\dot{\otimes}\, g_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, g_e)$$

In particular if we set $\mathscr{F} = \mathcal{O}_X$ then the direct limit of the following direct system

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X^{\otimes 2} \longrightarrow \mathcal{O}_X^{\otimes 3} \longrightarrow \cdots$$

is a sheaf of commutative $\mathcal{O}_X$-algebras. Of course this direct system is canonically isomorphic to the direct system

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \cdots$$

whose morphisms are multiplication by f. We denote the direct limit of this system by $\mathcal{O}_{X,(f)}$. This is a sheaf of commutative $\mathcal{O}_X$-algebras with multiplication $(d, m) \cdot (e, n) = (d + e, mn)$. If $\mathscr{F}$ is just a sheaf of $\mathcal{O}_X$-modules then we have another direct system of multiplication by f

$$\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \cdots \tag{4}$$

and the direct limit $\mathscr{F}_{(f)}$ becomes a sheaf of $\mathcal{O}_{X,(f)}$-modules with action $(d, r) \cdot (e, m) = (d+e, r \cdot m)$.

Definition 4. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ a global section. Taking direct limits over multiplication by f yields a sheaf of commutative $\mathcal{O}_X$-algebras $\mathcal{O}_{X,(f)}$. For any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ the same procedure yields a sheaf of $\mathcal{O}_{X,(f)}$-modules $\mathscr{F}_{(f)}$. A morphism of sheaves of $\mathcal{O}_X$-modules $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ induces a morphism of the direct systems (4) and taking direct limits a morphism of sheaves of $\mathcal{O}_{X,(f)}$-modules $\alpha_{(f)} : \mathscr{F}_{(f)} \longrightarrow \mathscr{G}_{(f)}$ defined by $(\alpha_{(f)})(d, x) = (d, \alpha_U(x))$.

The next few results show just how good the analogy is between passing from the sheaf $M^{\sim}$ to the sheaf $(M_f)^{\sim}$ and passing from the sheaf $\mathscr{F}$ to the sheaf $\mathscr{F}_{(f)}$. First we have to show how to write the localisation M_f as a direct limit.

Lemma 21. *Let A be a ring, M an A-module and $f \in A$. Consider the direct system of modules*

$$M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M \longrightarrow \cdots \tag{5}$$

determined by multiplication by f. There is a canonical isomorphism of A-modules natural in M

$$\varinjlim M \longrightarrow M_f$$
$$(k, m) \mapsto m/f^k$$

Proof. Just for convenience index the copies of M in the direct system in the following way $M^0 \longrightarrow M^1 \longrightarrow \cdots$. Define for $k \geq 0$ a morphism of A-modules $M^k \longrightarrow M_f$ by $m \mapsto m/f^k$. This is a cocone on the direct system, and one checks easily it is a colimit. In the direct limit definition of the localisation, it is the position in the sequence that corresponds to the degree of f in the denominator. So applying one of the morphisms in the system corresponds to multiplying top and bottom of a fraction by f. $\square$

If we replace f by a power f^k then the corresponding direct system is just a cofinal subset of (5), so the direct limit remains unchanged. The same argument works in the sheaf case. That is, given a scheme X, $f \in \Gamma(X, \mathcal{O}_X)$ and a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ we have for any $k > 0$ a canonical isomorphism of sheaves of modules $\mathscr{F}_{(f^k)} \longrightarrow \mathscr{F}_{(f)}$ natural in $\mathscr{F}$ defined by $(n, m) \mapsto (nk, m)$.

In fact, in the sense that we elaborate below, the "localisation" $\mathscr{F}_{(f)}$ is also natural in the global section f.

Remark 8. Let X be a scheme and $f, g \in \Gamma(X, \mathcal{O}_X)$ global sections. If $f \in \sqrt{g}$, by which we mean that f belongs to the radical of the ideal (g) generated by g in $\Gamma(X, \mathcal{O}_X)$, then we can write $f^k = bg$ for some $k > 0$ and $b \in \Gamma(X, \mathcal{O}_X)$. It follows easily that $X_f \subseteq X_g$ as open subsets.

Lemma 22. *Let X be a scheme, $f, g \in \Gamma(X, \mathcal{O}_X)$ global sections with $f \in \sqrt{g}$, and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. There is a canonical morphism of sheaves of modules natural in $\mathscr{F}$*

$$\mathscr{F}_{(g)} \longrightarrow \mathscr{F}_{(f)}$$

Proof. Firstly just suppose we are given h such that $f = bg$. We define a morphism of direct systems

$$\begin{array}{ccccccc}
\mathscr{F} & \xrightarrow{g} & \mathscr{F} & \xrightarrow{g} & \mathscr{F} & \xrightarrow{g} & \cdots \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b^2} & & \\
\mathscr{F} & \xrightarrow[f]{} & \mathscr{F} & \xrightarrow[f]{} & \mathscr{F} & \xrightarrow[f]{} & \cdots
\end{array}$$

which induces a morphism of sheaves of modules $\mathscr{F}_{(g)} \longrightarrow \mathscr{F}_{(f)}$ defined on an open set U by $(n, m) \mapsto (n, b|_U^n m)$. In general we can find $k > 0$ and b such that $f^k = bg$. Then we have a composite $\mathscr{F}_{(g)} \longrightarrow \mathscr{F}_{(f^k)} \cong \mathscr{F}_{(f)}$ defined by $(n, m) \mapsto (nk, b|_U^n m)$. It is straightforward to check that this morphism doesn't actually depend on $k > 0$ or b, so it is canonical. Naturality in $\mathscr{F}$ is also clear. $\square$

Proposition 23. *Let X be a concentrated scheme and $f \in \Gamma(X, \mathcal{O}_X)$ a global section. There is a canonical isomorphism of $\Gamma(X, \mathcal{O}_X)$-algebras $\theta : \Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X, \mathcal{O}_{X,(f)})$ and for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ and $i \geq 0$ there is a canonical isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules*

$$\vartheta : H^i(X, \mathscr{F})_f \longrightarrow H^i(X, \mathscr{F}_{(f)})$$

compatible with the ring morphism θ and natural in $\mathscr{F}$ and f.

Proof. For any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ we have the direct system of multiplication by f

$$\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \cdots \tag{6}$$

Applying $H^i(X,-)$ and taking direct limits we obtain an isomorphism of $\Gamma(X,\mathcal{O}_X)$-modules $\varinjlim H^i(X,\mathscr{F}) \longrightarrow H^i(X,\mathscr{F})_f$ defined by $(k,m)\mapsto m/f^k$. Using (COS,Theorem 26) (here we use the fact that X is concentrated) we have a canonical isomorphism of $\Gamma(X,\mathcal{O}_X)$-modules

$$\begin{gathered}\vartheta : H^i(X,\mathscr{F})_f \cong \varinjlim H^i(X,\mathscr{F}) \cong H^i(X,\mathscr{F}_{(f)})\\ a/f^k \mapsto (k,a)\end{gathered}$$

Taking $i=0$ we obtain in the same way, for every quasi-compact open $U\subseteq X$, a canonical isomorphism of $\Gamma(U,\mathcal{O}_X)$-modules

$$\begin{gathered}\theta : \Gamma(U,\mathscr{F})_{f|_U} \longrightarrow \Gamma(U,\mathscr{F}_{(f)})\\ a/f|_U^k \mapsto (k,a)\end{gathered}$$

In the case where $\mathscr{F}=\mathcal{O}_X$ it is clear that θ is a morphism of rings and therefore an isomorphism of $\Gamma(U,\mathcal{O}_X)$ algebras. In fact the canonical morphism of rings $\Gamma(U,\mathcal{O}_X)\longrightarrow\Gamma(U,\mathcal{O}_{X,(f)})$ sends $f|_U$ to a unit with inverse $(1,1)$, and θ is the induced morphism of rings. Taking $U=X$ proves the first statement of the proposition.

We can consider $\mathscr{F}_{(f)}$ as a sheaf of modules over the ringed space $(X,\mathcal{O}_{X,(f)})$, so the cohomology group $H^i(X,\mathscr{F}_{(f)})$ is canonically a $\Gamma(X,\mathcal{O}_{X,(f)})$-module. Compatibility of ϑ and θ is now trivial, given the explicit expressions we have for both morphisms. Naturality in $\mathscr{F}$ is obvious. By naturality in f we mean that for global sections $f,g\in\Gamma(X,\mathcal{O}_X)$ with $f\in\sqrt{g}$ the following diagram commutes

$$\begin{array}{ccc} H^i(X,\mathscr{F})_g & \longrightarrow & H^i(X,\mathscr{F}_{(g)})\\ \downarrow & & \downarrow\\ H^i(X,\mathscr{F})_f & \longrightarrow & H^i(X,\mathscr{F}_{(f)})\end{array}$$

But we can find $k>0$ and b such that $f^k=bg$, and the explicit construction of $\mathscr{F}_{(g)}\longrightarrow\mathscr{F}_{(f)}$ makes it easy to check that this diagram commutes. □

Remark 9. Let X be a concentrated scheme, $f\in\Gamma(X,\mathcal{O}_X)$ a global section and $\mathscr{F}$ a sheaf of modules. Let P denote the presheaf of modules $U\mapsto\Gamma(U,\mathscr{F})_{f|_U}$. For each quasi-compact open $U\subseteq X$ we have the isomorphism of $\Gamma(U,\mathcal{O}_X)$-modules $\theta:\Gamma(U,\mathscr{F})_{f|_U}\longrightarrow\Gamma(U,\mathscr{F}_{(f)})$ given in the proof of Proposition 23. This morphism is natural with respect to U, in the sense that for another quasi-compact open set $V\subseteq U$ the following diagram commutes

$$\begin{array}{ccc} \Gamma(U,\mathscr{F})_{f|_U} & \xrightarrow{\theta} & \Gamma(U,\mathscr{F}_{(f)})\\ \downarrow & & \downarrow\\ \Gamma(V,\mathscr{F})_{f|_V} & \xrightarrow[\theta]{} & \Gamma(V,\mathscr{F}_{(f)})\end{array}$$

In Proposition 23 the morphism θ was only defined for quasi-compact open sets U, but the same formula allows us to define it on any open set U. So we have a morphism of presheaves of $\mathcal{O}_X$-modules $\phi: P\longrightarrow\mathscr{F}_{(f)}$ with the property that θ_U is an isomorphism for any quasi-compact open set U.

5.2 Localisation as Restriction (Invertible sheaves)

Let X be a scheme, $\mathscr{L}$ an invertible sheaf and $f\in\Gamma(X,\mathscr{L})$ a global section, $\mathscr{F}$ a sheaf of modules on X and adopt the notation of the previous section. Let X_f be the open subset of X consisting of those points x where f generates the stalk $\mathscr{L}_x$ as an $\mathcal{O}_{X,x}$-module (MOS,Lemma 29). The open set X_f is not necessarily affine, but the inclusion $X_f\longrightarrow X$ is always an affine morphism We have a direct system of the form (2) for the sheaf of modules ${}_{X_f}\mathscr{F}$, and it follows from the next result that every morphism in this system is an isomorphism.

Lemma 24. *For $i \geq 1$ the canonical morphism of sheaves of modules*

$$\nu : {}_{X_f}\mathscr{F} \longrightarrow ({}_{X_f}\mathscr{F}) \otimes \mathscr{L}^{\otimes i}$$
$$\nu_U(a) = a \,\dot{\otimes}\, (f|_U)^{\otimes i}$$

is an isomorphism.

Proof. The global section f determines a morphism $\alpha : \mathcal{O}_X \longrightarrow \mathscr{L}$ which is an isomorphism when restricted to X_f. That is, we have a canonical isomorphism $\mathscr{L}|_{X_f} \cong \mathcal{O}_X|_{X_f}$. If $i : X_f \longrightarrow X$ is the inclusion then we deduce from the projection formula (MRS,Lemma 80) an isomorphism

$$\kappa : ({}_{X_f}\mathscr{F}) \otimes \mathscr{L}^{\otimes i} \cong i_*(\mathscr{F}|_{X_f} \otimes \mathscr{L}^{\otimes i}|_{X_f}) \cong i_*(\mathscr{F} \otimes \mathcal{O}_X|_{X_f}^{\otimes i}) \cong {}_{X_f}\mathscr{F}$$

It is easy to check that $\kappa\nu = 1$. Since we already know κ is an isomorphism, it is immedite that $\nu\kappa = 1$ as well, so the proof is complete. Just for interest's sake, observe that on a section of the form $a \,\dot{\otimes}\, f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_i$, where the f_j are sections of $\mathscr{L}$ over an open subset $U \subseteq X$, we have

$$\kappa_U(a \,\dot{\otimes}\, f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_i) = \left(\frac{f_1|_{U \cap X_f}}{f|_{U \cap X_f}} \cdots \frac{f_i|_{U \cap X_f}}{f|_{U \cap X_f}} \right) \cdot a$$

where for $1 \leq j \leq i$ the symbol $f_j|_{U \cap X_f} / f|_{U \cap X_f}$ denotes the unique element of $\Gamma(U \cap X_f, \mathcal{O}_X)$ which acts on $f|_{U \cap X_f} \in \Gamma(U \cap X_f, \mathscr{L})$ to give $f_j|_{U \cap X_f}$. □

It is therefore clear that the morphisms $({}_{X_f}\mathscr{F}) \otimes \mathscr{L}^{\otimes i} \longrightarrow {}_{X_f}\mathscr{F}$ are a direct limit of the system (2) for ${}_{X_f}\mathscr{F}$. Consider the natural morphism of sheaves of modules $\psi : \mathscr{F} \longrightarrow {}_{X_f}\mathscr{F}$. Then ψ induces a morphism of sheaves of modules

$$\phi : \varinjlim(\mathscr{F} \otimes \mathscr{L}^{\otimes n}) \longrightarrow \varinjlim({}_{X_f}\mathscr{F} \otimes \mathscr{L}^{\otimes n}) = {}_{X_f}\mathscr{F}$$
$$\phi_U((n, a \,\dot{\otimes}\, f_1 \,\dot{\otimes}\, \cdots \,\dot{\otimes}\, f_n)) = \left(\frac{f_1|_{U \cap X_f}}{f|_{U \cap X_f}} \cdots \frac{f_n|_{U \cap X_f}}{f|_{U \cap X_f}} \right) \cdot a|_{U \cap X_f}$$

which is natural in $\mathscr{F}$ in the obvious sense. If $\mathscr{F}$ is a sheaf of commutative $\mathcal{O}_X$-algebras then both of the sheaves involved here become sheaves of $\mathcal{O}_X$-algebras, and it is easy to check that ϕ is a morphism of $\mathcal{O}_X$-algebras.

Proposition 25. *Let X be a scheme and $\mathscr{F}$ a sheaf of modules on X. If $\mathscr{F}$ is quasi-coherent then for every invertible sheaf $\mathscr{L}$ and global section $f \in \Gamma(X, \mathscr{L})$ the canonical morphism*

$$\phi : \varinjlim(\mathscr{F} \otimes \mathscr{L}^{\otimes n}) \longrightarrow {}_{X_f}\mathscr{F}$$

is an isomorphism.

Proof. The key point is to show that ϕ is local, so we can reduce to X affine. Let $U \subseteq X$ be open, so that we have the global section $f|_U$ of $\mathscr{L}|_U$. It is clear that the open subset $X_{f|_U}$ of U is just $X_f \cap U$ and that ${}_{X_f \cap U}(\mathscr{F}|_U)$ is $({}_{X_f}\mathscr{F})|_U$. Comparing the direct systems it is not difficult to check that we have a commutative diagram of sheaves of modules on U

$$\begin{array}{ccc} \varinjlim(\mathscr{F} \otimes \mathscr{L}^{\otimes n})|_U & \xrightarrow{\phi_{\mathscr{L}, f|_U}} & ({}_{X_f}\mathscr{F})|_U \\ \Big\Downarrow & & \Big\Downarrow 1 \\ \varinjlim(\mathscr{F}|_U \otimes \mathscr{L}|_U^{\otimes n}) & \xrightarrow[\phi_{\mathscr{L}|_U, f|_U}]{} & {}_{X_f \cap U}(\mathscr{F}|_U) \end{array}$$

which shows that ϕ is local. Reducing to the affine case, we are left with the following algebra problem: suppose we are given a ring A, and A-module M and $f \in A$. We have a direct system of modules

$$M \longrightarrow M \otimes A \longrightarrow M \otimes A^{\otimes 2} \longrightarrow M \otimes A^{\otimes 3} \longrightarrow \cdots$$

defined by multiplication by f. There are morphisms of A-modules

$$M \otimes A^{\otimes i} \longrightarrow M_f$$
$$m \otimes a_1 \otimes \cdots \otimes a_i \mapsto \frac{a_1 \cdots a_i \cdot m}{f^i}$$

and we have to show that the induced morphism $\varinjlim(M \otimes A^{\otimes n}) \longrightarrow M_f$ is an isomorphism. This is just Lemma 21, so the proof is complete. □

Theorem 26. *Let $\mathscr{F}$ be a quasi-coherent sheaf on a concentrated scheme X and $\mathscr{L}$ be an invertible sheaf with global section $f \in \Gamma(X, \mathscr{L})$. There is a canonical isomorphism of abelian groups natural in $\mathscr{F}$ for $i \geq 0$*

$$\varinjlim H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \longrightarrow H^i(X_f, \mathscr{F}|_{X_f})$$

Proof. If we set $U = X_f$ then the inclusion $U \longrightarrow X$ is affine (RAS,Lemma 6), so by (COS,Corollary 28) there is a canonical isomorphism of abelian groups natural in $\mathscr{F}$ for $i \geq 0$

$$H^i(U, \mathscr{F}|_U) \longrightarrow H^i(X, {}_U\mathscr{F})$$

By Proposition 25 we have a canonical isomorphism natural in $\mathscr{F}$

$$\phi : \varinjlim(\mathscr{F} \otimes \mathscr{L}^{\otimes n}) \longrightarrow {}_U\mathscr{F}$$

where the direct limit is taken over the direct system (2) defined by multiplication by f. Applying (COS,Theorem 26) (X is concentrated and therefore its underlying space is quasi-noetherian) we have the desired canonical isomorphism of abelian groups

$$\varinjlim H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \cong H^i(X, \varinjlim(\mathscr{F} \otimes \mathscr{L}^{\otimes n})) \cong H^i(X, {}_U\mathscr{F}) \cong H^i(U, \mathscr{F}|_U) \qquad (7)$$

which one checks is natural in $\mathscr{F}$. □

5.3 Localisation as Restriction

Let X be a scheme, $f \in \Gamma(X, \mathcal{O}_X)$ a global section, and $\mathscr{F}$ a sheaf of modules on X. Specialising Lemma 24 to the case $\mathcal{L} = \mathcal{O}_X$ we have shown that in the following direct system of multiplication by f

$${}_{X_f}\mathscr{F} \longrightarrow {}_{X_f}\mathscr{F} \longrightarrow {}_{X_f}\mathscr{F} \longrightarrow \cdots$$

every morphism is an isomorphism. We deduce a canonical morphism of sheaves of modules $\phi : \mathscr{F}_{(f)} \longrightarrow {}_{X_f}\mathscr{F}$ natural in $\mathscr{F}$ defined by $\phi_U(n, a) = (f|_{U \cap X_f})^{-n} \cdot a|_{U \cap X_f}$. In the case $\mathscr{F} = \mathcal{O}_X$ this is actually a morphism of $\mathcal{O}_X$-algebras.

From Proposition 25 we know that $\phi : \mathscr{F}_{(f)} \longrightarrow {}_{X_f}\mathscr{F}$ is an isomorphism provided $\mathscr{F}$ is quasi-coherent. In particular we always have an isomorphism of $\mathcal{O}_X$-algebras $\mathcal{O}_{X,(f)} \longrightarrow {}_{X_f}\mathcal{O}_X$. The following result upgrades Proposition 25 slightly in our current context.

Corollary 27. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of modules on X. For any global section $f \in \Gamma(X, \mathcal{O}_X)$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{F}$*

$$\phi : \mathscr{F}_{(f)} \longrightarrow {}_{X_f}\mathscr{F}$$

which sends the action of $\mathcal{O}_{X,(f)}$ to the action of ${}_{X_f}\mathcal{O}_X$. This morphism is also natural in f.

Proof. The fact that ϕ is an isomorphism of sheaves of $\mathcal{O}_X$-modules is just Proposition 25. But we know from the previous section that $\mathscr{F}_{(f)}$ is also a sheaf of $\mathcal{O}_{X,(f)}$-modules, and ${}_{X_f}\mathscr{F}$ is certainly a sheaf of ${}_{X_f}\mathcal{O}_X$-modules. We claim that ϕ sends the action of $\mathcal{O}_{X,(f)}$ to the action of ${}_{X_f}\mathcal{O}_X$ in a way compatible with the canonical isomorphism of algebras $\mathcal{O}_{X,(f)} \cong {}_{X_f}\mathcal{O}_X$. This comes down to an explicit calculation, for which we use the explicit form of ϕ and the explicit actions given in the last section.

By naturality in f we mean that for global sections $f, g \in \Gamma(X, \mathcal{O}_X)$ with $f \in \sqrt{g}$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}_{(g)} & \longrightarrow & {}_{X_g}\mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{F}_{(f)} & \longrightarrow & {}_{X_f}\mathscr{F} \end{array}$$

which is easily checked by choosing specific $k > 0$ and b such that $f^k = bg$. □

We have just shown that the morphism $\mathscr{F}_{(f)} \longrightarrow {}_{X_f}\mathscr{F}$ preserves the additional module structure present on both sheaves. In our application, we will also need to know that the induced morphisms on cohomology preserve the additional structure. This will follow from the next general lemma.

Lemma 28. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{G}, \mathscr{H}$ sheaves of commutative $\mathcal{O}_X$-algebras and $\mathscr{M}, \mathscr{N}$ sheaves of modules over $\mathscr{G}, \mathscr{H}$ respectively. Suppose we have two morphisms of sheaves of $\mathcal{O}_X$-modules*

$$\psi : \mathscr{G} \longrightarrow \mathscr{H}, \quad \psi : \mathscr{M} \longrightarrow \mathscr{N}$$

where the former is a morphism of $\mathcal{O}_X$-algebras and the latter sends the action of $\mathscr{G}$ to the action of $\mathscr{H}$ using the morphism ψ. Then the induced morphism

$$H^i(X, \psi) : H^i(X, \mathscr{M}) \to H^i(X, \mathscr{N})$$

is a morphism of $\Gamma(X, \mathcal{O}_X)$-modules that sends the action of $\Gamma(X, \mathscr{G})$ to the action of $\Gamma(X, \mathscr{H})$.

Proof. Considering $(X, \mathscr{G})$ as a ringed space in its own right, the cohomology groups $H^i(X, \mathscr{M})$ become $\Gamma(X, \mathscr{G})$-modules, and the same can be said of $\mathscr{H}$ and $\mathscr{N}$. Given $\alpha \in \Gamma(X, \mathscr{G})$ let $\alpha : \mathscr{M} \longrightarrow \mathscr{M}$ and $\psi(\alpha) : \mathscr{N} \longrightarrow \mathscr{N}$ denote the endomorphisms of sheaves of abelian groups determined by the action of α and $\psi(\alpha)$ respectively. By assumption the following diagram commutes

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\phi} & \mathscr{N} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \psi(\alpha)} \\ \mathscr{M} & \xrightarrow[\phi]{} & \mathscr{N} \end{array}$$

Applying $H^i(X, -)$ and using the characterisation of the module structures given by (COS,Remark 4), we reach the desired conclusion. □

We can deduce from Theorem 26 that under certain hypotheses there is an isomorphism of abelian groups $H^i(X, \mathscr{F}_{(f)}) \cong H^i(X_f, \mathscr{F}|_{X_f})$. But for the application we have in mind, we will have to upgrade this to an isomorphism of modules.

Corollary 29. *Let $\mathscr{F}$ be a quasi-coherent sheaf on a concentrated scheme X and let $f \in \Gamma(X, \mathcal{O}_X)$ be a global section. For $i \geq 0$ there is a canonical isomorphism of $\Gamma(X, \mathcal{O}_X)_f$-modules natural in $\mathscr{F}$ and f*

$$H^i(X, \mathscr{F}_{(f)}) \longrightarrow H^i(X_f, \mathscr{F}|_{X_f})$$

Proof. First we have to explain what the module structures are. If we set $U = X_f$ then the group $H^i(U, \mathscr{F}|_U)$ has a canonical $\Gamma(U, \mathcal{O}_X)$-module structure, and therefore via the ring morphism $\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(U, \mathcal{O}_X)$ it acquires the required module structure. For the first group, observe that we can consider $\mathscr{F}_{(f)}$ as a sheaf of modules over the ringed space $(X, \mathcal{O}_{X,(f)})$, so the cohomology group $H^i(X, \mathscr{F}_{(f)})$ is canonically a $\Gamma(X, \mathcal{O}_{X,(f)})$-module, and by Proposition 23 this ring is canonically isomorphic to $\Gamma(X, \mathcal{O}_X)_f$.

From Corollary 27 and Lemma 28 we deduce that there is a natural isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$H^i(X, \mathscr{F}_{(f)}) \longrightarrow H^i(X, {}_U\mathscr{F})$$

sending the $\Gamma(X, \mathcal{O}_{X,(f)})$-module structure on the left to the $\Gamma(U, \mathcal{O}_X)$-module structure on the right (induced by considering ${}_U\mathscr{F}$ as a sheaf over ${}_U\mathcal{O}_X$). Since the inclusion $U \longrightarrow X$ is affine we can apply (COS,Corollary 28) to the quasi-coherent sheaf $\mathscr{F}|_U$ to deduce a natural isomorphism of abelian groups

$$H^i(U, \mathscr{F}|_U) \longrightarrow H^i(X, {}_U\mathscr{F}) \tag{8}$$

Both of these groups have canonical $\Gamma(U, \mathcal{O}_X)$-module structures. An element $t \in \Gamma(U, \mathcal{O}_X) = \Gamma(X, {}_U\mathcal{O}_X)$ induces a morphism of sheaves of abelian groups $\mathscr{F}|_U \longrightarrow \mathscr{F}|_U$, and naturality then implies that (8) is an isomorphism of $\Gamma(U, \mathcal{O}_X)$-modules. Putting these together we have the desired canonical isomorphism of $\Gamma(X, \mathcal{O}_X)_f$-modules

$$H^i(X, \mathscr{F}_{(f)}) \cong H^i(X, {}_U\mathscr{F}) \cong H^i(U, \mathscr{F}|_U)$$

which is natural with respect to morphisms of quasi-coherent sheaves.

By naturality in f we mean that for global sections $f, g \in \Gamma(X, \mathcal{O}_X)$ with $f \in \sqrt{g}$ the following diagram commutes

$$\begin{array}{ccc} H^i(X, \mathscr{F}_{(g)}) & \longrightarrow & H^i(X_g, \mathscr{F}|_{X_g}) \\ \downarrow & & \downarrow \\ H^i(X, \mathscr{F}_{(f)}) & \longrightarrow & H^i(X_f, \mathscr{F}|_{X_f}) \end{array}$$

where the vertical morphism on the right is defined in (COS,Remark 11). Commutativity of this square follows from the naturality of Corollary 27 and a close study of the proof of (COS,Corollary 28) and the definition in (COS,Remark 11). □

5.4 The Proof

Theorem 30. *Let $f : X \longrightarrow Y$ be a morphism of schemes where X is concentrated and $Y = SpecA$ is affine. Then for any quasi-coherent sheaf of modules $\mathscr{F}$ on X and $i \geq 0$ there is a canonical isomorphism of sheaves of modules on Y natural in $\mathscr{F}$*

$$\gamma : H^i(X, \mathscr{F})^{\sim} \longrightarrow R^i f_*(\mathscr{F})$$

Proof. We give $R^i f_*(\mathscr{F})$ the canonical $\mathcal{O}_Y$-module structure of Definition 2. We know that the abelian group $H^i(X, \mathscr{F})$ has a canonical $\Gamma(X, \mathcal{O}_X)$-module structure, and therefore also an A-module structure. For $g \in A$ we set $U = D(g)$ and let $h \in \Gamma(X, \mathcal{O}_X)$ be the image of g, so that $f^{-1}U = X_h$. Combining (COS,Lemma 12), Corollary 29 and Proposition 23 we have a canonical isomorphism of $\Gamma(X, \mathcal{O}_X)_h$-modules

$$H^i(X_h, \mathscr{F}) \cong H^i(X_h, \mathscr{F}|_{X_h}) \cong H^i(X, \mathscr{F}_{(h)}) \cong H^i(X, \mathscr{F})_h$$

Of course considering $H^i(X, \mathscr{F})$ as an A-module there is a canonical isomorphism of abelian groups $H^i(X, \mathscr{F})_h \cong H^i(X, \mathscr{F})_g$. So finally we have for each $g \in A$ a canonical isomorphism of $\Gamma(D(g), \mathcal{O}_Y)$-modules

$$\Gamma(D(g), H^i(X, \mathscr{F})^{\sim}) \cong H^i(X, \mathscr{F})_g \cong H^i(X_h, \mathscr{F}) = \Gamma(D(g), f_*\mathscr{H}^i(\mathscr{F}))$$

Naturality of all our results in g means that for another element $f \in A$ with $D(f) \subseteq D(g)$ the following diagram commutes

$$\begin{array}{ccc} \Gamma(D(g), H^i(X, \mathscr{F})^{\sim}) & \longrightarrow & \Gamma(D(g), f_*\mathscr{H}^i(\mathscr{F})) \\ \downarrow & & \downarrow \\ \Gamma(D(f), H^i(X, \mathscr{F})^{\sim}) & \longrightarrow & \Gamma(D(f), f_*\mathscr{H}^i(\mathscr{F})) \end{array}$$

We deduce that the presheaf $f_*\mathscr{H}^i(\mathscr{F})$ actually satisfies the sheaf condition with respect to open covers of affine open subsets of the form $D(g) \subseteq Y$, and therefore obtain a canonical isomorphism of $\Gamma(D(g), \mathcal{O}_Y)$-modules

$$\Gamma(D(g), H^i(X,\mathscr{F})^\sim) \cong \Gamma(D(g), f_*\mathscr{H}^i(\mathscr{F})) \cong \Gamma(D(g), \mathbf{a}f_*\mathscr{H}^i(\mathscr{F})) \cong \Gamma(D(g), R^if_*(\mathscr{F}))$$

which is natural with respect to inclusions of the form $D(f) \subseteq D(g)$. By glueing there is a unique morphism of sheaves of modules $\gamma : H^i(X,\mathscr{F})^\sim \longrightarrow R^if_*(\mathscr{F})$ which evaluates to this isomorphism on $D(g)$ for every $g \in A$. Clearly γ is the desired isomorphism, natural in $\mathscr{F}$. □

Corollary 31. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $R^if_*(\mathscr{F})$ is quasi-coherent for $i \geq 0$. Further, there is a canonical isomorphism of $\Gamma(U, \mathcal{O}_Y)$-modules*

$$H^i(f^{-1}U, \mathscr{F}|_{f^{-1}U}) \longrightarrow \Gamma(U, R^if_*\mathscr{F})$$

for any affine open $U \subseteq Y$.

Proof. Since the higher direct image is local (HDIS,Corollary 6), to show that $R^if_*\mathscr{F}$ is quasi-coherent we can reduce to the case where $Y = Spec(A)$ is affine. In particular this means that X is a concentrated scheme, so the result follows from Theorem 30. □

The famous theorem of Serre (COS,Theorem 14) tells us that quasi-coherent sheaves have vanishing higher cohomology on an affine scheme. On the much larger class of concentrated schemes something similar is true: quasi-coherent sheaves can have nonzero higher cohomology, but the number of nonzero groups is bounded by a fixed integer.

Lemma 32. *Let X be a concentrated scheme. There is an integer $d \geq 0$ such that $H^i(X,\mathscr{F}) = 0$ for every quasi-coherent sheaf $\mathscr{F}$ on X and $i > d$.*

Proof. Given a concentrated scheme X let $n(X)$ denote the smallest integer $n \geq 1$ such that X can be covered by n quasi-compact separated open subsets. Such an integer exists because X is quasi-compact, and any affine open subset is quasi-compact and separated.

If $n = 1$ then X is separated, and we can find a finite cover $\mathfrak{U}$ of X by open affines with affine intersections. If this cover has d elements then the Čech complex has only d nonzero terms (that is, $\mathscr{C}^e(\mathfrak{U},\mathscr{F}) = 0$ for any sheaf $\mathscr{F}$ and $e \geq d$). Since cohomology on X can be calculated using the Čech complex (COS,Theorem 35) we deduce that for any quasi-coherent sheaf $\mathscr{F}$ and $i > d-1$ we have $H^i(X,\mathscr{F}) = 0$ as required.

If $n > 1$ then let $X_1, \ldots, X_n$ be a cover of X by quasi-compact separated open subsets, and set $U = X_1$, $V = X_2 \cup \cdots \cup X_n$. Given a quasi-coherent sheaf $\mathscr{F}$ on X let $\mathscr{I}$ be an injective resolution of $\mathscr{F}$ as a sheaf of abelian groups. From the canonical short exact sequence (the Čech complex for $\mathscr{I}$ and the cover $\{U, V\}$)

$$0 \longrightarrow \Gamma(X,\mathscr{I}) \longrightarrow \Gamma(U,\mathscr{I}) \oplus \Gamma(V,\mathscr{I}) \longrightarrow \Gamma(U \cap V,\mathscr{I}) \longrightarrow 0$$

we deduce a long exact sequence

$$\cdots \longrightarrow H^{i-1}(U \cap V, \mathscr{F}|_{U\cap V}) \longrightarrow H^i(X,\mathscr{F}) \longrightarrow H^i(U,\mathscr{F}|_U) \oplus H^i(V,\mathscr{F}|_V) \longrightarrow \cdots$$

But $n(U) = 1, n(V) = n-1$ and $n(U \cap V) < n$ since X is quasi-separated so finite intersections of quasi-compact open subsets of X are quasi-compact. By the inductive hypothesis and the above long exact sequence, we reach the desired conclusion. □

One of Grothendieck's fundamental insights was that a concept in algebraic geometry only reaches its full potential when it is made relative with respect to a morphism of schemes. The relative version of cohomology of a sheaf is the higher direct image of the sheaf (taken over $Spec\mathbb{Z}$ this yields the usual cohomology groups). In this sense the next result is the relative version of Lemma 32.

Proposition 33. *Let $f : X \longrightarrow Y$ be a concentrated morphism of schemes with Y quasi-compact. There is an integer $d \geq 0$ such that for any quasi-coherent sheaf $\mathscr{F}$ on X and $i > d$, we have $R^i f_*(\mathscr{F}) = 0$.*

Proof. Since Y can be covered by a finite number of open affines, we can reduce to the case where $Y = SpecA$ is affine, in which case X is concentrated. So by Theorem 30 what we have to show is that there exists an integer $d \geq 0$ such that $H^i(X, \mathscr{F}) = 0$ for $\mathscr{F}$ quasi-coherent on X and $i > d$, which is precisely what we did in Lemma 32. □

Lemma 34. *Let $f : X \longrightarrow Y$ be an affine morphism of schemes and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $R^i f_*(\mathscr{F}) = 0$ for $i > 0$.*

Proof. We can reduce immediately to the case where X, Y are affine, where the result follows from Theorem 30 and Serre's theorem (COS,Theorem 14). □

Here are some more useful properties of the direct image functor.

Lemma 35. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the forgetful functor $F : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Ab}(X)$ preserves and reflects all colimits.*

We say that a continuous map of topological spaces $f : X \longrightarrow Y$ is *quasi-compact* if for any quasi-compact open subset $V \subseteq Y$ the open set $f^{-1}V$ is quasi-compact. By (CON,Corollary 2) there is no ambiguity with the same concept for morphisms of schemes.

Corollary 36. *Let $f : X \longrightarrow Y$ be a quasi-compact map of quasi-noetherian topological spaces, and let $\mathscr{F}$ be a quasi-flasque sheaf of abelian groups on X. Then $R^i f_*(\mathscr{F}) = 0$ for $i > 0$.*

Proof. Since Y has a basis of quasi-compact open sets, it is enough by Proposition 1 to show that $H^i(f^{-1}V, \mathscr{F}) = 0$ for $V \subseteq Y$ quasi-compact and $i > 0$. But $H^i(f^{-1}V, \mathscr{F}) \cong H^i(f^{-1}V, \mathscr{F}|_{f^{-1}V})$, and since $f^{-1}V$ is quasi-compact it is itself a quasi-noetherian space. So the desired result follows from (COS,Corollary 11). □

Proposition 37. *If $f : X \longrightarrow Y$ is a morphism of concentrated schemes then the additive functor $f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ preserves direct limits and coproducts.*

Proof. By (CON,Lemma 16) the morphism f is concentrated and in particular quasi-compact, so by (AC,Lemma 43) it is enough to prove the following statement: if $f : X \longrightarrow Y$ is a quasi-compact map of quasi-noetherian topological spaces then $f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ preserves direct limits..

Let $\{\mathscr{F}_\alpha, \varphi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a direct system of sheaves of abelian groups on X and $\varinjlim \mathscr{F}_\alpha$ a direct limit. We have to show that the canonical morphism of sheaves

$$\Phi : \varinjlim_\alpha f_*(\mathscr{F}_\alpha) \longrightarrow f_*(\varinjlim \mathscr{F}_\alpha)$$

is an isomorphism. For this it suffices to show that Φ_V is an isomorphism for $V \subseteq Y$ quasi-compact. We have by (COS,Proposition 23)

$$\begin{aligned} \Gamma(V, \varinjlim_\alpha f_*(\mathscr{F}_\alpha)) &\cong \varinjlim_\alpha \Gamma(V, f_*(\mathscr{F}_\alpha)) \\ &\cong \varinjlim_\alpha \Gamma(f^{-1}V, \mathscr{F}_\alpha) \\ &\cong \Gamma(f^{-1}V, \varinjlim_\alpha \mathscr{F}_\alpha) \\ &= \Gamma(V, f_*(\varinjlim_\alpha \mathscr{F}_\alpha)) \end{aligned}$$

This composite is equal to Φ_V, so the proof is complete. □

Corollary 38. *If $f : X \longrightarrow Y$ is a morphism of concentrated schemes then the additive functor $R^i f_*(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ preserves direct limits and coproducts for all $i \geq 0$.*

Proof. It is enough to prove the following statement: if $f : X \longrightarrow Y$ is a quasi-compact map of quasi-noetherian topological spaces then $R^i f_*(-) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ preserves direct limits. The proof of this statement is the same as (COS,Theorem 26), but we write down the proof anyway.

Let $\{\mathscr{F}_\alpha, \varphi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a direct system of sheaves of abelian groups on X. For each $\alpha \in \Lambda$ we have a short exact sequence

$$0 \longrightarrow \mathscr{F}_\alpha \longrightarrow \mathscr{F}_\alpha^d \longrightarrow \mathscr{G}_\alpha \longrightarrow 0$$

Taking direct limits, we have a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C} \longrightarrow \mathscr{G} \longrightarrow 0$$

where $\mathscr{F} = \varinjlim_\alpha \mathscr{F}_\alpha$ and $\mathscr{C} = \varinjlim_\alpha \mathscr{F}_\alpha^d$ is quasi-flasque by (COS,Corollary 25). Hence it is f_*-acyclic by Corollary 36. Using the long exact sequence of cohomology, we have the following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim f_*(\mathscr{F}_\alpha) & \longrightarrow & \varinjlim f_*(\mathscr{F}_\alpha^d) & \longrightarrow & \varinjlim f_*(\mathscr{G}_\alpha) & \longrightarrow & \varinjlim R^1 f_*(\mathscr{F}_\alpha) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*(\mathscr{F}) & \longrightarrow & f_*(\mathscr{C}) & \longrightarrow & f_*(\mathscr{G}) & \longrightarrow & R^1 f_*(\mathscr{F}) & \longrightarrow & 0 \end{array}$$

and for $i > 0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varinjlim R^i f_*(\mathscr{G}_\alpha) & \longrightarrow & \varinjlim R^{i+1} f_*(\mathscr{F}_\alpha) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^i f_*(\mathscr{G}) & \longrightarrow & R^{i+1} f_*(\mathscr{F}) & \longrightarrow & 0 \end{array}$$

By induction on i we are done, because Proposition 37 included the case $i = 0$. $\square$

6 Uniqueness of Cohomology

Let X be a concentrated scheme. Taking global sections defines three additive functors

$$\Gamma(X, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$$
$$\Gamma_m(X, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$$
$$\Gamma_{qc}(X, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$$

whose right derived functors we denote by $H^i(X, -), H^i_m(X, -)$ and $H^i_{qc}(X, -)$ respectively (note that since X is concentrated $\mathfrak{Qco}(X)$ has enough injectives (MOS,Proposition 66)). It would be very unsettling if these three functors were to define distinct cohomology theories. In (COS,Section 1.2) we showed that $H^i(X, -)$ and $H^i_m(X, -)$ agree for any scheme (even a ringed space). In this section we will show that under very mild hypotheses on X the third cohomology $H^i_{qc}(X, -)$ agrees with the other two.

More generally, for any concentrated morphism of schemes $f : X \longrightarrow Y$ with X concentrated we have three additive functors (CON,Proposition 18)

$$f_* : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$$
$$f_*^m : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$$
$$f_*^{qc} : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$$

with right derived functors $R^i f_*(-), R^i f_*^m(-)$ and $R^i f_*^{qc}(-)$. We showed in Section 2 that $R^i f_*(-)$ and $R^i f_*^m(-)$ agree, and the main result of this section shows that under the same mild hypotheses the third derived functor $R^i f_*^{qc}(-)$ agrees with the other two. This material is well-known, and one good reference is [3] which we follow closely.

If the scheme X is noetherian then a quasi-coherent sheaf $\mathscr{I}$ is injective in $\mathfrak{Qco}(X)$ if and only if it is injective in $\mathfrak{Mod}(X)$, so the claims are all completely trivial in this case:

Proposition 39. *Let $f : X \longrightarrow Y$ be a morphism of schemes with X noetherian. Then for any quasi-coherent sheaf $\mathscr{F}$ on X and $i \geq 0$ there is a canonical isomorphism of sheaves of modules $R^i f_*^{qc}(\mathscr{F}) \longrightarrow R^i f_*(\mathscr{F})$ natural in $\mathscr{F}$.*

Proof. Fix assignments of injective resolutions to $\mathfrak{Ab}(X)$ and $\mathfrak{Qco}(X)$, and define $R^i f_*(-) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$ as in Definition 2. Then we have two universal cohomological δ-functors

$$\{R^i f_*^{qc}(-)\}_{\geq 0} : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(Y)$$
$$\{R^i f_*(-)\}_{\geq 0} : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y)$$

Composing with the exact functors $V : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ and $v : \mathfrak{Qco}(Y) \longrightarrow \mathfrak{Mod}(Y)$ we have cohomological δ-functors $\{v \circ R^i f_*^{qc}(-)\}_{\geq 0}$ and $\{R^i f_*(-) \circ V\}_{\geq 0}$. The first is clearly universal. Universality of the second follows from the fact that injectives in $\mathfrak{Qco}(X)$ agree with injectives in $\mathfrak{Mod}(X)$. Since they agree in degree zero, we deduce a canonical isomorphism of δ-functors. In particular for $i \geq 0$ we have a canonical natural equivalence

$$v \circ R^i f_*^{qc}(-) \longrightarrow R^i f_*(-) \circ V$$

which is what we wanted to show. □

Before proceeding we have to define the "presheaf of cohomology" for the cohomology functors defined relative to quasi-coherent sheaves, in precisely the same was as we did in (COS,Section 1.3) for sheaves of abelian groups.

Definition 5. Let X be a concentrated scheme and $U \subseteq X$ an open subset. Let $H^i_{qc}(U, -)$ be the ith right derived functor of the left exact functor $\Gamma(U, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$. There is a canonical natural equivalence $H^0_{qc}(U, -) \cong \Gamma(U, -)$, and short exact sequences of quasi-coherent sheaves lead to long exact sequences in the usual manner.

Let $U \subseteq V$ be open subsets of X. Then restriction defines a natural transformation $\Gamma(V, -) \longrightarrow \Gamma(U, -)$ which leads to natural transformations $\mu^i_{V,U} : H^i_{qc}(V, -) \longrightarrow H^i_{qc}(U, -)$ for $i \geq 0$ as defined in (DF,Definition 11). This construction is functorial, in the sense that for open sets $W \subseteq U \subseteq V$ we have $\mu^i_{U,W} \circ \mu^i_{V,U} = \mu^i_{V,W}$ and $\mu^i_{U,U} = 1$ for any open set U. For a quasi-coherent sheaf $\mathscr{F}$ on X we define a presheaf of abelian groups $\mathscr{H}^i_{qc}(\mathscr{F})$ for $i \geq 0$ by $\Gamma(U, \mathscr{H}^i_{qc}(\mathscr{F})) = H^i_{qc}(U, \mathscr{F})$ with the restriction map $\Gamma(V, \mathscr{H}^i_{qc}(\mathscr{F})) \longrightarrow \Gamma(U, \mathscr{H}^i_{qc}(\mathscr{F}))$ given by $(\mu^i_{U,V})_{\mathscr{F}}$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of quasi-coherent sheaves then for $i \geq 0$ there is a morphism of presheaves of abelian groups

$$\mathscr{H}^i_{qc}(\phi) : \mathscr{H}^i_{qc}(\mathscr{F}) \longrightarrow \mathscr{H}^i_{qc}(\mathscr{G})$$
$$\mathscr{H}^i_{qc}(\phi)_U = H^i_{qc}(U, \phi)$$

This defines for $i \geq 0$ an additive functor $\mathscr{H}^i_{qc}(-) : \mathfrak{Qco}(X) \longrightarrow Ab(X)$ where $Ab(X)$ is the category of all presheaves of abelian groups on X. There is a canonical isomorphism of presheaves of abelian groups $\mathscr{F} \cong \mathscr{H}^0_{qc}(\mathscr{F})$ natural in $\mathscr{F}$ (DF,Lemma 43).

Suppose we have an exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

For an open subset $U \subseteq X$ and $i \geq 0$ we have the canonical connecting morphism $H^i_{qc}(U, \mathscr{F}'') \longrightarrow H^{i+1}_{qc}(U, \mathscr{F}')$ and since these are natural in U (DF,Proposition 44) we have a morphism of presheaves of abelian groups $\omega^i : \mathscr{H}^i_{qc}(\mathscr{F}'') \longrightarrow \mathscr{H}^{i+1}_{qc}(\mathscr{F}')$. These fit into a long exact sequence of *presheaves* of abelian groups

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow \mathscr{H}^1_{qc}(\mathscr{F}') \longrightarrow \mathscr{H}^1_{qc}(\mathscr{F}) \longrightarrow \mathscr{H}^1_{qc}(\mathscr{F}'') \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathscr{H}^n_{qc}(\mathscr{F}'') \longrightarrow \mathscr{H}^{n+1}_{qc}(\mathscr{F}') \longrightarrow \mathscr{H}^{n+1}_{qc}(\mathscr{F}) \longrightarrow \mathscr{H}^{n+1}_{qc}(\mathscr{F}'') \longrightarrow \cdots$$

Remark 10. Since the situation will arise several times in what follows, consider a scheme X which is quasi-compact and semi-separated (CON,Definition 4). Equivalently, X is a scheme which admits a finite open cover $\mathfrak{U}$ by affine open sets with affine pairwise intersections. If $\mathscr{F}$ is a quasi-coherent sheaf on X, then we have the canonical Čech resolution (COS,Theorem 35)

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \tag{9}$$

which is an exact sequence of quasi-coherent sheaves.

Let $V \subseteq X$ be an open subset whose inclusion $V \longrightarrow X$ is affine. Then V is also quasi-compact and semi-separated, and in particular both X and V are concentrated schemes (CON,Lemma 20). Let $\mathfrak{U}|_V = \{U \cap V\}_{U \in \mathfrak{U}}$ denote the restricted affine open cover of V, which still has affine pairwise intersections. It is clear that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})|_V = \mathscr{C}^p(\mathfrak{U}|_V, \mathscr{F}|_V)$, so the restriction to V of the Čech resolution (9) is the Čech resolution for $V, \mathfrak{U}|_V$ and $\mathscr{F}|_V$.

Proposition 40. *Let X be a quasi-compact semi-separated scheme with finite semi-separating open cover $\mathfrak{U}$, $\mathscr{F}$ a quasi-coherent sheaf on X and $V \subseteq X$ an open subset with affine inclusion. Then $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is acyclic with respect to $\Gamma_{qc}(V, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ for $p \geq 0$.*

Proof. Let $f : W \longrightarrow X$ be the inclusion of an affine open subset, and assume further that f is an affine morphism. Consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{Qco}(X) & & \\ & \overset{f_*}{\nearrow} & & \searrow^{\Gamma_{qc}(V,-)} & \\ \mathfrak{Qco}(W) & & \xrightarrow[\Gamma_{qc}(W \cap V, -)]{} & & \mathbf{Ab} \end{array}$$

in which $f_* : \mathfrak{Qco}(W) \longrightarrow \mathfrak{Qco}(X)$ is exact (CON,Lemma 19) and has an exact left adjoint (the restriction functor). In particular f_* preserves quasi-coherent injectives (AC,Proposition 25). This means that for any quasi-coherent sheaf $\mathscr{X}$ on W we have a Grothendieck spectral sequence

$$E_2^{pq} = H^p_{qc}(V, R^q f_*^{qc}(\mathscr{X})) \Longrightarrow H^{p+q}_{qc}(W \cap V, \mathscr{X})$$

Of course f_* is exact so its higher derived functors vanish, and therefore $E_2^{pq} = 0$ for $q > 0$. That is, the spectral sequence degenerates. From this we deduce an isomorphism for $p > 0$

$$H^p_{qc}(V, f_*(\mathscr{X})) \cong H^p_{qc}(W \cap V, \mathscr{X}) = 0$$

since $W \cap V$ is affine and therefore $\Gamma_{qc}(W \cap V, -)$ is exact. Since any $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is built from finite coproducts of sheaves of the form $f_*(\mathscr{X})$, this completes the proof that $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is acyclic for $\Gamma_{qc}(V, -)$. ∎

Setting $V = X$ in the next result shows that the three types of cohomology $H^i(X, -), H^i_m(X, -)$ and $H^i_{qc}(X, -)$ agree on a quasi-compact semi-separated scheme. We need the result in the stated generality to deal with the derived direct image functors later on.

Corollary 41. *Let X be a quasi-compact semi-separated scheme, $\mathscr{F}$ a quasi-coherent sheaf on X and $V \subseteq X$ an open subset with affine inclusion. For $i \geq 0$ there is an isomorphism of abelian groups natural in $\mathscr{F}$ and V*

$$H^i_{qc}(V, \mathscr{F}) \longrightarrow H^i(V, \mathscr{F})$$

Proof. To be clear, we mean the inclusion $i : V \longrightarrow X$ is an affine morphism, and naturality of the two cohomology groups in V is defined by Definition 5 and (COS,Section 1.3) respectively. Let $\mathfrak{U}$ be a finite semi-separating cover of X. By (COS,Theorem 35) the Čech sheaves give an exact sequence of quasi-coherent sheaves

$$\mathscr{C}(\mathfrak{U}, \mathscr{F}) : 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \tag{10}$$

Slightly modifying the proof of (COS,Theorem 35) one checks that each $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ is acyclic for the additive functor $\Gamma(V,-):\mathfrak{Ab}(X)\longrightarrow \mathbf{Ab}$, and we also know from Proposition 40 that this sheaf is acyclic for $\Gamma_{qc}(V,-):\mathfrak{Qco}(X)\longrightarrow \mathbf{Ab}$.

Suppose $\mathscr{I},\mathscr{J}$ are the chosen injective resolutions of $\mathscr{F}$ in $\mathfrak{Ab}(X),\mathfrak{Qco}(X)$ respectively. Then as usual the identity lifts to morphisms of complexes $\mathscr{C}(\mathfrak{U},\mathscr{F})\longrightarrow\mathscr{I}$ and $\mathscr{C}(\mathfrak{U},\mathscr{F})\longrightarrow\mathscr{J}$ whose images under $\Gamma(V,-)$ and $\Gamma_{qc}(V,-)$ respectively are quasi-isomorphisms (DTC2,Remark 14). In other words, we have an isomorphism for $i\geq 0$

$$H^i_{qc}(V,\mathscr{F})=H^i(\Gamma_{qc}(V,\mathscr{J}))\cong H^i(\Gamma(V,\mathscr{C}(\mathfrak{U},\mathscr{F})))\cong H^i(\Gamma(V,\mathscr{I}))=H^i(V,\mathscr{F})$$

which is easily checked to be natural in $\mathscr{F}$ and V. Observe that the isomorphism is canonical once we fix the finite semi-separating cover $\mathfrak{U}$. □

Corollary 42. *Let X be a quasi-compact semi-separated scheme, $\mathscr{F}$ a quasi-coherent sheaf on X and $V\subseteq X$ an open subset with affine inclusion. For $i\geq 0$ there is an isomorphism of abelian groups natural in $\mathscr{F}$*

$$H^i_{qc}(V,\mathscr{F})\longrightarrow H^i_{qc}(V,\mathscr{F}|_V)$$

Proof. To be clear, the left hand side uses the derived functors of $F=\Gamma(V,-):\mathfrak{Qco}(X)\longrightarrow\mathbf{Ab}$ and the right hand side the right derived functors of $G=\Gamma(V,-):\mathfrak{Qco}(V)\longrightarrow\mathbf{Ab}$. Let $\mathfrak{U}$ be a finite semi-separating cover of X. From Proposition 40 we know that the Čech resolution

$$\mathscr{C}(\mathfrak{U},\mathscr{F}):0\longrightarrow\mathscr{F}\longrightarrow\mathscr{C}^0(\mathfrak{U},\mathscr{F})\longrightarrow\mathscr{C}^1(\mathfrak{U},\mathscr{F})\longrightarrow\cdots \tag{11}$$

is an F-acyclic resolution of $\mathscr{F}$ in $\mathfrak{Qco}(X)$. Restricting this resolution to V we have $\mathscr{C}(\mathfrak{U}|_V,\mathscr{F}|_V)$ which by the same argument is a G-acyclic resolution of $\mathscr{F}|_V$ in $\mathfrak{Qco}(V)$, from which one deduces the desired isomorphism. □

Proposition 43. *Let $f:X\longrightarrow Y$ be a quasi-compact morphism of semi-separated schemes with X quasi-compact, and $\mathscr{F}$ a quasi-coherent sheaf on X. For every $i\geq 0$ there is a canonical isomorphism of sheaves of abelian groups on Y natural in $\mathscr{F}$*

$$\nu:\mathbf{a}f_*\mathscr{H}^i_{qc}(\mathscr{F})\longrightarrow R^if^{qc}_*(\mathscr{F})$$

In other words, $R^if^{qc}_*(\mathscr{F})$ is the sheafification of the presheaf

$$U\mapsto H^i_{qc}(f^{-1}U,\mathscr{F})$$

Proof. The assumptions mean that f is a concentrated morphism, so it makes sense to talk about the right derived functors of $f^{qc}_*:\mathfrak{Qco}(X)\longrightarrow\mathfrak{Qco}(Y)$. For every $i\geq 0$ we have the additive functor $\mathbf{a}f_*\mathscr{H}^i_{qc}(-)$ which is the composite of $\mathscr{H}^i_{qc}(-)$ with the direct image for presheaves $f_*:Ab(X)\longrightarrow Ab(Y)$ and the sheafification $\mathbf{a}:Ab(Y)\longrightarrow\mathfrak{Ab}(Y)$. Note that the functor $\mathbf{a}f_*$ is exact. Suppose we have a short exact sequence of quasi-coherent sheaves

$$0\longrightarrow\mathscr{F}'\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}''\longrightarrow 0$$

Then by Definition 5 we have a canonical connecting morphism $\omega^i:\mathscr{H}^i_{qc}(\mathscr{F}'')\longrightarrow\mathscr{H}^{i+1}_{qc}(\mathscr{F}')$ for each $i\geq 0$. The functors $\mathbf{a}f_*\mathscr{H}^i_{qc}(-)$ together with the morphisms $\mathbf{a}f_*\omega^i$ define a cohomological δ-functor between $\mathfrak{Qco}(X)$ and $\mathfrak{Ab}(Y)$. We claim that this δ-functor is universal.

Let $\mathscr{F}$ be a quasi-coherent sheaf on X and $\mathfrak{U}$ a finite semi-separating cover of X. Then $\mathscr{F}$ embeds into the quasi-coherent sheaf $\mathscr{X}=\mathscr{C}^0(\mathfrak{U},\mathscr{F})$. Let $\mathfrak{V}$ be a semi-separating affine *basis* of Y. For any $W\in\mathfrak{V}$ the inclusion $W\longrightarrow Y$ is affine, and therefore by pullback so is the inclusion $f^{-1}W\longrightarrow X$. But then by Proposition 40 we have for $i>0$

$$\Gamma(W,f_*\mathscr{H}^i_{qc}(\mathscr{X}))=H^i_{qc}(f^{-1}W,\mathscr{X})=0$$

Since the W form a basis we conclude that $\mathbf{a}f_*\mathscr{H}^i_{qc}(\mathscr{X})=0$ for $i>0$, so the functors $\mathbf{a}f_*\mathscr{H}^i_{qc}(-)$ are effaceable for $i>0$. It follows that the δ-functor is universal (DF,Theorem 74).

Composing the derived functors $R^i f_*^{qc}(\mathscr{F})$ with the exact forgetful functor $\mathfrak{Qco}(Y) \longrightarrow \mathfrak{Ab}(Y)$ we have another universal cohomological δ-functor between $\mathfrak{Qco}(X)$ and $\mathfrak{Ab}(Y)$ (DF,Definition 24). In degree zero we have a canonical isomorphism of sheaves of abelian groups natural in $\mathscr{F}$

$$\mathbf{a}f_*\mathscr{H}^0(\mathscr{F}) \cong \mathbf{a}(f_*\mathscr{F}) \cong f_*\mathscr{F} \cong R^0 f_*^{qc}(\mathscr{F})$$

This natural equivalence $\mathbf{a}f_*\mathscr{H}^0(-) \cong R^0 f_*^{qc}(-)$ induces a canonical isomorphism of cohomological δ-functors. In particular for each $i \geq 0$ we have the canonical natural equivalence ν, as required. □

We are now ready to prove our main result, which assures us that the three types of higher direct image $R^i f_*(-), R^i f_*^m(-)$ and $R^i f_*^{qc}(-)$ agree.

Proposition 44. *Let $f : X \longrightarrow Y$ be a quasi-compact morphism of semi-separated schemes with X quasi-compact, and $\mathscr{F}$ a quasi-coherent sheaf on X. For every $i \geq 0$ there is an isomorphism of sheaves of abelian groups on Y natural in $\mathscr{F}$*

$$\tau : R^i f_*^{qc}(\mathscr{F}) \longrightarrow R^i f_*(\mathscr{F})$$

Proof. Fix a finite semi-separating cover $\mathfrak{U}$ for X and a semi-separating affine basis $\mathfrak{V}$ for Y. For each $W \in \mathfrak{V}$ the inclusion $f^{-1}W \longrightarrow X$ is affine, so we have by Corollary 41 an isomorphism of abelian groups

$$\Gamma(W, f_*\mathscr{H}^i_{qc}(\mathscr{F})) = H^i_{qc}(f^{-1}W, \mathscr{F}) \longrightarrow H^i(f^{-1}W, \mathscr{F}) = \Gamma(W, f_*\mathscr{H}^i(\mathscr{F}))$$

This is natural in $\mathscr{F}$ and W, and induces an isomorphism on the stalks of the two presheaves. We can lift this isomorphism on stalks to an isomorphism of sheaves of abelian groups natural in $\mathscr{F}$

$$\mathbf{a}f_*\mathscr{H}^i_{qc}(\mathscr{F}) \longrightarrow \mathbf{a}f_*\mathscr{H}^i(\mathscr{F})$$

Composing with the isomorphisms of Proposition 43 and Proposition 8 we have the desired isomorphism of sheaves of abelian groups natural in $\mathscr{F}$

$$R^i f_*^{qc}(\mathscr{F}) \cong \mathbf{a}f_*\mathscr{H}^i_{qc}(\mathscr{F}) \cong \mathbf{a}f_*\mathscr{H}^i(\mathscr{F}) \cong R^i f_*(\mathscr{F})$$

which completes the proof. □

References

[1] R. Hartshorne, "Algebraic Geometry", *Springer GTM* Vol. 52 (1977).

[2] G. R. Kempf, "Some elementary proofs of basic theorems in the cohomology of quasicoherent sheaves", *Rocky Mountain J. Math.* 10 (1980), no.3, 637-645.

[3] R.W. Thomason, T. Trobaugh, "Higher Algebraic K-Theory of Schemes and of Derived Categories", The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60th birthday) Volume 3 pp. 247-435, Birkhäuser 1990.