Section 3.7 - The Serre Duality Theorem

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In this note we prove the Serre duality theorem for the cohomology of coherent sheaves on a projective scheme. First we do the case of projective space itself. Then on an arbitrary projective scheme X, we show that there is a coherent sheaf ω_X°, which plays the role in duality theory similar to the canonical sheaf of a nonsingular variety. In particular, if X is Cohen-Macaulay, it gives a duality theorem just like the one on projective space. Finally, if X is a nonsingular variety over an algebraically closed field we show that the dualising sheaf ω_X° agrees with the canonical sheaf ω_X.

Theorem 1. *Suppose $X = \mathbb{P}^n_k$ for some field k and $n \geq 1$ and set $\omega = \omega_{X/k}$. Then for any coherent sheaf of modules $\mathscr{F}$ on X we have*

(a) There is a canonical isomorphism of k-modules $H^n(X, \omega) \cong k$.

(b) There is a canonical perfect pairing of finite dimensional vector spaces over k

$$\tau : Hom(\mathscr{F}, \omega) \times H^n(X, \mathscr{F}) \longrightarrow k$$

(c) For $i \geq 0$ there is a canonical isomorphism of k-modules natural in $\mathscr{F}$

$$\eta : Ext^i(\mathscr{F}, \omega) \longrightarrow H^{n-i}(X, \mathscr{F})^\vee$$

Proof. (a) By (DIFF,Corollary 22) there is a canonical isomorphism of sheaves of modules $\omega \cong \mathcal{O}(-n-1)$. Composing $H^n(X, \omega) \cong H^n(X, \mathcal{O}(-n-1))$ with the isomorphism $H^n(X, \mathcal{O}(-n-1)) \cong k$ of (COS,Theorem 40)(b) gives the required canonical isomorphism of k-modules.

(b) The k-modules $Hom(\mathscr{F}, \omega)$ and $H^n(X, \mathscr{F})$ are finitely generated by (COS,Corollary 44) and (COS,Theorem 43), therefore both are free of finite rank. Given a morphism of sheaves of modules $\phi : \mathscr{F} \longrightarrow \omega$ there is a morphism of k-modules $H^n(X, \phi) : H^n(X, \mathscr{F}) \longrightarrow H^n(X, \omega)$, and we define a k-bilinear pairing

$$\tau' : Hom(\mathscr{F}, \omega) \times H^n(X, \mathscr{F}) \longrightarrow H^n(X, \omega)$$
$$\tau(\phi, a) = H^n(X, \phi)(a)$$

composing with the canonical isomorphism $H^n(X, \omega) \cong k$ of (a) we have another k-bilinear pairing $\tau : Hom(\mathscr{F}, \omega) \times H^n(X, \mathscr{F}) \longrightarrow k$. This corresponds under the bijection of (TES,Lemma 23) to a morphism of k-modules $\phi : Hom(\mathscr{F}, \omega) \longrightarrow H^n(X, \mathscr{F})^\vee$. To show that τ is perfect we have to show that ϕ is an isomorphism. We begin with the case $\mathscr{F} = \mathcal{O}(q)$ for some $q \in \mathbb{Z}$. Using (MPS,Corollary 17) we have a canonical isomorphism of k-modules

$$Hom(\mathscr{F}, \omega) \cong Hom(\mathcal{O}(q), \mathcal{O}(-n-1)) \cong \Gamma(X, \mathcal{O}(-q-n-1)) \cong H^0(X, \mathcal{O}(-q-n-1))$$

The perfect pairing of (COS,Theorem 40)(d) induces an isomorphism of k-modules

$$\phi' : H^0(X, \mathcal{O}(-q-n-1)) \longrightarrow H^n(X, \mathcal{O}(q))^\vee$$

which fits into the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}om(\mathscr{F}, \omega) & & \\ \Downarrow & \overset{\phi}{\searrow} & \\ H^0(X, \mathcal{O}(-q-n-1)) & \underset{\phi'}{\longrightarrow} & H^n(X, \mathscr{F})^\vee \end{array}$$

Therefore ϕ is an isomorphism for $\mathscr{F} = \mathcal{O}(q)$. Now assume that $\mathscr{F}$ is a coproduct of a finite number of $\mathcal{O}(q_i)$. Using the fact that cohomology commutes with coproducts (COS,Theorem 26) we see that $\phi : \mathcal{H}om(\mathscr{F}, \omega) \longrightarrow H^n(X, \mathscr{F})^\vee$ is the product of the isomorphisms $\mathcal{H}om(\mathcal{O}(q_i), \omega) \longrightarrow H^n(X, \mathcal{O}(q_i))^\vee$ and is therefore also an isomorphism.

Now let $\mathscr{F}$ be an arbitrary coherent sheaf. By (H, II 5.18) we can write $\mathscr{F}$ as the cokernel of a morphism of a morphism of coherent sheaves $\mathscr{E}_1 \longrightarrow \mathscr{E}_2$ with each $\mathscr{E}_i$ being a finite direct sum of sheaves $\mathcal{O}(q_i)$. That is, we have an exact sequence

$$\mathscr{E}_1 \longrightarrow \mathscr{E}_2 \longrightarrow \mathscr{F} \longrightarrow 0$$

Using (COS,Corollary 41) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}om(\mathscr{F}, \omega) & \longrightarrow & \mathcal{H}om(\mathscr{E}_2, \omega) & \longrightarrow & \mathcal{H}om(\mathscr{E}_1, \omega) \\ & & \downarrow & & \Downarrow & & \Downarrow \\ 0 & \longrightarrow & H^n(X, \mathscr{F})^\vee & \longrightarrow & H^n(X, \mathscr{E}_2)^\vee & \longrightarrow & H^n(X, \mathscr{E}_1)^\vee \end{array}$$

Therefore $\phi : \mathcal{H}om(\mathscr{F}, \omega) \longrightarrow H^n(X, \mathscr{F})^\vee$ is an isomorphism for any coherent $\mathscr{F}$, as required.
(c) The abelian category $\mathfrak{Mod}(X)$ is k-linear, so for every $i \geq 0$ we have a contravariant additive functor $Ext^i(-, \omega) : \mathfrak{Mod}(X) \longrightarrow k\mathbf{Mod}$ (EXT,Section 4.1). The following sequences of contravariant additive functors form contravariant cohomological δ-functors between the abelian categories $\mathfrak{Coh}(X)$ and $k\mathbf{Mod}$ (using (COS,Corollary 41)(i))

$$S : Ext^0(-, \omega), Ext^1(-, \omega), \ldots, Ext^n(-, \omega), Ext^{n+1}(-, \omega), \ldots$$
$$T : H^n(X, -)^\vee, H^{n-1}(X, -)^\vee, \ldots, H^0(X, -)^\vee, 0, 0, \ldots$$

We claim that both δ-functors are universal. It suffices to show that S^i, T^i are effaceable for $i \geq 1$. Given a coherent sheaf of modules $\mathscr{F}$, it follows from (H, II 5.18) and its proof that we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E} = \bigoplus_{j=1}^N \mathcal{O}(-q)$ with $q > 0$. Then for $i > 0$

$$Ext^i(\mathscr{E}, \omega) \cong \bigoplus Ext^i(\mathcal{O}(-q), \omega) \cong \bigoplus H^i(X, \omega(q)) \cong \bigoplus H^i(X, \mathcal{O}(q-n-1)) = 0$$

using (COS,Proposition 59), (MPS,Lemma 14), (COS,Proposition 54) and the calculations of (COS,Theorem 40). Similarly for $0 < i \leq n$ we have

$$H^{n-i}(X, \mathscr{E}) \cong \bigoplus H^{n-i}(X, \mathcal{O}(-q)) = 0$$

by (COS,Theorem 26) and (COS,Theorem 40). Therefore S^i, T^i are effaceable for $i \geq 1$, and therefore by (DF,Theorem 74) both S, T are universal.

The isomorphism $\phi : \mathcal{H}om(\mathscr{F}, \omega) \longrightarrow H^n(X, \mathscr{F})^\vee$ of k-modules defined in (b) is natural in the coherent sheaf $\mathscr{F}$, so there is a canonical natural equivalence of contravariant functors $\mathfrak{Coh}(X) \longrightarrow k\mathbf{Mod}$

$$\eta^0 : Ext^0(-, \omega) \longrightarrow \mathcal{H}om(-, \omega) \longrightarrow H^n(X, -)^\vee$$

This induces a canonical isomorphism of cohomological δ-functors $\eta : S \longrightarrow T$. That is, for each $i \geq 0$ and coherent sheaf $\mathscr{F}$ we have a canonical isomorphism of k-modules $\eta^i : Ext^i(\mathscr{F}, \omega) \longrightarrow H^{n-i}(X, \mathscr{F})^\vee$ natural in $\mathscr{F}$, as required. $\square$

Remark 1. Let X be a nonempty noetherian scheme of finite dimension n over a field k. Then for $0 \le i \le n$ let $T^i : \mathfrak{Mod}(X) \longrightarrow k\mathbf{Mod}$ be the contravariant additive functor $H^{n-i}(X,-)^\vee$. For $i > n$ we set $T^i = 0$. Given a short exact sequence of sheaves of modules on X

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

we have the connecting morphisms of k-modules $\delta^j : H^j(X,\mathscr{F}'') \longrightarrow H^{j+1}(X,\mathscr{F}')$ for $j \ge 0$. For $0 \le i < n$ we set $\omega^i = (\delta^{n-i-1})^\vee$ and for $i \ge n$ we set $\omega^i = 0$. Then using (COS,Theorem 30) we have an exact sequence of k-modules

$$0 \longrightarrow H^n(X,\mathscr{F}'')^\vee \longrightarrow H^n(X,\mathscr{F})^\vee \longrightarrow H^n(X,\mathscr{F}')^\vee \longrightarrow \cdots \longrightarrow H^0(X,\mathscr{F}')^\vee \longrightarrow 0$$

So the functors $\{T^i\}_{i\ge 0}$ form a contravariant cohomological δ-functor between $\mathfrak{Mod}(X)$ and $k\mathbf{Mod}$. We usually refer to this δ-functor by the sequence $H^n(X,-)^\vee, \ldots, H^0(X,-)^\vee, 0, 0, \ldots$ with the connecting morphisms implicit.

To generalise Theorem 1 to other schemes, we take properties (a) and (b) as our guide, and make the following definitions.

Definition 1. Let X be a proper scheme of finite dimension n over a field k. A *dualising sheaf* for X over k is a coherent sheaf ω_X° on X, together with a *trace* morphism of k-modules $t : H^n(X,\omega_X^\circ) \longrightarrow k$, such that for all coherent sheaves $\mathscr{F}$ on X, the natural k-bilinear pairing

$$\tau : Hom(\mathscr{F},\omega_X^\circ) \times H^n(X,\mathscr{F}) \longrightarrow H^n(X,\omega_X^\circ)$$
$$\tau(\psi,a) = H^n(X,\psi)(a)$$

followed by t corresponds under the bijection of (TES,Lemma 23) to an *isomorphism* of k-modules

$$Hom(\mathscr{F},\omega_X^\circ) \longrightarrow H^n(X,\mathscr{F})^\vee$$

which is certainly natural in $\mathscr{F}$.

Proposition 2. *With the notation of Definition 1, if a dualising sheaf exists it is unique. That is, if $(\omega,t),(\omega',t')$ are dualising sheaves then there is a unique isomorphism of sheaves of modules $\varphi : \omega \longrightarrow \omega'$ such that $t = t' \circ H^n(X,\varphi)$.*

Proof. Since ω' is dualising, we get an isomorphism $Hom(\omega,\omega') \cong H^n(X,\omega)^\vee$. Let $\varphi : \omega \longrightarrow \omega'$ correspond to $t \in H^n(X,\omega)^\vee$. That is, $t = t' \circ H^n(X,\varphi)$. Similarly, let $\psi : \omega' \longrightarrow \omega$ correspond to $t' \in H^n(X,\omega')^\vee$. Then $t \circ H^n(X,\psi\varphi) = t$ and therefore $\psi\varphi = 1$. Similarly we see that $\varphi\psi = 1$ so that φ is an isomorphism of sheaves of modules, which is clearly unique with the stated property. ∎

The question of existence of dualising sheaves is more difficult. In fact they exist for any X proper over k, but we will prove the existence here only for projective schemes. First we need some preliminary results.

Lemma 3. *Set $P = \mathbb{P}^n_k$ for some field k and let $i : X \longrightarrow P$ be a nonempty closed subscheme of codimension r. Then $\mathscr{E}xt^i_P(i_*\mathcal{O}_X,\omega_P) = 0$ for all $0 \le i < r$.*

Proof. A closed immersion is finite, so X is noetherian and $i_*\mathcal{O}_X$ coherent. Therefore the sheaf $\mathscr{F}^i = \mathscr{E}xt^i_P(i_*\mathcal{O}_X,\omega_P)$ is coherent for $i \ge 0$ (COS,Proposition 64). If $r = 0$ then the statement is vacuous, so assume $r \ge 1$ and $0 \le i < r$. After twisting by a suitably large integer q, the sheaf $\mathscr{F}^i$ will be generated by global sections (H, II 5.17). Thus to show $\mathscr{F}^i$ is zero, it will be sufficient to show that $\Gamma(P,\mathscr{F}^i(q)) = 0$ for all sufficiently large q. But by (COS,Proposition 59) and (COS,Proposition 66) we have for all sufficiently large q

$$\Gamma(P,\mathscr{F}^i(q)) \cong Ext^i_P(i_*\mathcal{O}_X,\omega_P(q)) \cong Ext^i_P((i_*\mathcal{O}_X)(-q),\omega_P)$$

By the projection formula (MRS,Lemma 80) there is a canonical isomorphism of sheaves of modules $(i_*\mathcal{O}_X)(-q) \cong i_*i^*\mathcal{O}(-q)$ and so finally an isomorphism for all sufficiently large q

$$\Gamma(P, \mathscr{F}^i(q)) \cong Ext^i_P(i_*i^*\mathcal{O}(-q), \omega_P)$$

On the other hand, by Theorem 1 this last Ext group is dual to $H^{n-i}(P, i_*i^*\mathcal{O}(-q))$, which is isomorphic as an abelian group to $H^{n-i}(X, i^*\mathcal{O}(-q))$ (COS,Lemma 27). Since $dimX = n - r < n - i$ we have $H^{n-i}(X, i^*\mathcal{O}(-q)) = 0$ by (COS,Theorem 30). Therefore $\Gamma(P, \mathscr{F}^i(q)) = 0$ for sufficiently large q, as required. $\square$

Remark 2. Let $\mathcal{A}$ be an abelian category, and suppose we have a positive cochain complex of injective objects

$$I : 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \longrightarrow \cdots$$

with differentials $d^i : I^i \longrightarrow I^{i+1}$. We say I is *exact up to the rth place* for $r \geq 0$ if the complex is exact at I^i for every $0 \leq i < r$. Suppose that this is the case (if $r = 0$ the condition says nothing, so assume $r \geq 1$ henceforth). Set $K^0 = I^0$ and for $i \geq 1$ set $K^i = Im(d^i)$ so that we have short exact sequences

$$\begin{gathered} 0 \longrightarrow K^0 \longrightarrow I^1 \longrightarrow K^1 \longrightarrow 0 \\ 0 \longrightarrow K^1 \longrightarrow I^2 \longrightarrow K^2 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow K^{r-1} \longrightarrow I^r \longrightarrow C \longrightarrow 0 \end{gathered}$$

where $I^r \longrightarrow C$ is the canonical cokernel of d^{r-1}. Since I^0 is injective the first sequence splits, which implies that K^1 is injective. In this way we show that $K^1, \ldots, K^{r-1}$ are all injective. Choose a splitting $\gamma : I^r \longrightarrow K^{r-1}$ of the last sequence, which makes I^r into a biproduct $K^{r-1} \oplus C$. Let $\mu : C \longrightarrow I^{r+1}$ and $j : I^{r-1} \longrightarrow K^{r-1}$ be the canonical morphisms. Then $d^{r-1} = j \oplus 0, d^r = 0 \oplus \mu$ where we consider I^{r-1} as the biproduct $I^{r-1} \oplus 0$ and I^{r+1} as $0 \oplus I^{r+1}$. We therefore have

$$Ker(d^r) = K^{r-1} \oplus Ker(\mu)$$

By definition $Im(d^{r-1}) = K^{r-1}$ so there is a canonical isomorphism $\tau : H^r(I) \longrightarrow Ker(\mu)$. One can check this is the unique morphism making the following diagram commute

$$\begin{array}{ccccc} & & Ker(d^r) & \longrightarrow & I^r \\ & \swarrow & & & \downarrow \\ H^r(I) & \underset{\tau}{\Longrightarrow} & Ker(\mu) & \longrightarrow & C \end{array}$$

In particular τ is independent of the choice of splitting γ, so it is a canonical isomorphism. In the same way, one shows that for any object B there is a unique isomorphism of abelian groups $\tau' : H^r(Hom(B, I)) \cong Hom(B, Ker\mu)$ making the following diagram commute

$$\begin{array}{ccccc} & & KerHom(B, d^r) & \longrightarrow & Hom(B, I^r) \\ & \swarrow & & & \downarrow \\ H^r(Hom(B, I)) & \underset{\tau'}{\Longrightarrow} & Hom(B, Ker\mu) & \longrightarrow & Hom(B, C) \end{array}$$

If $\mathcal{A}$ is R-linear for some ring R, then τ' will be an isomorphism of R-modules. It is clear that τ' is natural in B.

Remark 3. With the notation of Lemma 3 it follows from (HDIS,Proposition 15) that $R^p i^!(\omega_P) = 0$ for $0 \leq p < r$ (in particular, $i^!(\omega_P) = 0$ provided $r > 0$). It turns out that the first nonzero sheaf $R^r i^!(\omega_P)$ is the dualising sheaf we are looking for.

Lemma 4. *With the notation of Lemma 3, let* $\omega_X^\circ = R^r i^!(\omega_P)$*. Then for any sheaf of modules* $\mathscr{F}$ *on* X *there is a canonical isomorphism of* k*-modules natural in* $\mathscr{F}$

$$\chi : Hom_X(\mathscr{F}, \omega_X^\circ) \longrightarrow Ext_P^r(i_*\mathscr{F}, \omega_P)$$

Proof. By (HDIS,Corollary 19) the sheaf of modules ω_X° is coherent. Suppose we have an injective resolution of ω_P in $\mathfrak{Mod}(P)$

$$\mathscr{I} : 0 \longrightarrow \omega_P \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

Then by definition $Ext_P^i(i_*\mathscr{F}, \omega_P)$ is the ith cohomology group of the cochain complex of k-modules $Hom_P(i_*\mathscr{F}, \mathscr{I})$. But by adjointness the cochain complex $Hom_P(i_*\mathscr{F}, \mathscr{I})$ is canonically isomorphic to $Hom_X(\mathscr{F}, i^!\mathscr{I})$ (as a cochain complex of k-modules), so we have a canonical isomorphism of k-modules natural in $\mathscr{F}$ for $i \geq 0$

$$Ext_P^i(i_*\mathscr{F}, \omega_P) \cong H^i(Hom_X(\mathscr{F}, i^!\mathscr{I}))$$

Since $i^!$ has an exact left adjoint it preserves injectives, so $\mathscr{J}^i = i^!\mathscr{I}^i$ is an injective object of $\mathfrak{Mod}(X)$ for $i \geq 0$. It follows from Lemma 3 that the following sequence is exact up to the rth place

$$0 \longrightarrow \mathscr{H}om_P(i_*\mathcal{O}_X, \mathscr{I}^0) \longrightarrow \mathscr{H}om_P(i_*\mathcal{O}_X, \mathscr{I}^1) \longrightarrow \cdots$$

From (MRS,Proposition 96) and exactness of i^{-1} we infer that $H^i(\mathscr{J}) = 0$ for $0 \leq i < r$. That is, the complex $\mathscr{J}$ is exact up to the rth place. Using the notation of Remark 2 we have a canonical isomorphism of k-modules natural in $\mathscr{F}$

$$\begin{aligned} Ext_P^r(i_*\mathscr{F}, \omega_P) &\cong H^r(Hom_X(\mathscr{F}, \mathscr{J})) \\ &\cong Hom_X(\mathscr{F}, Ker\mu) \\ &\cong Hom_X(\mathscr{F}, H^r(\mathscr{J})) \\ &= Hom_X(\mathscr{F}, \omega_X^\circ) \end{aligned}$$

as required. □

Proposition 5. *Let* X *be a nonempty projective scheme over a field* k*. Then* X *has a dualising sheaf.*

Proof. A projective scheme over a field is proper and has finite dimension (since it embeds in $\mathbb{P}_k^n$ for some finite n) so it makes sense to talk about dualising sheaves on X. Let $i : X \longrightarrow \mathbb{P}_k^n$ be a closed immersion of k-schemes for some $n \geq 1$ and let $0 \leq r \leq n$ be the codimension of X in $P = \mathbb{P}_k^n$. We claim that the coherent sheaf ω_X° of Lemma 4 is a dualising sheaf for X. For any coherent sheaf of modules $\mathscr{F}$ on X we have by Lemma 4, Theorem 1 and (COS,Corollary 28) an isomorphism of k-modules natural in $\mathscr{F}$

$$\begin{aligned} Hom_X(\mathscr{F}, \omega_X^\circ) &\cong Ext_P^r(i_*\mathscr{F}, \omega_P) \\ &\cong H^{n-r}(P, i_*\mathscr{F})^\vee \\ &\cong H^m(X, \mathscr{F})^\vee \end{aligned}$$

where $m = n - r$ is the dimension of X. In particular, taking $\mathscr{F} = \omega_X^\circ$ the element $1 \in Hom_X(\omega_X^\circ, \omega_X^\circ)$ corresponds to a morphism of k-modules $t : H^m(X, \omega_X^\circ) \longrightarrow k$, which we take as our trace map. It is now clear that the pair (ω_X°, t) is a dualising sheaf for X. □

Definition 2. We say a nonempty noetherian topological space X is *equidimensional* if all the irreducible components of X have the same dimension, which must be equal to the dimension of X.

If X is an integral noetherian scheme of finite type over a field, then for every closed point $P \in X$ we have $dim\mathcal{O}_{X,P} = dimX$. If we do not require X to be integral, this condition characterises the equidimensional schemes (of which X is the simplest type, with only one irreducible component).

Lemma 6. *Let X be a noetherian scheme of finite dimension n which is of finite type over a field k. Then X is equidimensional if and only if for any closed point $P \in X$ we have $dim\mathcal{O}_{X,P} = n$.*

Proof. Suppose that X is equidimensional and let $P \in X$ be a closed point. Let Z be the irreducible closed set $\{P\}$. Since the codimension of Z in X is finite, we can find some irreducible component Y of X with $P \in Y$ and $codim(Z, Y) = codim(Z, X)$. Putting the reduced scheme structure on Y (which is therefore an integral scheme of finite type over k) we have $dim\mathcal{O}_{X,P} = dim\mathcal{O}_{Y,P} = n$, where the first equality follows from (FPOS,Lemma 1) and the second from (H, Ex.3.20) and the equidimension assumption.

For the converse, let $X = Y_1 \cup \cdots \cup Y_n$ be the irreducible components of X. It suffices to show that $dimY_1 = n$. We claim that there exists a closed point $P \in Y_1$ with $codim(\{P\}, Y_1) = codim(\{P\}, X)$. Suppose otherwise, and let $\{P\} = Z_0 \subset \cdots \subset Z_m$ be a chain of closed irreducible subsets of X of length $m = codim(\{P\}, X)$. The set Z_m must be contained in some Y_j, $j > 1$, and therefore $P \in Y_j$. It follows that every closed point of Y belongs to the union $Y_2 \cup \cdots \cup Y_n$ and therefore $Y_1 \subseteq Y_j$ for some $j > 1$ (VS,Proposition 14), which is a contradiction. This shows that we can find a closed point $P \in Y_1$ with the required property, in which case

$$n = dim\mathcal{O}_{X,P} = codim(\{P\}, X) = codim(\{P\}, Y_1) = dim\mathcal{O}_{Y_1,P} = dimY_1$$

since Y is an integral scheme of finite type over a field. This shows that X is equidimensional and completes the proof. □

Theorem 7. *Let X be a projective scheme of finite dimension n over a field k. Let ω_X° be a dualising sheaf on X, and let $\mathcal{O}(1)$ be a very ample invertible sheaf on X. Then*

(a) For any coherent sheaf of modules $\mathscr{F}$ on X and $i \geq 0$ there is a canonical morphism of k-modules natural in $\mathscr{F}$

$$\theta^i : Ext^i(\mathscr{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathscr{F})^\vee$$

(b) The following conditions are equivalent

(i) X is Cohen-Macaulay and equidimensional.

(ii) For any locally finitely free sheaf of modules $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F}(-q)) = 0$ for $0 \leq i < n$ and all sufficiently large q.

(iii) The morphisms θ^i of (a) are isomorphisms for all $i \geq 0$ and $\mathscr{F}$ coherent on X.

Proof. (a) Since k is a field the functor $(-)^\vee$ is exact (MRS,Lemma 73), and we have two contravariant cohomological δ-functors $\{Ext^i(-, \omega_X^\circ)\}_{i \geq 0}, \{H^{n-i}(X, -)^\vee\}_{i \geq 0}$ between $\mathfrak{Coh}(X)$ and k**Mod** (the latter δ-functor was defined in Remark 1). To show that the first δ-functor is universal, it suffices to show that $Ext^i(-, \omega_X^\circ)$ is effaceable for $i > 0$. But if $\mathscr{F}$ is coherent, we can write $\mathscr{F}$ as a quotient of a direct sum $\mathscr{E} = \bigoplus_{i=1}^N \mathcal{O}(-q)$ for all sufficiently large $q > 0$. Then using (COS,Proposition 59) and (COS,Proposition 54) we have an isomorphism

$$Ext^i(\mathscr{E}, \omega_X^\circ) \cong \bigoplus_{j=1}^N Ext^i(\mathcal{O}(-q), \omega_X^\circ) \cong \bigoplus_{j=1}^N H^i(X, \omega_X^\circ(q))$$

By (COS,Theorem 43) if we take q large enough, we have $H^i(X,\omega_X^\circ(q)) = 0$ for all $i > 0$ and therefore $Ext^i(\mathscr{E},\omega_X^\circ) = 0$, as required. Therefore $\{Ext^i(-,\omega_X^\circ)\}_{i\geq 0}$ is universal, and the natural equivalence

$$\theta^0 : Ext^0(-,\omega_X^\circ) \cong Hom(-,\omega_X^\circ) \cong H^n(X,-)^\vee$$

defined by the dualising sheaf induces a canonical morphism of cohomological δ-functors θ. In particular for every $i \geq 0$ we have a canonical natural transformation $\theta^i : Ext^i(-,\omega_X^\circ) \longrightarrow H^{n-i}(X,-)^\vee$, as required.

(b) Let $j : X \longrightarrow P$ be a closed immersion of k-schemes with $P = \mathbb{P}_k^N$ for some $N \geq 1$. We may also assume that $j^*\mathcal{O}(1) \cong \mathcal{O}(1)$. $(i) \Rightarrow (ii)$ For any locally finitely free sheaf of modules $\mathscr{F}$ on X, and any closed point $x \in X$ in the support of $\mathscr{F}$, we have using (MAT,Lemma 70), Lemma 6 and the hypothesis (i)

$$depth_{\mathcal{O}_{X,x}}\mathscr{F}_x = depth_{\mathcal{O}_{X,x}}\mathcal{O}_{X,x} = dim\mathcal{O}_{X,x} = n$$

Set $A = \mathcal{O}_{P,j(x)}$, which is a regular local ring of dimension N since $\mathbb{P}_k^N$ is nonsingular. Since we have a surjective local morphism of local rings $A \longrightarrow \mathcal{O}_{X,x}$ it is clear that $depth_A(j_*\mathscr{F})_{j(x)} = depth_{\mathcal{O}_{X,x}}\mathscr{F}_x = n$ (MAT,Lemma 69) and therefore $proj.dim_A((j_*\mathscr{F})_{j(x)}) = N-n$ (MAT,Corollary 132). Using the definition of projective dimension, (COS,Proposition 63) and (VS,Corollary 18) we see that for $l > N-n$ the functor $\mathscr{E}xt_P^l(j_*\mathscr{F},-)$ is zero on quasi-coherent sheaves.

Now assume that $0 \leq i < n$. Using Theorem 1 we find that $H^i(P,(j_*\mathscr{F})(-q))$ is dual to $Ext_P^{N-i}(j_*\mathscr{F},\omega_P(q))$. For sufficiently large q, this Ext is isomorphic to $\Gamma(P,\mathscr{E}xt_P^{N-i}(j_*\mathscr{F},\omega_P(q)))$ by (COS,Proposition 66). But this is zero for $N-i > N-n$, as we've just seen. In other words, $H^i(P,(j_*\mathscr{F})(-q)) = 0$ for $i < n$ and all sufficiently large q.

By the projection formula (MRS,Lemma 80) and (MRS,Proposition 95) we have an isomorphism $(j_*\mathscr{F})(-q) \cong j_*(\mathscr{F}(-q))$ and therefore $H^i(X,\mathscr{F}(-q)) \cong H^i(P,(j_*\mathscr{F})(-q)) = 0$ for $i < n$ and all sufficiently large q (COS,Corollary 28), which is what we wanted to show.

$(ii) \Rightarrow (i)$ Fix some $i > N-n$. We claim that $\mathscr{E}xt_P^i(j_*\mathcal{O}_X,\omega_P) = 0$. To show this, it suffices by (H, II 5.17) and (COS,Lemma 65) to show that $\Gamma(P,\mathscr{E}xt_P^i(j_*\mathcal{O}_X,\omega_P(q))) = 0$ for some $q > 0$. But for all sufficiently large q we have an isomorphism of abelian groups (COS,Proposition 66)

$$\begin{aligned}\Gamma(P,\mathscr{E}xt_P^i(j_*\mathcal{O}_X,\omega_P(q))) &\cong Ext_P^i(j_*\mathcal{O}_X,\omega_P(q))\\ &\cong H^{N-i}(P,j_*\mathcal{O}(-q))^\vee\\ &\cong H^{N-i}(X,\mathcal{O}(-q))^\vee = 0\end{aligned}$$

using (ii) with $\mathscr{F} = \mathcal{O}_X$ and duality for P. Given $x \in X$ we set $A = \mathcal{O}_{P,j(x)}$. Applying (COS,Proposition 63) to the equation $\mathscr{E}xt_P^i(j_*\mathcal{O}_X,\omega_P) = 0$ we deduce that $Ext_A^i(\mathcal{O}_{X,x},A) = 0$ for $i > N-n$ and also $proj.dim_A\mathcal{O}_{X,x} \leq N-n$ (MAT,Corollary 129). If x is a closed point then A is a regular local ring of dimension N, so using (MAT,Corollary 132), (MAT,Lemma 69) we deduce

$$depth_{\mathcal{O}_{X,x}}\mathcal{O}_{X,x} = depth_A\mathcal{O}_{X,x} \geq n \tag{1}$$

But always $depth_{\mathcal{O}_{X,x}}\mathcal{O}_{X,x} \leq dim\mathcal{O}_{X,x} \leq n$, so we must have equality. This shows that for every closed point $x \in X$ the noetherian local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay of dimension n. In particular X is Cohen-Macaulay (DIFF,Lemma 43) and also equidimensional by Lemma 6.

$(i) \Rightarrow (iii)$ To show that the θ^i are isomorphisms, it will be enough to show that the δ-functor $\{H^{n-i}(X,-)^\vee\}_{i\geq 0}$ is also universal, for which it suffices to show that $H^{n-i}(X,-)^\vee$ is effaceable for $i > 0$. So given a coherent sheaf $\mathscr{F}$, write it as a quotient of $\mathscr{E} = \bigoplus_{i=1}^N \mathcal{O}(-q)$, with q so large that $H^{n-i}(X,\mathcal{O}(-q)) = 0$ for $i > 0$ by (ii). Then $H^{n-i}(X,\mathscr{E})^\vee = 0$, as required.

$(iii) \Rightarrow (i)$ If θ^i is an isomorphism, then for any $\mathscr{F}$ locally finitely free on X and $0 \leq i < n$ we have $H^i(X,\mathscr{F}(-q)) \cong Ext^{n-i}(\mathscr{F}(-q),\omega_X^\circ)^\vee$. But this Ext is isomorphic to $H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X^\circ(q))$ by (COS,Proposition 54) and (COS,Proposition 59), so it is zero for all sufficiently large q by (COS,Theorem 43). □

Remark 4. In particular if X is nonsingular projective variety over a field k, or more generally a projective local complete intersection, then X is Cohen-Macaulay (DIFF,Proposition 44), so the θ^i are isomorphisms.

Corollary 8. *Let X be a projective Cohen-Macaulay scheme of equidimension n over a field k, and ω_X° a dualising sheaf on X. Then for any locally finitely free sheaf $\mathscr{F}$ on X and $0 \le i \le n$ there is a canonical isomorphism of k-modules natural in $\mathscr{F}$*

$$H^i(X,\mathscr{F}) \longrightarrow H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X^\circ)^\vee$$

Proof. We have the following isomorphism of k-modules, natural in $\mathscr{F}$, for any $0 \le i \le n$

$$\begin{aligned} H^i(X,\mathscr{F})^\vee &\cong Ext^{n-i}(\mathscr{F},\omega_X^\circ) \\ &\cong Ext^{n-i}(\mathcal{O}_X,\mathscr{F}^\vee \otimes \omega_X^\circ) \\ &\cong H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X^\circ) \end{aligned}$$

where we have used Theorem 7, (COS,Proposition 59) and (COS,Proposition 54). Taking duals, we have the required isomorphism (by (COS,Theorem 43) the k-module $H^i(X,\mathscr{F})$ is finitely generated, so we can apply (TES,Example 2)). □

Corollary 9. *Let X be a projective Cohen-Macaulay scheme of equidimension n over a field k, and ω_X° a dualising sheaf on X. Then for any locally finitely free sheaves $\mathscr{F},\mathscr{E}$ on X and $0 \le i \le n$ there is a canonical isomorphism of k-modules natural in both variables*

$$Ext^i(\mathscr{E},\mathscr{F}) \longrightarrow Ext^{n-i}(\mathscr{F},\mathscr{E} \otimes \omega_X^\circ)^\vee$$

Proof. We have the following isomorphism of k-modules, natural in both variables

$$\begin{aligned} Ext^i(\mathscr{E},\mathscr{F}) &\cong Ext^i(\mathcal{O}_X,\mathscr{E}^\vee \otimes \mathscr{F}) \\ &\cong H^i(X,\mathscr{E}^\vee \otimes \mathscr{F}) \\ &\cong H^{n-i}(X,(\mathscr{E}^\vee \otimes \mathscr{F})^\vee \otimes \omega_X^\circ)^\vee \\ &\cong H^{n-i}(X,\mathscr{E} \otimes \mathscr{F}^\vee \otimes \omega_X^\circ)^\vee \\ &\cong Ext^{n-i}(\mathcal{O}_X,\mathscr{F}^\vee \otimes \mathscr{E} \otimes \omega_X^\circ)^\vee \\ &\cong Ext^{n-i}(\mathscr{F},\mathscr{E} \otimes \omega_X^\circ)^\vee \end{aligned}$$

as required. □

Corollary 10. *Let X be an integral normal projective scheme of dimension ≥ 2 over a field k. Then for any locally finitely free sheaf $\mathscr{F}$ on X we have $H^1(X,\mathscr{F}(-q)) = 0$ for all sufficiently large q.*

Proof. If $x \in X$ is a closed point then $dim\mathcal{O}_{X,x} = dimX \ge 2$. If $U \cong SpecA$ is an affine open neighborhood of x, then x corresponds to a maximal ideal of height ≥ 2 in A. It follows that $depth_{\mathcal{O}_{X,x}}\mathscr{F}_x \ge 2$ for every closed point x in the support of $\mathscr{F}$ (DIFF,Theorem 42). So the result follows by the same method as in the proof of $(i) \Rightarrow (ii)$ in Theorem 7(b). □

Corollary 11. *If X is a nonsingular projective variety over an algebraically closed field k, then any dualising sheaf ω_X° is isomorphic as a sheaf of modules to the canonical sheaf ω_X.*

Corollary 12. *Let X be a nonsingular projective curve over an algebraically closed field k. Then $p_g(X) = rank_k H^1(X,\mathcal{O}_X)$.*

Proof. Let ω_X° be a dualising sheaf on X. Then by Theorem 7 and Corollary 11 we have an isomorphism of k-modules

$$\Gamma(X,\omega_X) \cong \Gamma(X,\omega_X^\circ) \cong Ext^0(\mathcal{O}_X,\omega_X^\circ) \cong H^1(X,\mathcal{O}_X)^\vee$$

from which it follows that $p_g(X) = rank_k H^1(X,\mathcal{O}_X)$, as required. □

Remark 5. Let X be as in Corollary 12. We now completely understand the cohomology groups $H^i(X,\mathcal{O}_X)$ for $i \ge 0$. These groups are zero for $i > 1$ and using (VS,Corollary 28) we have

$$\begin{aligned} rank_k H^0(X,\mathcal{O}_X) &= 1 \\ rank_k H^1(X,\mathcal{O}_X) &= p_g(X) \end{aligned}$$