Section 3.2 - Cohomology of Sheaves

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In this note we define cohomology of sheaves by taking the derived functors of the global section functor. As an application of general techniques of cohomology we prove the Grothendieck and Serre vanishing theorems. We introduce the Čech cohomology and use it to calculate cohomology of projective space.

The original reference for this material is EGA III, but most graduate students would probably encounter it in Hartshorne's book [Har77] where many proofs are given only for noetherian schemes, probably because the only known proofs in the general case utilised spectral sequences. Several years after Hartshorne's book was published there appeared a paper by Kempf [Kem80] giving very elegant and elementary proofs in the full generality of quasi-compact quasi-separated schemes. The proofs given here are a mix of those from Hartshorne's book and Kempf's paper.

Contents

1 Cohomology

We will define sheaf cohomology by deriving the global sections functor. In order to apply this machinery we need to introduce abelian categories and recall some facts about injectives. For the necessary background see our notes on Abelian Categories (AC) and Derived Functors (DF).

Definition 1. A preadditive category $\mathcal{A}$ is *abelian* if it has zero, finite products, kernels, cokernels, is normal and conormal, and has epi-mono factorisations. A subcategory $\mathcal{C}$ of $\mathcal{A}$ is an *abelian subcategory* if $\mathcal{C}$ is abelian (it inherits an additive structure from $\mathcal{A}$, so if it is abelian it must be with this structure) and the inclusion is exact. An abelian category $\mathcal{A}$ is *grothendieck* if it is cocomplete, has exact direct limits and a generator.

Theorem 1. *Any grothendieck abelian category $\mathcal{A}$ has the following properties*

(i) $\mathcal{A}$ is locally small and colocally small.

(ii) $\mathcal{A}$ *has enough injectives.*

(iii) $\mathcal{A}$ *has an injective cogenerator.*

(iv) $\mathcal{A}$ *is complete.*

Proof. (i) [Mit65] Theorem II 15.1 and Proposition I 14.2. (ii) [Mit65] Theorem III 3.2. (iii) [Mit65] Corollary III, 3.4. (iv) follows from (LOR,Corollary 27) or alternatively (AC,Corollary 24). □

Theorem 2. *Let* $T : \mathcal{A} \longrightarrow \mathcal{B}$ *be a functor between grothendieck abelian categories. Then*

(i) T *has a right adjoint if and only if it is colimit preserving.*

(ii) T *has a left adjoint if and only if it is limit preserving.*

Proof. Combine (AC,Theorem 22) and its dual (AC,Theorem 23) with Theorem 1. □

Example 1. The following abelian categories are grothendieck abelian, and therefore complete with enough injectives:

(i) The category of abelian groups **Ab** and the category of left modules R**Mod** over a ring R.

(ii) The category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on a topological space X.

(iii) The category $\mathfrak{Mod}(X)$ of sheaves of modules over a ringed space $(X, \mathcal{O}_X)$. In particular, the category of sheaves of modules over a scheme X.

Example 2. Let X be a scheme. See (MOS,Definition 1) for the definition of the full subcategories $\mathfrak{Qco}(X), \mathfrak{Coh}(X)$ and the proof that $\mathfrak{Qco}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$. If X is noetherian, then $\mathfrak{Coh}(X)$ is an abelian subcategory of $\mathfrak{Mod}(X)$.

Definition 2. Let X be a topological space. Let $\Gamma(X, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$ be the global section functor. This is a left exact additive covariant functor, and we define the *cohomology functors* $H^i(X, -)$ to be the right derived functors of $\Gamma(X, -)$. There is a canonical natural equivalence $H^0(X, -) \cong \Gamma(X, -)$. For any sheaf of abelian groups $\mathscr{F}$, the groups $H^i(X, \mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$. For any exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow \mathscr{H} \longrightarrow 0$$

we have the *long exact sequence of cohomology*

$$0 \longrightarrow \Gamma(X,\mathscr{G}) \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{H}) \longrightarrow H^1(X,\mathscr{G}) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^n(X,\mathscr{H}) \longrightarrow H^{n+1}(X,\mathscr{G}) \longrightarrow H^{n+1}(X,\mathscr{F}) \longrightarrow H^{n+1}(X,\mathscr{H}) \longrightarrow \cdots$$

which is natural in the exact sequence, in the sense that for every commutative diagram of sheaves of abelian groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{G}' & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{H}' & \longrightarrow & 0 \end{array}$$

the following diagrams commute for $n \geq 1$

$$\begin{array}{ccc} \Gamma(X,\mathscr{H}) & \longrightarrow & H^1(X,\mathscr{G}) \\ \downarrow & & \downarrow \\ \Gamma(X,\mathscr{H}') & \longrightarrow & H^1(X,\mathscr{G}') \end{array} \qquad \begin{array}{ccc} H^n(X,\mathscr{H}) & \longrightarrow & H^{n+1}(X,\mathscr{G}) \\ \downarrow & & \downarrow \\ H^n(X,\mathscr{H}') & \longrightarrow & H^{n+1}(X,\mathscr{G}') \end{array}$$

Remark 1. Note that even if X and $\mathscr{F}$ have some additional structure, e.g., X a scheme and $\mathscr{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathscr{F}$ simply as a sheaf of abelian groups on the underlying topological space X.

Recall ([Har77] II Ex.1.16) that a sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is surjective. This property is stable under isomorphism.

Lemma 3. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. If $\mathscr{F}$ is injective then it is flasque. In particular an injective sheaf of abelian groups is flasque.*

Proof. Associated with any open $U \subseteq X$ is an $\mathcal{O}_X$-module $\mathcal{O}_U$ with the property that there is a natural isomorphism $Hom(\mathcal{O}_U, \mathscr{F}) \cong \mathscr{F}(U)$ for any module $\mathscr{F}$ (see (MRS,Section 1.5)). Now let $\mathscr{J}$ be an injective $\mathcal{O}_X$-module, and let $V \subseteq U$ be open sets. There is a canonical monomorphism $\mathcal{O}_V \longrightarrow \mathcal{O}_U$. Since $\mathscr{J}$ is injective, the top row in the following commutative diagram is surjective

$$\begin{array}{ccc} Hom(\mathcal{O}_U, \mathscr{J}) & \longrightarrow & Hom(\mathcal{O}_V, \mathscr{J}) \\ \Downarrow & & \Downarrow \\ \mathscr{J}(U) & \longrightarrow & \mathscr{J}(V) \end{array}$$

Therefore $\mathscr{J}(U) \longrightarrow \mathscr{J}(V)$ is surjective and $\mathscr{J}$ is flasque. The statement also works for sheaves of abelian groups by (MRS,Lemma 12). ☐

Lemma 4. *Let U be an open subset of a topological space X. If a sheaf of abelian groups $\mathscr{F}$ on X is injective, then so is $\mathscr{F}|_U$.*

Proof. It follows from (SGR,Lemma 28) that the restriction functor $(-)|_U : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(U)$ has an exact left adjoint $j_! : \mathfrak{Ab}(U) \longrightarrow \mathfrak{Ab}(X)$ where $j : U \longrightarrow X$ is the inclusion. Therefore by (AC,Proposition 25) the functor $(-)|_U$ preserves injectives, as required. ☐

Proposition 5. *Let X be a topological space and $\mathscr{F}$ a flasque sheaf of abelian groups on X. Then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. Find a monomorphism $\mathscr{F} \longrightarrow \mathscr{I}$ where $\mathscr{I}$ is an injective object of $\mathfrak{Ab}(X)$, and let $\mathscr{G}$ be the quotient, so we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0$$

Then $\mathscr{F}$ is flasque by hypothesis, $\mathscr{I}$ is flasque by Lemma 3 and so $\mathscr{G}$ is flasque by ([Har77] II Ex.1.16*c*). By ([Har77] II Ex.1.16*b*) we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow 0$$

On the other hand, since $\mathscr{I}$ is injective we have $H^i(X, \mathscr{I}) = 0$ for $i > 0$. Then from the long exact sequence of cohomology, we get $H^1(X, \mathscr{F}) = 0$ and $H^i(X, \mathscr{F}) \cong H^{i-1}(X, \mathscr{G})$ for each $i \geq 2$. But $\mathscr{G}$ is also flasque, so by induction on i we get the result. ☐

Remark 2. The result tells us that flasque sheaves are acyclic for the functor $\Gamma(X, -)$ (see (DF,Definition 14) for the definition of acyclic objects). Hence we can calculate cohomology using flasque resolutions (DF,Proposition 54). The reader knowing more category theory can use (DTC2,Remark 14), and whenever we refer to the "canonical" isomorphism of something involving acyclic resolutions, it is this latter one we have in mind.

Lemma 6. *Let $(X, \mathcal{O}_X)$ be a ringed space, $x \in X$ a point and M an $\mathcal{O}_{X,x}$-module. Then $H^i(X, Sky_x(M)) = 0$ for $i > 0$.*

Proof. Suppose we are given an injective resolution of M as a $\mathcal{O}_{X,x}$-module

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \tag{1}$$

Then since $Sky_x(-)$ is exact and preserves injectives (MRS,Lemma 13) the following is an injective resolution of $Sky_x(M)$ in $\mathfrak{Mod}(X)$

$$0 \longrightarrow Sky_x(M) \longrightarrow Sky_x(I^0) \longrightarrow Sky_x(I^1) \longrightarrow \cdots$$

It is therefore also a flasque resolution of $Sky_x(M)$ as a sheaf of abelian groups. Applying $\Gamma(X, -)$ to this resolution we end up where we started: with the exact sequence (1). It follows that $H^i(X, Sky_x(M)) = 0$ for $i > 0$, as required. □

1.1 Quasi-flasque sheaves

Definition 3. A *full basis* of a topological space X is a nonempty collection $\mathfrak{B}$ of open subsets of X which is closed under finite intersections and has the property that for any open $U \subseteq X$ and $x \in U$ there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Definition 4. A topological space X is *quasi-noetherian* if it is quasi-compact and possesses a full basis $\mathfrak{B}$ consisting of quasi-compact open subsets. This property is stable under homeomorphism. Any noetherian topological space is quasi-noetherian. A quasi-compact open subset of a quasi-noetherian space is itself quasi-noetherian.

Lemma 7. *If X is a quasi-noetherian topological space and U, V quasi-compact open subsets, then $U \cap V$ is quasi-compact.*

Proof. Let $\mathfrak{B}$ be a full basis of quasi-compact open subsets. Given quasi-compact open sets U, V we can write $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cup \cdots \cup V_m$ for open sets $U_i, V_j \in \mathfrak{B}$. Then

$$U \cap V = \bigcup_{i,j} U_i \cap V_j$$

By hypothesis each $U_i \cap V_j$ is quasi-compact, so $U \cap V$ is a finite union of quasi-compact open sets, and is therefore itself quasi-compact. □

Lemma 8. *Let X be a scheme. The underlying space of X is quasi-noetherian if and only if X is concentrated.*

Proof. This follows immediately from Lemma 7 and (CON,Proposition 12). □

It will be necessary to generalise the notion of flasque sheaf. A sheaf of sets on a topological space X is *quasi-flasque* if the restriction $\Gamma(X, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F})$ is surjective for every *quasi-compact* open subset $U \subseteq X$. Clearly this property is stable under isomorphism and restriction. On a quasi-noetherian space quasi-flasque sheaves possess most of the nice properties of flasque sheaves. In the remainder of this section we check some of these properties.

Proposition 9. *Let X be a quasi-noetherian topological space and suppose we have a short exact sequence of sheaves of abelian groups*

$$0 \longrightarrow \mathscr{L}' \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}'' \longrightarrow 0$$

with $\mathscr{L}'$ quasi-flasque. Then the following sequence is exact

$$0 \longrightarrow \Gamma(X, \mathscr{L}') \longrightarrow \Gamma(X, \mathscr{L}) \longrightarrow \Gamma(X, \mathscr{L}'') \longrightarrow 0$$

Proof. Let s be a global section of $\mathscr{L}''$. Consider the family $\mathfrak{V}$ of quasi-compact open subsets $V \subseteq X$ such that $s|_V$ is represented by an element of $\Gamma(V, \mathscr{L})$. We need to see that X is contained in $\mathfrak{V}$. Since $\mathfrak{V}$ covers X and X is quasi-compact, it suffices to show that $\mathfrak{V}$ is closed under finite unions.

Let V_1 and V_2 be two members of $\mathfrak{B}$. Then $V_1 \cup V_2$ and $V_1 \cap V_2$ are both quasi-compact. Let $t_i \in \Gamma(V_i, \mathscr{L})$ represent $s|_{V_i}$. The difference of t_1 and t_2 restricted to $V_1 \cap V_2$ lifts to a global section u of $\mathscr{L}'$ as $\mathscr{L}'$ is quasi-flasque. By adding $u|_{V_2}$ to t_2 we may assume that t_1 and t_2 agree on the overlap $V_1 \cap V_2$. We may patch t_1 and t_2 together to get a section t of $\Gamma(V_1 \cup V_2, \mathscr{L})$ which represents $s|_{V_1 \cup V_2}$. Hence $V_1 \cup V_2 \in \mathfrak{V}$ and we are done. □

Corollary 10. *Let X be a quasi-noetherian topological space and suppose we have a short exact sequence of sheaves of abelian groups*

$$0 \longrightarrow \mathscr{L}' \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}'' \longrightarrow 0$$

If both $\mathscr{L}', \mathscr{L}$ are quasi-flasque, then so is $\mathscr{L}''$.

Proof. Given a quasi-compact open subset $U \subseteq X$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, \mathscr{L}') & \longrightarrow & \Gamma(X, \mathscr{L}) & \longrightarrow & \Gamma(X, \mathscr{L}'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(U, \mathscr{L}') & \longrightarrow & \Gamma(U, \mathscr{L}) & \longrightarrow & \Gamma(U, \mathscr{L}'') & \longrightarrow & 0 \end{array}$$

in which the first two columns are surjective. It follows that the third column is surjective, which is what we wanted to show. □

Corollary 11. *Let $\mathscr{L}$ be any quasi-flasque sheaf of abelian groups on a quasi-noetherian topological space X. Then $H^i(X, \mathscr{L}) = 0$ for $i > 0$.*

Proof. One uses the argument given in Proposition 5, together with Proposition 9 and Corollary 10. □

1.2 Module structure

Throughout this section let $(X, \mathcal{O}_X)$ be a ringed space and $A = \Gamma(X, \mathcal{O}_X)$. Fix assignments of injective resolutions $\mathcal{I}, \mathcal{J}$ to the objects of $\mathfrak{Mod}(X), \mathfrak{Ab}(X)$ respectively, with respect to which all right derived functors are calculated.

Taking global sections defines a left exact additive functor $\Gamma_A(X, -) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$ and we denote by $H^i_A(X, -)$ the right derived functors of $\Gamma_A(X, -)$ for $i \geq 0$. Let $U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Ab}(X)$ and $u : A\mathbf{Mod} \longrightarrow \mathbf{Ab}$ be the forgetful functors. Then the following diagram commutes

$$\begin{array}{ccc} \mathfrak{Mod}(X) & \xrightarrow{\Gamma_A(X,-)} & A\mathbf{Mod} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle u} \\ \mathfrak{Ab}(X) & \xrightarrow[\Gamma(X,-)]{} & \mathbf{Ab} \end{array} \qquad (2)$$

and we denote the composite by $G : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$. Since U, u are exact and U sends injective objects into right $\Gamma(X, -)$-acyclic objects of $\mathfrak{Ab}(X)$ by Lemma 3, we are in a position to apply (DF,Proposition 77) and (DF,Remark 2) to obtain a canonical natural equivalence for $i \geq 0$

$$\mu^i : u \circ H^i_A(X, -) \longrightarrow H^i(X, -) \circ U$$

Moreover given an exact sequence of sheaves of modules $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ the following diagram of abelian groups commutes for $i \geq 0$

$$\begin{array}{ccc} H^i_A(X, \mathscr{F}'') & \longrightarrow & H^{i+1}_A(X, \mathscr{F}') \\ \Downarrow & & \Downarrow \\ H^i(X, \mathscr{F}'') & \longrightarrow & H^{i+1}(X, \mathscr{F}') \end{array}$$

If $\mathcal{I}, \mathcal{I}'$ are two assignments of injective resolutions to the objects of $\mathfrak{Mod}(X)$ then for any sheaf of modules $\mathscr{F}$ the composite $H^i_{A,\mathcal{I}}(X, \mathscr{F}) \cong H^i(X, \mathscr{F}) \cong H^i_{A,\mathcal{I}'}(X, \mathscr{F})$ is just the evaluation of the canonical natural equivalence $H^i_{A,\mathcal{I}}(X, -) \cong H^i_{A,\mathcal{I}'}(X, -)$. This means that the A-module structure induced on $H^i(X, \mathscr{F})$ is independent of the choice of resolutions on $\mathfrak{Mod}(X)$.

Definition 5. Let $(X, \mathcal{O}_X)$ be a ringed space and $A = \Gamma(X, \mathcal{O}_X)$. Fix an assignment of injective resolutions $\mathcal{J}$ to the objects of $\mathfrak{Ab}(X)$. Then for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ and $i \geq 0$ the cohomology group $H^i(X, \mathscr{F})$ has a canonical A-module structure. For $i = 0$ this is the A-module structure induced by $H^0(X, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{G})$ is a morphism of A-modules, so we have an additive functor $H^i(X, -) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$. For a short exact sequence of sheaves of modules

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

the connecting morphism $\delta^i : H^i(X, \mathscr{F}'') \longrightarrow H^{i+1}(X, \mathscr{F}')$ is a morphism of A-modules for $i \geq 0$. So we have a long exact sequence of A-modules

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow H^1(X, \mathscr{F}') \longrightarrow \cdots$$

Remark 3. Let $(X, \mathcal{O}_X)$ be a ringed space and set $A = \Gamma(X, \mathcal{O}_X)$. Let $\mathscr{F}$ be a sheaf of modules on X and fix an assignments of injective resolutions $\mathcal{J}$ to the objects of $\mathfrak{Ab}(X)$. To calculate the A-module structure on the abelian group $H^i(X, \mathscr{F})$ you proceed as follows (using (DF,Remark 2)). Choose any injective resolution of $\mathscr{F}$ in $\mathfrak{Mod}(X)$

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{I}^2 \longrightarrow \cdots$$

and observe that this is a flasque resolution in $\mathfrak{Ab}(X)$. Suppose that the chosen injective resolution for $\mathscr{F}$ in $\mathfrak{Ab}(X)$ is

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{J}^0 \longrightarrow \mathscr{J}^1 \longrightarrow \mathscr{J}^2 \longrightarrow \cdots$$

Then we can lift the identity to a morphism of cochain complexes $\mathscr{I} \longrightarrow \mathscr{J}$ in $\mathfrak{Ab}(X)$ (DF,Theorem 19). Applying the functor $\Gamma(X, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$ and taking cohomology at i we obtain an isomorphism of abelian groups $H^i_A(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F})$, which induces the A-module structure on $H^i(X, \mathscr{F})$. If we fix an assignment of injective resolutions $\mathcal{I}$ to the objects of $\mathfrak{Mod}(X)$, then these isomorphisms define a natural equivalence $H^i_A(X, -) \cong H^i(X, -)$ of functors $\mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$.

Remark 4. With the notation of Remark 3 there is a simpler way to calculate the A-module structure. Any $a \in A$ gives an endomorphism of sheaves of abelian groups $a : \mathscr{F} \longrightarrow \mathscr{F}$ which induces a morphism of abelian groups $A : H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F})$. A quick calculation using Remark 3 shows that $A(x) = a \cdot x$, so this endomorphism also gives the A-module structure on $H^i(X, \mathscr{F})$. This makes it clear that given two assignments $\mathcal{I}, \mathcal{J}$ of injective resolutions to the objects of $\mathfrak{Ab}(X)$, the canonical isomorphism of abelian groups $H^i_{\mathcal{I}}(X, \mathscr{F}) \longrightarrow H^i_{\mathcal{J}}(X, \mathscr{F})$ is an isomorphism of A-modules.

1.3 Presheaf of Cohomology

Let X be a topological space, $U \subseteq X$ an open subset. Then let $H^i(U, -)$ be the ith right derived functor of the left exact functor $\Gamma(U, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$. There is a canonical natural equivalence

$H^0(U,-) \cong \Gamma(U,-)$. For any sheaf of abelian groups $\mathscr{F}$, the groups $H^i(U,\mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$ over U. We have the usual natural long exact sequence of cohomology over U.

Let $U \subseteq V$ be open subsets of X. Then restriction defines a natural transformation $\Gamma(V,-) \longrightarrow \Gamma(U,-)$ which leads to natural transformations $\mu^i_{V,U} : H^i(V,-) \longrightarrow H^i(U,-)$ for $i \geq 0$ as defined in (DF,Definition 11). This construction is functorial, in the sense that for open sets $W \subseteq U \subseteq V$ we have $\mu^i_{U,W} \circ \mu^i_{V,U} = \mu^i_{V,W}$ and $\mu^i_{U,U} = 1$ for any open set U. For a sheaf of abelian groups $\mathscr{F}$ on X we define a presheaf of abelian groups $\mathscr{H}^i(\mathscr{F})$ for $i \geq 0$ by $\Gamma(U,\mathscr{H}^i(\mathscr{F})) = H^i(U,\mathscr{F})$ with the restriction map $\Gamma(V,\mathscr{H}^i(\mathscr{F})) \longrightarrow \Gamma(U,\mathscr{H}^i(\mathscr{F}))$ given by $(\mu^i_{U,V})_{\mathscr{F}}$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of abelian groups then for $i \geq 0$ there is a morphism of presheaves of abelian groups

$$\mathscr{H}^i(\phi) : \mathscr{H}^i(\mathscr{F}) \longrightarrow \mathscr{H}^i(\mathscr{G})$$
$$\mathscr{H}^i(\phi)_U = H^i(U,\phi)$$

This defines for $i \geq 0$ an additive functor $\mathscr{H}^i(-) : \mathfrak{Ab}(X) \longrightarrow Ab(X)$ where $Ab(X)$ is the category of all presheaves of abelian groups on X. There is a canonical isomorphism of presheaves of abelian groups $\mathscr{F} \cong \mathscr{H}^0(\mathscr{F})$, which shows that $\mathscr{H}^0(\mathscr{F})$ is a sheaf (DF,Lemma 43). This isomorphism is clearly natural in $\mathscr{F}$.

Suppose we have an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

For an open subset $U \subseteq X$ and $i \geq 0$ we have the canonical connecting morphism $H^i(U,\mathscr{F}'') \longrightarrow H^{i+1}(U,\mathscr{F}')$ and since these are natural in U (DF,Proposition 44) we have a morphism of presheaves of abelian groups $\omega^i : \mathscr{H}^i(\mathscr{F}'') \longrightarrow \mathscr{H}^{i+1}(\mathscr{F}')$. These fit into a long exact sequence of *presheaves* of abelian groups

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow \mathscr{H}^1(\mathscr{F}') \longrightarrow \mathscr{H}^1(\mathscr{F}) \longrightarrow \mathscr{H}^1(\mathscr{F}'') \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathscr{H}^n(\mathscr{F}'') \longrightarrow \mathscr{H}^{n+1}(\mathscr{F}') \longrightarrow \mathscr{H}^{n+1}(\mathscr{F}) \longrightarrow \mathscr{H}^{n+1}(\mathscr{F}'') \longrightarrow \cdots$$

Lemma 12. *Let U be an open subset of a topological space X and let $\mathscr{F}$ be a sheaf of abelian groups on X. Then for $i \geq 0$ there is a canonical isomorphism of abelian groups $H^i(U,\mathscr{F}|_U) \cong H^i(U,\mathscr{F})$ natural in $\mathscr{F}$.*

Proof. Let $\mathcal{I},\mathcal{J}$ be assignments of injective resolutions to the objects of $\mathfrak{Ab}(U),\mathfrak{Ab}(X)$ respectively, with respect to which all right derived functors are calculated. Suppose the chosen injective resolution for $\mathscr{F}$ is

$$I : 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

Then applying the exact functor $(-)|_U$, which by Lemma 4 preserves injectives, we have an injective resolution of $\mathscr{F}|_U$

$$I|_U : 0 \longrightarrow \mathscr{F}|_U \longrightarrow \mathscr{I}^0|_U \longrightarrow \mathscr{I}^1|_U \longrightarrow \cdots$$

There is a canonical isomorphism of abelian groups $H^i(U,\mathscr{F}|_U) \cong H^i(\Gamma(U,I|_U)) = H^i(U,\mathscr{F})$. This isomorphism is clearly natural in $\mathscr{F}$. ∎

For the rest of this section let $(X,\mathcal{O}_X)$ be a ringed space. If we fix an open subset $U \subseteq X$ and set $A = \Gamma(U,\mathcal{O}_X)$ then the theory of Section 1.2 applies, so that for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ the cohomology group $H^i(U,\mathscr{F})$ becomes an A-module in a canonical way. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $H^i(U,\mathscr{F}) \longrightarrow H^i(U,\mathscr{G})$ is a morphism of A-modules, so we have an additive functor $H^i(U,-) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$. From a short exact sequence of sheaves of modules, we get a long exact sequence of A-modules. The isomorphism $H^i(U,\mathscr{F}|_U) \cong H^i(U,\mathscr{F})$ of Lemma 12 is also an isomorphism of A-modules.

Suppose that we have open sets $U \subseteq V$ and a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$. One checks that the morphism of abelian groups $H^i(V,\mathscr{F}) \longrightarrow H^i(U,\mathscr{F})$ sends the action of $\Gamma(V,\mathcal{O}_X)$ to the action of $\Gamma(U, \mathcal{O}_X)$. In other words, $\mathscr{H}^i(\mathscr{F})$ is a presheaf of $\mathcal{O}_X$-modules for $i \geq 0$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $\mathscr{H}^i(\phi)$ is a morphism of presheaves of modules, so we have an additive functor $\mathscr{H}^i(-) : \mathfrak{Mod}(X) \longrightarrow Mod(X)$, where $Mod(X)$ is the category of presheaves of $\mathcal{O}_X$-modules. For any short exact sequence of sheaves of modules the connecting morphism ω^i is a morphism of presheaves of modules, so we have a long exact sequence of presheaves of modules.

2 A Vanishing Theorem of Serre

Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Given an open subset $U \subseteq X$ let $i : U \longrightarrow X$ be the inclusion. We denote by ${}_U\mathscr{F}$ the sheaf of abelian groups $i_*(\mathscr{F}|_U)$. There is a canonical morphism $\mathscr{F} \longrightarrow {}_U\mathscr{F}$ natural in $\mathscr{F}$. If $\mathscr{F}$ is flasque this is an epimorphism.

Proposition 13. *Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Assume that X admits a full basis $\mathfrak{U}$ and integer $n > 0$ such that for every $U \in \mathfrak{U}$ we have $H^i(U, \mathscr{F}|_U) = 0$ for $0 < i < n$. Then given $\alpha \in H^n(X, \mathscr{F})$ there is a nonempty cover $\mathfrak{V}$ of X by elements of $\mathfrak{U}$ such that the image of α in $H^n(X, {}_V\mathscr{F})$ is zero for all $V \in \mathfrak{V}$.*

Proof. Let $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$ be a short exact sequence of sheaves where $\mathscr{G}$ is flasque. Then $H^i(U, \mathscr{G}|_U) = 0$ for $i > 0$ and any open $U \subseteq X$. By the long exact sequence of cohomology we have an exact sequence

$$0 \longrightarrow \Gamma(U, \mathscr{F}|_U) \longrightarrow \Gamma(U, \mathscr{G}|_U) \longrightarrow \Gamma(U, \mathscr{H}|_U) \longrightarrow H^1(U, \mathscr{F}|_U) \longrightarrow 0 \tag{3}$$

and isomorphisms

$$H^i(U, \mathscr{H}|_U) \longrightarrow H^{i+1}(U, \mathscr{F}|_U) \quad \text{for } i > 0 \tag{4}$$

For any open subset V of X we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_V\mathscr{F} & \longrightarrow & {}_V\mathscr{G} & \longrightarrow & {}_V\mathscr{H} & & \end{array}$$

where ${}_V\mathscr{G}$ is a flasque sheaf. The image of $\mathscr{H}$ and the image of ${}_V\mathscr{G}$ in ${}_V\mathscr{H}$ coincide: denote this image by $\mathscr{K}$. Repeating the above for the sequence $0 \longrightarrow {}_V\mathscr{F} \longrightarrow {}_V\mathscr{G} \longrightarrow \mathscr{K} \longrightarrow 0$ we have an exact sequence

$$0 \longrightarrow \Gamma(X, {}_V\mathscr{F}) \longrightarrow \Gamma(X, {}_V\mathscr{G}) \longrightarrow \Gamma(X, \mathscr{K}) \longrightarrow H^1(X, {}_V\mathscr{F}) \longrightarrow 0 \tag{5}$$

and isomorphisms

$$H^i(X, \mathscr{K}) \longrightarrow H^{i+1}(X, {}_V\mathscr{F}) \quad \text{for } i > 0 \tag{6}$$

For $U = X$ the sequences (3) and (5) fit into a commutative diagram. By definition we have an inclusion $\Gamma(X, \mathscr{K}) \subseteq \Gamma(X, {}_V\mathscr{H}) = \Gamma(V, \mathscr{H})$.

Assume first that $n > 1$. Then if V, W belong to $\mathfrak{U}$ the sequence (3) for $U = V \cap W$ shows that $\Gamma(U, {}_V\mathscr{G}) \longrightarrow \Gamma(U, {}_V\mathscr{H})$ is surjective. Hence, $\mathscr{K} = {}_V\mathscr{H}$. Furthermore, (4) shows that $\mathscr{H}$ satisfies the condition of the proposition for $n - 1$. We can use the isomorphisms (6) to complete the inductive step.

So we need only consider the case $n = 1$. By the sequence (3) for $U = X$, our element α of $H^1(X, \mathscr{F})$ is $\delta(\beta)$ for some $\beta \in \Gamma(X, \mathscr{H})$. By (5) for any $V \in \mathfrak{U}$ the image of α in $H^1(X, {}_V\mathscr{F})$ is zero if and only if the image of β in $\Gamma(X, \mathscr{K}) \subseteq \Gamma(V, \mathscr{H})$ lifts to an element of $\Gamma(X, {}_V\mathscr{G}) = \Gamma(V, \mathscr{G})$. As $\mathfrak{U}$ consists of arbitrarily small subsets, there is no problem finding a subcovering $\mathfrak{V}$ satisfying the last condition because $\mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$ is exact. □

Theorem 14 (Serre). *Let $\mathscr{F}$ be a quasi-coherent sheaf on an affine scheme X. Then for any $i > 0$ we have $H^i(X, \mathscr{F}) = 0$.*

Proof. First we prove the case $i = 1$. There is an exact sequence of sheaves of modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0$$

with $\mathscr{I}$ injective and therefore flasque. The long exact cohomology sequence is

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow 0$$

But we know from [Har77] II.5.6 that the connecting morphism is zero, from which it follows that $H^1(X, \mathscr{F}) = 0$ as required.

Now assume the theorem has been proven for $0 < i < n$. That is, $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ on affine schemes X and $0 < i < n$. Let $\mathfrak{U}$ the full basis of X consisting of affine open subsets X_f for $f \in \Gamma(X, \mathcal{O}_X)$. Either by observing that the inclusion $X_f \longrightarrow X$ is quasi-compact and applying (MOS,Proposition 58), or just by recognising it as $(\Gamma(X, \mathscr{F})_f)^{\sim}$, we see that ${}_{X_f}\mathscr{F}$ is quasi-coherent. The inductive hypothesis means that the conditions of Proposition 13 are satisfied for the cover $\mathfrak{U}$ and integer n. Therefore, for any given element α of $H^n(X, \mathscr{F})$, we may find a covering $V_1, \ldots, V_p$ of X by members of $\mathfrak{U}$ such that the image of α in

$$H^n(X, \bigoplus_{1 \le j \le p} {}_{V_j}\mathscr{F})$$

is zero. From the long exact exact cohomology sequence of the short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{1 \le j \le p} {}_{V_j}\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

we deduce that α is in the image $\delta(H^{n-1}(X, \mathscr{G}))$. But $H^{n-1}(X, \mathscr{G}) = 0$ by the inductive hypothesis, so the proof is complete. ∎

Corollary 15. *Let X be a scheme and $U \subseteq X$ an affine open subset. Then the additive functor $\Gamma(U, -) : \mathfrak{Qco}(X) \longrightarrow \mathbf{Ab}$ is exact.*

Remark 5. Let us make a trivial remark. If X is a noetherian scheme then it is in particular a Zariski space ([Har77] II Ex.3.17). Therefore using ([Har77] I Ex.1.7) we see that every nonempty closed subset $Y \subseteq X$ contains at least one closed point.

Theorem 16. *Let X be a noetherian scheme. Then the following conditions are equivalent:*

(i) X is affine;

(ii) $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves of modules $\mathscr{F}$ and $i > 0$;

(iii) $H^1(X, \mathscr{F}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.

Proof. $(i) \Rightarrow (ii)$ is Theorem 14 and $(ii) \Rightarrow (iii)$ is trivial, so we only have to prove $(iii) \Rightarrow (i)$. We use the criterion of ([Har77] II Ex.2.17). First we show that X can be covered by open affine subsets of the form X_f, with $f \in A = \Gamma(X, \mathcal{O}_X)$. Let P be a closed point of X, let U be an open affine neighborhood of P, and let $Y = X \setminus U$. Then we have an exact sequence

$$0 \longrightarrow \mathscr{J}_{Y \cup \{P\}} \longrightarrow \mathscr{J}_Y \longrightarrow k(P) \longrightarrow 0$$

where $\mathscr{J}_Y$ and $\mathscr{J}_{Y \cup \{P\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{P\}$, respectively. The quotient is the skyscraper sheaf $k(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$ by (MRS,Remark 4). Now from the exact sequence of cohomology, and hypothesis (iii), we get an exact sequence

$$\Gamma(X, \mathscr{J}_Y) \longrightarrow \Gamma(X, k(P)) \longrightarrow H^1(X, \mathscr{J}_{Y \cup \{P\}}) = 0$$

So there is an element $f \in \Gamma(X, \mathscr{J}_Y) \subseteq \Gamma(X, \mathcal{O}_X)$ which goes to 1 in $k(P)$. That is, $germ_P f - 1 \in \mathfrak{m}_P$ and therefore by construction $P \in X_f \subseteq U$. Furthermore, $X_f = U_{f|_U}$ so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f. By quasi-compactness and Remark 5, we can cover X with a finite number of these, corresponding to $f_1, \dots, f_r \in A$. Let $\alpha : \mathcal{O}_X^r \longrightarrow \mathcal{O}_X$ be the morphism out of the coproduct induced by the f_i. Since the X_{f_i} cover X, this is an epimorphism of sheaves of modules. Let $\mathscr{F}$ be the kernel, so we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{O}_X \longrightarrow 0$$

For each $1 \le i \le r$ let $\mathcal{O}_X^i \longrightarrow \mathcal{O}_X^r$ be the monomorphism embedding in the first i coordinates, and consider $\mathcal{O}_X^i$ as a submodule of $\mathcal{O}_X^r$ in this way. For $1 \le i < r$ we have $\mathcal{O}_X^i \subseteq \mathcal{O}_X^{i+1}$ and a canonical isomorphism $\mathcal{O}_X^{i+1}/\mathcal{O}_X^i \cong \mathcal{O}_X$. We filter $\mathscr{F}$ as follows

$$\mathscr{F} = \mathscr{F} \cap \mathcal{O}_X^r \supseteq \mathscr{F} \cap \mathcal{O}_X^{r-1} \supseteq \cdots \supseteq \mathscr{F} \cap \mathcal{O}_X$$

For $1 \le i < r$ we have $\mathcal{O}_X^i \cap (\mathscr{F} \cap \mathcal{O}_X^{i+1}) = \mathscr{F} \cap \mathcal{O}_X^i$. Therefore we have a monomorphism (see [Mit65] I 16.6) $(\mathscr{F} \cap \mathcal{O}_X^{i+1})/(\mathscr{F} \cap \mathcal{O}_X^i) \longrightarrow \mathcal{O}_X^{i+1}/\mathcal{O}_X^i \cong \mathcal{O}_X$, showing that each quotient of the filtration is isomorphic to a coherent sheaf of ideals on X. Using the hypothesis (iii) and the long exact cohomology sequence, we climb up the filtration and deduce that $H^1(X, \mathscr{F}) = 0$. But this shows that $\alpha_X : \Gamma(X, \mathcal{O}_X^r) \longrightarrow \Gamma(X, \mathcal{O}_X)$ is surjective, which tells us that $f_1, \dots, f_r$ generate the unit ideal of A, as required. □

2.1 Cohomology of a Noetherian Affine Scheme

In the approach to Serre's vanishing theorem given in Hartshorne, one works with noetherian schemes and the key point is showing that if I is an injective A-module, then the sheaf $\widetilde{I}$ on X is flasque. Although this no longer plays any part in the proof, some of these results are still useful when we work with noetherian schemes.

Proposition 17 (Krull's Theorem). *Let A be a noetherian ring, $\mathfrak{a}$ an ideal, M a finitely generated A-module and N a submodule of M. Then the $\mathfrak{a}$-adic topology on N is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n > 0$ there exists $k \ge n$ such that $\mathfrak{a}^n N \supseteq N \cap \mathfrak{a}^k M$.*

Proof. See [AM69] Theorem 10.11. □

Definition 6. Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal and M an A-module. Then we define the following submodule of M

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M \,|\, \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$$

In other words, m belongs to $\Gamma_{\mathfrak{a}}(M)$ if and only if its annihilator is an open ideal in the $\mathfrak{a}$-adic topology on A.

Remark 6. Let A be a ring, set $X = SpecA$ and let M be an A-module with $\mathscr{F} = M^\sim$. If $m \in M$ then $Supp(\dot{m}) = V(Ann(m))$. If M is finitely generated then $Supp(\mathscr{F}) = V(Ann(M))$. It follows easily that the support of a coherent sheaf on a noetherian scheme is closed.

Let X be a topological space, $Z \subseteq X$ a closed subset and $\mathscr{F}$ a sheaf of abelian groups on X. Then recall ([Har77] II Ex.1.20) that $\Gamma_Z(X, \mathscr{F}) = \{s \in \mathscr{F}(X) \,|\, Supp(s) \subseteq Z\}$ is a subgroup of $\mathscr{F}(X)$, and we have a subsheaf $\mathscr{H}_Z^0(\mathscr{F})$ of $\mathscr{F}$ defined by

$$\Gamma(V, \mathscr{H}_Z^0(\mathscr{F})) = \{s \in F(V) \,|\, Supp(s) \subseteq Z \cap V\}$$

If $(X, \mathcal{O}_X)$ is a ringed space and $\mathscr{F}$ a sheaf of modules, then $\mathscr{H}_Z^0(\mathscr{F})$ is a submodule of $\mathscr{F}$.

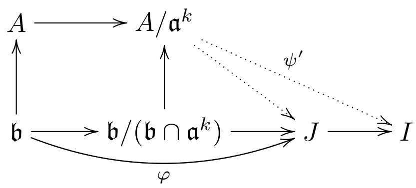
Lemma 18. *Let A be a noetherian ring, $\mathfrak{a} \subseteq A$ an ideal and M an A-module. Set $X = SpecA$ and let $\mathscr{F} = M^\sim$. Then there is a canonical isomorphism of sheaves of modules $\Gamma_{\mathfrak{a}}(M)^\sim \cong \mathscr{H}_Z^0(\mathscr{F})$ where $Z = V(\mathfrak{a})$.*

Proof. By ([Har77] II Ex1.20) the submodule $\mathscr{H}_Z^0(\mathscr{F})$ is the kernel of the unit $\mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)$ where $U = X \setminus Z$ is the inclusion. By [Har77] II.5.8 both these sheaves are quasi-coherent, and by [Har77] II.5.7 so is $\mathscr{H}_Z^0(\mathscr{F})$. Therefore there is a canonical isomorphism $\mathscr{H}^0(\mathscr{F}) \cong \Gamma(X, \mathscr{H}_Z^0(\mathscr{F}))^\sim$, and to complete the proof it suffices to produce a canonical isomorphism of

A-modules $\Gamma_{\mathfrak{a}}(M) \cong \Gamma(X, \mathscr{H}^0(\mathscr{F}))$. So we need only show that the isomorphism $M \cong \Gamma(X, \mathscr{F})$ identifies elements of $\Gamma_{\mathfrak{a}}(M)$ with elements of $\Gamma(X, \mathscr{H}^0_Z(\mathscr{F}))$. But for $m \in M$ we have $Supp(\dot{m}) \subseteq Z = V(\mathfrak{a})$ if and only if $V(Ann(m)) = Supp(m) \subseteq V(\mathfrak{a})$, which is if and only if $\sqrt{\mathfrak{a}} \subseteq \sqrt{Ann(m)}$. Since A is noetherian the ideal $\mathfrak{a}$ is finitely generated, so it is not hard to check that $\sqrt{\mathfrak{a}} \subseteq \sqrt{Ann(m)}$ if and only if $\mathfrak{a}^n \subseteq Ann(m)$ for some $n > 0$. This shows that $m \in \Gamma_{\mathfrak{a}}(M)$ if and only if $\dot{m} \in \Gamma(X, \mathscr{H}^0_Z(\mathscr{F}))$, and completes the proof. □

Lemma 19. *Let A be a noetherian ring, $\mathfrak{a} \subseteq A$ an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.*

Proof. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any morphism $\varphi : \mathfrak{b} \longrightarrow J$, there exists a morphism $\psi : A \longrightarrow J$ extending $\mathfrak{b}$ (see Stenstrom Proposition I, 6.5). Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some power of $\mathfrak{a}$, so there exists $n > 0$ such that $\mathfrak{a}^n \varphi(\mathfrak{b}) = 0$, or equivalently, $\varphi(\mathfrak{a}^n \mathfrak{b}) = 0$. Applying Proposition 17 to the inclusion $\mathfrak{b} \subseteq A$, we find that there is a $k \geq n$ with $\mathfrak{a}^n \mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^k$. Hence $\varphi(\mathfrak{b} \cap \mathfrak{a}^k) = 0$, and so the morphism $\varphi : \mathfrak{b} \longrightarrow J$ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^k)$. Now we consider the following diagram



Since I is injective, the map $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^k) \longrightarrow I$ extends to a morphism ψ'. But the image of ψ' is annihilated by $\mathfrak{a}^k$, so it is contained in J. Composing with $A \longrightarrow A/\mathfrak{a}^k$ we obtain the desired morphism $\psi : A \longrightarrow J$ extending φ. □

Lemma 20. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$ the canonical morphism $I \longrightarrow I_f$ is surjective.*

Proof. For each $i > 0$ let $\mathfrak{b}_i$ be the annihilator of f^i in A. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ and since A is noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \cdots$. Now let $\theta : I \longrightarrow I_f$ be the canonical morphism and let $x \in I_f$ be any element. Then by definition, there is $y \in I$ and $n \geq 0$ such that $x = y/f^n$. We define the map φ from the ideal (f^{n+r}) to I by sending f^{n+r} to $f^r y$. This is possible, because the annihilator of f^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$, and $\mathfrak{b}_r$ annihilates $f^r y$. Since I is injective, φ extends to a morphism $\psi : A \longrightarrow I$, which corresponds to an element $z = \psi(1)$. Then $f^{n+r} z = f^r y$. But this implies that $\theta(z) = y/f^n = x$. Hence θ is surjective. □

Proposition 21. *Let A be a noetherian ring and set $X = SpecA$. If I is an injective A-module then the sheaf of modules $I^{\sim}$ is flasque.*

Proof. We will use noetherian induction (MAT,Lemma 43) on the support of I. To be a little clearer, let $\mathscr{P}$ be the following statement about a closed subset Y of X:

($\mathscr{P}$) For a injective module I with $Supp(I) \subseteq Y$ the sheaf of modules $I^{\sim}$ is flasque.

The proposition is this statement in the case $Y = X$. By noetherian induction, to prove the proposition it suffices to show that for any closed subset Y of X, if $\mathscr{P}$ is true for every *proper* closed subset of Y then it is true for Y. So in the rest of the proof let Y be a fixed closed subset of X with this property, and let I be an injective A-module with $Supp(I) \subseteq Y$.

First of all, we can assume $Y = \overline{Supp(I)}$ since otherwise $Supp(I)$ is contained in a proper closed subset of Y, in which case $I^{\sim}$ is flasque by the inductive hypothesis. If Y is empty or consists of a single closed point of X, then $I^{\sim}$ is a skyscraper sheaf which is obviously flasque.

In the general case, to show that $I^{\sim}$ is flasque, it will be sufficient to show, for any open set $U \subseteq X$ that $\Gamma(X, I^{\sim}) \longrightarrow \Gamma(U, I^{\sim})$ is surjective. If $Y \cap U = \emptyset$, there is nothing to prove. If

$Y \cap U \neq \emptyset$, we can find an $f \in A$ such that the open set $X_f = D(f)$ is contained in U and $X_f \cap Y \neq \emptyset$. Let $Z = X \setminus X_f$ and consider the following commutative diagram

$$\begin{array}{ccccc} \Gamma(X, \widetilde{I}) & \longrightarrow & \Gamma(U, \widetilde{I}) & \longrightarrow & \Gamma(X_f, \widetilde{I}) \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma_Z(X, \widetilde{I}) & \longrightarrow & \Gamma_Z(U, \widetilde{I}) & \longrightarrow & \Gamma_Z(X_f, \widetilde{I}) \end{array}$$

Given a section $s \in \Gamma(U, I^{\sim})$, we consider its image $s' \in \Gamma(X_f, I^{\sim})$. By Lemma 20 there is $t \in \Gamma(X, I^{\sim})$ restricting to s'. Let t' be the restriction of t to $\Gamma(U, I^{\sim})$. Then $s - t'$ goes to zero in $\Gamma(X_f, I^{\sim})$ so it has support in Z. Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X, I^{\sim}) \longrightarrow \Gamma_Z(U, I^{\sim})$ is surjective. That is, we have to show that the sheaf $\mathscr{H}^0_Z(I^{\sim})$ of Remark 6 is flasque.

If $J = \Gamma(X, \mathscr{H}^0_Z(I^{\sim}))$ then we know from the proof of Lemma 18 that there is an isomorphism of A-modules $J \cong \Gamma_{\mathfrak{a}}(I)$ where $\mathfrak{a} = (f)$. By Lemma 19 the module J is injective, and $Supp(J) = Supp(\mathscr{H}^0_Z(I^{\sim}))$ is contained in $Y \cap Z$ which is strictly smaller than Y. Therefore by the inductive hypothesis, $J^{\sim}$ is flasque. Since $J^{\sim} \cong \mathscr{H}^0_Z(I^{\sim})$ we see that $\mathscr{H}^0_Z(I^{\sim})$ is flasque and the proof is complete. □

Corollary 22. *Let X be a noetherian scheme, $\mathscr{F}$ a quasi-coherent sheaf of modules on X. Then there is a monomorphism $\mathscr{F} \longrightarrow \mathscr{G}$ where $\mathscr{G}$ is a flasque quasi-coherent sheaf of modules.*

Proof. Cover X with a finite number of open affines U_i and for each i let $f_i : Spec(A_i) \longrightarrow X$ be the canonical open immersion, which factors as an isomorphism $g_i : Spec(A_i) \longrightarrow U_i$ followed by the inclusion, and let M_i an A_i-module for which there is an isomorphism $\mathscr{F}|_{U_i} \longrightarrow (g_i)_*(M_i{}^{\sim})$. Embed M_i in an injective A-module I_i so that we have a monomorphism $\mathscr{F}|_{U_i} \longrightarrow (g_i)_*(I_i{}^{\sim})$. By adjointness we have a morphism $\mathscr{F} \longrightarrow (f_i)_*(I_i{}^{\sim})$. Since I_i is injective, the sheaf $I_i{}^{\sim}$ is flasque by Proposition 21 and therefore so is the coproduct $\mathscr{G} = \bigoplus_{i=1}^n (f_i)_*(I_i{}^{\sim})$ (since this is finite, we can take the pointwise coproduct). The induced morphism $\mathscr{F} \longrightarrow \mathscr{G}$ is clearly a monomorphism and $\mathscr{G}$ is quasi-coherent by [Har77] II.5.8, so the proof is complete. □

Remark 7. If X is noetherian then $\mathfrak{Qco}(X)$ is a grothendieck abelian category (MOS,Proposition 66) and therefore has enough injectives. It is also true that a quasi-coherent sheaf of modules is injective in $\mathfrak{Qco}(X)$ if and only if it is injective in $\mathfrak{Mod}(X)$ (MOS,Proposition 68). Therefore every quasi-coherent sheaf $\mathscr{F}$ can be embedded in a quasi-coherent sheaf $\mathscr{G}$ which is injective in $\mathfrak{Mod}(X)$ (and therefore flasque). This gives an alternative proof of Corollary 22, but it requires some sophisticated category theory.

Using these results, we can also strengthen Proposition 21. If A is a noetherian ring and $X = SpecA$, then for any injective A-module I, the sheaf $I^{\sim}$ is injective in $\mathfrak{Qco}(X)$ and therefore also in $\mathfrak{Mod}(X)$.

3 A Vanishing Theorem of Grothendieck

Proposition 23. *Let $\{\mathscr{F}_\alpha, \varphi_{\alpha\beta}\}_{\alpha \in \Lambda}$ be a direct system of sheaves of abelian groups on a quasi-noetherian space X. Then for any quasi-compact open subset U of X, the canonical morphism*

$$\varinjlim_{\alpha} \Gamma(U, \mathscr{F}_\alpha) \longrightarrow \Gamma(U, \varinjlim_{\alpha} \mathscr{F}_\alpha)$$

*is an isomorphism. In words, the additive functor $\Gamma(U, -) : \mathfrak{Ab}(X) \longrightarrow$ **Ab** preserves direct limits and coproducts.*

Proof. Let P be the canonical direct limit in the category of presheaves of abelian groups, so $P(U) = \varinjlim_\alpha \Gamma(U, \mathscr{F}_\alpha)$ for any open $U \subseteq X$. We show that P satisfies the sheaf condition with respect to open covers of quasi-compact open sets.

Suppose we are given an open set U and a finite open cover $V_1, \dots, V_n$ of U, together with $a \in P(U)$ satisfying $a|_{V_i} = 0$ for each $i \in I$. Write $a = (\alpha, m)$ for some $m \in \Gamma(U, \mathscr{F}_\alpha)$. Then $a|_{V_i} = 0$ for $1 \le i \le n$ so we can find a single index β with $(\varphi_{\alpha\beta})_U(m)|_{V_i} = 0$ for $1 \le i \le n$. Since $\mathscr{F}_\beta$ is a sheaf, we have $(\varphi_{\alpha\beta})_U(m) = 0$ and therefore $a = 0$ in $P(U)$ as required.

Now to show we can amalgamate matching families. Let U be a quasi-compact open subset and $\{V_i\}_{i\in I}$ a nonempty open cover, together with $a_i \in P(V_i)$ which agree on overlaps. Let $V_1, \dots, V_n$ be a finite subcover. We can write the $a_1, \dots, a_n$ in the form $a_i = (\alpha, m_i)$ for some fixed $\alpha \in \Lambda$ and $m_i \in \Gamma(V_i, \mathscr{F}_\alpha)$. We can also find a fixed $\beta \in \Lambda$ such that $(\varphi_{\alpha\beta})_{V_i}(m_i)|_{V_i \cap V_j} = (\varphi_{\alpha\beta})_{V_j}(m_j)|_{V_i\cap V_j}$. Since $\mathscr{F}_\beta$ this has a unique amalgamation $m \in \Gamma(U, \mathscr{F}_\beta)$. If we set $a = (\beta, m)$ then it is clear that $a|_{V_i} = a_i$ for $i \in I$ and that a is the unique element of $P(U)$ with this property.

Now let $\psi : P \longrightarrow \varinjlim_\alpha \mathscr{F}_\alpha$ be the canonical morphism of presheaves of abelian groups, where the sheaf $\varinjlim_\alpha \mathscr{F}_\alpha$ is the direct limit in $\mathfrak{Ab}(X)$. The morphism ψ is just the sheafification of P, and we claim that ψ_U is an isomorphism for any quasi-compact open $U \subseteq X$.

It is easy to see that ψ_U is injective for any quasi-compact open U. To see that it is surjective, let $t \in \Gamma(U, \varinjlim_\alpha \mathscr{F}_\alpha)$ be given. Since X is quasi-noetherian, we can find an open cover $\{W_i\}_{i\in I}$ of U by quasi-compact open sets W_i, together with $a_i \in P(W_i)$ such that $t|_{W_i} = \dot{a}_i$. One checks that $a_i|_{W_i\cap W_j} = a_j|_{W_i\cap W_j}$ (here we use the fact that $W_i \cap W_j$ is quasi-compact). By the above discussion, there exists $a \in P(U)$ with $a|_{W_i} = a_i$ for each $i \in I$, and it is clear that $t = \psi_U(a)$ as required. This shows that $\Gamma(U, -)$ commutes with direct limits, and it therefore commutes with coproducts by (AC,Lemma 43). ∎

As an immediate consequence we have

Lemma 24. *Let $\{\mathscr{F}_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in\Lambda}$ be a direct system of sheaves of abelian groups on a noetherian space X. Then for any open subset U of X, the canonical morphism*

$$\varinjlim_\alpha \Gamma(U, \mathscr{F}_\alpha) \longrightarrow \Gamma(U, \varinjlim_\alpha \mathscr{F}_\alpha)$$

is an isomorphism. In words, the additive functor $\Gamma(U, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$ preserves direct limits and coproducts.

Remark 8. Let $(X, \mathcal{O}_X)$ be a quasi-noetherian ringed space and $U \subseteq X$ a quasi-compact open subset. It follows from Proposition 23 that the sheaves $\mathcal{O}_U$ of (MRS,Section 1.5) is *compact* in the category $\mathfrak{Mod}(X)$. It is easy to check that the functor $\Gamma_{X\setminus U}(X, -)$ preserves coproducts, so the sheaf $\mathcal{O}_{X\setminus U}$ is also compact.

Corollary 25. *The direct limit of any direct system of quasi-flasque sheaves of abelian groups on a quasi-noetherian space is quasi-flasque. The direct limit of any direct system of flasque sheaves of abelian groups on a noetherian space is flasque*

Remark 9. Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups. In ([Har77] Ex1.16e) we defined the sheaf $\mathscr{F}^d$ of discontinuous sections of $\mathscr{F}$, which is flasque. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of abelian groups, then $\phi^d_U(s)(x) = \phi_x(s(x))$ gives a morphism of sheaves of abelian groups $\phi^d : \mathscr{F}^d \longrightarrow \mathscr{G}^d$ and this defines an additive functor $(-)^d : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(X)$. There is a canonical monomorphism of sheaves of abelian groups $\mathscr{F} \longrightarrow \mathscr{F}^d$ natural in $\mathscr{F}$.

Theorem 26. *Let $\{\mathscr{F}_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in\Lambda}$ be a direct system of sheaves of abelian groups on a quasi-noetherian space X. Then the canonical morphism*

$$\varinjlim_\alpha H^i(X, \mathscr{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathscr{F}_\alpha)$$

is an isomorphism for $i \ge 0$. In words, the additive functor $H^i(X, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$ preserves direct limits and coproducts.

Proof. For each $\alpha \in \Lambda$ we have a short exact sequence

$$0 \longrightarrow \mathscr{F}_\alpha \longrightarrow \mathscr{F}^d_\alpha \longrightarrow \mathscr{G}_\alpha \longrightarrow 0$$

Taking direct limits, we have a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C} \longrightarrow \mathscr{G} \longrightarrow 0$$

where $\mathscr{F} = \varinjlim_\alpha \mathscr{F}_\alpha$ and $\mathscr{C} = \varinjlim_\alpha \mathscr{F}_\alpha^d$ is quasi-flasque by Corollary 25. Hence it has no higher cohomology by Corollary 11. Using the long exact sequence of cohomology, we have the following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \varinjlim \Gamma(X, \mathscr{F}_\alpha) & \longrightarrow & \varinjlim \Gamma(X, \mathscr{F}_\alpha^d) & \longrightarrow & \varinjlim \Gamma(X, \mathscr{G}_\alpha) & \longrightarrow & \varinjlim H^1(X, \mathscr{F}_\alpha) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \Gamma(X, \mathscr{F}) & \longrightarrow & \Gamma(X, \mathscr{C}) & \longrightarrow & \Gamma(X, \mathscr{G}) & \longrightarrow & H^1(X, \mathscr{F}) & \longrightarrow 0 \end{array}$$

and for $i > 0$

$$\begin{array}{ccccc} 0 \longrightarrow & \varinjlim H^i(X, \mathscr{G}_\alpha) & \longrightarrow & \varinjlim H^{i+1}(X, \mathscr{F}_\alpha) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^i(X, \mathscr{G}) & \longrightarrow & H^{i+1}(X, \mathscr{F}) & \longrightarrow 0 \end{array}$$

By induction on i we are done, because Proposition 23 included the case $i = 0$. □

Lemma 27. *Let Y be a closed subset of a topological space X with inclusion $j : Y \longrightarrow X$, and let $\mathscr{F}$ be a sheaf of abelian groups on Y. Then for $i \geq 0$ there is a canonical isomorphism of abelian groups $H^i(Y, \mathscr{F}) \cong H^i(X, j_*\mathscr{F})$ natural in $\mathscr{F}$.*

Proof. Let $\mathcal{I}, \mathcal{J}$ be assignments of injective resolutions to the objects of $\mathfrak{Ab}(Y), \mathfrak{Ab}(X)$ respectively, with respect to which all right derived functors are calculated. Suppose the chosen injective resolution for $\mathscr{F}$ is

$$I : 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

Then applying the functor $j_* : \mathfrak{Ab}(Y) \longrightarrow \mathfrak{Ab}(X)$ we have a flasque resolution of $j_*\mathscr{F}$

$$j_*I : 0 \longrightarrow j_*\mathscr{F} \longrightarrow j_*\mathscr{I}^0 \longrightarrow j_*\mathscr{I}^1 \cdots$$

since j_* is exact in this case. By Remark 2 there is a canonical isomorphism of abelian groups natural in $\mathscr{F}$ for $i \geq 0$

$$H^i(X, j_*\mathscr{F}) \cong H^i(\Gamma(X, j_*I)) = H^i(Y, \mathscr{F})$$

which is what we wanted to show. □

Remark 10. The same proof applies just as well to the case where Y, X are arbitrary topological spaces and $j : Y \longrightarrow X$ is a continuous map which induces a homeomorphism of Y with a closed subset of X. That is, there is a canonical isomorphism of abelian groups $H^i(Y, \mathscr{F}) \cong H^i(X, j_*\mathscr{F})$ for any sheaf of abelian groups $\mathscr{F}$ and $i \geq 0$, and moreover this isomorphism is natural in $\mathscr{F}$. In particular this is true when j is a homeomorphism.

If $j : Y \longrightarrow X$ is a closed immersion of schemes over an affine scheme $SpecA$, then using Remark 4 and naturality we see that the isomorphism $H^i(Y, \mathscr{F}) \cong H^i(X, j_*\mathscr{F})$ is an isomorphism of A-modules for any sheaf of modules $\mathscr{F}$ on Y. More generally this is true for any *affine* morphism provided $\mathscr{F}$ is quasi-coherent.

Corollary 28. *Let $f : X \longrightarrow Y$ be an affine morphism of schemes and $\mathscr{F}$ a quasi-coherent sheaf on X. Then there is a canonical isomorphism of abelian groups natural in $\mathscr{F}$ for $i \geq 0$*

$$H^i(X, \mathscr{F}) \longrightarrow H^i(Y, f_*\mathscr{F})$$

Moreover if $f : X \longrightarrow Y$ is a morphism of schemes over an affine scheme $Spec(A)$, this is an isomorphism of A-modules.

Proof. Choose assignments of injective resolutions to $\mathfrak{Ab}(X)$ and $\mathfrak{Ab}(Y)$, and suppose that the chosen resolution for $\mathscr{F}$ is

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots \tag{7}$$

We claim that the following complex

$$0 \longrightarrow f_*\mathscr{F} \longrightarrow f_*\mathscr{I}^0 \longrightarrow f_*\mathscr{I}^1 \longrightarrow \cdots \tag{8}$$

is a flasque resolution of $f_*\mathscr{F}$ as a sheaf of abelian groups. The sheaves $f_*\mathscr{I}^i$ are trivially flasque, so it suffices to show that this sequence is exact. For this it suffices to show it is exact after we apply $\Gamma(V, -)$ for every affine open subset $V \subseteq Y$. Fix some such affine open subset V. Since f is affine, $f^{-1}V$ is an affine open subset of X. We can split (7) into a series of short exact sequences

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{K}^0 \longrightarrow 0$$
$$0 \longrightarrow \mathscr{K}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{K}^1 \longrightarrow 0$$
$$\vdots$$

Restricting the first sequence to $f^{-1}V$ and using the fact that the higher cohomology of $\mathscr{F}|_{f^{-1}V}$ vanishes by Theorem 14, we deduce an exact sequence

$$0 \longrightarrow \Gamma(f^{-1}V, \mathscr{F}) \longrightarrow \Gamma(f^{-1}V, \mathscr{I}^0) \longrightarrow \Gamma(f^{-1}V, \mathscr{K}^0) \longrightarrow 0$$

and isomorphisms $H^i(f^{-1}V, \mathscr{I}^0|_{f^{-1}V}) \cong H^i(f^{-1}V, \mathscr{K}^0|_{f^{-1}V})$ for $i > 0$. Since the sheaf $\mathscr{I}^0$ is flasque all its restrictions are flasque, so we deduce that the higher cohomology groups of $\mathscr{K}^0|_{f^{-1}V}$ all vanish. Proceeding in this way, we deduce a short exact sequence for $i \geq 0$

$$0 \longrightarrow \Gamma(f^{-1}V, \mathscr{K}^i) \longrightarrow \Gamma(f^{-1}V, \mathscr{I}^{i+1}) \longrightarrow \Gamma(f^{-1}V, \mathscr{K}^{i+1}) \longrightarrow 0$$

Piecing all of these sequences back together, we see that the complex obtained by applying $\Gamma(V, -)$ to (8) is exact, as required. That is, we have shown that (8) is a flasque resolution of $f_*\mathscr{F}$. From Remark 2 we deduce a canonical isomorphism of abelian groups for $i \geq 0$

$$H^i(X, \mathscr{F}) = H^i(\Gamma(X, \mathscr{I})) = H^i(\Gamma(Y, f_*\mathscr{I})) \cong H^i(Y, f_*\mathscr{F})$$

naturality with respect to morphisms of quasi-coherent sheaves of modules $\mathscr{F} \longrightarrow \mathscr{G}$ is easily checked (actually the isomorphism is natural with respect to morphisms of sheaves of *abelian groups* $\mathscr{F} \longrightarrow \mathscr{G}$ where $\mathscr{F}, \mathscr{G}$ are quasi-coherent sheaves of modules). Suppose finally that $f : X \longrightarrow Y$ is a morphism of A-schemes for some ring A. Using the characterisation of the module action given in Remark 4 it is clear that $H^i(X, \mathscr{F}) \longrightarrow H^i(Y, f_*\mathscr{F})$ is an isomorphism of A-modules. □

Remark 11. In Section 1.3 when we defined the presheaf of cohomology, it was as $U \mapsto H^i(U, \mathscr{F})$ where $H^i(U, -) = R^i\Gamma(U, -)$, not as $U \mapsto H^i(U, \mathscr{F}|_U)$, because in the latter case it wasn't clear how to define restriction. But at least for affine open subsets we can now do so. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf on X. Given open subsets $V \subseteq U$ with affine inclusion $q : V \longrightarrow U$ we have a canonical morphism of abelian groups for $i \geq 0$

$$H^i(U, \mathscr{F}|_U) \longrightarrow H^i(U, q_*(\mathscr{F}|_V)) \cong H^i(V, \mathscr{F}|_V)$$

using Corollary 28 and the canonical morphism $\mathscr{F}|_U \longrightarrow q_*(\mathscr{F}|_V)$. In fact this morphism of abelian groups sends the action of $\Gamma(U, \mathcal{O}_X)$ to the action of $\Gamma(V, \mathcal{O}_X)$ in a way compatible with restriction. Moreover the isomorphism $H^i(U, \mathscr{F}) \cong H^i(U, \mathscr{F}|_U)$ of Lemma 12 is now natural with respect to affine inclusions of open sets.

Remark 12. Let X be an irreducible topological space, $\mathcal{Z}$ the constant sheaf of abelian groups with $\Gamma(U, \mathcal{Z}) = \mathbb{Z}$ for nonempty open U, which is a sheaf since X is irreducible. For open $U \subseteq X$ write $\mathcal{Z}_U = j_!(\mathcal{Z}|_U)$ where $j : U \longrightarrow X$ is the inclusion. If $\mathscr{F}$ is any sheaf of abelian groups on X, there is a canonical bijection $\Gamma(U, \mathscr{F}) \cong \mathcal{H}om(\mathcal{Z}_U, \mathscr{F})$ for any open subset $U \subseteq X$, natural in $\mathscr{F}$.

In particular for open sets $V \subseteq U$ there is a canonical monomorphism $\mathcal{Z}_V \longrightarrow \mathcal{Z}_U$, so we can consider $\mathcal{Z}_V$ as a subsheaf of $\mathcal{Z}_U$.

Lemma 29. *Let X be an irreducible topological space with nonempty open subset U. If $\mathscr{R}$ is a nonzero subsheaf of $\mathcal{Z}_U$ then there is a nonempty open subset $V \subseteq U$ and a monomorphism $\mathcal{Z}_V \longrightarrow \mathscr{R}$ which restricts to an isomorphism on V.*

Proof. Since $\mathscr{R}$ is nonzero, at least one of the groups $\Gamma(V, \mathscr{R})$ for $V \subseteq U$ must be nonzero. Use the monomorphism $\mathscr{R}|_U \longrightarrow \mathcal{Z}_U|_U \cong \mathcal{Z}|_U$ to identify $\mathscr{R}|_U$ which a nonzero subsheaf $\mathscr{T}$ of $\mathcal{Z}|_U$. Let d be the smallest positive integer occurring in the groups $\Gamma(W, \mathscr{T})$ for $W \subseteq U$, and let $V \subseteq U$ be an open set with $d \in \Gamma(V, \mathscr{T})$. If $m \in \Gamma(V, \mathscr{R})$ corresponds to d then the morphism $\mathcal{Z}_V \longrightarrow \mathscr{R}$ determined by m has the required properties. $\square$

For the proof of the next Theorem we introduce some convenient notation. If X is a topological space, $U \subseteq X$ open and $Y = X \setminus U$, then for a sheaf of abelian groups $\mathscr{F}$ on X we write $\mathscr{F}_Y = i_*(i^{-1}\mathscr{F})$ and $\mathscr{F}_U = j_!(\mathscr{F}|_U)$ where $i : Y \longrightarrow Z, j : U \longrightarrow X$ are the inclusions. By (SGR,Lemma 26) we have an exact sequence

$$0 \longrightarrow \mathscr{F}_U \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_Y \longrightarrow 0 \tag{9}$$

In particular this shows that if $\mathscr{F}|_U = 0$ then there is a canonical isomorphism $\mathscr{F} \cong \mathscr{F}_Y$, and similarly there is a canonical isomorphism $\mathscr{F}_U \cong \mathscr{F}$ if $i^{-1}\mathscr{F} = 0$.

Theorem 30. *Let X be a nonempty noetherian topological space of finite dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F}) = 0$.*

Proof. First we make some observations in the case where X is irreducible.

(A) Suppose that X is irreducible of dimension $n = 0$. In this case X has the discrete topology so the functor $\Gamma(X, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$ is an isomorphism. In particular it is exact, so for any sheaf of abelian groups $\mathscr{F}$ on X we have $H^i(X, \mathscr{F}) = 0$ for $i > 0$, showing that the result is true in this case.

(B) Suppose that X is irreducible of dimension $n \geq 1$. By (SGR,Lemma 13) and Theorem 26 it will be sufficient to prove vanishing of cohomology for $\mathscr{F}$ finitely generated (in the sense of (SGR,Definition 9)). Given a finitely generated sheaf of abelian groups $\mathscr{F}$, let $\mu(\mathscr{F})$ be the smallest integer $t \geq 1$ for which there exists a set of t sections generating $\mathscr{F}$. If $t \geq 2$ find sections $a_1, \dots, a_t$ generating $\mathscr{F}$ (so there are open sets U_i with $a_i \in \mathscr{F}(U_i)$). Let $\mathscr{F}'$ be the submodule generated by a_1. It is clear that $\mathscr{G} = \mathscr{F}/\mathscr{F}'$ is generated by the images of the $a_2, \dots, a_t$ (for example, using (SGR,Lemma 14)), so $\mu(\mathscr{F}') = 1, \mu(\mathscr{G}) \leq t - 1$ and we have an exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

Using the long exact sequence of cohomology and induction on $\mu(\mathscr{F})$, we reduce to the case where $\mathscr{F}$ is generated by a single section a over some open set U. There is a canonical epimorphism $\mathcal{Z}_U \longrightarrow \mathscr{F}$ corresponding to the element a, and therefore an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathcal{Z}_U \longrightarrow \mathscr{F} \longrightarrow 0$$

Again using the long exact sequence of cohomology, we see that to prove the result for X it suffices to prove it in the special case where $\mathscr{F}$ is a subsheaf of $\mathcal{Z}_U$ for some open U.

Next we show that we can reduce to the case where X is irreducible of lower dimension.

(C) Fix an integer $e \geq 0$ and suppose that the theorem is true whenever X is *irreducible* and $\dim X \leq e$. We prove the theorem for arbitrary X of dimension $\leq e$ by induction on the number of irreducible components r. The case $r = 1$ is the hypothesis, so assume $r \geq 2$ and let $Y_1, \dots, Y_r$ be the irreducible components of X. Set $Y = Y_1$ and $U = X \setminus Y$. If necessary rearrange the Y_i so that for some $2 \leq s \leq r$ the irreducible components $Y_2, \dots, Y_s$ are precisely those with $Y_i \cap U \neq \emptyset$. It is easy to see that $\overline{U} = Y_2 \cup \dots \cup Y_s$ is the irreducible decomposition of $\overline{U}$. Since $\overline{U}, Y$ are nonempty noetherian topological spaces of dimension

$\leq e$, it follows from the inductive hypothesis, Lemma 27 and (SGR,Lemma 29) that for any sheaf of abelian groups $\mathscr{F}$ on X and $i > dimX$

$$H^i(X, \mathscr{F}_Y) \cong H^i(Y, i^{-1}\mathscr{F}) = 0$$
$$H^i(X, \mathscr{F}_U) \cong H^i(\overline{U}, k_!\mathscr{F}|_U) = 0$$

where $k : U \longrightarrow \overline{U}$ is the inclusion. Using the long exact cohomology sequence of (9) we see that $H^i(X, \mathscr{F}) = 0$ for $i > dimX$, as required.

The main part of the proof is by induction on the dimension $n \geq 0$. Combining $(A), (C)$ gives the case $n = 0$, so suppose that $n \geq 1$ and that the theorem is true for all X of dimension $< n$. Then by (C) the theorem would hold for all X of dimension n once we show it holds in the special case where X is irreducible. So we have reduced to proving the following statement:

(D) Let X be an irreducible noetherian topological space of dimension $n \geq 1$, and suppose that the theorem holds for all nonempty noetherian topological spaces of dimension $< n$. Then the theorem is true for X.

Proof of (D). By (B) we reduce to the special case where $\mathscr{F}$ is a subsheaf of $\mathscr{Z}_U$ for some open U. Suppose for the moment that we can show $H^i(X, \mathscr{Z}_U) = 0$ for $i > n$ and any open set U. Let U an open subset of X and $\mathscr{F}$ any subsheaf of $\mathscr{Z}_U$. If $\mathscr{F} = 0$ the result is trivial, so we can assume U is nonempty and $\mathscr{F}$ nonzero. Then by Lemma 29 there is a nonempty open subset $V \subseteq U$ and a monomorphism $\mathscr{Z}_V \longrightarrow \mathscr{F}$ which restricts to an isomorphism on V. If $V = X$ then we are done, so assume V is proper. We have an exact sequence

$$0 \longrightarrow \mathscr{Z}_V \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{Z}_V \longrightarrow 0 \tag{10}$$

Set $Z = X \setminus V$, $\mathscr{Q} = \mathscr{F}/\mathscr{Z}_V$ and let $c : Z \longrightarrow X$ be the inclusion. Then Z is a proper nonempty closed subset of X, so $dimZ < n$ by ([Har77] Ex.1.10) and therefore by assumption the theorem holds for Z. Since $\mathscr{Q}|_V = 0$ there is a canonical isomorphism $\mathscr{Q} \cong \mathscr{Q}_Z$ so using Lemma 27 we have for $i > n$

$$H^i(X, \mathscr{Q}) \cong H^i(X, \mathscr{Q}_Z) \cong H^i(Z, c^{-1}\mathscr{Q}) = 0$$

Using the long exact cohomology sequence of (10) and the assumption that $H^i(X, \mathscr{Z}_V) = 0$ for $i > n$, we see that $H^i(X, \mathscr{F}) = 0$ for $i > n$, as required.

Now it only remains to prove the theorem in the case where $\mathscr{F} = \mathscr{Z}_U$ for some open set U. If U is empty this is trivial, so assume U is nonempty and let $Y = X \setminus U$ be the proper complement. Then we have an exact sequence

$$0 \longrightarrow \mathscr{Z}_U \longrightarrow \mathscr{Z} \longrightarrow \mathscr{Z}_Y \longrightarrow 0 \tag{11}$$

Suppose for the moment that U is proper. Then Y is a proper nonempty closed subset of X, so $dimY < n$ and therefore by assumption the theorem holds for Y. In particular $H^i(X, \mathscr{Z}_Y) = 0$ for $i \geq n$. Using the long exact cohomology sequence of (11) we reduce to the case $\mathscr{F} = \mathscr{Z}$. If $U = X$ then $\mathscr{Z}_U \cong \mathscr{Z}$, so we reduce to this case directly. But $\mathscr{Z}$ is trivially flasque, so we have $H^i(X, \mathscr{Z}) = 0$ for $i > n$ and the proof is complete. □

4 Cech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space X, with respect to a given open covering of X. We will prove that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in Section 1. The value of this result is that it gives a practical method for computing cohomology of quasi-coherent sheaves on a scheme.

Let X be a topological space, $\mathfrak{U} = \{U_i\}_{i \in I}$ a nonempty open cover of X with totally ordered index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by

$U_{i_0,\dots,i_p}$. Now let $\mathscr{F}$ be a sheaf of abelian groups on X. We define a complex $C(\mathfrak{U}, \mathscr{F})$ of abelian groups as follows. For each $p \geq 0$, let

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0,\dots,i_p})$$

Thus an element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is determined by giving an element

$$\alpha_{i_0,\cdots,i_p} \in \mathscr{F}(U_{i_0,\dots,i_p})$$

for each $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I. We define the coboundary morphism $d^p : C^p \longrightarrow C^{p+1}$ for $p \geq 0$ by setting

$$(d^p\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \tag{12}$$

where the notation $\widehat{i_k}$ means omit i_k. It is straightforward to check that $d^{p+1} \circ d^p = 0$ for $p \geq 0$, so we have a positive cochain complex of abelian groups.

Remark 13. If $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ for $p \geq 0$, it is sometimes convenient to have the symbol $\alpha_{i_0,\dots,i_p}$ defined for *all* $(p+1)$-tuples of elements of I. If there is a repeated index in the set $\{i_0, \dots, i_p\}$, we define $\alpha_{i_0,\dots,i_p} = 0$. If the indices are all distinct, we define $\alpha_{i_0,\dots,i_p} = (-1)^{sgn(\sigma)}\alpha_{\sigma(i_0),\dots,\sigma(i_p)}$ where σ is the unique permutation with $\sigma(i_0) < \cdots < \sigma(i_p)$. Obviously this is unambiguous if $i_0 < \cdots < i_p$ is already strictly ascending. This defines a morphism of abelian groups $C^p(\mathfrak{U}, \mathscr{F}) \longrightarrow D^p(\mathfrak{U}, \mathscr{F})$ where $D^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0,\dots,i_p} \mathscr{F}(U_{i_0,\dots,i_p})$. The formula (12) gives a morphism of abelian groups $d^p : D^p \longrightarrow D^{p+1}$ for $p \geq 0$ and it is clear that $d^{p+1} \circ d^p$ for $p \geq 0$ in this case as well.

Definition 7. Let X be a topological space, $\mathfrak{U} = \{U_i\}_{i \in I}$ a nonempty open cover of X with totally ordered index set I. For any sheaf of abelian groups $\mathscr{F}$ on X and $p \geq 0$ we define the pth *Čech cohomology group of $\mathscr{F}$ with respect to the covering* $\mathfrak{U}$, to be the abelian group

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = H^p(C(\mathfrak{U}, \mathscr{F}))$$

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of abelian groups then there is an induced morphism of cochain complexes $C(\mathfrak{U}, \phi) : C(\mathfrak{U}, \mathscr{F}) \longrightarrow C(\mathfrak{U}, \mathscr{G})$ and therefore a morphism of abelian groups $\check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^p(\mathfrak{U}, \mathscr{G})$ for each $p \geq 0$. For each $p \geq 0$ this defines an additive functor $\check{H}^p(\mathfrak{U}, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{Ab}$.

Remark 14. Keeping X and $\mathfrak{U}$ fixed, if $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ is a short exact sequence of sheaves of abelian groups on X, we do *not* in general get a long exact sequence of Čech cohomology groups. In other words, the functors $\check{H}^p(\mathfrak{U}, -)$ do not form a δ-functor.

Lemma 31. *Let $X, \mathfrak{U}, \mathscr{F}$ be as in Definition 7. Then there is a canonical isomorphism of abelian groups $\check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F})$ natural in $\mathscr{F}$.*

Proof. The group $\check{H}^0(\mathfrak{U}, \mathscr{F})$ is the kernel of $d : C^0(\mathfrak{U}, \mathscr{F}) \longrightarrow C^1(\mathfrak{U}, \mathscr{F})$. Saying a sequence of sections $\{s_i\}_{i \in I}$ belongs to this kernel is equivalent to saying that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Sending such a sequence to its unique amalgamation defines the isomorphism of abelian groups $\check{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F})$ with inverse $s \mapsto \{s|_{U_i}\}_{i \in I}$. This isomorphism is clearly natural in $\mathscr{F}$. □

Example 3. Let k be a field and set $X = \mathbb{P}^1_k$. Set $U_0 = D_+(x_0)$ and $U_1 = D_+(x_1)$, so we have canonical isomorphisms $U_0 \cong Speck[x]$ and $U_1 \cong Speck[y]$ (where x corresponds to the section x_1/x_0 on U_0 and y corresponds to the section x_0/x_1 on U_1). Let $\mathscr{F}$ be the sheaf of abelian groups $\Omega = \Omega_{X/k}$ on X. By (DIFF,Proposition 9) we have canonical isomorphisms

$$\Gamma(U_0, \Omega) \cong \Omega_{k[x]/k}$$
$$\Gamma(U_1, \Omega) \cong \Omega_{k[y]/k}$$

These are free modules on dx, dy respectively. The isomorphism $U_0 \cong Speck[x]$ identifies the open set $U_0 \cap U_1$ with $D(x)$ so there is a canonical isomorphism of $U_0 \cap U_1$ with $Speck[x, x^{-1}]$ and we have another isomorphism

$$\Gamma(U_0 \cap U_1, \Omega) \cong \Omega_{k[x,x^{-1}]/k} \cong (\Omega_{k[x]/k})_x$$

This is a free module on $dx/1$. Let $\varphi : k[y] \longrightarrow k[x, x^{-1}]$ be the morphism of k-algebras defined by $y \mapsto 1/x$. Then $dy \mapsto -dx/x^2$ defines a morphism of $k[y]$-modules $\Omega_{k[y]/k} \longrightarrow (\Omega_{k[x]/k})_x$ and together with the localisation morphism $\Omega_{k[x]/k} \longrightarrow (\Omega_{k[x]/k})_x$ we have a commutative diagram

$$\begin{array}{ccccc} \Omega_{k[x]/k} & \longrightarrow & (\Omega_{k[x]/k})_x & \longleftarrow & \Omega_{k[y]/k} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma(U_0, \Omega) & \longrightarrow & \Gamma(U_0 \cap U_1, \Omega) & \longleftarrow & \Gamma(U_1, \Omega) \end{array}$$

Let $\mathfrak{U}$ be the cover $\{U_0, U_1\}$. Then the Čech cohomology of Ω with respect to $\mathfrak{U}$ is canonically isomorphic to the cohomology of the following sequence of abelian groups

$$\cdots \longrightarrow 0 \longrightarrow \Omega_{k[x]/k} \oplus \Omega_{k[y]/k} \stackrel{\theta}{\longrightarrow} (\Omega_{k[x]/k})_x \longrightarrow 0 \longrightarrow \cdots$$

where $\theta(f(x)dx, g(y)dy) = (f(x) + 1/x^2 g(1/x))dx$. The kernel of θ is therefore in bijection with pairs of polynomials $(f(x), g(y))$ with $f(x) + 1/x^2 g(1/x) = 0$ in $k[x, x^{-1}]$. A little calculation shows that this is only possible if $f = g = 0$, so $Ker(\theta) = 0$ and therefore $\check{H}^0(\mathfrak{U}, \Omega) = 0$.

To compute H^1, observe that we have a commutative diagram of abelian groups

$$\begin{array}{ccc} \Omega_{k[x]/k} \oplus \Omega_{k[y]/k} & \longrightarrow & (\Omega_{k[x]/k})_x \\ \Downarrow & & \Downarrow \\ k[x] \oplus k[y] & \underset{\psi}{\longrightarrow} & k[x, x^{-1}] \end{array}$$

where ψ is the morphism of k-modules $(f(x), g(y)) \mapsto f(x) + 1/x^2 g(1/x)$. Since $k[x, x^{-1}]$ is the ring of Laurent polynomials in x, it is a free k-module on the basis $\{x^n \mid n \in \mathbb{Z}\}$. The image of ψ is the k-submodule generated by $\{x^n \mid n \in \mathbb{Z}\} \setminus \{x^{-1}\}$. Therefore the quotient $k[x, x^{-1}]/Im\psi$ is a free k-module on the basis $x^{-1} + Im\psi$ and there is an isomorphism of abelian groups $\check{H}^1(\mathfrak{U}, \Omega) \cong k$. So finally

$$\check{H}^i(\mathfrak{U}, \Omega) = \begin{cases} 0 & i = 0 \\ k & i = 1 \\ 0 & i > 1 \end{cases}$$

In particular Lemma 31 shows that $\Gamma(X, \Omega) = 0$.

Definition 8. Let X be a topological space, $\mathfrak{U} = \{U_i\}_{i \in I}$ a nonempty open cover of X with totally ordered index set I. For any open set $V \subseteq X$ let $f : V \longrightarrow X$ denote the inclusion. If $\mathscr{F}$ is a sheaf of abelian groups on X then we define a sheaf of abelian groups for $p \geq 0$ by

$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0, \ldots, i_p}})$$

For $p \geq 0$ we define a morphism of sheaves of abelian groups

$$d^p : \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^{p+1}(\mathfrak{U}, \mathscr{F})$$

$$(d^p)_V(\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \widehat{i_k}, \ldots, i_{p+1}}|_{V \cap U_{i_0, \ldots, i_{p+1}}}$$

As before we check that $d^{p+1} \circ d^p = 0$ for $p \geq 0$ so we have a positive cochain complex of sheaves of abelian groups. Note that by construction for each $p \geq 0$ we have $\Gamma(X, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})) = C^p(\mathfrak{U}, \mathscr{F})$ and on global sections d^p is the differential of (12).

If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of abelian groups then there is an induced morphism of cochain complexes $\mathscr{C}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}(\mathfrak{U}, \mathscr{G})$ which on global sections is the morphism of cochain complexes $C(\mathfrak{U}, \mathscr{F}) \longrightarrow C(\mathfrak{U}, \mathscr{G})$ of Definition 7. This defines an additive functor $\mathscr{C}(\mathfrak{U}, -) : \mathfrak{Ab}(X) \longrightarrow \mathbf{coCh}\mathfrak{Ab}(X)$.

Lemma 32. *Let $X, \mathfrak{U}, \mathscr{F}$ be as in Definition 8. Then the cochain complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is a resolution of $\mathscr{F}$. That is, there is a canonical morphism of sheaves of abelian groups $\varepsilon : \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ making the following sequence exact*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \tag{13}$$

This resolution is natural in $\mathscr{F}$, in the sense that for a morphism of sheaves of abelian groups $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{C}^0(\mathfrak{U}, \mathscr{G}) & \longrightarrow & \mathscr{C}^1(\mathfrak{U}, \mathscr{G}) & \longrightarrow & \cdots \end{array}$$

Proof. The canonical morphisms $\mathscr{F} \longrightarrow f_*(\mathscr{F}|_{U_i})$ induce the morphism of sheaves of abelian groups $\varepsilon : \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ and the sheaf axioms show that the sequence (13) is exact in the first two nonzero places. We show (13) is exact everywhere by showing that for every $x \in X$ the following positive cochain of abelian groups is exact

$$0 \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})_x \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F})_x \longrightarrow \mathscr{C}^2(\mathfrak{U}, \mathscr{F})_x \longrightarrow \cdots \tag{14}$$

Choose an index j with $x \in U_j$ and fix $p \geq 1$. We define a morphism of abelian groups

$$\Sigma^p : \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x \longrightarrow \mathscr{C}^{p-1}(\mathfrak{U}, \mathscr{F})_x$$

as follows. An element $z \in \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x$ can be represented by a section $\alpha \in \Gamma(V, \mathscr{C}^p(\mathfrak{U}, \mathscr{F}))$ over a neighborhood V of x, which we may choose so small that $V \subseteq U_j$. We define an element of $\Gamma(V, \mathscr{C}^{p-1}(\mathfrak{U}, \mathscr{F}))$ by $\sigma(\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}}$ using the notational convention of Remark 13. This makes sense because $V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{j,i_0,\ldots,i_{p-1}}$. One checks that the element $\Sigma^p(z) = (V, \sigma(\alpha))$ is independent of the representative (V, α) chosen, and that it defines a morphism of abelian groups. We also define a morphism of abelian groups

$$\Sigma^0 : \mathscr{C}^0(\mathfrak{U}, \mathscr{F})_x \longrightarrow \mathscr{F}_x$$
$$(V, \alpha) \mapsto (V \cap U_j, \alpha_j)$$

Take Σ^i to be zero for $i < 0$. It is tedious but straightforward to check that $1 = d^{p-1}\Sigma^p + \Sigma^{p+1}d^p$ for all $p \in \mathbb{Z}$. Therefore the identity and zero endomorphisms of the cochain complex (14) are homotopic, and therefore have the same effect on cohomology. It is therefore clear that (14) is exact, as required. Naturality in $\mathscr{F}$ is obvious. □

Proposition 33. *Let X be a topological space, $\mathfrak{U}$ a nonempty open cover, and let $\mathscr{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$.*

Proof. Consider the resolution (13) of $\mathscr{F}$ given by Lemma 32. Since $\mathscr{F}$ is flasque, the sheaves $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ are flasque for each $p \geq 0$. Indeed, for any $i_0, \ldots, i_p$, $\mathscr{F}|_{U_{i_0,\ldots,i_p}}$ is a flasque sheaf on $U_{i_0,\ldots,i_p}$, f_* preserves flasque sheaves, and a product of flasque sheaves is flasque. So by Remark 2 we can use this resolution to compute the cohomology groups of $\mathscr{F}$. But $\mathscr{F}$ is flasque, so $H^p(X, \mathscr{F}) = 0$ for $p > 0$. On the other hand, the answer given by this resolution is

$$H^p(\Gamma(X, \mathscr{C}(\mathfrak{U}, \mathscr{F}))) = \check{H}^p(\mathfrak{U}, \mathscr{F})$$

So we conclude that $\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > 0$. □

Lemma 34. *Let X be a topological space, $\mathfrak{U}$ an nonempty open cover, $\mathscr{F}$ a sheaf of abelian groups on X. Then for each $p \geq 0$ there is a canonical morphism of abelian groups natural in $\mathscr{F}$*

$$\nu : \check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F})$$

Proof. Fix an assignment of injective resolutions $\mathcal{I}$ with respect to which cohomology is calculated. Suppose the chosen injective resolution for $\mathscr{F}$ is

$$\mathscr{I} : 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

By (DF,Theorem 19) there is a morphism of cochain complexes $\mathscr{C}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{I}$ which induces the identity on $\mathscr{F}$ and is unique up to homotopy. Applying the functor $\Gamma(X, -)$ and taking cohomology we get the desired morphism ν. Naturality in $\mathscr{F}$ is clear. In the case $p = 0$ this is the composition of the isomorphism $\check{H}^0(\mathfrak{U}, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$ of Lemma 31 with the canonical isomorphism $\Gamma(X, \mathscr{F}) \cong H^0(X, \mathscr{F})$. □

Remark 15. If in Lemma 34 we have two assignments of injective resolutions $\mathcal{I}, \mathcal{J}$ to the objects of $\mathfrak{Ab}(X)$, then the canonical isomorphism of abelian groups $H^p_{\mathcal{I}}(X, \mathscr{F}) \cong H^p_{\mathcal{J}}(X, \mathscr{F})$ makes the following diagram commute

$$\begin{array}{ccc} \check{H}^p(\mathfrak{U}, \mathscr{F}) & \xrightarrow{\nu_{\mathcal{I}}} & H^p_{\mathcal{I}}(X, \mathscr{F}) \\ & \searrow^{\nu_{\mathcal{J}}} & \Downarrow \\ & & H^p_{\mathcal{J}}(X, \mathscr{F}) \end{array}$$

Remark 16. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathfrak{U} = \{U_i\}_{i \in I}$ a nonempty open cover of X with totally ordered index set I, and $\mathscr{F}$ a sheaf of modules on X. Then for $p \geq 0$ the sheaves $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ are canonically sheaves of $\mathcal{O}_X$-modules and $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is a cochain complex in $\mathfrak{Mod}(X)$. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $\mathscr{C}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}(\mathfrak{U}, \mathscr{G})$ is a morphism of cochain complexes in $\mathfrak{Mod}(X)$, so we have an additive functor $\mathscr{C}(\mathfrak{U}, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{coCh}\mathfrak{Mod}(X)$. The morphism $\varepsilon : \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ of Lemma 32 is a morphism of sheaves of modules, so we have an exact sequence in $\mathfrak{Mod}(X)$

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots$$

Set $A = \Gamma(X, \mathcal{O}_X)$. The above shows that for $p \geq 0$ the Čech cohomology group $\check{H}^p(\mathfrak{U}, \mathscr{F})$ has a canonical A-module structure. If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of modules then $\check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^p(\mathfrak{U}, \mathscr{G})$ is a morphism of A-modules. The morphism $\nu : \check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F})$ of Lemma 34 is also a morphism of A-modules.

Theorem 35. *Let $\mathfrak{U}$ be a finite nonempty cover of a scheme X by open affines which have affine intersections. If $\mathscr{F}$ is a quasi-coherent sheaf on X, then*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots$$

is a $\Gamma(X, -)$-acyclic resolution of $\mathscr{F}$ by quasi-coherent sheaves, and the canonical morphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$\nu : \check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F})$$

is an isomorphism for $p \geq 0$.

Proof. By assumption, an intersection $V = U_1 \cap \cdots \cap U_n$ of elements of the cover is affine and the inclusion $f : V \longrightarrow X$ is an affine morphism. An affine morphism is concentrated, so the sheaf $f_*(\mathscr{F}|_V)$ is quasi-coherent (CON,Proposition 18) and from Corollary 28 we deduce that for $i > 0$

$$0 = H^i(V, \mathscr{F}|_V) \cong H^i(X, f_*(\mathscr{F}|_V))$$

since by Theorem 14 a quasi-coherent sheaf has vanishing higher cohomology on an affine scheme. The sheaf $f_*(\mathscr{F}|_V)$ is therefore quasi-coherent and $\Gamma(X, -)$-acyclic, so the same is true of any finite coproduct of such sheaves. This shows that the sheaf of modules $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is quasi-coherent and $\Gamma(X, -)$-acyclic for $p \geq 0$, as claimed. It now follows from (DTC2,Remark 14) that the canonical morphism ν is an isomorphism for $p \geq 0$. □

Remark 17. The reference to (DTC2,Remark 14) in the proof of Theorem 35 is unkind to the reader. There is a more "elementary" proof of the quoted result using spectral sequences, but this might also be unkind. The author's recommendation for the reader unfamiliar with derived categories is to read the alternative proof given in Section 4.1 (which is anyway good enough to read Hartshorne's book) and worry about removing the noetherian hypothesis later.

Remark 18. Observe that under the hypotheses of Theorem 35 the monomorphism of sheaves of modules $\mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ embeds $\mathscr{F}$ as a subobject of a quasi-coherent $\Gamma(X, -)$-acyclic sheaf of modules (cf. Corollary 22).

Corollary 36. *Let X be a concentrated scheme, $\mathfrak{U}$ as in Theorem 35 and $\{\mathscr{F}_\alpha\}_{\alpha \in \Lambda}$ a nonempty family of quasi-coherent sheaves on X. The canonical morphism of $\Gamma(X, \mathcal{O}_X)$-modules*

$$\bigoplus_\alpha \check{H}^p(\mathfrak{U}, \mathscr{F}_\alpha) \longrightarrow \check{H}^p(\mathfrak{U}, \bigoplus_\alpha \mathscr{F}_\alpha)$$

is an isomorphism for $p \geq 0$.

Proof. The coproduct $\bigoplus_\alpha \mathscr{F}_\alpha$ is quasi-coherent (MOS,Proposition 25), so the result follows Theorem 35 and Theorem 26. ☐

Example 4. Let k be a field and set $A = k[x, y]$, $X = \mathbb{A}^2 = Spec A$. Let $U = X \setminus \{(0,0)\}$ and $U_0 = D(x), U_1 = D(y)$ so that $\{U_0, U_1\}$ is an affine open cover of U. It is trivial that U is a noetherian separated scheme, so we can use Theorem 35 to calculate the cohomology of the quasi-coherent sheaf $\mathcal{O} = \mathcal{O}_X|_U$. The Čech cohomology is the cohomology of the following complex of abelian groups

$$0 \longrightarrow A_x \times A_y \xrightarrow{\theta} A_{xy} \longrightarrow 0$$

where $\theta(a/x^n, b/y^m) = bx^m/(xy)^m - ay^n/(xy)^n$. Therefore $H^i(U, \mathcal{O}) = 0$ for $i > 1$. It is easy to check that $A_{xy} = k[x, y]_{xy}$ is a free k-module on the basis $\{x^i y^j \,|\, i, j \in \mathbb{Z}\}$, and the image of θ is the k-submodule generated by the set $\{x^i y^j \,|\, i \geq 0 \text{ or } j \geq 0\}$. Therefore the quotient $A_{xy}/Im\theta$ is isomorphic as a k-module to the free k-module on the set $\{x^i y^j \,|\, i < 0 \text{ and } j < 0\}$ which is in particular infinite-dimensional. Therefore $H^1(U, \mathcal{O}) \neq 0$, which gives another proof that U is not affine.

4.1 Proof from Hartshorne

There is an alternative proof of Theorem 35 given in Hartshorne. But it only works for noetherian separated schemes, and in any case it seems to me that the proof is incomplete. Here we use an easy result (DTC,Proposition 10) from the theory of derived categories to patch the gap. This proof works for infinite affine open covers, but the reader can probably skip this section on a first reading.

Theorem 37. *Let X be a noetherian separated scheme, $\mathfrak{U}$ an open affine cover of X, and let $\mathscr{F}$ be a quasi-coherent sheaf of modules on X. The canonical morphism of $\Gamma(X, \mathcal{O}_X)$-modules*

$$\nu : \check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F})$$

is an isomorphism for $p \geq 0$.

Proof. Fix an assignment of injective resolutions $\mathcal{I}$ to the objects of $\mathfrak{Ab}(X)$ with respect to which cohomology is calculated. For $p = 0$ we have an isomorphism by Lemma 31. For the general case, embed $\mathscr{F}$ in a flasque, quasi-coherent sheaf of modules $\mathscr{G}$ using Corollary 22 so we have an exact sequence of quasi-coherent sheaves of modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{R} \longrightarrow 0 \tag{15}$$

For each $i_0 < \cdots < i_p$ the open set $U_{i_0,\ldots,i_p}$ is affine, since it is a finite intersection of affine open subsets of a separated scheme ([Har77] II Ex.4.3). Therefore by [Har77] II.5.6 or Theorem 14 (or even more simply (MOS,Lemma 5)) we have an exact sequence

$$0 \longrightarrow \mathscr{F}(U_{i_0,\ldots,i_p}) \longrightarrow \mathscr{G}(U_{i_0,\ldots,i_p}) \longrightarrow \mathscr{H}(U_{i_0,\ldots,i_p}) \longrightarrow 0$$

Taking products, we find that the corresponding sequence of Čech complexes

$$0 \longrightarrow C(\mathfrak{U},\mathscr{F}) \longrightarrow C(\mathfrak{U},\mathscr{G}) \longrightarrow C(\mathfrak{U},\mathscr{R}) \longrightarrow 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. The proof works by comparing this sequence with the other long exact sequence of cohomology, so we need to establish a basis for such a comparison. Let $\mathscr{I}', \mathscr{I}''$ be the assigned injective resolutions of $\mathscr{F}, \mathscr{R}$ respectively, and using (DF,Corollary 40) produce an injective resolution $\mathscr{I}$ of $\mathscr{G}$ fitting into a short exact sequence

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}'' \longrightarrow 0$$

with the morphisms lifting $\mathscr{F} \longrightarrow \mathscr{G}$ and $\mathscr{G} \longrightarrow \mathscr{R}$ respectively. By Lemma 32 and (DF,Theorem 19) we can lift the identities to give morphisms of cochain complexes $g : \mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{I}', f : \mathscr{C}(\mathfrak{U},\mathscr{G}) \longrightarrow \mathscr{I}$ and $e : \mathscr{C}(\mathfrak{U},\mathscr{R}) \longrightarrow \mathscr{I}''$. We have a diagram of cochain complexes in $\mathfrak{Ab}(X)$ with the bottom row exact

$$\begin{array}{ccccccccc} & & \mathscr{C}(\mathfrak{U},\mathscr{F}) & \xrightarrow{u} & \mathscr{C}(\mathfrak{U},\mathscr{G}) & \xrightarrow{q} & \mathscr{C}(\mathfrak{U},\mathscr{R}) & & \\ & & \downarrow g & & \downarrow f & & \downarrow e & & \\ 0 & \longrightarrow & \mathscr{I}' & \xrightarrow[u']{} & \mathscr{I} & \xrightarrow[q']{} & \mathscr{I}'' & \longrightarrow & 0 \end{array} \tag{16}$$

There exists homotopies $\Sigma : fu \longrightarrow u'g$ and $\Theta : q'f \longrightarrow eq$ (DF,Theorem 19) and also a homotopy of homotopies $\vartheta : \Theta u \longrightarrow q'\Sigma$ (DTC,Proposition 10). The same can therefore be said of the result of applying $\Gamma(X, -)$ to (16)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\mathfrak{U},\mathscr{F}) & \longrightarrow & C(\mathfrak{U},\mathscr{G}) & \longrightarrow & C(\mathfrak{U},\mathscr{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X,\mathscr{I}') & \longrightarrow & \Gamma(X,\mathscr{I}) & \longrightarrow & \Gamma(X,\mathscr{I}'') & \longrightarrow & 0 \end{array}$$

The connecting morphisms of the bottom row are the canonical connecting morphisms $H^p(X,\mathscr{R}) \longrightarrow H^{p+1}(X,\mathscr{F})$, so by (DTC,Theorem 9) we have a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \check{H}^p(\mathfrak{U},\mathscr{F}) & \longrightarrow & \check{H}^p(\mathfrak{U},\mathscr{G}) & \longrightarrow & \check{H}^p(\mathfrak{U},\mathscr{R}) & \longrightarrow & \check{H}^{p+1}(\mathfrak{U},\mathscr{F}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^p(X,\mathscr{F}) & \longrightarrow & H^p(X,\mathscr{G}) & \longrightarrow & H^p(X,\mathscr{R}) & \xrightarrow[\delta^p]{} & H^{p+1}(X,\mathscr{F}) & \longrightarrow & \cdots \end{array} \tag{17}$$

Since $\mathscr{G}$ is flasque, its Čech cohomology vanishes for $p > 0$ by Proposition 33 so we have an exact sequence

$$0 \longrightarrow \check{H}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \check{H}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \check{H}^1(\mathfrak{U},\mathscr{F}) \longrightarrow 0$$

and isomorphisms $\check{H}^p(\mathfrak{U},\mathscr{R}) \longrightarrow \check{H}^{p+1}(\mathfrak{U},\mathscr{F})$ for $p \geq 1$. Now comparing with (17), using the case $p = 0$ and Proposition 5 we conclude that the natural map

$$\nu : \check{H}^1(\mathfrak{U},\mathscr{F}) \longrightarrow H^1(X,\mathscr{F})$$

is an isomorphism. Since $\mathscr{F}$ was an aritrary quasi-coherent sheaf, the same must be true for $\mathscr{R}$, so it follows that $\check{H}^2(\mathfrak{U},\mathscr{F}) \longrightarrow H^2(\mathfrak{U},\mathscr{F})$ is an isomorphism. Proceeding in this way, we see that $\nu : \check{H}^p(\mathfrak{U},\mathscr{F}) \longrightarrow H^p(X,\mathscr{F})$ is an isomorphism for all $p \geq 0$. □

Corollary 38. *Let X be a noetherian separated scheme, $\mathfrak{U}$ an open affine cover of X, and suppose we have a short exact sequence of quasi-coherent sheaves of modules on X*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0 \tag{18}$$

Then the corresponding sequence of Čech cochain complexes is exact

$$0 \longrightarrow C(\mathfrak{U}, \mathscr{F}') \longrightarrow C(\mathfrak{U}, \mathscr{F}) \longrightarrow C(\mathfrak{U}, \mathscr{F}'') \longrightarrow 0 \tag{19}$$

and the two long exact sequences of cohomology are isomorphic

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \check{H}^p(\mathfrak{U}, \mathscr{F}') & \longrightarrow & \check{H}^p(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \check{H}^p(\mathfrak{U}, \mathscr{F}'') & \longrightarrow & \check{H}^{p+1}(\mathfrak{U}, \mathscr{F}') & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & H^p(X, \mathscr{F}') & \longrightarrow & H^p(X, \mathscr{F}) & \longrightarrow & H^p(X, \mathscr{F}'') & \longrightarrow & H^{p+1}(X, \mathscr{F}') & \longrightarrow \cdots \end{array}$$

Proof. It follows from the proof of Theorem 35 that (19) is an exact sequence of cochain complexes, so we have the long exact sequence in the first row of (19). The argument given in the proof of Theorem 35 shows that the diagram commutes. □

Remark 19. One consequence of Theorem 37 is that for a noetherian separated scheme X the conclusion of Corollary 36 holds for *any* affine open cover $\mathfrak{U}$.

5 The Cohomology of Projective Space

In this section we make explicit calculations of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine cover. These explicit calculations form the basis for various general results about cohomology of coherent sheaves on projective varieties. See (TES,Definition 9) for the definition of a *perfect pairing*.

Lemma 39. *Let A be a ring and $n \geq 1$. Then $A[x_1, \ldots, x_n]_{x_1 \cdots x_n}$ is a free A-module on the basis $\{x_1^{i_1} \cdots x_n^{i_n} \,|\, i_1, \ldots, i_n \in \mathbb{Z}\}$.*

Proof. Set $Q = A[x_1, \ldots, x_n]_{x_1 \cdots x_n}$. The images of the x_j in Q are units, so it makes sense to write x_j^k for any $k \in \mathbb{Z}$. It is easy to see that the products $x_1^{i_1} \cdots x_n^{i_n}$ for $i_1, \ldots, i_n \in \mathbb{Z}$ span Q as an A-module. To see that they are linearly independent, suppose we have a nonempty linear combination (over distinct sequences of exponents)

$$\alpha_1 \cdot x_1^{i_{11}} \cdots x_n^{i_{1n}} + \cdots + \alpha_r \cdot x_1^{i_{r1}} \cdots x_n^{i_{rn}} = 0$$

Let N be a positive integer strictly larger than the absolute value of all the exponents in this sum. Multiply through by $(x_1 \cdots x_n)^N$ and we have a linear combination of distinct monomials in $A[x_1, \ldots, x_n]$ (since A is not necessarily a domain, we may need to multiply by another power of $x_1 \cdots x_n$ to translate back to the polynomial ring). Since monomials in a polynomial ring are linearly independent, we see that $\alpha_1 = \cdots = \alpha_r = 0$, as required. □

Remark 20. Let A be a nonzero ring, $r \geq 1$ and set $S = A[x_0, \ldots, x_r]$. Let f be a nonempty product $x_{i_1} \cdots x_{i_p}$ for some arbitrary sequence $i_1, \ldots, i_p$. For any index $0 \leq k \leq r$ there is a canonical isomorphism of S-algebras $S_{fx_k} \cong (S_f)_{x_k/1}$. Let $(S_f)_{x_k}$ denote the localisation $T^{-1}S_f$ as an S-module, where $T = \{1, x_k, x_k^2, \ldots\}$. Then there is an obvious isomorphism of S-modules $(S_f)_{x_k} \cong (S_f)_{x_k/1}$. So finally we have an isomorphism of S-modules $S_{fx_k} \cong (S_f)_{x_k}$.

Remark 21. Let A be a noetherian ring and set $X = \mathbb{P}^r_A$. Let $\mathscr{F}$ be the quasi-coherent sheaf of modules $\bigoplus_{n \in \mathbb{Z}} \mathcal{O}(n)$ (MOS,Proposition 25) and set $S = A[x_0, \ldots, x_r]$. There is a canonical isomorphism of rings $A \cong \Gamma(X, \mathcal{O}_X)$ (AAMPS,Proposition 15) so by Definition 5 there is a canonical A-module structure on the group $H^i(X, \mathscr{T})$ for any $i \geq 0$ and sheaf of modules $\mathscr{T}$.

By Theorem 26 cohomology commutes with arbitrary coproducts on the noetherian topological space X (TPC,Proposition 2), so there is a canonical isomorphism of A-modules $H^i(X,\mathscr{F}) \cong \bigoplus_{n\in\mathbb{Z}} H^i(X,\mathcal{O}(n))$ that makes $H^i(X,\mathscr{F})$ into a graded A-module. Given $r \in S_m, x \in H^i(X,\mathcal{O}(n))$ for $m \geq 0, n \in \mathbb{Z}$ there is an induced morphism of A-modules

$$R : H^i(X,\mathcal{O}(n)) \longrightarrow H^i(X,\mathcal{O}(m+n))$$

and we define $r \cdot x = R(x)$. This makes $H^i(X,\mathscr{F})$ into a graded S-module. Let $\mathfrak{U} = \{U_0,\ldots,U_r\}$ be the canonical affine open cover, where $U_i = D_+(x_i)$. Then by Corollary 36 there is a canonical isomorphism of A-modules $\check{H}^i(\mathfrak{U},\mathscr{F}) \cong \bigoplus_{n\in\mathbb{Z}} \check{H}^i(\mathfrak{U},\mathcal{O}(n))$. We make $\check{H}^i(\mathfrak{U},\mathscr{F})$ into a graded S-module as above, so that the canonical isomorphism $H^i(X,\mathscr{F}) \cong \check{H}^i(\mathfrak{U},\mathscr{F})$ of Theorem 35 is an isomorphism of graded S-modules.

By Theorem 26 for any open set $U \subseteq X$ we have a canonical isomorphism of A-modules $\Gamma(U,\mathscr{F}) \cong \bigoplus_{n\in\mathbb{Z}} \Gamma(U,\mathcal{O}(n))$. Therefore $\Gamma(U,\mathscr{F})$ becomes a graded S-module in a canonical way, and there is an isomorphism of graded S-modules for any $p \geq 0$ and $i_0 < \cdots < i_p$

$$\begin{aligned}\Gamma(U_{i_0,\cdots,i_p},\mathscr{F}) &\cong \bigoplus_{n\in\mathbb{Z}} \Gamma(U_{i_0,\cdots,i_p},\mathcal{O}(n)) \cong \bigoplus_{n\in\mathbb{Z}} S(n)_{(x_{i_0}\cdots x_{i_p})} \\ &= \bigoplus_{n\in\mathbb{Z}} (S_{x_{i_0}\cdots x_{i_p}})_n \cong S_{x_{i_0}\cdots x_{i_p}}\end{aligned} \tag{20}$$

For a nonempty subset $\{j_0,\ldots,j_q\} \subseteq \{i_0,\ldots,i_p\}$ the restriction map $\mathscr{F}(U_{j_0,\ldots,j_q}) \longrightarrow \mathscr{F}(U_{i_0,\ldots,i_p})$ corresponds to the canonical ring morphism $\mu_{(j_0,\ldots,j_q),(i_0,\ldots,i_p)} : S_{x_{j_0}\cdots x_{j_q}} \longrightarrow S_{x_{i_0}\cdots x_{i_p}}$. Therefore we have a commutative diagram of graded S-modules

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^0(\mathfrak{U},\mathscr{F}) & \longrightarrow & C^1(\mathfrak{U},\mathscr{F}) & \longrightarrow & \cdots & \longrightarrow & C^r(\mathfrak{U},\mathscr{F}) & \longrightarrow & 0 \\ & & \Downarrow & & \Downarrow & & & & \Downarrow & & \\ 0 & \longrightarrow & \prod_{i_0} S_{x_{i_0}} & \longrightarrow & \prod_{i_0<i_1} S_{x_{i_0}x_{i_1}} & \longrightarrow & \cdots & \longrightarrow & S_{x_0\cdots x_r} & \longrightarrow & 0 \end{array} \tag{21}$$

We are now prepared for first major calculation of cohomology on projective space.

Theorem 40. *Let A be a nonzero noetherian ring and let $X = \mathbb{P}^r_A$ for some $r \geq 1$. Then*

(a) $H^i(X,\mathcal{O}(m)) = 0$ for $0 < i < r$ and all $m \in \mathbb{Z}$.

(b) For $m \geq 0$ the A-module $H^r(X,\mathcal{O}(-m-r-1))$ is free of rank $\binom{m+r}{r}$.

(c) There is a canonical isomorphism of A-modules $H^r(X,\mathcal{O}(-r-1)) \cong A$.

(d) There is a canonical perfect pairing of finite free A-modules for every $m \in \mathbb{Z}$

$$\tau : H^0(X,\mathcal{O}(m)) \times H^r(X,\mathcal{O}(-m-r-1)) \longrightarrow A$$

Proof. We adopt the notation of Remark 21. By Lemma 39, $S_{x_0\cdots x_r}$ is the free A-module with basis $x_0^{l_0}\cdots x_r^{l_r}$ with $l_i \in \mathbb{Z}$. The image of $d^{r-1} : \prod_{k=0}^r S_{x_0\cdots\widehat{x_k}\cdots x_r} \longrightarrow S_{x_0\cdots x_r}$ is the A-submodule generated by those basis elements for which at least one $l_i \geq 0$. From (21) we obtain a canonical isomorphism of graded S-modules of $\check{H}^r(\mathfrak{U},\mathscr{F})$ with $S_{x_0\cdots x_r}/Imd^{r-1}$, which is a free A-module with basis consisting of the "negative" monomials

$$\{x_0^{l_0}\cdots x_r^{l_r} \,|\, l_i < 0 \text{ for each } i\}$$

The quotient has the grading $deg(x_0^{l_0}\cdots x_r^{l_r}) = \sum l_i$. Using Theorem 35 we see that for $n \in \mathbb{Z}$ the A-module $H^r(X,\mathcal{O}(n))$ is a free A-module on the following basis (possibly empty)

$$\{x_0^{l_0}\cdots x_r^{l_r} \,|\, l_i < 0 \text{ for each } i \text{ and } \sum_i l_i = n\} \tag{22}$$

If $n > -r-1$ then this set is empty, so $H^r(X, \mathcal{O}(n)) = 0$. If $m \geq 0$ then are precisely $\binom{m+r}{r}$ basis elements for $H^r(X, \mathcal{O}(-m-r-1))$ which proves (b) and also the special case (c). If $\sum_i l_i = n$ and $x_0^{l_0} \cdots x_r^{l_r}$ is the corresponding basis element of $H^r(X, \mathcal{O}(n))$, then observe that as elements of the graded S-module $H^r(X, \mathscr{F})$ we have

$$(x_0^{s_i} \cdots x_r^{s_r}) \cdot (x_0^{l_0} \cdots x_r^{l_r}) = x_0^{s_i + l_i} \cdots x_r^{s_r + l_r}$$

where the right hand side denotes the basis element of $H^r(X, \mathcal{O}(n+m))$ where $m = \sum_i s_i$.

(d) Elements of $H^0(X, \mathcal{O}(m))$ are in bijection with elements of S_m (AAMPS,Corollary 16), so we can define a A-bilinear pairing

$$\tau' : H^0(X, \mathcal{O}(m)) \times H^r(X, \mathcal{O}(-m-r-1)) \longrightarrow H^r(X, \mathcal{O}(-r-1))$$
$$\tau(r, x) = r \cdot x$$

Let τ be the composite of τ' with the canonical isomorphism of (c). We claim that τ is perfect. If $m < 0$ then $H^0(X, \mathcal{O}(m))$ and $H^r(X, \mathcal{O}(-m-r-1))$ are both zero, so in this case τ is trivially perfect.

If $m \geq 0$ then $H^r(X, \mathcal{O}(-m-r-1))$ is a free A-module of rank $\binom{m+r}{r}$ by (b). The A-module $H^0(X, \mathcal{O}(m))$ is free of the same rank with basis given by all monomials in S of degree m. We know from (b) that $H^r(X, \mathcal{O}(-r-1))$ is free of rank 1 on the basis $x_0^{-1} \cdots x_r^{-1}$. The only pairs of basis elements giving a nonzero value are the following

$$\tau(x_0^{m_0} \cdots x_r^{m_r}, x_0^{-m_0-1} \cdots x_r^{-m_r-1}) = 1$$

By (TES,Lemma 25) this is enough to show that τ is perfect.

It remains to prove the statement (a). First we show that $H^p(X, \mathscr{F})_{x_r} = 0$ for $p > 0$. Let $\mathfrak{J} = \{U_0 \cap U_r, U_1 \cap U_r, \ldots, U_r \cap U_r\}$, which is an affine open cover of the affine noetherian scheme U_r. Combining Theorem 35 and Theorem 16 we know that $\check{H}^p(\mathfrak{J}, \mathscr{F}|_{U_r}) = 0$ for $p > 0$. By Remark 20 we have isomorphisms of S-modules $C^p(\mathfrak{J}, \mathscr{F}|_{U_r}) \cong (\prod_{i_0 < \cdots < i_p} S_{x_{i_0} \cdots x_{i_p}})_{x_r}$ which fit into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(\mathfrak{J}, \mathscr{F}|_{U_r}) & \longrightarrow & C^1(\mathfrak{J}, \mathscr{F}|_{U_r}) & \longrightarrow \cdots \longrightarrow & C^r(\mathfrak{J}, \mathscr{F}|_{U_r}) & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \\
0 & \longrightarrow & (\prod_{i_0} S_{x_{i_0}})_{x_r} & \longrightarrow & (\prod_{i_0 < i_1} S_{x_{i_0} x_{i_1}})_{x_r} & \longrightarrow \cdots \longrightarrow & (S_{x_0 \cdots x_r})_{x_r} & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \\
0 & \longrightarrow & C^0(\mathfrak{U}, \mathscr{F})_{x_r} & \longrightarrow & C^1(\mathfrak{U}, \mathscr{F})_{x_r} & \longrightarrow \cdots \longrightarrow & C^r(\mathfrak{U}, \mathscr{F})_{x_r} & \longrightarrow & 0
\end{array}$$

Since $\check{H}^p(\mathfrak{J}, \mathscr{F}|_{U_r}) = 0$ for $p > 0$ the top row is exact at $C^1(\mathfrak{J}, \mathscr{F}|_{U_r}), \ldots, C^r(\mathfrak{J}, \mathscr{F}|_{U_r})$ and therefore the bottom row is exact at $C^1(\mathfrak{U}, \mathscr{F})_{x_r}, \ldots, C^r(\mathfrak{U}, \mathscr{F})_{x_r}$. Localisation is exact, so it commutes with cohomology and we conclude that $\check{H}^p(\mathfrak{U}, \mathscr{F})_{x_r} = 0$ and therefore $H^p(X, \mathscr{F})_{x_r} = 0$ for $p > 0$, as claimed.

We now prove (a) by induction on r. If $r = 1$ there is nothing to prove, so assume $r > 1$ and set $T = A[x_0, \ldots, x_{r-1}]$. There is a canonical surjective morphism of graded A-algebras $\varphi : S \longrightarrow T$ defined by $x_r \mapsto 0$ which induces a closed immersion $\Phi : \mathbb{P}_A^{r-1} \longrightarrow \mathbb{P}_A^r$ of A-schemes with closed image $H = V(x_r)$. Write $\mathcal{O}_H$ for the quasi-coherent sheaf of modules $\Phi_*(\mathcal{O}_{\mathbb{P}_A^{r-1}})$ and observe that we have a short exact sequence of graded S-modules

$$0 \longrightarrow S(-1) \xrightarrow{x_r} S \xrightarrow{\varphi} T \longrightarrow 0$$

Applying the exact functor $\widetilde{-}$ we have a short exact sequence of quasi-coherent sheaves of modules on $X = \mathbb{P}_A^r$ (using (MPS,Proposition 7))

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

For $m \in \mathbb{Z}$ twisting is exact, so we have another short exact sequence

$$0 \longrightarrow \mathcal{O}(m-1) \longrightarrow \mathcal{O}(m) \longrightarrow \mathcal{O}_H(m) \longrightarrow 0 \tag{23}$$

Since $\mathcal{O}_H(m) \cong \Phi_*(\mathcal{O}(m))$ ([Har77] II 5.12c) it follows from the inductive hypothesis and Remark 10 that $H^i(X, \mathcal{O}_H(m)) \cong H^i(\mathbb{P}^{r-1}_A, \mathcal{O}(m)) = 0$ for $0 < i < r-1$. We also have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X, \mathcal{O}(m-1)) & \longrightarrow & \Gamma(X, \mathcal{O}(m)) & \longrightarrow & \Gamma(X, \mathcal{O}_H(m)) & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & & \Downarrow & & \\
0 & \longrightarrow & S_{m-1} & \longrightarrow & S_m & \longrightarrow & T_m & \longrightarrow & 0
\end{array}$$

where the bottom row is trivially exact and therefore so is the top row. From the long exact cohomology sequence of (23) we see obtain isomorphisms $H^i(X, \mathcal{O}(m-1)) \longrightarrow H^i(X, \mathcal{O}(m))$ for $0 < i < r-1$ and an exact sequence

$$0 \longrightarrow H^{r-1}(X, \mathcal{O}(m-1)) \longrightarrow H^{r-1}(X, \mathcal{O}(m)) \longrightarrow H^{r-1}(X, \mathcal{O}_H(m)) \longrightarrow H^r(X, \mathcal{O}(m-1))$$

We want to show that $H^{r-1}(X, \mathcal{O}(m-1)) \longrightarrow H^{r-1}(X, \mathcal{O}(m))$ is an isomorphism, for which it suffices to show that $\delta^{r-1} : H^{r-1}(X, \mathcal{O}_H(m)) \longrightarrow H^r(X, \mathcal{O}(m-1))$ is injective. By Corollary 38 it suffices to show that the connecting morphism $\omega^{r-1} : \check{H}^{r-1}(\mathfrak{U}, \mathcal{O}_H(m)) \longrightarrow \check{H}^r(\mathfrak{U}, \mathcal{O}(m-1))$ for Čech cohomology is injective. Let $\mathfrak{J}$ be the canonical affine open cover $\{U_0, \ldots, U_{r-1}\}$ of $\mathbb{P}^{r-1}_A$. Since $\Phi^{-1}U_r = \emptyset$ there is a canonical isomorphism of complexes of A-modules $C(\mathfrak{U}, \mathcal{O}_H(m)) \cong C(\mathfrak{J}, \mathcal{O}(m))$ and therefore an isomorphism of A-modules $\check{H}^p(\mathfrak{U}, \mathcal{O}_H(m)) \cong \check{H}^p(\mathfrak{J}, \mathcal{O}(m))$ for $p \geq 0$. In particular the A-module

$$\check{H}^{r-1}(\mathfrak{U}, \mathcal{O}_H(m)) \cong \check{H}^{r-1}(\mathfrak{J}, \mathcal{O}(m))$$

is zero if $m > -r$ (in which case ω^{r-1} is trivially injective) and otherwise is free on the set of monomials $x_0^{l_0} \cdots x_{r-1}^{l_{r-1}}$ with all $l_j < 0$ and $l_1 + \cdots + l_{r-1} = m$. Similarly $\check{H}^r(\mathfrak{U}, \mathcal{O}(m-1))$ is the free A-module on the set of monomials $x_0^{l_0} \cdots x_r^{l_r}$ with all $l_j < 0$ and $l_1 + \cdots + l_r = m-1$. One checks that the connecting morphism ω^{r-1} is the morphism of A-modules defined by $x_0^{l_0} \cdots x_{r-1}^{l_{r-1}} \mapsto (-1)^r x_0^{l_0} \cdots x_{r-1}^{l_{r-1}} x_r^{-1}$, which is clearly injective. This shows that for $0 < i < r$ and any $m \in \mathbb{Z}$ the canonical morphism

$$x_r : H^i(X, \mathcal{O}(m-1)) \longrightarrow H^i(X, \mathcal{O}(m))$$

is an isomorphism. Now suppose that for some $0 < i < r$ and $m \in \mathbb{Z}$ that $H^i(X, \mathcal{O}(m))$ contains a nonzero element v. Since $H^i(X, \mathscr{F})_{x_r} = 0$ for some $k > 0$ the composite

$$H^i(X, \mathcal{O}(m)) \xrightarrow{x_r} H^i(X, \mathcal{O}(m+1)) \xrightarrow{x_r} \cdots \xrightarrow{x_r} H^i(X, \mathcal{O}(m+k))$$

sends v to zero. But we have shown that these morphisms are all isomorphisms, which contradicts the assumption that $v \neq 0$. Therefore $H^i(X, \mathcal{O}(m)) = 0$ and the proof is complete. □

Corollary 41. *Let A be a noetherian ring and let $X = \mathbb{P}^r_A$ for some $r \geq 1$. Then*

(i) For any quasi-coherent sheaf of modules $\mathscr{F}$ we have $H^i(X, \mathscr{F}) = 0$ for $i > r$.

*(ii) The functor $H^r(X, -) : \mathfrak{Qco}(X) \longrightarrow A$***Mod** *is right exact.*

Proof. (i) By Theorem 35 we can calculate cohomology using Čech cohomology and the canonical affine open cover $\mathfrak{U} = \{U_0, \ldots, U_r\}$. Since this cover only contains $r+1$ elements, the cohomology of a quasi-coherent sheaf must vanish for $i > r$ (alternatively if k is a field then this holds for *any* sheaf of abelian groups $\mathscr{F}$ by Theorem 30).

(ii) Suppose we have a short exact sequence of quasi-coherent sheaves of modules

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

Then using the long exact cohomology sequence and the fact that $H^{r+1}(X, \mathscr{F}') = 0$ we see that the following sequence of A-modules is exact

$$H^r(X, \mathscr{F}') \longrightarrow H^r(X, \mathscr{F}) \longrightarrow H^r(X, \mathscr{F}'') \longrightarrow 0$$

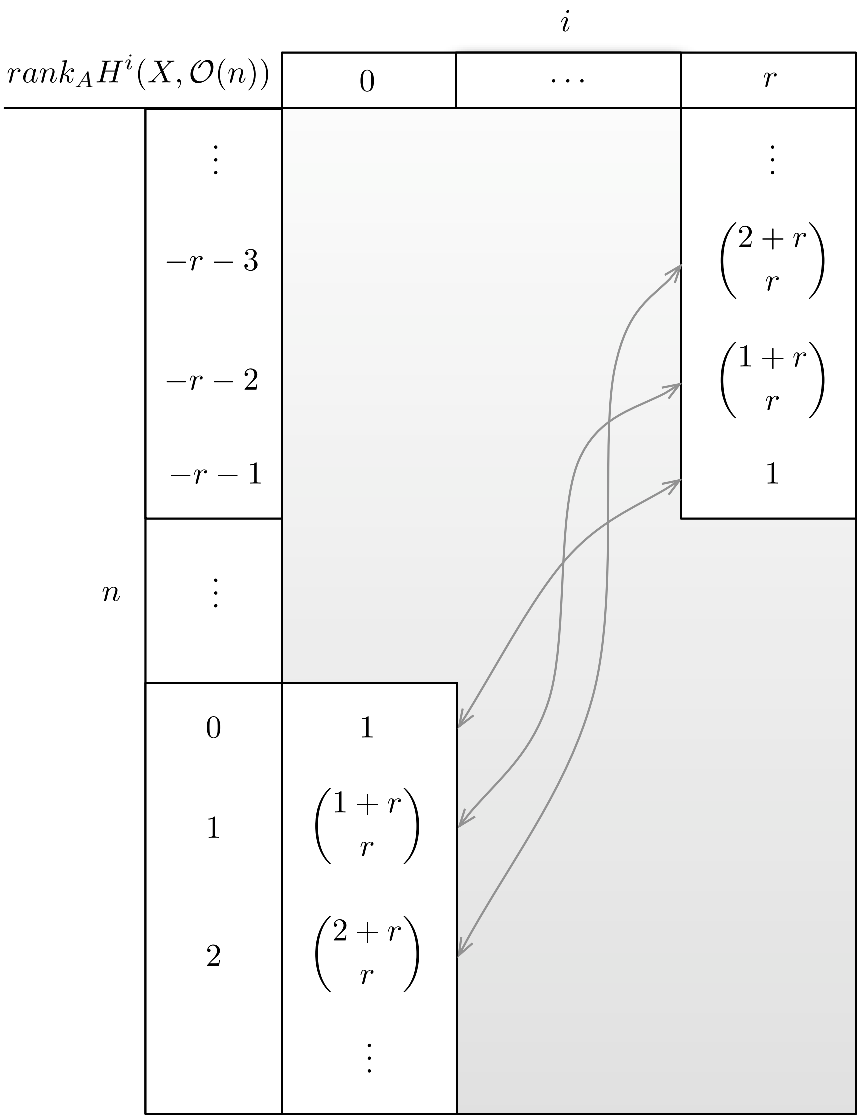
that is, the functor $H^r(X, -)$ is right exact when restricted to $\mathfrak{Qco}(X)$. $\square$

Corollary 42. *Let A be a nonzero noetherian ring and let $X = \mathbb{P}^r_A$ for some $r \geq 1$. Then for $i \geq 0, m \in \mathbb{Z}$ the free A-module $H^i(X, \mathcal{O}(m))$ is zero unless $i = 0$ or $i = r$. In those cases we have for $m \geq 0$*

$$rank_A H^0(X, \mathcal{O}(m)) = rank_A H^r(X, \mathcal{O}(-m-r-1)) = \binom{m+r}{r}$$

with $H^0(X, \mathcal{O}(m)) = H^r(X, \mathcal{O}(-m-r-1)) = 0$ for $m < 0$. In particular $H^i(X, \mathcal{O}) = 0$ for $i > 0$.

These results are reflected in the following table.



In particular Corollary 42 implies that for any $i \geq 0$ and $m \in \mathbb{Z}$ the A-module $H^i(X, \mathcal{O}(m))$ is finitely generated. Also, there is $N > 0$ such that for all for any $i > 0$ and $n \geq N$ we have $H^i(X, \mathcal{O}(n)) = 0$. These results generalise to closed subschemes of projective space.

Theorem 43. *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}(1)$ be a very ample invertible sheaf on X over $SpecA$. Let $\mathscr{F}$ be a coherent sheaf of modules on X. Then*

(a) For each $i \geq 0$, $H^i(X, \mathscr{F})$ is a finitely generated A-module.

(b) There is an integer $N > 0$ such that $H^i(X, \mathscr{F}(n)) = 0$ for each $i > 0$ and $n \geq N$.

Proof. Since $\mathcal{O}(1)$ is a very ample sheaf on X over $SpecA$, there is a closed immersion $i : X \longrightarrow \mathbb{P}^r_A$ of A-schemes for some $r \geq 1$, such that $\mathcal{O}(1) \cong i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$ (see the argument of [Har77] II.5.16.1). If $\mathscr{F}$ is coherent on X, then $i_*\mathscr{F}$ is coherent on $\mathbb{P}^r_A$ ([Har77] II Ex.5.5) and the cohomology is the same by Remark 10. Using (MRS,Lemma 80) we have an isomorphism for $n > 0$

$$i_*(\mathscr{F}(n)) = i_*(\mathscr{F} \otimes \mathcal{O}(1)^{\otimes n}) \cong i_*(\mathscr{F} \otimes i^*(\mathcal{O}_{\mathbb{P}^r_A}(n))) \cong i_*\mathscr{F} \otimes \mathcal{O}_{\mathbb{P}^r_A}(n) = (i_*\mathscr{F})(n)$$

Thus we reduce to the case $X = \mathbb{P}^r_A$. In this case, (a) and (b) are true for any sheaf of the form $\mathcal{O}(q), q \in \mathbb{Z}$, as we observed above. Hence the same is true for any finite direct sum of such sheaves.

To prove (a) for arbitrary coherent sheaves, we use descending induction on i. For $i > r$ we have $H^i(X, \mathscr{F}) = 0$ by Theorem 35, since X can be covered by $r+1$ open affines, so the result is trivial in this case.

In general, given a coherent sheaf $\mathscr{F}$ on X, we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E}$ which is a finite direct sum of sheaves $\mathcal{O}(q_i)$ for various integers q_i ([Har77] II 5.18). Therefore we have a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0 \tag{24}$$

We get a long exact sequence of A-modules

$$\cdots \longrightarrow H^i(X, \mathscr{E}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^{i+1}(X, \mathscr{R}) \longrightarrow \cdots$$

Now the module on the left is finitely generated because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$, as remarked above. The module on the right is finitely generated by the induction hypothesis. Since A is noetherian, we conclude that $H^i(X, \mathscr{F})$ is a finitely generated A-module, which proves (a).

In (b) we are only dealing with a finite number of integers $0 < i \leq r$ so it suffices to prove for a fixed integer i that there exists $N > 0$ with $H^i(X, \mathscr{F}(n)) = 0$ for all $n \geq N$. We do this by descending induction, starting with $i = r$. Given $n > 0$ we can twist (24) and obtain a long exact sequence

$$\cdots \longrightarrow H^i(X, \mathscr{E}(n)) \longrightarrow H^i(X, \mathscr{F}(n)) \longrightarrow H^{i+1}(X, \mathscr{R}(n)) \longrightarrow \cdots$$

For large enough n, the module on the left vanishes since (b) is true for $\mathscr{E}$, and the module on the right vanishes by the inductive hypothesis. Therefore $H^i(X, \mathscr{F}(n)) = 0$ for large enough n, and the proof is complete. □

Remark 22. As a special case of (a), we see that for any coherent sheaf $\mathscr{F}$ on X, $\Gamma(X, \mathscr{F})$ is a finitely generated A-module. This generalises and gives another proof of ([Har77] II 5.19).

Remark 23. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}$ a sheaf of modules. If $U \subseteq X$ is open then we say $\mathscr{F}$ is *generated by global sections over* U if for every $x \in X$ the stalk $\mathscr{F}_x$ is generated as an $\mathcal{O}_{X,x}$-module by the images of the global sections. This property is stable under isomorphism, and $\mathscr{F}$ is generated by global sections iff. it is generated by global sections over the open set X. If $\mathscr{F}, \mathscr{G}$ are both generated by global sections over U then so is their tensor product $\mathscr{F} \otimes \mathscr{G}$.

Corollary 44. *Let X be a projective scheme over a noetherian ring A and let $\mathscr{F}, \mathscr{G}$ be coherent sheaves of modules on X. Then $Hom(\mathscr{F}, \mathscr{G})$ is a finitely generated A-module.*

Proof. By (MOS,Corollary 44) the sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is coherent. Therefore by Remark 22 the A-module $\Gamma(X, \mathscr{H}om(\mathscr{F}, \mathscr{G})) = Hom(\mathscr{F}, \mathscr{G})$ is finitely generated. □

As an application of these results, we give a cohomological criterion for an invertible sheaf to be ample. The idea is that an invertible sheaf is ample if it behaves like the twisting sheaf in Theorem 43(b).

Proposition 45. *Let A be a noetherian ring, and let X be a proper scheme over $SpecA$. Let $\mathscr{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*

(ii) *For each coherent sheaf $\mathscr{F}$ on X there is an integer $N > 0$ such that $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$ for each $i > 0$ and $n \geq N$.*

Proof. $(i) \Rightarrow (ii)$ If $\mathscr{L}$ is ample on X then for some $m > 0$ the sheaf $\mathscr{L}^{\otimes m}$ is very ample over $SpecA$ (PM,Theorem 19). Since X is proper over $SpecA$, it is necessarily projective ([Har77] II 5.16.1). By applying Theorem 43 to each of the following coherent sheaves

$$\mathscr{F}, \mathscr{F} \otimes \mathscr{L}^{\otimes 2}, \cdots, \mathscr{F} \otimes \mathscr{L}^{\otimes (m-1)}$$

we obtain an integer $N > 0$ so large that $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes (mn+j)}) = 0$ for any $i > 0, 0 \leq j \leq m-1$ and $n \geq N$. If we take $M = mN$ then any integer $k \geq M$ can be written in the form $mn + j$ for $n \geq N$ and $0 \leq j \leq m-1$, and therefore $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes k})$ for all $i > 0$ and $k \geq M$, as required.

$(ii) \Rightarrow (i)$ If X is empty this is trivial, so assume otherwise. Then X admits a closed point P by Remark 5. Let $\mathscr{J}_P$ be the ideal sheaf of the closed subset $\{P\}$ and let $\mathscr{F}$ be a coherent sheaf. Then by (MRS,Remark 4) and (MRS,Lemma 54) there is an exact sequence

$$0 \longrightarrow \mathscr{J}_P\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \otimes k(P) \longrightarrow 0$$

where $k(P)$ is the skyscraper sheaf of $\mathcal{O}_{X,P}/\mathfrak{m}_P$ at P. For any $n > 0$ tensoring with $\mathscr{L}^{\otimes n}$ gives an exact sequence

$$0 \longrightarrow \mathscr{J}_P\mathscr{F} \otimes \mathscr{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n} \otimes k(P) \longrightarrow 0$$

By (MOS,Corollary 12) and the hypothesis (ii) there is an integer $N_P > 0$ such that for all $n \geq N_P$ we have $H^1(X, \mathscr{J}_P\mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$. Therefore the map

$$\Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n} \otimes k(P))$$

is surjective for all $n \geq N_P$. Setting $\mathscr{G}_n = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ it follows from (MRS,Lemma 14) that there is a canonical isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules $\Gamma(X, \mathscr{G}_n \otimes k(P)) \cong \mathscr{G}_{n,P}/\mathfrak{m}_P\mathscr{G}_{n,P}$. Hence by Nakayama's lemma the canonical map $\Gamma(X, \mathscr{G}_n) \longrightarrow \mathscr{G}_{n,P}$ is surjective, and $(\mathscr{F} \otimes \mathscr{L}^{\otimes n})_P$ can be generated by a finite number of global sections. Therefore for each $n \geq N_P$ there is an open neighborhood U_n of P such that the global sections of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ generate the sheaf at every point of U_n (DIFF,Proposition 25).

In particular, taking $\mathscr{F} = \mathcal{O}_X$, we find that there is an integer $N_1 > 0$ and an open neighborhood V of P such that $\mathscr{L}^{\otimes N_1}$ is generated by global sections over V. On the other hand, for each $r = 0, 1, \ldots, N_1 - 1$ the above argument gives an open neighborhood U_r of P such that $\mathscr{F} \otimes \mathscr{L}^{\otimes (N_P + r)}$ is generated by global sections over U_r. Now let

$$U_P = V \cap U_0 \cap \cdots \cap U_{N_1 - 1}$$

Then over U_P, all of the sheaves $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ for $n \geq N_P$ are generated by global sections. Indeed, any such sheaf is isomorphic to a tensor product of the form

$$(\mathscr{F} \otimes \mathscr{L}^{\otimes (N_P + r)}) \otimes (\mathscr{L}^{\otimes N_1})^{\otimes m}$$

for suitable $0 \leq r < N_1$ and $m \geq 0$.

Now cover X by a finite number of the open sets U_P for various closed points P, and let N be the maximum of all the corresponding N_P. Then $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for all $n \geq N$, which shows that $\mathscr{L}$ is ample and completes the proof. ∎

Lemma 46. *Let X be a projective scheme over a noetherian ring A and let $\mathscr{F} \longrightarrow \mathscr{G}$ be an epimorphism of coherent sheaves. Then there exists $N > 0$ such that for all $n \geq N$ the morphism $\Gamma(X, \mathscr{F}(n)) \longrightarrow \Gamma(X, \mathscr{G}(n))$ is surjective.*

Proof. Let $\mathscr{K}$ be the kernel of ψ. Using the long exact cohomology sequence of

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

it suffices to produce an integer $N > 0$ such that $H^1(X, \mathscr{K}(n)) = 0$ for all $n \geq N$, which is an immediate consequence of Theorem 43(b). $\square$

Lemma 47. *Let X be a projective scheme over a noetherian ring A and suppose we have an exact sequence of coherent sheaves for some $r \geq 3$*

$$\mathscr{F}^1 \longrightarrow \mathscr{F}^2 \longrightarrow \cdots \longrightarrow \mathscr{F}^r$$

Then there exists $N > 0$ such that for all $n \geq N$ the following sequence is exact

$$\Gamma(X, \mathscr{F}^1(n)) \longrightarrow \Gamma(X, \mathscr{F}^2(n)) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathscr{F}^r(n))$$

Proof. It is clearly enough to prove this in the case $n = 3$, and looking at the image of $\mathscr{F}^1 \longrightarrow \mathscr{F}^2$ we reduce to Lemma 46. $\square$

Definition 9. Let X be a projective scheme over a field k and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i rank_k H^i(X, \mathscr{F})$$

Since X is finite dimensional the cohomology groups vanish for i large, and by Theorem 43 all the k-modules $H^i(X, \mathscr{F})$ are finitely generated, so this defintion makes sense. Isomorphic coherent sheaves have the same Euler characteristic.

Lemma 48. *Let X be a projective scheme over a field k and suppose we have an exact sequence of coherent sheaves $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$. Then we have $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.*

Lemma 49. *Let $f : Y \longrightarrow X$ be a closed subscheme of a projective scheme X over a field k and let $\mathscr{F}$ be a coherent sheaf on Y. Then $\chi_X(f_* \mathscr{F}) = \chi_Y(\mathscr{F})$.*

Proof. This is an immediate consequence of Corollary 28. $\square$

6 Ext Groups and Sheaves

In this section we develop the properties of Ext groups and sheaves, which we will need for the duality theorem. Throughout the beginning of this section $(X, \mathcal{O}_X)$ is an arbitrary ringed space. For any sheaf of modules $\mathscr{F}$ we have left exact functors

$$Hom(\mathscr{F}, -) : \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$$
$$\mathscr{H}om(\mathscr{F}, -) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$

Since $\mathfrak{Mod}(X)$ has enough injectives we can define the right derived functors of these functors. The right derived functors of $Hom(\mathscr{F}, -)$ are denoted $Ext^i(\mathscr{F}, -)$ and are the usual Ext groups for an abelian category (EXT,Definition 1). We also define

Definition 10. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathscr{F}$ a sheaf of modules. The right derived functors of $\mathscr{H}om(\mathscr{F}, -)$ are denoted $\mathscr{E}xt^i(\mathscr{F}, -)$ for $i \geq 0$. This is an additive functor $\mathscr{E}xt^i(\mathscr{F}, -)$: $\mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$. This definition depend on the choice of assignment of injective resolutions, so for another sheaf of modules $\mathscr{G}$ and $i \geq 0$ we have a sheaf of modules $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ determined up to canonical isomorphism. There is a canonical natural equivalence $\mathscr{H}om(\mathscr{F}, -) \cong \mathscr{E}xt^0(\mathscr{F}, -)$ and for a short exact sequence of sheaves of modules

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

we have a long exact sequence of sheaves of modules

$$0 \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{G}') \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{G}'') \longrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}') \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}') \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}'') \longrightarrow \mathscr{E}xt^{i+1}(\mathscr{F},\mathscr{G}') \longrightarrow \cdots$$

which is natural in the usual way with respect to morphisms of short exact sequences. If it is necessary to avoid confusion, we write $\mathscr{E}xt^i_X(\mathscr{F},-)$ to indicate the ringed space X. If $\mathscr{I}$ is an injective object of $\mathfrak{Mod}(X)$ then $\mathscr{E}xt^i(\mathscr{F},\mathscr{I}) = 0$ for all $i > 0$.

Definition 11. Let $(X, \mathcal{O}_X)$ be a ringed space and $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ a morphism of sheaves of modules. There is an induced natural transformation $\alpha : \mathscr{H}om(\mathscr{F}',-) \longrightarrow \mathscr{H}om(\mathscr{F},-)$ which for a sheaf of modules $\mathscr{G}$ is given by $\alpha_{\mathscr{G}} = \mathscr{H}om(\phi,\mathscr{G})$. This induces a natural transformation of the right derived functors $R^i\alpha : \mathscr{E}xt^i(\mathscr{F}',-) \longrightarrow \mathscr{E}xt^i(\mathscr{F},-)$ for $i \geq 0$ (DF,Definition 11) and we denote by $\mathscr{E}xt^i(\phi,\mathscr{G})$ the morphism $(R^i\alpha)_{\mathscr{G}} : \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ for a sheaf of modules $\mathscr{G}$. This defines an additive *contravariant* functor for $i \geq 0$

$$\mathscr{E}xt^i(-,\mathscr{G}) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$

There is a canonical natural equivalence $\mathscr{E}xt^0(-,\mathscr{G}) \cong \mathscr{H}om(-,\mathscr{G})$ and for any exact sequence of sheaves of modules $0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$ the following diagram commutes (DF,Proposition 44)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}') & \longrightarrow & \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) & \longrightarrow & \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}'') & \longrightarrow & \mathscr{E}xt^{i+1}(\mathscr{F}',\mathscr{G}') & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathscr{E}xt^i(\mathscr{F},\mathscr{G}') & \longrightarrow & \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) & \longrightarrow & \mathscr{E}xt^i(\mathscr{F},\mathscr{G}'') & \longrightarrow & \mathscr{E}xt^{i+1}(\mathscr{F},\mathscr{G}') & \longrightarrow \cdots
\end{array} \tag{25}$$

Proposition 50. *For $i \geq 0$ and morphisms of sheaves of modules $\phi : \mathscr{F} \longrightarrow \mathscr{F}'$ and $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ we have*

$$\mathscr{E}xt^i(\mathscr{F},\psi)\mathscr{E}xt^i(\phi,\mathscr{G}) = \mathscr{E}xt^i(\phi,\mathscr{G}')\mathscr{E}xt^i(\mathscr{F}',\psi) \tag{26}$$

It follows that $\mathscr{E}xt^i$ defines a bifunctor $\mathfrak{Mod}(X)^{op} \times \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ for $i \geq 0$, with $\mathscr{E}xt^i(\phi,\psi) : \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}')$ given by the equivalent expressions in (26). The partial functors are the functors $\mathscr{E}xt^i(\mathscr{F},-)$ and $\mathscr{E}xt^i(-,\mathscr{G})$ defined above.

Proof. This follows for arbitrary ϕ and monomorphisms (or epimorphisms) ψ by commutativity of (25). Since $\mathfrak{Mod}(X)$ has epi-mono factorisations it then follows for arbitrary ψ. The bifunctor $\mathscr{E}xt^i(-,-)$ is defined relative to an assignment of injective resolutions $\mathcal{I}$. If $\mathcal{J}$ is another such assignment then the associated bifunctor is canonically naturally equivalent to the one defined for $\mathcal{I}$. ☐

Lemma 51. *If $\mathscr{I}$ is an injective object of $\mathfrak{Mod}(X)$ and $U \subseteq X$ an open subset, then $\mathscr{I}|_U$ is an injective object of $\mathfrak{Mod}(U)$.*

Proof. That is, the additive functor $(-)|_U : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(U)$ preserves injectives. Since this functor has an exact left adjoint given by extension by zero the result is immediate (AC,Proposition 25). ☐

Lemma 52. *If $\mathscr{I}$ is an injective object of $\mathfrak{Mod}(X)$ and $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ a short exact sequence of sheaves of modules, then the following sequence is exact*

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}'',\mathscr{I}) \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{I}) \longrightarrow \mathscr{H}om(\mathscr{F}',\mathscr{I}) \longrightarrow 0$$

That is, the contravariant functor $\mathscr{H}om(-,\mathscr{I}) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$ is exact.

Proof. In the proof of (MRS,Lemma 72) we showed that $\mathscr{H}om(-,\mathscr{I})$ sends cokernels to kernels, so it suffices to show that $\mathscr{H}om(-,\mathscr{I})$ sends monomorphisms to epimorphisms. Let $\mathscr{F}' \longrightarrow \mathscr{F}$ be a monomorphism. To show $\mathscr{H}om(\mathscr{F},\mathscr{I}) \longrightarrow \mathscr{H}om(\mathscr{F}',\mathscr{I})$ is an epimorphism, it is certainly enough to show that $Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U,\mathscr{I}|_U) \longrightarrow Hom_{\mathcal{O}_X|_U}(\mathscr{F}'|_U,\mathscr{I}|_U)$ is surjective for every open set $U \subseteq X$. Since this follows immediately from Lemma 51 the proof is complete. $\square$

For a short exact sequence of sheaves of modules $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ the correspoding sequence of natural transformations $\mathscr{H}om(\mathscr{F}'',-) \longrightarrow \mathscr{H}om(\mathscr{F},-) \longrightarrow \mathscr{H}om(\mathscr{F}',-)$ is exact on injectives by Lemma 52 (see (DF,Definition 12) for what we mean by *exact on injectives*). Therefore for any sheaf of modules $\mathscr{G}$ there are canonical connecting morphisms $\mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{E}xt^{i+1}(\mathscr{F}'',\mathscr{G})$ for $i \geq 0$ fitting into a long exact sequence (DF,Proposition 45)

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}'',\mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{E}xt^1(\mathscr{F}'',\mathscr{G}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathscr{E}xt^i(\mathscr{F}'',\mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) \longrightarrow \mathscr{E}xt^{i+1}(\mathscr{F}'',\mathscr{G}) \longrightarrow \cdots \tag{27}$$

This long exact sequence is natural with respect to morphisms of the short exact sequence and morphisms in the second variable (see (DF,Proposition 45) for precise statements).

Remark 24. In summary, for a ringed space $(X,\mathcal{O}_X)$ and an assignment of injective resolutions $\mathcal{I}$ to the objects of $\mathfrak{Mod}(X)$ and $i \geq 0$ we have canonical bifunctors

$$\mathscr{E}xt^i(-,-) : \mathfrak{Mod}(X)^{\text{op}} \times \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$
$$Ext^i(-,-) : \mathfrak{Mod}(X)^{\text{op}} \times \mathfrak{Mod}(X) \longrightarrow \mathbf{Ab}$$

For both of these bifunctors short exact sequences in either variable lead to a long exact sequence which is natural with respect to morphisms of the exact sequence and morphisms in the remaining variable. There are canonical natural equivalences of bifunctors $Ext^0(-,-) \cong Hom(-,-)$ and $\mathscr{E}xt^0(-,-) \cong \mathscr{H}om(-,-)$.

Proposition 53. *For any open subset $U \subseteq X$ and sheaves of modules $\mathscr{F},\mathscr{G}$ on X there is a canonical isomorphism of sheaves of modules natural in both variables for $i \geq 0$*

$$\mathscr{E}xt^i_X(\mathscr{F},\mathscr{G})|_U \cong \mathscr{E}xt^i_U(\mathscr{F}|_U,\mathscr{G}|_U)$$

Proof. Fix assignments of injective resolutions $\mathcal{I},\mathcal{J}$ to the objects of $\mathfrak{Mod}(U),\mathfrak{Mod}(X)$ respectively. Let $\mathscr{F},\mathscr{G}$ be given and suppose the injective resolutions for $\mathscr{G},\mathscr{G}|_U$ are respectively

$$\mathscr{J} : 0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{J}^0 \longrightarrow \mathscr{J}^1 \longrightarrow \mathscr{J}^2 \longrightarrow \cdots$$
$$\mathscr{I} : 0 \longrightarrow \mathscr{G}|_U \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{I}^2 \longrightarrow \cdots$$

Since the functor $(-)|_U$ is exact and preserves injectives, $\mathscr{J}|_U$ is an injective resolution of $\mathscr{G}|_U$ and there is an induced morphism of cochain complexes $\mathscr{J}|_U \longrightarrow \mathscr{I}$ lifting the identity, which is therefore a homotopy equivalence. This gives a canonical isomorphism of sheaves of modules $H^i(\mathscr{H}om(\mathscr{F}|_U,\mathscr{J}|_U)) \cong \mathscr{E}xt^i(\mathscr{F}|_U,\mathscr{G}|_U)$ for $i \geq 0$. But there is an equality of cochain complexes $\mathscr{H}om(\mathscr{F}|_U,\mathscr{J}|_U) = \mathscr{H}om(\mathscr{F},\mathscr{J})|_U$ and a canonical isomorphism of sheaves of modules $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})|_U \cong H^i(\mathscr{H}om(\mathscr{F},\mathscr{J})|_U)$ for $i \geq 0$ by (DF,Proposition 61). So finally our isomorphism is the composite

$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G})|_U \cong H^i(\mathscr{H}om(\mathscr{F},\mathscr{J})|_U) = \mathscr{H}om(\mathscr{F}|_U,\mathscr{J}|_U) \cong \mathscr{E}xt^i(\mathscr{F}|_U,\mathscr{G}|_U)$$

which is easily checked to be natural in both variables. $\square$

Remark 25. There are canonical natural equivalences $\mathscr{E}xt^0(\mathcal{O}_X,-) \cong \mathscr{H}om(\mathcal{O}_X,-) \cong 1$ so for any sheaf of modules $\mathscr{G}$ there is a canonical isomorphism of sheaves of modules $\mathscr{E}xt^0(\mathcal{O}_X,\mathscr{G}) \cong \mathscr{G}$ natural in $\mathscr{G}$. Also $\mathscr{E}xt^i(\mathcal{O}_X,\mathscr{G}) = 0$ for $i > 0$ since the functor $\mathscr{H}om(\mathcal{O}_X,-) \cong 1$ is exact.

Proposition 54. *For any sheaf of modules $\mathscr{G}$ and $i \geq 0$ there is a canonical isomorphism of A-modules $Ext^i(\mathcal{O}_X, \mathscr{G}) \cong H^i(X, \mathscr{G})$ natural in $\mathscr{G}$, where $A = \mathcal{O}_X(X)$.*

Proof. Let $\Gamma_A(X, -), Hom(\mathcal{O}_X, -) : \mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$ be the canonical functors. There is a canonical natural equivalence $Hom(\mathcal{O}_X, -) \cong \Gamma_A(X, -)$ which induces natural equivalences $R^i Hom(\mathcal{O}_X, -) \cong H^i_A(X, -)$ for $i \geq 0$ (notation of Section 1.2). We have a canonical natural equivalence $H^i_A(X, -) \cong H^i(X, -)$ and an equality $R^i Hom(\mathcal{O}_X, -) = Ext^i(\mathcal{O}_X, -)$. Putting these together, we have for $i \geq 0$ a canonical natural equivalence $Ext^i(\mathcal{O}_X, -) \cong H^i(X, -)$ between functors $\mathfrak{Mod}(X) \longrightarrow A\mathbf{Mod}$, as required. □

Definition 12. If $\mathscr{F}$ is a sheaf of modules then a *locally free resolution* of $\mathscr{F}$ is an exact sequence

$$\mathscr{L} : \cdots \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{L}_0 \longrightarrow \mathscr{F} \longrightarrow 0 \tag{28}$$

where each $\mathscr{L}_i$ is a locally finitely free sheaf of modules (MRS,Definition 14).

Proposition 55. *Suppose $\mathscr{F}$ is a sheaf of modules with locally free resolution (28). Then for any sheaf of modules $\mathscr{G}$ and $i \geq 0$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{G}$*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \cong H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G}))$$

Proof. To be clear, $\mathscr{H}om(\mathscr{L}, \mathscr{G})$ denotes the cochain complex of sheaves of modules

$$0 \longrightarrow \mathscr{H}om(\mathscr{L}_0, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{L}_1, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{L}_2, \mathscr{G}) \longrightarrow \cdots$$

Given a morphism $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ the morphisms $\mathscr{H}om(\mathscr{L}_i, \psi) : \mathscr{H}om(\mathscr{L}_i, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{L}_i, \mathscr{G}')$ give a morphism of cochain complexes $\mathscr{H}om(\mathscr{L}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{L}, \mathscr{G}')$ and therefore a morphism of sheaves of modules $H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G})) \longrightarrow H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G}'))$ which defines an additive functor for $i \geq 0$

$$H^i(\mathscr{H}om(\mathscr{L}, -)) : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$

Suppose we have an exact sequence of sheaves of modules $0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$. Then for each $i \geq 0$ the functor $\mathscr{H}om(\mathscr{L}_i, -)$ is exact (MOS,Lemma 37) so we have an exact sequence of cochain complexes

$$0 \longrightarrow \mathscr{H}om(\mathscr{L}, \mathscr{G}') \longrightarrow \mathscr{H}om(\mathscr{L}, \mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{L}, \mathscr{G}'') \longrightarrow 0$$

and therefore a canonical connecting morphism $\omega^i : H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G}'')) \longrightarrow H^{i+1}(\mathscr{H}om(\mathscr{L}, \mathscr{G}'))$ for each $i \geq 0$. It is not difficult to check that this defines a cohomological δ-functor

$$\{H^i(\mathscr{H}om(\mathscr{L}, -))\}_{i \geq 0} : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X) \tag{29}$$

If $\mathscr{I}$ is injective then the contravariant functor $\mathscr{H}om(-, \mathscr{I})$ is exact by Lemma 52, so the cochain complex $\mathscr{H}om(\mathscr{L}, \mathscr{I})$ is acyclic and therefore $H^i(\mathscr{H}om(\mathscr{L}, \mathscr{I})) = 0$ for $i > 0$. This shows that the cohomological δ-functor (29) is universal (DF,Theorem 74). There is a canonical natural equivalence

$$\psi^0 : H^0(\mathscr{H}om(\mathscr{L}, -)) \cong \mathscr{H}om(\mathscr{F}, -) \cong \mathscr{E}xt^0(\mathscr{F}, -)$$

Since $\{\mathscr{E}xt^i(\mathscr{F}, -)\}_{i \geq 0}$ is another universal cohomological δ-functor, there must exist canonical natural equivalences $\psi^i : \mathscr{E}xt^i(\mathscr{F}, -) \longrightarrow H^i(\mathscr{H}om(\mathscr{L}, -))$ for $i \geq 1$, as required. □

Example 5. Let X be a projective scheme over a noetherian ring A. Then by ([Har77] II 5.18) any coherent sheaf $\mathscr{F}$ on X is a quotient of a locally finitely free sheaf. Thus any coherent sheaf on X has a locally free resolution (28). So Proposition 55 tells us that we can calculate $\mathscr{E}xt$ by taking locally free resolutions in the first variable.

Corollary 56. *Let $\mathscr{L}, \mathscr{G}$ be sheaves of modules with $\mathscr{L}$ locally finitely free. Then $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ for all $i > 0$.*

With this result we can make a slight improvement on (MRS,Proposition 93)

Corollary 57. *Given a short exact sequence* $0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{M} \longrightarrow \mathscr{N} \longrightarrow 0$ *of sheaves of modules with* $\mathscr{N}$ *locally finitely free, the sequence of duals is also exact*

$$0 \longrightarrow \mathscr{N}^\vee \longrightarrow \mathscr{M}^\vee \longrightarrow \mathscr{L}^\vee \longrightarrow 0$$

Proof. This follows immediately from the long exact sequence of $\mathscr{E}xt$ in the first variable (27) and Corollary 56, with $\mathscr{G} = \mathcal{O}_X$. ☐

Lemma 58. *If* $\mathscr{L}, \mathscr{I}$ *are sheaves of modules with* $\mathscr{L}$ *locally finitely free and* $\mathscr{I}$ *injective then* $\mathscr{L} \otimes \mathscr{I}$ *is also injective.*

Proof. We must show that the functor $Hom(-, \mathscr{L} \otimes \mathscr{I})$ is exact. But for a sheaf of modules $\mathscr{F}$ we have an isomorphism of abelian groups natural in $\mathscr{F}$ (MRS,Proposition 74) (MRS,Proposition 75) (MRS,Proposition 76)

$$Hom(\mathscr{F}, \mathscr{L} \otimes \mathscr{I}) \cong Hom(\mathscr{F}, (\mathscr{L}^\vee)^\vee \otimes \mathscr{I}) \cong Hom(\mathscr{F}, \mathscr{H}om(\mathscr{L}^\vee, \mathscr{I})) \cong Hom(\mathscr{F} \otimes \mathscr{L}^\vee, \mathscr{I})$$

Therefore $Hom(-, \mathscr{L} \otimes \mathscr{I})$ is exact, since it is naturally equivalent to the composite of the exact functors $- \otimes \mathscr{L}^\vee$ and $Hom(-, \mathscr{I})$. ☐

Proposition 59. *Let* $\mathscr{L}, \mathscr{F}, \mathscr{G}$ *be sheaves of modules with* $\mathscr{L}$ *locally finitely free. Then for* $i \geq 0$ *there is a canonical isomorphism of* $\mathcal{O}_X(X)$*-modules natural in all three variables*

$$Ext^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \cong Ext^i(\mathscr{F}, \mathscr{L}^\vee \otimes \mathscr{G}) \tag{30}$$

and for $i \geq 0$ *there are canonical isomorphisms of sheaves of modules natural in all three variables*

$$\begin{aligned} \mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) &\cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{L}^\vee \otimes \mathscr{G}) \\ \mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) &\cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \mathscr{L}^\vee \end{aligned} \tag{31}$$

Proof. The functors $\{Ext^i(\mathscr{F}, \mathscr{L}^\vee \otimes -)\}_{i \geq 0}$ are a cohomological δ-functor between $\mathfrak{Mod}(X)$ and $A\mathbf{Mod}$ since tensoring with $\mathscr{L}^\vee$ is exact (we set $A = \mathcal{O}_X(X)$). For $i > 0$ these functors vanish on injectives by Lemma 58, so this is a universal cohomological δ-functor. Since there is a canonical natural equivalence

$$\begin{aligned} \psi^0 : Ext^0(\mathscr{F} \otimes \mathscr{L}, -) &\cong Hom(\mathscr{F} \otimes \mathscr{L}, -) \cong Hom(\mathscr{F}, \mathscr{H}om(\mathscr{L}, -)) \\ &\cong Hom(\mathscr{F}, \mathscr{L}^\vee \otimes -) \cong Ext^0(\mathscr{F}, \mathscr{L}^\vee \otimes -) \end{aligned}$$

we obtain canonical natural equivalences $\psi^i : Ext^i(\mathscr{F} \otimes \mathscr{L}, -) \longrightarrow Ext^i(\mathscr{F}, \mathscr{L}^\vee \otimes -)$ for $i \geq 1$. This gives the canonical isomorphisms (30) natural in $\mathscr{G}$. We now check naturality in $\mathscr{L}$. Let $\alpha : \mathscr{L} \longrightarrow \mathscr{M}$ be a morphism of locally finitely free sheaves of modules. This induces natural transformations for $i \geq 0$

$$\begin{aligned} \beta^i &: Ext^i(\mathscr{F} \otimes \mathscr{M}, -) \longrightarrow Ext^i(\mathscr{F} \otimes \mathscr{L}, -) \\ \gamma^i &: Ext^i(\mathscr{F}, \mathscr{M}^\vee \otimes -) \longrightarrow Ext^i(\mathscr{F}, \mathscr{L}^\vee \otimes -) \end{aligned}$$

Using naturality of the long exact Ext sequence in the second variable with respect to morphisms in the first variable, one checks that in fact β, γ are morphisms of cohomological δ-functors. To prove that (30) is natural in $\mathscr{L}$ we have to show that the following diagram of cohomological δ-functors and their morphisms commutes

$$\begin{array}{ccc} \{Ext^i(\mathscr{F} \otimes \mathscr{M}, -)\}_{i \geq 0} & \xrightarrow{\beta} & \{Ext^i(\mathscr{F} \otimes \mathscr{L}, -)\}_{i \geq 0} \\ {\scriptstyle \psi}\Downarrow & & \Downarrow{\scriptstyle \psi} \\ \{Ext^i(\mathscr{F}, \mathscr{M}^\vee \otimes -)\}_{i \geq 0} & \xrightarrow[\gamma]{} & \{Ext^i(\mathscr{F}, \mathscr{L}^\vee \otimes -)\}_{i \geq 0} \end{array}$$

By universality we need only check commutativity in degree 0, which is easy. Naturalty of (30) in $\mathscr{F}$ follows similarly. For the sheaf $\mathscr{E}xt$, we have three universal cohomological δ-functors between $\mathfrak{Mod}(X)$ and $\mathfrak{Mod}(X)$

$$\{\mathscr{E}xt^i(\mathscr{F}\otimes\mathscr{L},-)\}_{i\geq 0}, \{\mathscr{E}xt^i(\mathscr{F},\mathscr{L}^\vee\otimes -)\}_{i\geq 0}, \{\mathscr{E}xt^i(\mathscr{F},-)\otimes\mathscr{L}^\vee\}_{i\geq 0}$$

The canonical natural equivalences (MRS,Proposition 77) (MRS,Proposition 75)

$$\begin{aligned}\varphi^0 : \mathscr{E}xt^0(\mathscr{F}\otimes\mathscr{L},-) &\cong \mathscr{H}om(\mathscr{F}\otimes\mathscr{L},-)\cong \mathscr{H}om(\mathscr{F},\mathscr{H}om(\mathscr{L},-))\\ &\cong \mathscr{H}om(\mathscr{F},\mathscr{L}^\vee\otimes -)\cong \mathscr{E}xt^0(\mathscr{F},\mathscr{L}^\vee\otimes -)\end{aligned}$$

and (MRS,Proposition 75) (MRS,Proposition 77)

$$\begin{aligned}\phi^0 : \mathscr{E}xt^0(\mathscr{F}\otimes\mathscr{L},-) &\cong \mathscr{H}om(\mathscr{F}\otimes\mathscr{L},-)\cong \mathscr{H}om(\mathscr{L}\otimes\mathscr{F},-)\\ &\cong \mathscr{H}om(\mathscr{L},\mathscr{H}om(\mathscr{F},-))\cong \mathscr{L}^\vee\otimes\mathscr{H}om(\mathscr{F},-)\\ &\cong \mathscr{H}om(\mathscr{F},-)\otimes\mathscr{L}^\vee\cong \mathscr{E}xt^0(\mathscr{F},-)\otimes\mathscr{L}^\vee\end{aligned}$$

induce canonical natural equivalences for $i\geq 1$

$$\begin{aligned}\varphi^i &: \mathscr{E}xt^i(\mathscr{F}\otimes\mathscr{L},-)\longrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{L}^\vee\otimes -)\\ \phi^i &: \mathscr{E}xt^i(\mathscr{F}\otimes\mathscr{L},-)\longrightarrow \mathscr{E}xt^i(\mathscr{F},-)\otimes\mathscr{L}^\vee\end{aligned}$$

as required. One checks naturality of these isomorphisms in the other two variables in the same way as above. □

Proposition 60. *Let $f:(X,\mathcal{O}_X)\longrightarrow(Y,\mathcal{O}_Y)$ be an isomorphism of ringed spaces. For sheaves of modules $\mathscr{F},\mathscr{G}$ on X and $i\geq 0$ there is a canonical isomorphism of sheaves of modules natural in $\mathscr{G}$*

$$\kappa : f_*\mathscr{E}xt^i_X(\mathscr{F},\mathscr{G})\longrightarrow \mathscr{E}xt^i_Y(f_*\mathscr{F},f_*\mathscr{G})$$

Proof. Let $\mathscr{F}$ be a sheaf of modules on X. Then the right derived functors of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},-)$ and $\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathscr{F},-)$ form universal cohomological δ-functors

$$\begin{aligned}\{\mathscr{E}xt^i_X(\mathscr{F},-)\}_{i\geq 0} &: \mathfrak{Mod}(X)\longrightarrow\mathfrak{Mod}(X)\\ \{\mathscr{E}xt^i_Y(f_*\mathscr{F},-)\}_{i\geq 0} &: \mathfrak{Mod}(Y)\longrightarrow\mathfrak{Mod}(Y)\end{aligned}$$

Since the functor $f_*:\mathfrak{Mod}(X)\longrightarrow\mathfrak{Mod}(Y)$ is an isomorphism it is in particular exact, and we obtain two more universal cohomological δ-functors (DF,Definition 24) (DF,Theorem 74)

$$\{f_*\mathscr{E}xt^i_X(\mathscr{F},-)\}_{i\geq 0}, \{\mathscr{E}xt^i_Y(f_*\mathscr{F},f_*(-))\}_{i\geq 0} : \mathfrak{Mod}(X)\longrightarrow\mathfrak{Mod}(Y)$$

We know from (MRS,Proposition 86) that there is a canonical natural equivalence

$$\psi^0 : f_*\mathscr{E}xt^0_X(\mathscr{F},-)\cong f_*\mathscr{H}om(\mathscr{F},-)\cong \mathscr{H}om(f_*\mathscr{F},f_*(-))\cong \mathscr{E}xt^0(f_*\mathscr{F},f_*(-))$$

which induces canonical natural equivalences $\psi^i : f_*\mathscr{E}xt^i(\mathscr{F},-)\longrightarrow \mathscr{E}xt^i(f_*\mathscr{F},f_*(-))$ for all $i>0$, as required. □

Lemma 61. *Let $X=SpecA$ be a noetherian affine scheme, I an injective A-module and $\mathscr{F}$ a coherent sheaf of modules on X. Then $\mathscr{E}xt^i(\mathscr{F},I^\sim)=0$ for $i>0$.*

Proof. We may as well assume $\mathscr{F}$ is $M^\sim$for a finitely generated A-module M. Since A is noetherian, we can find a finite free resolution of M

$$L:\cdots\longrightarrow A^{n_1}\longrightarrow A^{n_0}\longrightarrow M\longrightarrow 0$$

applying the functor $\widetilde{-}$ leads to a locally free resolution $L^\sim$of $\mathscr{F}$. By Proposition 55 and (MOS,Proposition 42) we have a canonical isomorphism for $i\geq 0$

$$\mathscr{E}xt^i(\mathscr{F},\widetilde{I})\cong H^i(\mathscr{H}om(\widetilde{L},\widetilde{I}))\cong H^i(Hom_A(L,I)^\sim)$$

But since I is injective the cochain complex $Hom_A(L,I)$ is acyclic, which shows that $\mathscr{E}xt^i(\mathscr{F},I^\sim)$ is a zero sheaf for $i>0$. □

Proposition 62. *Let $X = SpecA$ be an affine scheme and let M, N be A-modules. Then for $i \geq 0$ there is a canonical morphism of sheaves of modules natural in N*

$$\lambda^i : Ext^i_A(M, N)^{\sim} \longrightarrow \mathscr{E}xt^i(\widetilde{M}, \widetilde{N})$$

If A is noetherian and M finitely generated then this is an isomorphism.

Proof. Let M be an A-module and for $i \geq 0$ let $Ext^i_A(M, -)$ be the right derived functors of the left exact functor $Hom_A(M, -) : A\mathbf{Mod} \longrightarrow A\mathbf{Mod}$. So we have two universal cohomological δ-functors

$$\{Ext^i_A(M, -)\}_{i\geq 0} : A\mathbf{Mod} \longrightarrow A\mathbf{Mod}$$
$$\{\mathscr{E}xt^i(\widetilde{M}, -)\}_{i\geq 0} : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(X)$$

Since the functor $\widetilde{-} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$ is exact, we obtain two more cohomological δ-functors (DF,Remark 2) $\{Ext^i_A(M, -)^{\sim}\}_{i\geq 0}, \{\mathscr{E}xt^i(M^{\sim}, (-)^{\sim})\}_{i\geq 0} : A\mathbf{Mod} \longrightarrow \mathfrak{Mod}(X)$. The former is clearly universal (DF,Theorem 74). There is a canonical natural transformation (MOS,Proposition 42)

$$\lambda^0 : Ext^0_A(M, -)^{\sim} \cong Hom_A(M, -)^{\sim} \longrightarrow \mathscr{H}om(\widetilde{M}, \widetilde{-}) \cong \mathscr{E}xt^0(\widetilde{M}, \widetilde{-})$$

which induces natural transformations $\lambda^i : Ext^i_A(M, -)^{\sim} \longrightarrow \mathscr{E}xt^i(M^{\sim}, (-)^{\sim})$ for $i \geq 1$. If A is noetherian and M finitely generated then Lemma 61 implies that the cohomological δ-functor $\{\mathscr{E}xt^i(M^{\sim}, (-)^{\sim})\}_{i\geq 0}$ is universal and further λ^0 is a natural equivalence (MOS,Proposition 42). Therefore the natural transformations λ^i are all natural equivalences, and the proof is complete. □

Proposition 63. *Let X be a noetherian scheme, $\mathscr{F}, \mathscr{G}$ sheaves of modules on X with $\mathscr{F}$ coherent, and let $x \in X$ be a point. Then for $i \geq 0$ we have an isomorphism of $\mathcal{O}_{X,x}$-modules*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x \cong Ext^i_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

Proof. Let U be an affine open neighborhood of x, so that by Proposition 53 we have isomorphisms of $\mathcal{O}_{X,x}$-modules natural in both variables for $i \geq 0$

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x \cong \mathscr{E}xt^i(\mathscr{F}|_U, \mathscr{G}|_U)_x$$
$$Ext^i_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) \cong Ext^i_{(\mathcal{O}_X|_U)_x}((\mathscr{F}|_U)_x, (\mathscr{G}|_U)_x)$$

We can therefore reduce to the case where X is affine. Then $\mathscr{F}$ has a locally free resolution $\mathscr{L}$ which on the stalks at x gives a free resolution $\mathscr{L}_x$ of $\mathscr{F}_x$. Then one can calculate the $\mathcal{O}_{X,x}$-module $Ext^i(\mathscr{F}_x, \mathscr{G}_x)$ as the cohomology of the following sequence

$$0 \longrightarrow Hom(\mathscr{L}^0_x, \mathscr{G}_x) \longrightarrow Hom(\mathscr{L}^1_x, \mathscr{G}_x) \longrightarrow \cdots$$

which by (MRS,Proposition 89) is isomorphic as a cochain complex of $\mathcal{O}_{X,x}$-modules to

$$0 \longrightarrow \mathscr{H}om(\mathscr{L}^0, \mathscr{G})_x \longrightarrow \mathscr{H}om(\mathscr{L}^1, \mathscr{G})_x \longrightarrow \cdots$$

Since the stalk functor $(-)_x : \mathfrak{Mod}(X) \longrightarrow \mathcal{O}_{X,x}\mathbf{Mod}$ is exact, we have by Proposition 55 an isomorphism of $\mathcal{O}_{X,x}$-modules

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x \cong H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G}))_x \cong H^i(\mathscr{H}om(\mathscr{L}, \mathscr{G})_x) \cong Ext^i(\mathscr{F}_x, \mathscr{G}_x)$$

as required. □

Proposition 64. *Let X be a noetherian scheme, $\mathscr{F}, \mathscr{G}$ quasi-coherent sheaves of modules. Then*

(i) If $\mathscr{F}$ is coherent then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent for $i \geq 0$.

(ii) If $\mathscr{F}, \mathscr{G}$ are both coherent then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent for $i \geq 0$.

Proof. Given a point $x \in X$ let U be an affine open neighborhood of x with canonical isomorphism $f : U \longrightarrow Spec\mathcal{O}_X(U)$. Since $\mathscr{F}(U)$ is a finitely generated module it follows from Proposition 53, Proposition 60 and Proposition 62 that we have a canonical isomorphism for $i \geq 0$

$$\begin{aligned} f_*(\mathscr{E}xt^i(\mathscr{F},\mathscr{G})|_U) &\cong f_*(\mathscr{E}xt^i(\mathscr{F}|_U,\mathscr{G}|_U)) \\ &\cong \mathscr{E}xt^i(f_*\mathscr{F}|_U, f_*\mathscr{G}|_U) \\ &\cong \mathscr{E}xt^i(\mathscr{F}(U)^\sim, \mathscr{G}(U)^\sim) \\ &\cong Ext^i_{\mathcal{O}_X(U}(\mathscr{F}(U),\mathscr{G}(U))^\sim \end{aligned}$$

which shows that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is a quasi-coherent sheaf of modules. *(ii)* If $\mathscr{G}$ is coherent then $\mathscr{G}(U)$ is finitely generated, and therefore by (EXT,Proposition 9) the module $Ext^i_{\mathcal{O}_X(U)}(\mathscr{F}(U),\mathscr{G}(U))$ is also finitely generated. This implies that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is coherent, and completes the proof. □

Lemma 65. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample invertible sheaf, and let $\mathscr{F},\mathscr{G}$ be sheaves of modules on X. Then for $n \in \mathbb{Z}$ and $i \geq 0$ there are canonical isomorphisms of sheaves of modules natural in $\mathscr{F},\mathscr{G}$*

$$\mathscr{H}om(\mathscr{F},\mathscr{G}(n)) \cong \mathscr{H}om(\mathscr{F}(-n),\mathscr{G}) \cong \mathscr{H}om(\mathscr{F},\mathscr{G})(n)$$
$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n)) \cong \mathscr{E}xt^i(\mathscr{F}(-n),\mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F},\mathscr{G})(n)$$

Proof. As usual, for $n \in \mathbb{Z}$ we write $\mathcal{O}(n)$ for the invertible sheaf $\mathcal{O}(1)^{\otimes n}$, with the convention of (MRS,Proposition 85) for negative n. In that case we can define the additive functor $-(n) = -\otimes \mathcal{O}(n)$. For any $n \in \mathbb{Z}$ there is a canonical isomorphism of sheaves of modules $\mathcal{O}(n)^\vee \cong \mathcal{O}(-n)$ (MRS,Lemma 84). The result now follows immediately from Proposition 59. □

Proposition 66. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample invertible sheaf, and let $\mathscr{F},\mathscr{G}$ be coherent sheaves on X. Then given $i \geq 0$ there is $N > 0$ such that for every $n \geq N$ we have an isomorphism of abelian groups*

$$Ext^i(\mathscr{F},\mathscr{G}(n)) \cong \Gamma(X, \mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n)))$$

Proof. If $i = 0$ this is true for any $\mathscr{F},\mathscr{G},n$. If $\mathscr{F} = \mathcal{O}_X$ then the left hand side is $H^i(X,\mathscr{G}(n))$ by Proposition 54. So for sufficiently large n and $i > 0$ we have $Ext^i(\mathscr{F},\mathscr{G}(n)) = 0$ by Theorem 43. On the other hand, the right hand side is always zero for $i > 0$, so we have the result for $\mathscr{F} = \mathcal{O}_X$. If $\mathscr{F}$ is locally finitely free then by Proposition 59 we have an isomorphism for $i \geq 0, n > 0$

$$Ext^i(\mathscr{F},\mathscr{G}(n)) \cong Ext^i(\mathcal{O}_X, (\mathscr{F}^\vee \otimes \mathscr{G})(n))$$

so using Corollary 56 we reduce to the case $\mathscr{F} = \mathcal{O}_X$. In particular for large enough n and any $i > 0$ we have $Ext^i(\mathscr{F},\mathscr{G}(n)) = 0$.

Finally, if $\mathscr{F}$ is an arbitrary coherent sheaf, write it as a quotient of a locally finitely free sheaf $\mathscr{E}$ ([Har77] II 5.18) so we have an exact sequence of coherent sheaves

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

Since $\mathscr{E}$ is locally free, there exists $N > 0$ such that for all $n \geq N$ we have an exact sequence

$$0 \longrightarrow Hom(\mathscr{F},\mathscr{G}(n)) \longrightarrow Hom(\mathscr{E},\mathscr{G}(n)) \longrightarrow Hom(\mathscr{R},\mathscr{G}(n)) \longrightarrow Ext^1(\mathscr{F},\mathscr{G}(n)) \longrightarrow 0$$

and isomorphisms $Ext^i(\mathscr{R},\mathscr{G}(n)) \cong Ext^{i+1}(\mathscr{F},\mathscr{G}(n))$ for all $i > 0$. Using Proposition 64 and (MOS,Corollary 44) we obtain an exact sequence of coherent modules

$$0 \longrightarrow \mathscr{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{E},\mathscr{G}) \longrightarrow \mathscr{H}om(\mathscr{R},\mathscr{G}) \longrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) \longrightarrow 0$$

Replacing $\mathscr{G}$ by $\mathscr{G}(n)$ we get isomorphisms $\mathscr{E}xt^i(\mathscr{R},\mathscr{G}(n)) \cong \mathscr{E}xt^{i+1}(\mathscr{F},\mathscr{G}(n))$ for all $i > 0$ and $n \in \mathbb{Z}$. Combining Lemma 65 and Lemma 47 we find $M > 0$ such that for all $n \geq M$ the rows in the following commutative diagram are exact

$$\begin{array}{ccccccc}
Hom(\mathscr{E},\mathscr{G}(n)) & \longrightarrow & Hom(\mathscr{R},\mathscr{G}(n)) & \longrightarrow & Ext^1(\mathscr{F},\mathscr{G}(n)) & \longrightarrow & 0 \\
\Downarrow & & \Downarrow & & \vdots & & \\
\Gamma(X,\mathscr{H}om(\mathscr{E},\mathscr{G}(n))) & \longrightarrow & \Gamma(X,\mathscr{H}om(\mathscr{R},\mathscr{G}(n))) & \longrightarrow & \Gamma(X,\mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n))) & \longrightarrow & 0
\end{array}$$

There is an induced isomorphism of abelian groups $Ext^1(\mathscr{F},\mathscr{G}(n)) \longrightarrow \Gamma(X,\mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n)))$ for every $n \geq M$, which proves the result for $i = 1$. But $\mathscr{R}$ is also coherent, so by induction we get the general result. □

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