Section 2.9 - Formal Schemes

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October 5, 2006

One feature which clearly distinguishes the theory of schemes from the older theory of varieties is the possibility of having nilpotent elements in the structure sheaf of a scheme. In particular, if Y is a closed subvariety of a variety X, defined by a sheaf of ideals $\mathscr{I}$, then for any $n \geq 1$ we can consider the closed subscheme Y_n defined by the nth power $\mathscr{I}^n$ of the sheaf of ideals $\mathscr{I}$. For $n \geq 2$, this is a scheme with nilpotent elements. It carries information about Y together with the infinitesimal properties of the embedding of Y in X.

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1 Inverse Limits

Definition 1. A *preorder* is a nonempty small category in which every morphism set has at most one element. This is equivalent to giving a set with a binary relation $\leq$ which is reflexive and transitive and we freely identify the two, writing $i \leq j$ if $Hom(i,j) \neq \emptyset$. A *directed set* is a preorder with the property that for every i, j there is k with $i \leq k$ and $j \leq k$.

If I is a directed set, a *direct system* over I in a category $\mathcal{A}$ is a functor $I \longrightarrow \mathcal{A}$. This consists of the following data: an assignment of an object A_i of $\mathcal{A}$ to every object $i \in I$, and a morphism $\pi_{ij} : A_i \longrightarrow A_j$ to every relation $i \leq j$ with the property that $\pi_{ii} = 1$ for all i and $\pi_{jk}\pi_{ij} = \pi_{ik}$ for all $i \leq j \leq k$. A *direct limit* of this direct system is a colimit of the functor, often denoted $\varinjlim_{i \in I} A_i$.

Definition 2. An *inverse directed set* is a preorder with the property that for every i, j there is k with $k \leq i$ and $k \leq j$ (that is, its dual is a directed set). If I is an inverse directed set, a *inverse system* over I in a category $\mathcal{A}$ is a functor $I \longrightarrow \mathcal{A}$. This consists of the following data: an assignment of an object A_i of $\mathcal{A}$ to every object $i \in I$, and a morphism $\pi_{ij} : A_i \longrightarrow A_j$ to every relation $i \leq j$ with the property that $\pi_{ii} = 1$ and $\pi_{jk}\pi_{ij} = \pi_{ik}$ for all $i \leq j \leq k$. An *inverse limit* of this inverse system is a limit of the functor, often denoted $\varprojlim_{i \in I} A_i$.

Definition 3. If I is an inverse directed set then a nonempty subset $J \subseteq I$ is called *final* if for every $i \in I$ there is $j \in J$ with $j \leq i$. The set J with the induced order is also an inverse directed set. Let $\mathcal{A}$ be a category, $\{A_i, \pi_{ij}\}_{i \in I}$ an inverse system over I and $\{A_j, \pi_{jk}\}_{j \in J}$ the restricted inverse system over J. If the morphisms $\rho_j : A \longrightarrow A_j$ are a limit for the latter inverse system, then define $\rho_i : A \longrightarrow A_i$ for arbitrary $i \in I$ by choosing $j \in J$ with $j \leq i$ and composing ρ_j with $\pi_{ij} : A_i \longrightarrow A_j$. This doesn't depend on the $j \in J$ chosen, and the morphisms $\rho_i : A \longrightarrow A_i$ are easily checked to be a limit. In other words, $\varprojlim_{j\in J} A_j = \varprojlim_{i\in I} A_i$.

Let $\{A_i, \pi_{ij}\}_{i\in I}$ be an inverse system of sets and define

$$A = \{(a_i) \in \prod_{i\in I} A_i \,|\, \pi_{ij}(a_i) = a_j \text{ for all } i \leq j\}$$

Observe that this set may be empty. The projections induce functions $p_i : A \longrightarrow A_i$ for each $i \in I$ and it is not hard to check that this is a limit for the inverse system, so $A = \varprojlim_{i\in I} A_i$ and we call it the *canonical inverse limit*. If the A_i are abelian groups (resp. rings) and the π_{ij} are group (resp. ring) morphisms, then A is a subgroup (resp. subring) of the product and the p_i are an inverse limit of abelian groups (resp. rings). If the A_i are modules over a ring R and the π_{ij} are R-module morphisms then A is an R-submodule of the product and the p_i are an inverse limit of modules.

Definition 4. Let I be the inverse directed set consisting of the integers $n \geq 1$ with a single morphism $m \longrightarrow n$ if and only if $n \leq m$ (so all arrows are directed towards 1). Throughout this section, an *inverse system* of abelian groups (resp. rings, modules) is an inverse system over I in **Ab** (resp. **Rng**, R**Mod**). This is a collection of abelian groups $A_n, n \geq 1$ together with morphisms of abelian groups $\varphi_{m,n} : A_m \longrightarrow A_n$ for $n \leq m$ with the property that $\varphi_{n,t}\varphi_{m,n} = \varphi_{m,t}$ for $t \leq n \leq m$ (similarly for rings and modules). We will denote the inverse system by $(A_n, \varphi_{m,n})$ or just (A_n) with the φ understood. The *inverse limit* of (A_n) is the canonical inverse limit defined above

$$\begin{aligned}\varprojlim A_n &= \{(a_n) \in \prod_{n\geq 1} A_n \,|\, \varphi_{m,n}(a_m) = a_n \text{ for all } m \geq n\} \\ &= \{(a_n) \in \prod_{n\geq 1} A_n \,|\, \varphi_{n+1,n}(a_{n+1}) = a_n \text{ for all } n \geq 1\}\end{aligned}$$

A *morphism* of inverse systems $\phi : (A_n) \longrightarrow (B_n)$ is a family of morphisms $\phi_n : A_n \longrightarrow B_n$ such that $\varphi^B_{m,n}\phi_m = \phi_n\varphi^A_{m,n}$ for all $m \geq n$. This defines the functor category $[I, \mathbf{Ab}]$ (resp. $[I, \mathbf{Rng}], [I, R\mathbf{Mod}]$). Such a morphism of inverse systems induces a morphism of the direct limits $\varprojlim A_n \longrightarrow \varprojlim B_n$ defined by $(a_n) \mapsto (\phi_n(a_n))$. This defines a functor $\varprojlim(-) : [I, \mathbf{Ab}] \longrightarrow \mathbf{Ab}$ (similarly for rings, modules).

In the abelian categories $[I, \mathbf{Ab}], [I, R\mathbf{Mod}]$ a sequence

$$0 \longrightarrow (A_n) \xrightarrow{\phi} (B_n) \xrightarrow{\psi} (C_n) \longrightarrow 0$$

is exact if and only if for every $n \geq 1$ the following sequence is exact

$$0 \longrightarrow A_n \xrightarrow{\phi_n} B_n \xrightarrow{\psi_n} C_n \longrightarrow 0$$

The functor $\varprojlim(-)$ has a left adjoint, and therefore preserves monomorphisms and all limits. In particular, the following sequence of direct limits is exact

$$0 \longrightarrow \varprojlim A_n \xrightarrow{\varprojlim \phi} \varprojlim B_n \xrightarrow{\varprojlim \psi} \varprojlim C_n$$

to give a criterion for exactness of $\varprojlim$ on the right, we make the following definition.

Definition 5. An inverse system $(A_n, \varphi_{m,n})$ satisfies the *Mittag-Leffler condition* (ML) if for each $n \geq 1$ there exists $n_0 \geq n$ such that $Im\varphi_{i,n} = Im\varphi_{j,n}$ whenever $i, j \geq n_0$. In that case for $n \geq 1$ let $A'_n \subseteq A_n$ denote the *stable image* $Im\varphi_{m,n}$ for large m. For $m \geq n$ let $\varphi'_{m,n} : A'_m \longrightarrow A'_n$ be the induced morphism. It is clear that $(A'_n, \varphi'_{m,n})$ is an inverse system with all $\varphi'_{m,n}$ surjective. The canonical morphism of inverse systems $(A'_n) \longrightarrow (A_n)$ induces an isomorphism of the inverse limits $\varprojlim A'_n \longrightarrow \varprojlim A_n$. In particular the image of the projection $\varprojlim A_n \longrightarrow A_n$ is contained in A'_n.

Definition 6. Let X be a topological space and $\mathcal{C}$ the category of sheaves of abelian groups on X. Then inverse limits exist in $\mathcal{C}$. Let $(\mathscr{F}_n, \varphi_{m,n})$ be an inverse system in $\mathcal{C}$ over the canonical inverse directed set I defined above. That is, $\mathscr{F}_n$ is a sheaf of abelian groups and $\varphi_{m,n} : \mathscr{F}_m \longrightarrow \mathscr{F}_n$ a morphism of sheaves of abelian groups with $\varphi_{m,m} = 1$ and $\varphi_{n,t}\varphi_{m,n} = \varphi_{m,t}$. Then $\mathscr{F} = \varprojlim \mathscr{F}_n$ is the pointwise inverse limit, $\Gamma(U, \mathscr{F}) = \varprojlim \Gamma(U, \mathscr{F}_n)$ with the induced restriction and pointwise projections. If $(X, \mathcal{O}_X)$ is a ringed space and $(\mathscr{F}_n, \varphi_{m,n})$ an inverse system in $\mathfrak{Mod}(X)$ then we define the canonical inverse limit $\varprojlim \mathscr{F}_n$ in the same way.

Definition 7. Let X be a topological space. The category $\mathcal{C}$ of sheaves of rings on X is complete and cocomplete. Limits are constructed pointwise, so if $(\mathcal{O}_n, \varphi_{m,n})$ is an inverse system in $\mathcal{C}$ then the pointwise direct limit $\Gamma(U, \mathcal{O}) = \varprojlim \Gamma(U, \mathcal{O}_n)$ defines a sheaf of rings on X which is the direct limit of the inverse system in $\mathcal{C}$.

2 Completion

Throughout this note *group* means abelian group and *ring* means commutative ring. See (TR,Definition 1), (TR,Definition 3) for the definition of topological groups and rings.

Definition 8. Let A be a topological group. A *Cauchy sequence* in A is a sequence $(a_n)_{n\geq 1}$ of elements of A with the property that for every open neighborhood U of 0, there exists $N \geq 1$ such that for all $\mu, \nu \geq N$ we have $a_\mu - a_\nu \in U$. Two Cauchy sequences $(a_n), (b_n)$ are *equivalent* if the sequence $(a_n - b_n)$ converges to zero (that is, for every open neighborhood U of 0 there is $N \geq 1$ such that for $\mu \geq N$, $a_\mu - b_\mu \in U$). Let A^c denote the set of equivalence classes of Cauchy sequences under this relation, which is an abelian group with $(a_n) + (b_n) = (a_n + b_n)$. Sending $a \in A$ to the constant sequence (a) defines a morphism of abelian groups $\phi : A \longrightarrow A^c$. If $U \subseteq A$ is open then we say a Cauchy sequence $(a_n)_{n\geq 1}$ is *eventually in* U if there exists $N \geq 1$ such that $a_\mu \in U$ for all $\mu \geq N$.

Definition 9. Let X be a topological space. If $x \in X$ then a *fundamental system of neighborhoods of* x is a nonempty set $\mathcal{S}$ of open neighborhoods of x with the property that if U is open and $x \in U$, then there is $V \in \mathcal{S}$ with $V \subseteq U$.

Definition 10. A *linear topological group* is a topological group A which admits a fundamental system of neighborhoods of 0 consisting of subgroups (necessarily open). A *linear topological ring* is a topological ring A which admits a fundamental system of neighborhoods of 0 consisting of ideals (necessarily open).

Lemma 1. *Let A be a topological group, U an open subgroup and $(a_n), (b_n)$ equivalent Cauchy sequences. If one of these sequences is eventually in U then so is the other.*

Proof. Suppose that $N \geq 1$ is such that $a_\mu \in U$ for all $\mu \geq N$. We may also assume N is so large that $b_\mu - a_\mu \in U$ for all $\mu \geq N$. It is therefore clear that $b_\mu \in U$ for all $\mu \geq N$, as required. □

Definition 11. Let A be a topological group, $\phi : A \longrightarrow A^c$ the canonical morphism of abelian groups. We say that A is *separated* if ϕ is injective (equivalent conditions: (a) the topology on A is Hausdorff (b) if a Cauchy sequence has a limit, it is unique). We say that A is *complete* if ϕ is surjective (equivalently, every Cauchy sequence in A converges).

Definition 12. Let A be an abelian group and suppose we have a sequence of subgroups

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \tag{1}$$

We say a nonempty subset $U \subseteq A$ is *open* if and only if for every $x \in U$ there is $n \geq 0$ with $x + A_n \subseteq U$. In particular U is a neighborhood of 0 if and only if it contains some A_n. This makes A into a topological group. A sequence $(a_n)_{n\geq 1}$ is Cauchy iff. for every $n \geq 0$ there exists $N \geq 1$ such that for all $\mu, \nu \geq N$ we have $a_\mu - a_\nu \in A_n$. Two Cauchy sequences $(a_n), (b_n)$ are equivalent iff. for every $n \geq 0$ there is $N \geq 1$ such that for $\mu \geq N$, $a_\mu - b_\mu \in A_n$.

Lemma 2. *Let A be a topological ring. Then the group A^c of Cauchy sequences becomes a ring with product $(a_n)(b_n) = (a_n b_n)$. The canonical map $A \longrightarrow A^c$ is a morphism of rings.*

Proof. Let $(a_n), (b_n)$ be Cauchy sequences. The first task is to show that $(a_n b_n)$ is Cauchy. Let an open neighborhood U of 0 be given. Let P be an open neighborhood of 0 with $P+P+P+P \subseteq U$. Let $V \subseteq P$ be an open neighborhood of 0 with $V \cdot V \subseteq P$ and find $N \geq 1$ such that for all $\mu, \nu \geq N$ we have $a_\mu - a_\nu \in V, b_\mu - b_\nu \in V$. Let W be an open neighborhood of 0 with $a_N W \subseteq V, b_N W \subseteq V$. Let $M \geq N$ be such that for all $\mu, \nu \geq M$ we have $a_\mu - a_\nu \in W, b_\mu - b_\nu \in W$. Then for $\mu, \nu \geq M$ we have

$$\begin{aligned} a_\mu b_\mu - a_\nu b_\nu &= a_\mu b_\mu - a_\mu b_\nu + a_\mu b_\nu - a_\nu b_\nu \\ &= a_\mu(b_\mu - b_\nu) + b_\nu(a_\mu - a_\nu) \\ &= (a_\mu - a_N)(b_\mu - b_\nu) + (b_\nu - b_N)(a_\mu - a_\nu) + a_N(b_\mu - b_\nu) + b_N(a_\mu - a_\nu) \end{aligned}$$

each summand belongs to P, and therefore $a_\mu b_\mu - a_\nu b_\nu \in U$ as required. Using similar arguments one checks that this product is well-defined on equivalence classes of of Cauchy sequences. It is then not difficult to check that this definition makes A^c into a ring. $\square$

Definition 13. Let A, B be topological groups. A *morphism of topological groups* $\phi : A \longrightarrow B$ is a continuous morphism of abelian groups. We denote by **AbTop** the category of topological groups. If $(a_n)_{n\geq 1}$ is a Cauchy sequence in A then $(\phi(a_n))_{n\geq 1}$ is a Cauchy sequence in B, and this defines a morphism of abelian groups $\phi^c : A^c \longrightarrow B^c$ fitting into the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A^c & \xrightarrow[\phi^c]{} & B^c \end{array}$$

This defines a functor **AbTop** $\longrightarrow$ **Ab**. A *morphism of topological rings* $\phi : A \longrightarrow B$ is a continuous morphism of rings. We denote by **RngTop** the category of topological rings. In this case $\phi^c : A^c \longrightarrow B^c$ is a morphism of rings, so we have a functor **RngTop** $\longrightarrow$ **Rng**.

Definition 14. Let A be a topological group. If $U \subseteq A$ is an open subgroup then U is a topological group and the inclusion $U \longrightarrow A$ is a morphism of topological groups. We have an induced morphism of groups $U^c \longrightarrow A^c$ which is clearly injective. The image of this morphism is the set of all Cauchy sequences in A that are eventually in U.

Now suppose that A is a linear topological group, and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be the set of all open subgroups of A. By assumption this is a fundamental system of neighborhoods of 0. Then the set $\{U_\lambda^c\}_{\lambda\in\Lambda}$ of subgroups of A^c satisfies the conditions of (TR,Proposition 2) so A^c becomes a linear topological group in a canonical way. The morphism of abelian groups $A \longrightarrow A^c$ is continuous, and if $\phi : A \longrightarrow B$ is a continuous morphism of linear topological groups then $\phi^c : A^c \longrightarrow B^c$ is also continuous. If **LAbTop** denotes the category of linear topological groups then we have a functor $(-)^c :$ **LAbTop** $\longrightarrow$ **LAbTop**.

Definition 15. Let A be a linear topological ring. If $\mathfrak{a} \subseteq A$ is an open ideal then the subgroup $\mathfrak{a}^c$ of the ring A^c is an ideal. The set $\{\mathfrak{a}_\lambda\}_\lambda$ of all open ideals of A is a final subset of the set of all open subgroups. Therefore the topology on A^c given in Definition 14 agrees with the topology induced by the ideals $\{\mathfrak{a}_\lambda^c\}_\lambda$, and as a consequence A^c is a linear topological ring. The ring morphism $A \longrightarrow A^c$ is continuous.

If $\phi : A \longrightarrow B$ is a continuous morphism of linear topological rings then $\phi^c : A^c \longrightarrow B^c$ is also continuous. Therefore if **LRngTop** denotes the category of linear topological rings we have a functor $(-)^c :$ **LRngTop** $\longrightarrow$ **LRngTop**.

Remark 1. Let A be a linear topological group, $i : A \longrightarrow A^c$ the canonical morphism of linear topological groups. If $(a_n)_{n\geq 1}$ is a Cauchy sequence in A then $(i(a_n))_{n\geq 1}$ is a Cauchy sequence in A^c. We claim that it converges to $(a_n)_{n\geq 1}$ (that is, the sequence of Cauchy sequences

$i(a_1), i(a_2), \ldots$ converges to the Cauchy sequence $(a_n)_{n\geq 1})$. If V is an open neighborhood of $(a_n)_{n\geq 1}$ then by definition there is an open subgroup $U \subseteq A$ with $(a_n)_{n\geq 1} + U^c \subseteq V$. So it suffices to show that there exists $N \geq 1$ with $i(a_\mu) - (a_n)_{n\geq 1} \in U^c$ for all $\mu \geq N$. Of course this is the sequence

$$i(a_\mu) - (a_n)_{n\geq 1} = (a_\mu - a_1, a_\mu - a_2, \ldots, a_\mu - a_{\mu-1}, 0, a_\mu - a_{\mu+1}, \ldots)$$

so we need only make N so large that $a_\mu - a_\nu \in U$ for all $\mu, \nu \geq N$. This shows that the sequence $(i(a_n))_{n\geq 1}$ converges to $(a_n)_{n\geq 1}$, as claimed.

Lemma 3. *Let A be a linear topological group. The linear topological group A^c is separated.*

Proof. First we show that A^c is separated, or what is the same, the morphism of abelian groups $A^c \longrightarrow (A^c)^c$ is injective. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in A mapping to zero in $(A^c)^c$. This implies that for every open subgroup $U \subseteq A$ we have $(a_n)_{n\geq 1} \in U^c$. Therefore we can find $N \geq 1$ such that $a_\mu \in U$ for all $\mu \geq N$, which means that $(a_n)_{n\geq 1} = 0$ in A^c, as required. □

Lemma 4. *Let $\phi : A \longrightarrow B$ be a morphism of linear topological groups. Then ϕ^c is the unique morphism of linear topological groups making the following diagram commute*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A^c & \xrightarrow[\phi^c]{} & B^c \end{array} \tag{2}$$

Proof. Let $\psi : A^c \longrightarrow B^c$ be another continuous morphism of abelian groups making the diagram commute. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in A. Then in the topological group A^c, the element $(a_n)_{n\geq 1}$ is the limit of the Cauchy sequence $(i(a_n))_{n\geq 1}$. Therefore $\psi((a_n)_{n\geq 1})$ is a limit of the Cauchy sequence $(\psi(i(a_n)))_{n\geq 1} = (j\phi(a_n))_{n\geq 1}$. But we already know $\phi^c((a_n)_{n\geq 1})$ is a limit for this Cauchy sequence, and since A^c is separated we conclude that $\psi((a_n)_{n\geq 1}) = \phi^c((a_n)_{n\geq 1})$, as required. □

Remark 2. If $\phi : A \longrightarrow B$ is a morphism of linear topological rings then the morphism of linear topological rings $\phi^c : A^c \longrightarrow B^c$ is the unique morphism of linear topological rings making (2) commute.

Lemma 5. *Let A be a linear topological group. The morphism of linear topological groups $i : A \longrightarrow A^c$ is an isomorphism* of linear topological groups *if and only if A is separated and complete.*

Proof. That is, if $i : A \longrightarrow A^c$ is an isomorphism of abelian groups, it is also a homeomorphism. It suffices to show that if U is an open subgroup of A, then $i(U)$ is an open subgroup of A^c. Suppose we could show that the topological group U were separated and complete. Then the morphism $U \longrightarrow U^c$ would be bijective and it would follow that $i(U) = U^c$ is open. Since U is trivially separated, it suffices to show that a Cauchy sequence $(u_n)_{n\geq 1}$ in A with $u_n \in U$ for all $n \geq 1$ must converge to an element of U. Suppose to the contrary that $u_n \longrightarrow x$, where $x \notin U$. Then the open set $x + U$ must be disjoint from U, so it is impossible for $(u_n)_{n\geq 1}$ to converge to x. Therefore $x \in U$ and the proof is complete. □

Definition 16. Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal. Then we have a sequence of ideals

$$A \supseteq \mathfrak{a} \supseteq \mathfrak{a}^2 \supseteq \cdots \supseteq \mathfrak{a}^n \supseteq \cdots$$

With the topology of Definition 12, A is a linear topological ring (TR,Proposition 4) and $\phi : A \longrightarrow A^c$ is a morphism of linear topological rings. Let M be an A-module and consider the sequence of subgroups

$$M \supseteq \mathfrak{a}M \supseteq \mathfrak{a}^2 M \supseteq \cdots \supseteq \mathfrak{a}^n M \supseteq \cdots$$

if we give M the topology of Definition 12 then M is a topological left A-module. The completion M^c becomes a A^c-module via $(a_n) \cdot (m_n) = (a_n \cdot m_n)$. This defines an additive functor

$$(-)^c : A\mathbf{Mod} \longrightarrow A^c\mathbf{Mod}$$

where for a morphism of A-modules $f : M \longrightarrow N$ the morphism of A^c-modules $f^c : M^c \longrightarrow N^c$ is defined by $(a_n) \mapsto (f(a_n))$. This functor preserves finite products (equivalently, finite coproducts) and epimorphisms.

Let A be a topological abelian group whose topology is defined by a sequence of subgroups (1). Then for integers $m, n \geq 1$ with $m \geq n$ there is a canonical epimorphism of abelian groups $\varphi_{m,n} : A/A_m \longrightarrow A/A_n$. It is clear that $(A/A_n, \varphi_{m,n})$ is an inverse system.

Fix $m \geq 1$ and a Cauchy sequence $\alpha = (a_n)$ in A. The residues $a_1 + A_m, a_2 + A_m, \cdots$ eventually stabilise to some element $\sigma_m(\alpha) \in A/A_m$, and this gives a well-defined morphism of abelian groups $\sigma_m : A^c \longrightarrow A/A_m$. This induces an isomorphism of abelian groups

$$\begin{aligned} \theta : A^c &\longrightarrow \varprojlim A/A_n \\ \alpha &\mapsto (\sigma_n(\alpha)) \end{aligned}$$

In particular if A is a ring and $\mathfrak{a} \subseteq A$ an ideal, with $A_n = \mathfrak{a}^n$ for $n \geq 1$ then for $m \geq n$ the morphism $A/\mathfrak{a}^m \longrightarrow A/\mathfrak{a}^n$ is a morphism of rings, so $\varprojlim A/\mathfrak{a}^n$ acquires a canonical ring structure. It is not hard to check that we have an isomorphism of rings

$$\begin{aligned} \theta : A^c &\longrightarrow \varprojlim A/\mathfrak{a}^n \\ \alpha &\mapsto (\sigma_n(\alpha)) \end{aligned}$$

The ring morphisms $A \longrightarrow A/\mathfrak{a}^n$ induce a morphism of rings $A \longrightarrow \varprojlim A/\mathfrak{a}^n$ defined by $a \mapsto (a + \mathfrak{a}^n)$ fitting into the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A^c \\ & \searrow & \Downarrow \theta \\ & & \varprojlim A/\mathfrak{a}^n \end{array}$$

If M is an A-module with $A_n = \mathfrak{a}^n M$ then $\varprojlim M/\mathfrak{a}^n M$ becomes a $\varprojlim A/\mathfrak{a}^n$-module via $(r_n + \mathfrak{a}^n) \cdot (x_n + \mathfrak{a}^n M) = (r_n \cdot x_n + \mathfrak{a}^n M)$. If $f : M \longrightarrow N$ is a morphism of A-modules then $(x_n + \mathfrak{a}^n M) \mapsto (f(x_n) + \mathfrak{a}^n M)$ defines a morphism of $\varprojlim A/\mathfrak{a}^n$-modules $\varprojlim M/\mathfrak{a}^n M \longrightarrow \varprojlim N/\mathfrak{a}^n N$. In this case we have an isomorphism of abelian groups compatible with the ring isomorphism $A^c \cong \varprojlim A/\mathfrak{a}^n$

$$\begin{aligned} \theta : M^c &\longrightarrow \varprojlim M/\mathfrak{a}^n M \\ \alpha &\mapsto (\sigma_n(\alpha)) \end{aligned}$$

Moreover this isomorphism is natural in M, in the sense that if $f : M \longrightarrow N$ is a morphism of A-modules then the following diagram commutes

$$\begin{array}{ccc} M^c & \Longrightarrow & \varprojlim M/\mathfrak{a}^n M \\ {\scriptstyle f^c}\downarrow & & \downarrow \\ N^c & \Longrightarrow & \varprojlim N/\mathfrak{a}^n N \end{array}$$

Definition 17. Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal. Then we denote the ring $\varprojlim A/\mathfrak{a}^n$ by $\widehat{A}$ and call it the $\mathfrak{a}$-*adic completion* of A. So there is a canonical ring morphism $A \longrightarrow \widehat{A}$. If M is an A-module then the $\widehat{A}$-module $\varprojlim M/\mathfrak{a}^n M$ is called the $\mathfrak{a}$-*adic completion* of M and is denoted $\widehat{M}$. Completion defines an additive functor

$$\widehat{(-)} : A\mathbf{Mod} \longrightarrow \widehat{A}\mathbf{Mod}$$

There is a canonical morphism of A-modules $M \longrightarrow \widehat{M}$ defined by $m \mapsto (m + \mathfrak{a}^n M)$. This induces a morphism of $\widehat{A}$-modules $M \otimes_A \widehat{A} \longrightarrow \widehat{M}$ natural in M.

Theorem 6. *Let A be a noetherian ring and $\mathfrak{a}$ an ideal of A. Then with all completions $\mathfrak{a}$-adic, we have*

(a) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of finitely generated A-modules. Then the sequence $0 \longrightarrow \widehat{M'} \longrightarrow \widehat{M} \longrightarrow \widehat{M''} \longrightarrow 0$ of $\widehat{A}$-modules is exact.

(b) If M is finitely generated then $M \otimes_A \widehat{A} \longrightarrow \widehat{M}$ is an isomorphism.

(c) $\widehat{A}$ is a flat noetherian A-algebra.

(d) If $\mathfrak{b}$ is an ideal of A then $\widehat{\mathfrak{b}} \longrightarrow \widehat{A}$ is a monomorphism with image $\mathfrak{b}\widehat{A}$. A sequence $(a_n) \in \varprojlim A/\mathfrak{a}^n$ belongs to $\mathfrak{b}\widehat{A}$ if and only if $a_n \in (\mathfrak{b} + \mathfrak{a}^n)/\mathfrak{a}^n$ for $n \geq 1$. If there is no chance of confusion we denote this ideal simply by $\widehat{\mathfrak{b}}$.

(e) For any ideal $\mathfrak{b}$ and $n \geq 1$ we have $\widehat{\mathfrak{b}}^n = \widehat{\mathfrak{b}^n}$ and the canonical ring morphism $A/\mathfrak{a}^n \longrightarrow \widehat{A}/\widehat{\mathfrak{a}}^n$ is an isomorphism.

(f) Let $\mathfrak{b}$ be an ideal with $\mathfrak{b} \supseteq \mathfrak{a}^n$ for some $n \geq 1$. Then the isomorphism $A/\mathfrak{a}^n \cong \widehat{A}/\widehat{\mathfrak{a}}^n$ identifies the ideals $\mathfrak{b}/\mathfrak{a}^n$ and $\widehat{\mathfrak{b}}/\widehat{\mathfrak{a}}^n$. Therefore we have a canonical ring isomorphism $A/\mathfrak{b} \cong \widehat{A}/\widehat{\mathfrak{b}}$ for any ideal $\mathfrak{b}$ open in the $\mathfrak{a}$-adic topology.

(g) The ideal $\widehat{\mathfrak{a}}$ is contained in the Jacobson radical of $\widehat{A}$. In particular if $x \in \widehat{\mathfrak{a}}$ then $1 - x$ is a unit. An element $y \in \widehat{A}$ is a unit if and only if the image in $\widehat{A}/\widehat{\mathfrak{a}}$ is a unit.

(h) The ring isomorphism $\widehat{A} \cong A^c$ is an isomorphism of linear topological rings, where we give $\widehat{A}$ the $\widehat{\mathfrak{a}}$-adic topology. Therefore $\widehat{A}$ is separated and complete.

Proof. (a) A & M Chapter 10, Proposition 10.12. (b) A & M Chapter 10, Proposition 10.13. (c) A & M Chapter 10, Proposition 10.14 and Theorem 10.26. (d) Since $\mathfrak{b}$ is finitely generated, it follows from $(b), (c)$ that $\widehat{\mathfrak{b}} \longrightarrow \widehat{A}$ is a monomorphism with image equal to the image of $\mathfrak{b} \otimes_A \widehat{A} \longrightarrow \widehat{A}$, which is $\mathfrak{b}\widehat{A}$. It is clear that if a sequence (a_n) belongs to the ideal $\mathfrak{b}\widehat{A}$ then it must have the stated form. The converse is not trivial! It is a consequence of the Artin-Rees Lemma (A & M Theorem 10.11 to be precise) that the filtrations $\mathfrak{a}^n\mathfrak{b}$ and $\mathfrak{a}^n \cap \mathfrak{b}$ of the ideal $\mathfrak{b}$ have bounded difference. That is, there exists an integer $k \geq 0$ such that $\mathfrak{a}^{n+k} \cap \mathfrak{b} \subseteq \mathfrak{a}^n\mathfrak{b}$ for all $n \geq 1$.

Given a sequence $(b_n + \mathfrak{a}^n)$ of $\widehat{A}$ with $b_n \in \mathfrak{b}$ for $n \geq 1$ we can replace b_n by b_{n+k} for $n \geq 1$ without changing the sequence (since $b_{n+k} - b_n \in \mathfrak{a}^n$ by definition). The effect of this is to reduce to the case where $b_m - b_n \in \mathfrak{a}^{n+k}$ for all $m \geq n$. Therefore by construction of k, we have $b_m - b_n \in \mathfrak{a}^n\mathfrak{b}$ for $m \geq n$, which shows that $(b_n + \mathfrak{a}^n\mathfrak{b})$ is a well-defined element of the completion $\widehat{\mathfrak{b}} = \varprojlim_n \mathfrak{b}/\mathfrak{a}^n\mathfrak{b}$. The image of this element under $\widehat{\mathfrak{b}} \longrightarrow \widehat{A}$ is our original sequence, so the proof is complete. (e) These claims follow from A & M Chapter 10, Proposition 10.15. (f) It is clear that the isomorphism $A/\mathfrak{a}^n \cong \widehat{A}/\widehat{\mathfrak{a}}^n$ maps the ideal $\mathfrak{b}/\mathfrak{a}^n$ into the ideal $\widehat{\mathfrak{b}}/\widehat{\mathfrak{a}}^n$. Conversely, let $(b_m + \mathfrak{a}^m)$ be a sequence in $\widehat{\mathfrak{b}}$ with $b_m \in \mathfrak{b}$ for $m \geq 1$. Using the same trick as in (d) we can assume that $b_m - b_t \in \mathfrak{a}^{t+n}$ for $m \geq t$. Set $b = b_n$ and observe that since $b_m - b = b_m - b_n \in \mathfrak{a}^n$ (for $m \geq n$ this is trivial, and for $n > m$ we have $b_m - b_n = -(b_n - b_m) \in \mathfrak{a}^{m+n} \subseteq \mathfrak{a}^n$) we have $(b_m) - (b) = (b_m - b) \in \widehat{\mathfrak{a}}^n$. This shows that every element of the ideal $\widehat{\mathfrak{b}}/\widehat{\mathfrak{a}}^n$ is in the image of $\mathfrak{b}/\mathfrak{a}^n$, as required. The other claim now follows easily.

(g) Follows immediately from A & M Proposition 10.15.

(h) Using (d) we see that $A^c \cong \widehat{A}$ identifies the ideals $\mathfrak{b}^c$ and $\widehat{\mathfrak{b}}$ for any open ideal $\mathfrak{b}$ of A. So it is clear that $A^c \cong \widehat{A}$ is an isomorphism of linear topological rings. We have shown in our A & M notes that A^c is separated and complete, so the same can be said of $\widehat{A}$. □

3 Adic Rings

Definition 18. Let A be a topological ring. An element $x \in A$ is *topologically nilpotent* if 0 is a limit of the sequence $(x^n)_{n \geq 1}$.

Definition 19. If A is a linear topological ring and $\mathfrak{a} \subseteq A$ an ideal, then we say $\mathfrak{a}$ is an *ideal of definition* if $\mathfrak{a}$ is open and if for each open neighborhood V of 0, there is an integer $n > 0$ such that $\mathfrak{a}^n \subseteq V$. We say that a linear topological ring A is *preadmissible* if there exists an ideal of definition in A. We say that A is *admissible* if is preadmissible and if it is separated and complete.

It is clear that if $\mathfrak{a}$ is an ideal of definition in a linear topological ring A, and $\mathfrak{b}$ any open ideal of A, then $\mathfrak{a} \cap \mathfrak{b}$ is also an ideal of definition. The ideals of definition in a preadmissible ring A form a fundamental system of neighborhoods of 0.

Lemma 7. *Let A be a linear topological ring. Then*

(i) An element $x \in A$ is topologically nilpotent if and only if for each open ideal $\mathfrak{b}$ of A the image of x in $A/\mathfrak{b}$ is nilpotent. The set $\mathcal{I}$ of topological nilpotents in A is an ideal.

(ii) Suppose A is preadmissible and that $\mathfrak{a}$ is an ideal of definition of A. An element $x \in A$ is topologically nilpotent if and only if the image in $A/\mathfrak{a}$ is nilpotent. The ideal $\mathcal{I}$ is the inverse image of the nilradical of $A/\mathfrak{a}$ and is an open ideal.

Proof. (i) Immediate from the definitions. To prove (ii) it suffices to remark that for each open neighborhood V of 0 in A, there exists $n > 0$ with $\mathfrak{a}^n \subseteq V$. If $x \in A$ is such that $x^m \in \mathfrak{a}$, then $x^{mq} \in V$ for $q \geq n$, therefore x is topologically nilpotent. □

Proposition 8. *Let A be a preadmissible ring, $\mathfrak{a}$ an ideal of definition for A. Then*

(i) For an open ideal $\mathfrak{b}$ of A to be an ideal of definition, it is necessary and sufficient that there exist $n > 0$ with $\mathfrak{b}^n \subseteq \mathfrak{a}$.

(ii) For $x \in A$ to be contained in an ideal of definition, it is necessary and sufficient that x be topologically nilpotent.

Proof. (i) If $\mathfrak{b}^n \subseteq \mathfrak{a}$, then for an open neighborhood V of 0, there exists m such that $\mathfrak{a}^m \subseteq V$, therefore $\mathfrak{b}^{mn} \subseteq V$. (ii) The condition is clearly necessary. To see it is sufficient, assume that x is topologically nilpotent. Set $\mathfrak{b} = \mathfrak{a} + Ax$. It is clear that $\mathfrak{b}$ is an open ideal, and if $n \geq 1$ is such that $x^n \in \mathfrak{a}$ then $\mathfrak{b}^n \subseteq \mathfrak{a}$ and therefore by (a), $\mathfrak{b}$ is an ideal of definition. □

Corollary 9. *If A is a preadmissible ring and $\mathfrak{p}$ an open prime ideal, then $\mathfrak{p}$ contains every ideal of definition of A.*

Corollary 10. *Let A be a preadmissible ring. Then the following properties of an ideal of definition $\mathfrak{a}$ are equivalent:*

(a) $\mathfrak{a}$ contains every other ideal of definition (we say it is the largest *ideal of definition);*

(b) $\mathfrak{a}$ is not properly contained in any other ideal of definition;

(c) The ring $A/\mathfrak{a}$ is reduced.

For there to exist an ideal of definition $\mathfrak{a}$ with these properties, it is necessary and sufficient that the nilradical of $A/\mathfrak{b}$ be nilpotent for some ideal of definition $\mathfrak{b}$. In that case $\mathfrak{a}$ is equal to the ideal $\mathcal{I}$ of topological nilpotents of A and is therefore unique.

Proof. It is clear that $(a) \Rightarrow (b)$. $(b) \Rightarrow (c)$ If $\mathfrak{a}$ is maximal among all ideals of definition then from the proof of Proposition 8 we deduce that $\mathfrak{a}$ must contain every topological nilpotent of A and therefore by Lemma 7 the ring $A/\mathfrak{a}$ must be reduced. $(c) \Rightarrow (a)$ If $A/\mathfrak{a}$ is reduced then $\mathcal{I} \subseteq \mathfrak{a}$. But by Proposition 8 every ideal of definition is contained in $\mathcal{I}$. This implies (a) and also shows that $\mathfrak{a} = \mathcal{I}$.

We have already shown that if an ideal of definition $\mathfrak{a}$ with these equivalent properties exists, then $A/\mathfrak{a}$ has nilpotent nilradical and $\mathfrak{a} = \mathcal{I}$. Now suppose that $\mathfrak{b}$ is an ideal of definition such that $A/\mathfrak{b}$ has nilpotent nilradical. This implies that for some $n \geq 1$, $\mathcal{I}^n \subseteq \mathfrak{b}$. Therefore by Proposition 8(i) the open ideal $\mathcal{I}$ is an ideal of definition. It clearly has the required property. □

Corollary 11. *A noetherian preadmissible ring admits a largest ideal of definition.*

Corollary 12. *If A is a preadmissible ring and $\mathfrak{a}$ an ideal of definition such that the powers $\mathfrak{a}^n$ $(n > 0)$ form a fundamental system of open neighborhoods of 0, then so do the powers $\mathfrak{b}^n$ for any other ideal of definition $\mathfrak{b}$.*

Proof. Suppose that $\mathfrak{a}$ is an ideal of definition. The set $\{\mathfrak{a}^n \,|\, n \geq 1\}$ is a fundamental system of open neighborhoods of 0 if and only if $\mathfrak{a}^n$ is open for $n \geq 1$. If this is the case, then an arbitrary ideal $\mathfrak{b}$ is open if and only if $\mathfrak{a}^n \subseteq \mathfrak{b}$ for some $n \geq 1$. In particular, the product of open ideals is open. So it is clear that if $\mathfrak{b}$ is another ideal of definition, the powers $\mathfrak{b}^n$ must form a fundamental system of open neighborhoods of 0. □

Definition 20. We say a preadmissible ring A is *preadic* if there exists an ideal of definition $\mathfrak{a}$ in A such that all the powers $\mathfrak{a}^n$ $(n > 0)$ are open (equivalently, the set $\{\mathfrak{a}^n \,|\, n \geq 1\}$ is a fundamental system of neighborhoods of 0). We say a ring is *adic* if it is a preadic ring which is separated and complete.

Remark 3. Let A be a preadic ring. It follows from Corollary 12 that for any ideal of definition $\mathfrak{a}$ the powers $\mathfrak{a}^n$ are open and therefore $\{\mathfrak{a}^n \,|\, n \geq 1\}$ is a fundamental system of open neighborhoods of 0. In other words, the topology on A is the one arising from the following subgroup sequence (see Definition 12)

$$A \supseteq \mathfrak{a} \supseteq \mathfrak{a}^2 \supseteq \cdots \supseteq \mathfrak{a}^n \supseteq \cdots$$

We therefore say that A is $\mathfrak{a}$-*preadic* (or $\mathfrak{a}$-*adic* if A is adic) and we say that the topology on A is the $\mathfrak{a}$-*preadic* (resp. $\mathfrak{a}$-*adic*) topology. Since the ideal of defintion $\mathfrak{a}$ is arbitrary, the preadic topology does not depend on the choice of ideal of definition.

Let $\mathfrak{b}$ be an arbitrary ideal. Then $\mathfrak{b}$ is open if and only if $\mathfrak{a}^n \subseteq \mathfrak{b}$ for some $n \geq 1$, and $\mathfrak{b}$ is an ideal of definition if and only if there exists integers $m, n \geq 1$ with $\mathfrak{b} \supseteq \mathfrak{a}^n \supseteq \mathfrak{b}^m$. If $\mathfrak{b}, \mathfrak{c}$ are open ideals in A then the ideals $\mathfrak{b}\mathfrak{c}, \mathfrak{b} + \mathfrak{c}$ and $\mathfrak{b} \cap \mathfrak{c}$ are also open.

The set $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ of ideals of definition in A is a fundamental system of neighborhoods of 0. Given any ideal of definition $\mathfrak{a}$, the powers $\mathfrak{a}^n$ are open and are therefore themselves ideals of definition by Proposition 8(i). It is easy to see that the set $\{\mathfrak{a}^n \,|\, n \geq 1\}$ is a final subset of the inverse directed set $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$.

Example 1. Given a ring A and an ideal $\mathfrak{a} \subseteq A$ we make A into a topological ring as in Definition 16. Then it is clear that A is a preadic ring with ideal of definition $\mathfrak{a}$.

3.1 Complete Rings of Fractions

Lemma 13. *Let A be a ring, $\{\mathfrak{b}_\mu\}_{\mu \in \Lambda}$ a nonempty set of ideals in A. Suppose that for every pair of indices μ, λ there exists τ with $\mathfrak{b}_\tau \subseteq \mathfrak{b}_\mu \cap \mathfrak{b}_\lambda$. Then there is a unique topology on A making A into a topological ring in such a way that $\{\mathfrak{b}_\mu\}$ is a fundamental system of neighborhoods of 0.*

Proof. Follows immediately from (TR,Proposition 4). □

Let A be a preadmissible ring, $(\mathfrak{a}_\lambda)_{\lambda \in \Lambda}$ the fundamental system of open neighborhoods of 0 consisting of all ideals of definition of A. Let $u_\lambda : A \longrightarrow A_\lambda = A/\mathfrak{a}_\lambda$ be the canonical ring morphism, and for $\mathfrak{a}_\mu \subseteq \mathfrak{a}_\lambda$ let $u_{\lambda\mu} : A_\mu \longrightarrow A_\lambda$ be the canonical ring morphism. Set $S_\lambda = u_\lambda(S)$ and observe that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The $u_{\lambda\mu}$ induce ring morphisms $S_\mu^{-1}A_\mu \longrightarrow S_\lambda^{-1}A_\lambda$, and these form an inverse system of rings. We denote by $A\{S^{-1}\}$ the inverse limit of this inverse system. The morphisms $A \longrightarrow A_\lambda \longrightarrow S_\lambda^{-1}A_\lambda$ induce a ring morphism $\ell : A \longrightarrow A\{S^{-1}\}$.

Let $\mathfrak{b}$ be an ideal of A. For $\mu \in \Lambda$ we have the ideal $\mathfrak{b}'_\mu = (\mathfrak{b} + \mathfrak{a}_\mu)/\mathfrak{a}_\mu$ of A_μ. If $\mathfrak{a}_\mu \subseteq \mathfrak{a}_\lambda$ it is clear that $u_{\lambda\mu}(\mathfrak{b}'_\mu) = \mathfrak{b}'_\lambda$. Let $\mathfrak{b}_\mu$ denote the ideal $\mathfrak{b}'_\mu(S_\mu^{-1}A_\mu)$. The ring morphism $S_\mu^{-1}A_\mu \longrightarrow S_\lambda^{-1}A_\lambda$ maps the ideal $\mathfrak{b}_\mu$ onto the ideal $\mathfrak{b}_\lambda$. Therefore these ideals form an inverse system, and we denote by $\mathfrak{b}\{S^{-1}\}$ the ideal of $A\{S^{-1}\}$ given by the inverse limit $\varprojlim \mathfrak{b}_\mu$. If $\mathfrak{b} \subseteq \mathfrak{c}$ then $\mathfrak{b}\{S^{-1}\} \subseteq \mathfrak{c}\{S^{-1}\}$ and it is clear that $\ell(\mathfrak{b}) \subseteq \mathfrak{b}\{S^{-1}\}$.

In particular we have the ideals $\mathfrak{a}_\lambda\{S^{-1}\}$. The set $\{\mathfrak{a}_\lambda\{S^{-1}\}\}_{\lambda\in\Lambda}$ satisfies the conditions of Lemma 13 and therefore $A\{S^{-1}\}$ is canonically a topological ring with $\{\mathfrak{a}_\lambda\{S^{-1}\}\}_{\lambda\in\Lambda}$ a fundamental system of neighborhoods of 0. The ring morphism $\ell : A \longrightarrow A\{S^{-1}\}$ is continuous, and if $\mathfrak{b}$ is an open ideal of A then $\mathfrak{b}\{S^{-1}\}$ is an open ideal of $A\{S^{-1}\}$.

Definition 21. Let A be a preadmissible ring, $S \subseteq A$ a multiplicatively closed subset. We have defined a topological ring $A\{S^{-1}\}$ together with a continuous morphism of rings $A \longrightarrow A\{S^{-1}\}$. We call $A\{S^{-1}\}$ the *complete localisation of A with respect to S*. If $f \in A$ and $S = \{1, f, f^2, \dots\}$ then we denote $A\{S^{-1}\}$ by $A_{\{f\}}$. In this case if $\mathfrak{b}$ is an ideal of A we write $\mathfrak{b}_{\{f\}}$ for $\mathfrak{b}\{S^{-1}\}$.

Continuing the above discussion, we also have the ideals $S^{-1}\mathfrak{a}_\lambda = \mathfrak{a}_\lambda(S^{-1}A)$ of the ring $S^{-1}A$, and this family of ideals also satisfies the conditions of Lemma 13. Therefore $S^{-1}A$ becomes a topological ring with $\{S^{-1}\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ a fundamental system of neighborhoods of 0.

Proposition 14. *Let A be a preadmissible ring, S and $(\mathfrak{a}_\lambda)_{\lambda\in\Lambda}$ as above. Then there is a canonical isomomorphism of rings $A\{S^{-1}\} \longrightarrow \varprojlim_\lambda S^{-1}A/S^{-1}\mathfrak{a}_\lambda$.*

Proof. This follows from the following simple calculation

$$A\{S^{-1}\} = \varprojlim_\lambda S_\lambda^{-1}A_\lambda = \varprojlim_\lambda S_\lambda^{-1}(A/\mathfrak{a}_\lambda) \cong \varprojlim_\lambda S^{-1}A/S^{-1}\mathfrak{a}_\lambda$$

where $S_\lambda^{-1}(A/\mathfrak{a}_\lambda) \longrightarrow S^{-1}A/S^{-1}\mathfrak{a}_\lambda$ is the canonical isomorphism $(x+\mathfrak{a}_\lambda)/(s+\mathfrak{a}_\lambda) \mapsto x/s + \mathfrak{a}_\lambda$. Observe that for an ideal $\mathfrak{b}$ of A this isomorphism identifies $\mathfrak{b}\{S^{-1}\}$ with the inverse limit $\varprojlim_\lambda S^{-1}(\mathfrak{b}+\mathfrak{a}_\lambda)/S^{-1}\mathfrak{a}_\lambda$. □

Proposition 15. *Let A be a preadic ring, $S \subseteq A$ a multiplicatively closed subset and $\mathfrak{a}$ an ideal of definition. Then there is a canonical isomorphism of rings $A\{S^{-1}\} \longrightarrow \widehat{S^{-1}A}$, where the latter ring is the $S^{-1}\mathfrak{a}$-adic completion. If A is noetherian then for any ideal $\mathfrak{b}$ this isomorphism identifies the ideals $\mathfrak{b}\{S^{-1}\}$ and $\widehat{S^{-1}\mathfrak{b}}$ and is an isomorphism of linear topological rings.*

Proof. Let $(\mathfrak{a}_\lambda)_{\lambda\in\Lambda}$ be as above. The set $\{\mathfrak{a}^n \,|\, n \geq 1\}$ is final in the inverse directed set $(\mathfrak{a}_\lambda)_{\lambda\in\Lambda}$, so we have a canonical isomorphism of rings (be aware of the subtle difficulty when changing from $\varprojlim_\lambda$ to $\varprojlim_n$ that arises since we may have $\mathfrak{a}^n \subseteq \mathfrak{a}^m$ for $m > n$)

$$A\{S^{-1}\} \cong \varprojlim_\lambda S^{-1}A/S^{-1}\mathfrak{a}_\lambda \cong \varprojlim_n S^{-1}A/(S^{-1}\mathfrak{a})^n = \widehat{S^{-1}A}$$

Now suppose that A is noetherian and let $\mathfrak{b}$ be an ideal of A. We already know that the first isomorphism identifies $\mathfrak{b}\{S^{-1}\}$ with the ideal $\varprojlim_\lambda S^{-1}(\mathfrak{b}+\mathfrak{a}_\lambda)/S^{-1}\mathfrak{a}_\lambda$. The second isomorphism clearly identifies this latter ideal with $\varprojlim_n S^{-1}(\mathfrak{b}+\mathfrak{a}^n)/S^{-1}\mathfrak{a}^n$, which is the ideal consisting of all sequences (a_n) with $a_n \in S^{-1}(\mathfrak{b}+\mathfrak{a}^n)/S^{-1}\mathfrak{a}^n$, which by Theorem 6$(d)$ is precisely the ideal $\widehat{S^{-1}\mathfrak{b}}$. It is now not hard to check that the isomorphism $A\{S^{-1}\} \cong \widehat{S^{-1}A}$ is a homeomorphism, where we give $\widehat{S^{-1}A}$ the $\widehat{S^{-1}\mathfrak{a}}$-adic topology. □

Remark 4. Let A be a preadic ring. Let S be the multiplicatively closed set $S = \{1\}$ and write $A_{\{1\}}$ for $A\{S^{-1}\}$. Let $\mathfrak{a}$ be an ideal of definition of A. Then Proposition 15 defines a canonical isomorphism of rings $A_{\{1\}} \longrightarrow \widehat{A}$, where the completion is $\mathfrak{a}$-adic.

Corollary 16. *Let A be a preadic ring, $S \subseteq A$ a multiplicatively closed subset. If A is noetherian then $A\{S^{-1}\}$ is a flat noetherian A-algebra.*

Proof. Choose an ideal of definition $\mathfrak{a}$ for A and let $\widehat{S^{-1}A}$ denote the $S^{-1}\mathfrak{a}$-adic completion. By Theorem 6(c), $\widehat{S^{-1}A}$ is a flat noetherian A-algebra. Since the morphism $A \longrightarrow S^{-1}A$ is flat, by transitivity of flatness we see that $\widehat{S^{-1}A}$ is flat over A. The result now follows from Proposition 15. □

Proposition 17. *Let A be a noetherian preadic ring, $S \subseteq A$ a multiplicatively closed subset. Then*

(i) *If* $\mathfrak{b}$ *is an ideal then* $\mathfrak{b}\{S^{-1}\} = \mathfrak{b} \cdot A\{S^{-1}\}$.

(ii) *If* $\mathfrak{b}, \mathfrak{c}$ *are ideals then we have*

$$\begin{aligned} \mathfrak{b}\{S^{-1}\} \cdot \mathfrak{c}\{S^{-1}\} &= (\mathfrak{bc})\{S^{-1}\} \\ \mathfrak{b}\{S^{-1}\} + \mathfrak{c}\{S^{-1}\} &= (\mathfrak{b} + \mathfrak{c})\{S^{-1}\} \\ \mathfrak{b}^n\{S^{-1}\} &= (\mathfrak{b}\{S^{-1}\})^n \quad \text{for } n \geq 1 \end{aligned}$$

(iii) *If* $\mathfrak{b}$ *is an open ideal then there is a canonical isomorphism of rings* $A\{S^{-1}\}/\mathfrak{b}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{b})$. *If* $\mathfrak{c} \supseteq \mathfrak{b}$ *is another ideal then this isomorphism identifies the ideals* $\mathfrak{c}\{S^{-1}\}/\mathfrak{b}\{S^{-1}\}$ *and* $S^{-1}(\mathfrak{c}/\mathfrak{b})$.

Proof. (*i*) Choose an ideal of definition $\mathfrak{a}$. By Proposition 15 we have a commutative diagram

$$\begin{array}{ccc} A\{S^{-1}\} & \Longrightarrow & \widehat{S^{-1}A} \\ \uparrow & & \uparrow \\ A & \longrightarrow & S^{-1}A \end{array} \tag{3}$$

By Theorem 6 we have $\widehat{S^{-1}\mathfrak{b}} = S^{-1}\mathfrak{b} \cdot \widehat{S^{-1}A}$, and it is not hard to see this is the ideal generated by the image of $\mathfrak{b}$ under $A \longrightarrow S^{-1}A \longrightarrow \widehat{S^{-1}A}$. Since the top isomorphism identifies $\mathfrak{b}\{S^{-1}\}$ and $\widehat{S^{-1}\mathfrak{b}}$ it follows that $\mathfrak{b}\{S^{-1}\}$ is the smallest ideal containing the image of $\mathfrak{b}$. That is, $\mathfrak{b}\{S^{-1}\} = \mathfrak{b} \cdot A\{S^{-1}\}$. The statements of (*ii*) follow directly from this fact and elementary properties of expanding ideals.

(*iii*) Choose an ideal of definition $\mathfrak{a}$ for A. Then $\mathfrak{b}$ is open in the $\mathfrak{a}$-adic topology on A. By Proposition 15 there is a ring isomorphism $A\{S^{-1}\} \cong \widehat{S^{-1}A}$ identifying $\mathfrak{b}\{S^{-1}\}$ and $\widehat{S^{-1}\mathfrak{b}}$ (completions are $S^{-1}\mathfrak{a}$-adic). Clearly the ideal $S^{-1}\mathfrak{b}$ of $S^{-1}A$ is open under the $S^{-1}\mathfrak{a}$-adic topology, so using Theorem 6(*f*) we have a ring isomorphism

$$A\{S^{-1}\}/\mathfrak{b}\{S^{-1}\} \cong \widehat{S^{-1}A}/\widehat{S^{-1}\mathfrak{b}} \cong S^{-1}A/S^{-1}\mathfrak{b} \cong S^{-1}(A/\mathfrak{b})$$

as required. It is not hard to check that this isomorphism identifies the ideals $\mathfrak{c}\{S^{-1}\}/\mathfrak{b}\{S^{-1}\}$ and $S^{-1}(\mathfrak{c}/\mathfrak{b})$ for any ideal $\mathfrak{c}$ containing $\mathfrak{b}$. ☐

Remark 5. Let A be a noetherian preadic ring, $S \subseteq A$ a multiplicatively closed subset. Then from (3) we have a canonical morphism of rings $\psi : S^{-1}A \longrightarrow A\{S^{-1}\}$. The proof of Proposition 17 shows that if $\mathfrak{b}$ is an open ideal of A then ψ maps $S^{-1}\mathfrak{b}$ into $\mathfrak{b}\{S^{-1}\}$ and the induced ring morphism $S^{-1}A/S^{-1}\mathfrak{b} \longrightarrow A\{S^{-1}\}/\mathfrak{b}\{S^{-1}\}$ is an isomorphism. In particular, if A is discrete (that is, has the discrete topology) then $\mathfrak{b} = 0$ is open and ψ itself is an isomorphism.

Corollary 18. *Let* A *be a noetherian preadic ring and* $S \subseteq A$ *a multiplicatively closed subset. Then* $A\{S^{-1}\}$ *is a noetherian adic ring.*

Proof. We know from Corollary 16 that $A\{S^{-1}\}$ is noetherian. By definition $A\{S^{-1}\}$ is a linear topological ring. If $\mathfrak{a}$ is an ideal of definition of A then by definition $\mathfrak{a}\{S^{-1}\}$ is open. If $\{\mathfrak{a}_\lambda\}$ is the set of ideals of definition of A then $\{\mathfrak{a}_\lambda\{S^{-1}\}\}$ is a fundamental system of neighborhoods of 0 in $A\{S^{-1}\}$. Therefore if V is an open neighborhood of 0, we have $\mathfrak{a}_\lambda\{S^{-1}\} \subseteq V$ for some λ. But there is $n \geq 1$ with $\mathfrak{a}^n \subseteq \mathfrak{a}_\lambda$, and therefore $(\mathfrak{a}\{S^{-1}\})^n = \mathfrak{a}^n\{S^{-1}\} \subseteq V$. This shows that $\mathfrak{a}\{S^{-1}\}$ is an ideal of definition and since $(\mathfrak{a}\{S^{-1}\})^n = \mathfrak{a}^n\{S^{-1}\}$ all the powers are open, so $A\{S^{-1}\}$ is preadic. It only remains to show that $A\{S^{-1}\}$ is separated and complete. This follows immediately from Proposition 15. ☐

Proposition 19. *Let* A *be a noetherian preadic ring and* $S \subseteq A$ *a multiplicatively closed subset. Then*

(i) *Every open ideal of* $A\{S^{-1}\}$ *is of the form* $\mathfrak{b}\{S^{-1}\}$ *for an open ideal* $\mathfrak{b}$ *of* A.

(ii) *If* $\mathfrak{b}$ *is an open ideal of* A *then* $\mathfrak{b}\{S^{-1}\} = A\{S^{-1}\}$ *if and only if* $\mathfrak{b} \cap S \neq \emptyset$.

(iii) *The map* $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ *defines a bijection between the open prime ideals of* $A\{S^{-1}\}$ *and the open prime ideals of* A *not meeting* S.

(iv) *If* $\mathfrak{p}$ *is an open prime ideal not meeting* S *then the quotient field of* $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ *is* A*-isomorphic to the quotient field of* $A/\mathfrak{p}$.

Proof. (i) If $\mathfrak{h}$ is an open ideal of $A\{S^{-1}\}$ then $\mathfrak{h} \supseteq \mathfrak{a}\{S^{-1}\}$ for some ideal of definition $\mathfrak{a}$ of A. Then under the isomorphism $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{a})$ of Proposition 17 the ideal $\mathfrak{h}$ is identified with an ideal of $S^{-1}(A/\mathfrak{a})$. It is elementary that every such ideal is of the form $S^{-1}(\mathfrak{b}/\mathfrak{a})$ for an ideal $\mathfrak{b}$ of A containing $\mathfrak{a}$. Then $\mathfrak{b}$ is an open ideal and $\mathfrak{h} = \mathfrak{b}\{S^{-1}\}$ by Proposition 17(iii), as required.

Let $\mathcal{L}, \mathcal{L}'$ denote the partially ordered sets of open ideals of $A, A\{S^{-1}\}$ respectively. We have shown that $\mathfrak{b} \mapsto \mathfrak{b}\{S^{-1}\}$ defines a surjective map $\alpha : \mathcal{L} \longrightarrow \mathcal{L}'$. If $\ell : A \longrightarrow A\{S^{-1}\}$ is the canonical continuous morphism of rings, then $\mathfrak{h} \mapsto \ell^{-1}\mathfrak{h}$ defines a map $\beta : \mathcal{L}' \longrightarrow \mathcal{L}$. Since α is surjective, using Proposition 17(i) it is not hard to see that $(\ell^{-1}\mathfrak{h})\{S^{-1}\} = \mathfrak{h}$. That is, $\alpha\beta = 1$.

(ii) If $\mathfrak{b}$ is open then it follows from Proposition 17(iii) that $\mathfrak{b}\{S^{-1}\}$ is improper if and only if $S^{-1}(A/\mathfrak{b}) = 0$, which is if and only if $\mathfrak{b} \cap S \neq \emptyset$.

(iii) If $\mathfrak{p}$ is an open prime ideal of A then $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{p})$ so it is clear that provided $\mathfrak{p} \cap S = \emptyset$, $\mathfrak{p}\{S^{-1}\}$ is an open prime ideal of $A\{S^{-1}\}$. We claim that if $\mathfrak{b}, \mathfrak{q}$ are open ideals of A with $\mathfrak{q}$ prime and $\mathfrak{q} \cap S = \emptyset$ then $\mathfrak{b} \subseteq \mathfrak{q}$ if and only if $\mathfrak{b}\{S^{-1}\} \subseteq \mathfrak{q}\{S^{-1}\}$. One implication is trivial. For the other, let $\mathfrak{a}$ be an ideal of definition contained in the open set $\mathfrak{b} \cap \mathfrak{q}$. Using the isomorphism $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{a})$ of Proposition 17(iii) we see that $S^{-1}(\mathfrak{b}/\mathfrak{a}) \subseteq S^{-1}(\mathfrak{q}/\mathfrak{a})$ and it follows that $\mathfrak{b} \subseteq \mathfrak{q}$, as required. In particular, the map α is injective on the set of open prime ideals not meeting S. If $\mathfrak{h}$ is an open prime ideal of $A\{S^{-1}\}$ then $\mathfrak{p} = \ell^{-1}\mathfrak{h}$ is an open prime ideal of A and we already know that $\mathfrak{h} = \mathfrak{p}\{S^{-1}\}$ (by (ii) this implies that $\mathfrak{p} \cap S = \emptyset$), which shows that the map $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ defines a bijection between the set of open prime ideals of A not meeting S and the set of open prime ideals of $A\{S^{-1}\}$.

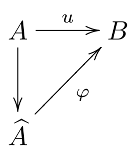
(iv) We know that there is an isomorphism of A-algebras $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{p})$ so the claim is easily checked. ☐

Proposition 20. *Let* A *be a noetherian preadic ring,* $\mathfrak{p}$ *an open prime ideal of* A *and set* $S = A \backslash \mathfrak{p}$. *Then* $A\{S^{-1}\}$ *is a local noetherian ring with residue field canonically isomorphic to the quotient field of* $A/\mathfrak{p}$.

Proof. Since $\mathfrak{p}$ is an open prime ideal we have $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{p})$, which is the quotient field of $A/\mathfrak{p}$. This shows that $\mathfrak{p}\{S^{-1}\}$ is maximal. Any other open maximal ideal of $A\{S^{-1}\}$ is of the form $\mathfrak{q}\{S^{-1}\}$ for an open prime ideal $\mathfrak{q}$ of A with $\mathfrak{q} \cap S = \emptyset$ (using Proposition 19(iii)). Therefore $\mathfrak{q} \subseteq \mathfrak{p}$ and consequently $\mathfrak{q}\{S^{-1}\} = \mathfrak{p}\{S^{-1}\}$. So to complete the proof it suffices to show that every maximal ideal of $A\{S^{-1}\}$ is open.

Let $\mathfrak{a}$ be an ideal of definition of A. We show that every maximal ideal of $A\{S^{-1}\}$ contains $\mathfrak{a}\{S^{-1}\}$. By Proposition 15 it suffices to show that every maximal ideal of $\widehat{S^{-1}A}$ contains $\widehat{S^{-1}\mathfrak{a}}$, which follows from Theorem 6(g). ☐

Proposition 21. *Let* A *be a noetherian preadic ring with ideal of definition* $\mathfrak{a}$. *If* B *is a separated, complete linear topological ring and* $u : A \longrightarrow B$ *a morphism of linear topological rings, then there is a unique morphism of linear topological rings* $\varphi : \widehat{A} \longrightarrow B$ *making the following diagram commute*

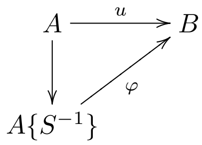


Proof. Let $\widehat{A}$ be the $\mathfrak{a}$-adic completion of A, which is a linear topological ring with the $\widehat{\mathfrak{a}}$-adic topology. It follows from Theorem 6(h) that we have an isomorphism of linear topological rings

$A^c \cong \widehat{A}$. By hypothesis and Lemma 5 the morphism $B \longrightarrow B^c$ is an isomorphism of linear topological rings.

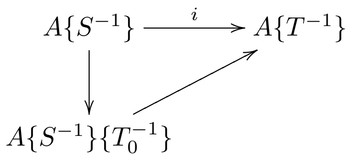
Given a morphism of topological rings $u : A \longrightarrow B$ the factorisation $\varphi : \widehat{A} \longrightarrow B$ is the composite of $\widehat{A} \cong A^c$ with $u^c : A^c \longrightarrow B^c$ and the isomorphism $B^c \cong B$. This is a morphism of linear topological rings making the diagram commute, and uniqueness follows from Lemma 4. Explicitly, the image of an element $(a_n + \mathfrak{a}^n) \in \varprojlim_n A/\mathfrak{a}^n$ under φ is the unique limit of the Cauchy sequence $(u(a_n))_{n \geq 1}$ in B. □

Proposition 22. *Let A be a noetherian preadic ring, $S \subseteq A$ a multiplicatively closed subset. If B is a separated, complete linear topological ring and $u : A \longrightarrow B$ a morphism of linear topological rings sending S to units, then there is a unique morphism of linear topological rings $\varphi : A\{S^{-1}\} \longrightarrow B$ making the following diagram commute*



Proof. Let $\mathfrak{a}$ be an ideal of definition of A. Then $A\{S^{-1}\}, S^{-1}A$ are noetherian preadic topological rings with ideals of definition $S^{-1}\mathfrak{a}, \mathfrak{a}\{S^{-1}\}$ respectively (using Corollary 18 for $A\{S^{-1}\}$). Let $u : A \longrightarrow B$ be a morphism of linear topological rings sending S to units. Then the induced morphism of rings $u' : S^{-1}A \longrightarrow B$ is easily checked to be continuous. By Proposition 21 we have an induced morphism of linear topological rings $\widehat{S^{-1}A} \longrightarrow B$ (completion is $S^{-1}\mathfrak{a}$-adic). Composing this with the isomorphism of linear topological rings $A\{S^{-1}\} \cong \widehat{S^{-1}A}$ of Proposition 15 we have our factorisation $\varphi : A\{S^{-1}\} \longrightarrow B$. Uniqueness follows from uniqueness of the factorisation in Proposition 15. □

Remark 6. Let A be a noetherian preadic ring, $S \subseteq T$ multiplicatively closed subsets of A. Then $A \longrightarrow A\{T^{-1}\}$ is a morphism of linear topological rings sending S to units, so there is a unique morphism of linear topological rings $i : A\{S^{-1}\} \longrightarrow A\{T^{-1}\}$ which is also a morphism of A-algebras. Now $A\{S^{-1}\}$ is a noetherian adic ring, and we let T_0 denote the image of T in $A\{S^{-1}\}$. The morphism i sends T_0 to units, so there is a unique morphism of linear topological rings $A\{S^{-1}\}\{T_0^{-1}\} \longrightarrow A\{T^{-1}\}$ making the following diagram commute



Using the uniqueness properties of these morphisms, it is easy to check that this is an isomorphism of linear topological rings.

Corollary 23. *Let A be a noetherian preadic ring, $S \subseteq T$ multiplicatively closed subsets of A. Then the canonical ring morphism $A\{S^{-1}\} \longrightarrow A\{T^{-1}\}$ is flat.*

Proof. This follows immediately from Remark 6 and Corollary 16, since $A\{S^{-1}\}$ is a noetherian adic ring. □

3.2 Local Completion

Definition 22. Let A be a preadmissible ring. For $f \in A$ we denote by $\mathfrak{D}(f)$ the set of all *open* prime ideals of A not containing f.

Lemma 24. *Let A be a noetherian preadic ring. If $f, g \in A$ then $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$ if and only if the ring morphism $A \longrightarrow A_{\{g\}}$ sends f to a unit.*

Proof. Let $\mathfrak{a}$ be an ideal of definition of A. Using Theorem 6(g) and Proposition 15 we see that an element of $A_{\{g\}}$ is a unit iff. its image in $A_{\{g\}}/\mathfrak{a}_{\{g\}} \cong A_g/\mathfrak{a}_g$ is a unit. Therefore f maps to a unit in $A_{\{g\}}$ if and only if $(f) + \mathfrak{a}_g = A_g$. This is clearly equivalent to the condition $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$, as required. ☐

Definition 23. Let A be a noetherian preadic ring. If $f, g \in A$ with $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$ then f maps to a unit under the morphism of linear topological rings $A \longrightarrow A_{\{g\}}$. Combining Corollary 18 and Proposition 22 we have a canonical morphism of linear topological rings $A_{\{f\}} \to A_{\{g\}}$, which is also a morphism of A-algebras. Using the uniqueness condition of Proposition 22 it is straightforward to check that this ring morphism is the one induced between the inverse limits $\varprojlim_\lambda (A/\mathfrak{a}_\lambda)_f \longrightarrow \varprojlim_\lambda (A/\mathfrak{a}_\lambda)_g$ by the canonical ring morphisms $(A/\mathfrak{a}_\lambda)_f \longrightarrow (A/\mathfrak{a}_\lambda)_g$, where $\{\mathfrak{a}_\lambda\}_\lambda$ is the inverse directed set of all ideals of definition.

Let S be a multiplicatively closed subset of A. Then the elements $f \in A$ become a directed set with the relation $f \leq g$ iff. $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$. The rings $A_{\{f\}}$ and the induced ring morphisms $A_{\{f\}} \longrightarrow A_{\{g\}}$ for $f \leq g$ are a direct system of rings. We denote the direct limit $\varinjlim_{f \in S} A_{\{f\}}$ by $A_{\{S\}}$. There is a canonical morphism of rings $A \longrightarrow A_{\{S\}}$ given by the composite $A \longrightarrow A_{\{f\}} \longrightarrow A_{\{S\}}$, which does not depend on the chosen $f \in S$. For each $f \in S$ we have by Remark 6 a canonical morphism of linear topological A-algebras $A_{\{f\}} \longrightarrow A\{S^{-1}\}$. These morphisms are compatible with the direct system, so there is an induced morphism of A-algebras $A_{\{S\}} \longrightarrow A\{S^{-1}\}$.

To prove the main result of this section, we need one technical lemma.

Remark 7. Let I be a directed set, $\{A_\mu, \varphi_{\mu\lambda}\}$ a direct system of rings and $\{M_\mu, \theta_{\mu\lambda}\}$ a direct system of abelian groups over I. Suppose that M_λ is an A_λ-module for every λ in such a way that $\theta_{\mu\lambda}(r \cdot m) = \varphi_{\mu\lambda}(r) \cdot \theta_{\mu\lambda}(m)$ for $r \in A_\mu, m \in M_\mu$ and $\mu \leq \lambda$. Then $M = \varinjlim_\lambda M_\lambda$ is a $A = \varinjlim_\lambda A_\lambda$-module via $(\lambda, r) \cdot (\lambda, m) = (\lambda, r \cdot m)$.

Suppose that $\{N_\mu, \zeta_{\mu\lambda}\}$ is another direct system of modules in the above sense, with direct limit $N = \varinjlim_\lambda N_\lambda$. Then the modules $M_\mu \otimes_{A_\mu} N_\mu$ together with the morphisms $\theta_{\mu\lambda} \otimes \zeta_{\mu\lambda}$ give another direct system of modules. The canonical morphism of abelian groups $M_\mu \otimes_{A_\mu} N_\mu \longrightarrow M \otimes_A N$ is compatible with the ring morphism $A_\mu \longrightarrow A$. These morphisms form a cocone for the direct system, and this is the universal cocone (among all cocones into an A-module whose morphisms are compatible with $A_\mu \longrightarrow A$). To see this, suppose we are given an A-module G and morphisms of abelian groups $\beta_\mu : M_\mu \otimes_{A_\mu} N_\mu \longrightarrow G$ compatible with the ring morphisms $A_\mu \longrightarrow A$ and the morphisms of the direct system. It is straightforward to check that there is a well-defined morphism A-modules

$$M \otimes_A N \longrightarrow G$$
$$(\mu, m) \otimes (\lambda, n) \mapsto \beta_\tau(\theta_{\mu\tau}(m) \otimes \zeta_{\lambda\tau}(n))$$

where $\mu, \lambda \leq \tau$. This is clearly the unique factorisation of the morphisms β_μ through $M \otimes_A N$. As a particular case we can take $G = \varinjlim_\lambda (M_\lambda \otimes_{A_\lambda} N_\lambda)$. Then we get a morphism of A-modules

$$\rho : (\varinjlim_\lambda M_\lambda) \otimes_A (\varinjlim_\lambda N_\lambda) \longrightarrow \varinjlim_\lambda (M_\lambda \otimes_{A_\lambda} N_\lambda)$$
$$(\mu, m) \otimes (\lambda, n) \mapsto (\tau, \theta_{\mu\tau}(m) \otimes \zeta_{\lambda\tau}(n))$$

using the uniqueness property of this morphism, it is not difficult to see that ρ is an isomorphism of A-modules.

Lemma 25. *Let $\{A_\mu, \varphi_{\mu\lambda}\}$ and $\{M_\mu, \theta_{\mu\lambda}\}$ be as in Remark 7. If M_λ is a flat A_λ for module every $\lambda \in I$, then M is a flat A-module.*

Proof. By (TOR,Proposition 15) it suffices to show for a finitely generated ideal $\mathfrak{a}$ of A that the canonical morphism of A-modules $\mathfrak{a} \otimes_A M \longrightarrow M$ is injective. If $\mathfrak{a}$ is a finitely generated ideal, then it is easy to see that $\mathfrak{a} = \mathfrak{a}_\kappa A$ for some index κ and finitely generated ideal $\mathfrak{a}_\kappa$ of A_κ. For $\mu \geq \kappa$ we set $\mathfrak{a}_\mu = \mathfrak{a}_\kappa A_\mu$ and otherwise we set $\mathfrak{a}_\mu = 0$. With the induced morphisms, this is a direct system of modules. The A-module $\varinjlim_\lambda \mathfrak{a}_\lambda$ is canonically A-isomorphic to $\mathfrak{a}$. Using Remark 7 and flatness of the individual M_λ we have an isomorphism of A-modules

$$\mathfrak{a} \otimes_A M \cong (\varinjlim_\lambda \mathfrak{a}_\lambda) \otimes_A (\varinjlim_\lambda M_\lambda) \cong \varinjlim_\lambda (\mathfrak{a}_\lambda \otimes_{A_\lambda} M_\lambda) \cong \varinjlim_\lambda (\mathfrak{a}_\lambda M_\lambda) \cong \mathfrak{a}M$$

it follows that the multiplication $\mathfrak{a} \otimes_A M \longrightarrow M$ is injective, as required. □

Proposition 26. *Let A be a noetherian preadic ring, $\mathfrak{p}$ an open prime ideal of A and set $S = A \backslash \mathfrak{p}$. Then $A_{\{S\}}$ is a local noetherian ring and the ring morphism $A_{\{S\}} \longrightarrow A\{S^{-1}\}$ is a faithfully flat local morphism. The residue field of $A_{\{S\}}$ is canonically isomorphic to the quotient field of $A/\mathfrak{p}$.*

Proof. First we show that $A_{\{S\}}$ is a local ring. For each $f \in S$ we have the open prime ideal $\mathfrak{p}_{\{f\}}$ of $A_{\{f\}}$ and $\mathfrak{m} = \varinjlim_{f \in S} \mathfrak{p}_{\{f\}}$ is a proper ideal of $A_{\{S\}}$. We show that $\mathfrak{m}$ is the unique maximal ideal by showing that every $x \notin \mathfrak{m}$ is a unit.

Let $\mathfrak{a}$ be an ideal of definition of A contained in $\mathfrak{p}$, so that $\mathfrak{a}_{\{f\}}$ is a proper ideal of definition of $A_{\{f\}}$ for every $f \in S$. If $x \notin \mathfrak{m}$ then x is the image of $z \notin \mathfrak{p}_{\{f\}}$ for some $f \in S$. Under the ring isomorphism $A_{\{f\}}/\mathfrak{a}_{\{f\}} \cong A_f/\mathfrak{a}_f$ of Proposition 17(*iii*) the residue of z is identified with an element $a/f^n + \mathfrak{a}_f$ where $a \notin \mathfrak{p}$ and $n \geq 1$. Set $g = af$. Clearly $g \in S$ and $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$. The following diagram commutes (see the explicit construction in the proof of Proposition 21)

$$\begin{array}{ccc} A_f & \longrightarrow & A_{\{f\}} \\ \downarrow & & \downarrow \\ A_g & \longrightarrow & A_{\{g\}} \end{array}$$

Since a/f^n maps to a unit in A_g it follows that under the induced morphism $A_{\{f\}}/\mathfrak{a}_{\{f\}} \longrightarrow A_{\{g\}}/\mathfrak{a}_{\{g\}}$ the residue of z maps to a unit. Using Theorem 6(g) and Proposition 15 we see that an element of $A_{\{g\}}$ is a unit iff. its image in $A_{\{g\}}/\mathfrak{a}_{\{g\}}$ is a unit. Therefore the image of z under $A_{\{f\}} \longrightarrow A_{\{g\}}$ is a unit, and it follows immediately that x is a unit in $A_{\{S\}}$ as required. This shows that $A_{\{S\}}$ is a local ring.

Next we show that the canonical ring morphism $A_{\{S\}} \longrightarrow A\{S^{-1}\}$ is a local morphism of local rings. This amounts to showing that for $f \in S$ the morphism of A-algebras $A_{\{f\}} \longrightarrow A\{S^{-1}\}$ sends $\mathfrak{p}_{\{f\}}$ into $\mathfrak{p}\{S^{-1}\}$. This is trivial since both ideals are the expansion of $\mathfrak{p} \subseteq A$, so $A_{\{S\}} \longrightarrow A\{S^{-1}\}$ is local. By Corollary 23 the ring morphism $A_{\{f\}} \longrightarrow A\{S^{-1}\}$ is flat for every $f \in S$. Taking direct limits and using Lemma 25 we see that the ring morphism $A_{\{S\}} \longrightarrow A\{S^{-1}\}$ is flat. By (MAT,Corollary 35) it is faithfully flat and so by (MAT,Proposition 37) and Corollary 16, $A_{\{S\}}$ is noetherian. We have an injective morphism of the residue fields $A_{\{S\}}/\mathfrak{m} \longrightarrow A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$. This latter ring is isomorphic to $S^{-1}A/S^{-1}\mathfrak{p}$, so any element of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ corresponds to a residue of the form $a/f + S^{-1}\mathfrak{p}$ for some $f \in S$. It is therefore clear that $A_{\{S\}}/\mathfrak{m} \longrightarrow A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is an isomorphism of rings. Composing with the canonical isomorphism $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\} \cong S^{-1}(A/\mathfrak{p})$ we have the desired canonical isomorphism of the residue field of $A_{\{S\}}$ with the quotient field of $A/\mathfrak{p}$. □

4 Formal Schemes

4.1 Affine Formal Schemes

Remark 8. Let A be a ring. If $\mathfrak{a} \subseteq A$ is an ideal then $V(\mathfrak{a}) = Supp(A/\mathfrak{a})$. The equivalence relation $\mathfrak{a} \sim \mathfrak{b}$ iff. $V(\mathfrak{a}) = V(\mathfrak{b})$ on the ideals of A is such that each equivalence class contains a unique radical ideal. That is, $V(\mathfrak{a}) = V(\mathfrak{b})$ iff. $\mathfrak{a}, \mathfrak{b}$ have the same radical. If A is noetherian every ideal contains a power of its radical, so if $V(\mathfrak{a}) = V(\mathfrak{b})$ there are $n, m \geq 1$ with $\mathfrak{a}^n \subseteq \mathfrak{b}$ and $\mathfrak{b}^m \subseteq \mathfrak{a}$.

Let A be a preadmissible ring. If $\mathfrak{a}$ is an ideal of definition of A then $V(\mathfrak{a})$ is the set of all *open* prime ideals of A. Therefore all ideals of definition have the same radical, equal to the intersection of all open prime ideals. We denote the subspace of $SpecA$ consisting of all open prime ideals by $\mathfrak{X}$. Let $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ be the fundamental system of neighborhoods of 0 consisting of all the ideals of definition. For each λ let $\mathcal{O}_\lambda$ denote the sheaf of rings on $\mathfrak{X}$ induced by the structure sheaf of $Spec(A/\mathfrak{a}_\lambda)$. For $\mathfrak{a}_\mu \subseteq \mathfrak{a}_\lambda$ the canonical morphism $A/\mathfrak{a}_\mu \longrightarrow A/\mathfrak{a}_\lambda$ induces a morphism of sheaves of rings $u_{\lambda\mu} : \mathcal{O}_\mu \longrightarrow \mathcal{O}_\lambda$ and $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is an inverse system of sheaves of rings. Let $\mathcal{O}_{\mathfrak{X}}$ denote the inverse limit sheaf of rings, so for an open subset $U \subseteq \mathfrak{X}$ we have $\Gamma(U, \mathcal{O}_{\mathfrak{X}}) = \varprojlim_\lambda \Gamma(U, \mathcal{O}_\lambda)$.

Definition 24. Let A be a preadmissible ring. The *affine formal scheme* of A, denoted $Spf(A)$, is the closed subspace $\mathfrak{X}$ of $Spec(A)$ consisting of all open prime ideals of A, together with the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$. Therefore $Spf(A) = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a ringed space. If $f \in A$ then we let $\mathfrak{D}(f)$ denote the set of all open prime ideals of A not containing f. That is, $\mathfrak{D}(f) = \mathfrak{X} \cap D(f)$.

Remark 9. Let A be a preadmissible ring with ideal of definition $\mathfrak{a}$. Then

- If $\mathfrak{b}$ is an open ideal, then $V(\mathfrak{b}) \subseteq \mathfrak{X}$.
- If $\mathfrak{b}$ is any ideal, then $\mathfrak{b} + \mathfrak{a}$ is an open ideal, and $V(\mathfrak{b}) \cap \mathfrak{X} = V(\mathfrak{b}) \cap V(\mathfrak{a}) = V(\mathfrak{b} + \mathfrak{a})$.
- Therefore the subspace topology on $\mathfrak{X}$ is equal to the following topology: the closed subsets of $\mathfrak{X}$ are of the form $V(\mathfrak{b})$ for open ideals $\mathfrak{b}$ of A.

Definition 25. Let A, B be preadmissible rings, $\phi : A \longrightarrow B$ a continuous morphism of rings. Then the induced map of spaces $\Phi : Spf(B) \longrightarrow Spf(A)$ defined by $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ is continuous. Let $\{\mathfrak{a}_\lambda\}_\lambda$ and $\{\mathfrak{b}_\alpha\}_\alpha$ be the ideals of definition of A, B respectively. Given α, let λ be such that $\mathfrak{a}_\lambda \subseteq \phi^{-1}\mathfrak{b}_\alpha$. Then we have an induced morphism of schemes $i : Spec(B/\mathfrak{b}_\alpha) \longrightarrow Spec(A/\mathfrak{a}_\lambda)$ making the following diagram of topological spaces commute

$$\begin{array}{ccc} Spf(B) & \xrightarrow{\Phi} & Spf(A) \\ {\scriptstyle g_\alpha}\Big\Uparrow & & \Big\Uparrow{\scriptstyle f_\lambda} \\ Spec(B/\mathfrak{b}_\alpha) & \xrightarrow[i]{} & Spec(A/\mathfrak{a}_\lambda) \end{array}$$

Pushing forward along f_λ the morphism of sheaves of rings $i^\#$ gives a morphism of sheaves of rings $\mathcal{O}_{A,\lambda} \longrightarrow \Phi_*\mathcal{O}_{B,\alpha}$ on $Spf(A)$ (where $\mathcal{O}_{B,\alpha} = (g_\alpha)_*\mathcal{O}_{Spec(B/\mathfrak{b}_\alpha)}$ and $\mathcal{O}_{A,\lambda} = (f_\lambda)_*\mathcal{O}_{Spec(A/\mathfrak{a}_\lambda)}$). It is not difficult to see that the morphism of sheaves of rings

$$\mathcal{O}_{Spf(A)} \longrightarrow \Phi_*\mathcal{O}_{B,\alpha} = \varprojlim_\lambda \mathcal{O}_{A,\lambda} \longrightarrow \mathcal{O}_{A,\lambda} \longrightarrow \Phi_*\mathcal{O}_{B,\alpha}$$

does not depend on the chosen ideal of definition $\mathfrak{a}_\lambda$ contained in $\phi^{-1}\mathfrak{b}_\alpha$. Therefore we have a canonical morphism of sheaves of rings $\Phi^\# : \mathcal{O}_{Spf(A)} \longrightarrow \varprojlim_\alpha \Phi_*\mathcal{O}_{B,\alpha} = \Phi_*\mathcal{O}_{Spf(B)}$. We denote by $Spf(\phi)$ the morphism of ringed spaces $(\Phi, \Phi^\#) : Spf(B) \longrightarrow Spf(A)$. Clearly $Spf(1) = 1$ and $Spf(\phi\psi) = Spf(\psi) \circ Spf(\phi)$.

Lemma 27. *Let A be a preadmissible ring, $f \in A$. Then $\mathfrak{D}(f) = \emptyset$ if and only if f is topologically nilpotent.*

Proof. Let $\mathfrak{a}$ be an ideal of definition. By Lemma 7(ii) the ideal $\mathcal{I}$ of all topological nilpotents is equal to the intersection of all open prime ideals of A. So it is clear that $\mathfrak{D}(f) = \emptyset$ if and only if $f \in \mathcal{I}$. □

Proposition 28. *Let A be a noetherian preadic ring. For $f \in A$ the canonical morphism of ringed spaces $Spf(A_{\{f\}}) \longrightarrow Spf(A)$ is an open immersion with image $\mathfrak{D}(f)$.*

Proof. In Definition 25 we associated to the continuous ring morphism $\phi : A \longrightarrow A_{\{f\}}$ a morphism of ringed spaces $\Phi : Spf(A_{\{f\}}) \longrightarrow Spf(A)$. We claim that this map induces a homeomorphism of $Spf(A_{\{f\}})$ with the open subset $\mathfrak{D}(f)$ of $Spf(A)$ and that the induced morphism of ringed spaces $(Spf(A_{\{f\}}), \mathcal{O}_{Spf(A_{\{f\}})}) \longrightarrow (\mathfrak{D}(f), \mathcal{O}_{Spf(A)}|_{\mathfrak{D}(f)})$ is an isomorphism.

It follows from Proposition 19 that Φ induces a bijection $\Phi' : Spf(A_{\{f\}}) \longrightarrow \mathfrak{D}(f)$ (with inverse $\mathfrak{p} \mapsto \mathfrak{p}_{\{f\}}$). Using Remark 9, Proposition 19(i) and the proof of Proposition 19(iii) we see that this map is a homeomorphism. It remains to show that the morphism $\Phi^{\#}|_U : \mathcal{O}_{Spf(A)}|_U \longrightarrow \Phi_* \mathcal{O}_{Spf(A_{\{f\}})}|_U$ of sheaves of rings on $U = \mathfrak{D}(f)$ is an isomorphism.

Let $\mathfrak{a}$ be an ideal of definition of A. Then $\mathfrak{a}_{\{f\}}$ is an ideal of definition of $A_{\{f\}}$ by the proof of Corollary 18. Therefore the powers $\{\mathfrak{a}^n \,|\, n \geq 1\}$ and $\{\mathfrak{a}^n_{\{f\}} \,|\, n \geq 1\}$ form final subsets of the fundamental systems of ideals of definition in $A, A_{\{f\}}$ respectively. Let $\mathcal{O}_{A,\mathfrak{a}^n}$ denote the pushforward along the homeomorphism $Spec(A/\mathfrak{a}^n) \longrightarrow Spf(A)$ of the structure sheaf, and define $\mathcal{O}_{A_{\{f\}},\mathfrak{a}^n_{\{f\}}}$ similarly. We reduce to showing that the canonical morphism $\mathcal{O}_{A,\mathfrak{a}^n} \longrightarrow \Phi_* \mathcal{O}_{A_{\{f\}},\mathfrak{a}^n_{\{f\}}}$ restricts to an isomorphism on U. That is, we have to show that for $n \geq 1$ the ring morphism $A/\mathfrak{a}^n \longrightarrow A_{\{f\}}/\mathfrak{a}^n_{\{f\}}$ induces an open immersion $Spec(A_{\{f\}}/\mathfrak{a}^n_{\{f\}}) \longrightarrow Spec(A/\mathfrak{a}^n)$. This is immediate, since by Proposition 17(iii) we have $A_{\{f\}}/\mathfrak{a}^n_{\{f\}} \cong (A/\mathfrak{a}^n)_f$. Therefore $Spf(A_{\{f\}}) \longrightarrow Spf(A)$ is an open immersion and the proof is complete. □

Proposition 29. *Let A be a noetherian preadic ring with affine formal scheme $\mathfrak{X} = Spf(A)$. For $f \in A$ there is a canonical isomorphism of rings $\Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}) \cong A_{\{f\}}$. For $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$ the following diagram commutes*

$$\begin{array}{ccc} \Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}) & \Longrightarrow & A_{\{f\}} \\ \downarrow & & \downarrow \\ \Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{X}}) & \Longrightarrow & A_{\{g\}} \end{array} \tag{4}$$

Proof. Let $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ be the inverse directed set of all ideals of definition of A. For $\lambda \in \Lambda$ let $f_\lambda : Spec(A/\mathfrak{a}_\lambda) \longrightarrow \mathfrak{X} = V(\mathfrak{a}_\lambda)$ be the canonical homeomorphism. Then $\mathcal{O}_\lambda = (f_\lambda)_* \mathcal{O}_{Spec(A/\mathfrak{a}_\lambda)}$ and for $f \in A$ this homeomorphism identifies $D(f + \mathfrak{a}_\lambda)$ with $\mathfrak{D}(f) \subseteq \mathfrak{X}$. Therefore for $f \in A$ we have a canonical isomorphism of rings

$$\Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}) = \varprojlim_{\lambda} \Gamma(\mathfrak{D}(f), \mathcal{O}_\lambda) = \varprojlim_{\lambda} \Gamma(D(f + \mathfrak{a}_\lambda), \mathcal{O}_{Spec(A/\mathfrak{a}_\lambda)}) \cong \varprojlim_{\lambda} (A/\mathfrak{a}_\lambda)_f = A_{\{f\}}$$

Commutativity of (4) is easily checked. In particular for any ideal of definition $\mathfrak{a}$ there is a canonical isomorphism of rings $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \cong A_{\{1\}} \cong \widehat{A}$, where the completion is $\mathfrak{a}$-adic. □

Corollary 30. *Let A be a noetherian preadic ring. Then the affine formal scheme $\mathfrak{X} = Spf(A)$ is a locally ringed space. For an open prime ideal $\mathfrak{p} \in \mathfrak{X}$ the local ring $\mathcal{O}_{\mathfrak{X},\mathfrak{p}}$ is noetherian with residue field canonically isomorphic to $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.*

Proof. Let $\mathfrak{p}$ be an open prime ideal of A and denote by S the multiplicatively closed set of all $f \in A$ with $f \notin \mathfrak{p}$ (that is, $\mathfrak{p} \in \mathfrak{D}(f)$). This is a directed set under the relation $f \leq g$ iff. $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$. The open sets $\mathfrak{D}(f)$ of $\mathfrak{X}$ are a cofinal subset of the set of all open neighborhoods of $\mathfrak{p}$, so by Proposition 29 there is a canonical isomorphism of rings

$$\mathcal{O}_{\mathfrak{X},\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \Gamma(U, \mathcal{O}_{\mathfrak{X}}) \cong \varinjlim_{f \in S} \Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}) \cong \varinjlim_{f \in S} A_{\{f\}} = A_{\{S\}}$$

It now follows from Proposition 26 that $\mathcal{O}_{\mathfrak{X},\mathfrak{p}}$ is a local noetherian ring with residue field canonically isomorphic to $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. □

4.2 General Formal Schemes

Lemma 31. *Let X be a noetherian scheme, $\mathscr{J}, \mathscr{K}$ coherent sheaves of ideals with $Supp(\mathcal{O}_X/\mathscr{J}) = Supp(\mathcal{O}_X/\mathscr{K})$. Then there integers $m, n \geq 1$ with $\mathscr{J}^n \subseteq \mathscr{K}, \mathscr{K}^m \subseteq \mathscr{J}$.*

Proof. Using (MOS,Proposition 11), (MOS,Corollary 12) and (MOS,Proposition 13) one reduces to the case where $X = SpecA$ is affine, and $\mathscr{J} = \widetilde{\mathfrak{a}}, \mathscr{K} = \widetilde{\mathfrak{b}}$. Then $Supp(\mathcal{O}_X/\widetilde{\mathfrak{a}}) = V(\mathfrak{a})$ so by assumption $\mathfrak{a}, \mathfrak{b}$ must have the same radical. Since A is noetherian there exists $m, n \geq 1$ with $\mathfrak{a}^n \subseteq \mathfrak{b}$ and $\mathfrak{b}^m \subseteq \mathfrak{a}$. Applying $\widetilde{-}$ gives the desired result. □

Definition 26. Let X be a noetherian scheme, $Y \subseteq X$ a closed subset. Let $\Phi_{Y,X}$ denote the set of all coherent sheaves of ideals $\mathscr{J}$ on X with $Supp(\mathcal{O}_X/\mathscr{J}) = Y$. In particular this set contains the following coherent sheaf of ideals

$$\mathscr{J}_Y(U) = \{f \in \mathcal{O}_X(U) \,|\, f \text{ vanishes on } Y \cap U\}$$

Therefore $\Phi_{Y,X}$ is nonempty and we partially order this set by inclusion. By Lemma 31 for any pair $\mathscr{J}, \mathscr{K} \in \Phi_{Y,X}$ we have $\mathscr{J}^n \subseteq \mathscr{K}, \mathscr{K}^m \subseteq \mathscr{J}$ for some $m, n \geq 1$.

Lemma 32. *Let Y be a closed subset of a noetherian scheme X. Then $\Phi_{Y,X}$ is an inverse directed set and if $\mathscr{J} \in \Phi_{Y,X}$ then the set of powers $\mathscr{J}^n$ is final in $\Phi_{Y,X}$.*

Proof. If $\mathscr{J}, \mathscr{K} \in \Phi$ then the intersection $\mathscr{J} \cap \mathscr{K}$ is coherent and $Supp(\mathcal{O}_X/\mathscr{J} \cap \mathscr{K}) = Y$ by (SI,Definition 3). This shows that Φ is an inverse directed set. Given $\mathscr{J} \in \Phi$ the powers $\mathscr{J}^n$ for $n \geq 1$ are coherent sheaves of ideals (MOS,Corollary 12), and it is clear that $Supp(\mathcal{O}_X/\mathscr{J}^n) = Y$. Lemma 31 shows that the set $\{\mathscr{J}^n \,|\, n \geq 1\}$ is final in Φ. ☐

Lemma 33. *Let I be an inverse directed set, $\{R_i, \pi_{ij}\}$ an inverse system of rings and $\{M_i, \rho_{ij}\}$ an inverse system of abelian groups. Suppose that for each $i \in I$ there is an R_i-module structure on M_i with the property that for $i \leq j$, $m \in M_i, r \in R_i$ we have $\rho_{ij}(r \cdot m) = \pi_{ij}(r) \cdot \rho_{ij}(m)$. Then there is a canonical $\varprojlim R_i$-module structure on $\varprojlim M_i$.*

Proof. It is not difficult to check that $(r_i) \cdot (m_i) = (r_i \cdot m_i)$ defines a $\varprojlim R_i$-module structure on $\varprojlim M_i$ with the property that the morphism of abelian groups $\varprojlim M_i \longrightarrow M_j$ maps the action of $\varprojlim R_i$ to the action of R_j. ☐

We begin by defining the completion of a scheme along a closed subscheme. For technical reasons we will limit our discussion to noetherian schemes.

Definition 27. Let X be a noetherian scheme and $Y \subseteq X$ a closed subset with inclusion $j : Y \longrightarrow X$. Let $\mathscr{F}$ be a sheaf of modules on X. Associated to every coherent sheaf of ideals $\mathscr{J} \in \Phi_{Y,X}$ is a sheaf of modules $\mathscr{F}/\mathscr{J}\mathscr{F}$, and for $\mathscr{J}, \mathscr{K} \in \Phi_{Y,X}$ with $\mathscr{J} \subseteq \mathscr{K}$ there is a morphism of sheaves of modules

$$\mathscr{F}/\mathscr{J}\mathscr{F} \to \mathscr{F}/\mathscr{K}\mathscr{F}$$
$$a \dotplus \Gamma(U, \mathscr{J}\mathscr{F}) \mapsto a \dotplus \Gamma(U, \mathscr{K}\mathscr{F})$$

This defines an inverse system in $\mathfrak{Mod}(X)$ over the inverse directed set $\Phi_{Y,X}$. The inverse limit in $\mathfrak{Mod}(X)$ is computed pointwise, so $\Gamma(U, \varprojlim_\Phi(\mathscr{F}/\mathscr{J}\mathscr{F})) = \varprojlim_\Phi \Gamma(U, \mathscr{F}/\mathscr{J}\mathscr{F})$. We call the sheaf of abelian groups $j^{-1}(\varprojlim_\Phi(\mathscr{F}/\mathscr{J}\mathscr{F}))$ on Y the *completion of $\mathscr{F}$ along Y* and denote it by $\mathscr{F}_{/Y}$.

- For $\mathscr{F} = \mathcal{O}_X$ the inverse system is an inverse system of sheaves of rings $\mathcal{O}_X/\mathscr{J}$, so the inverse limit $\varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})$ is a sheaf of rings. Therefore $\mathcal{O}_{X/Y}$ is a sheaf of rings. The morphisms $\mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{J}$ induce a morphism of sheaves of rings $\mathcal{O}_X \longrightarrow \varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})$.
- Observe that for $\mathscr{J} \in \Phi_{Y,X}$ the sheaf of abelian groups $\mathscr{F}/\mathscr{J}\mathscr{F}$ has a canonical structure as a sheaf of $\mathcal{O}_X/\mathscr{J}$-modules with $(r \dotplus \Gamma(U, \mathscr{J})) \cdot (a \dotplus \Gamma(U, \mathscr{J}\mathscr{F})) = ra \dotplus \Gamma(U, \mathscr{J}\mathscr{F})$.
- Using Lemma 32 we make $\varprojlim_\Phi(\mathscr{F}/\mathscr{J}\mathscr{F})$ into a sheaf of $\varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})$-modules with $(r_\mathscr{J}) \cdot (m_\mathscr{J}) = (r_\mathscr{J} \cdot m_\mathscr{J})$. Therefore $\mathscr{F}_{/Y}$ becomes a $\mathcal{O}_{X/Y}$-module in a canonical way.
- If $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules then for $\mathscr{J} \in \Phi_{Y,X}$ there is a morphism of $\mathcal{O}_X$-modules $\mathscr{F}/\mathscr{J}\mathscr{G} \longrightarrow \mathscr{G}/\mathscr{J}\mathscr{G}$. This defines a morphism between the inverse systems, and there is an induced morphism of $\mathcal{O}_X$-modules $\varprojlim_\Phi(\mathscr{F}/\mathscr{J}\mathscr{F}) \longrightarrow \varprojlim_\Phi(\mathscr{G}/\mathscr{J}\mathscr{G})$. This is also a morphism of $\varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})$-modules, so we have an additive functor

$$\varprojlim_\Phi((-)/\mathscr{J}) : \mathfrak{Mod}(\mathcal{O}_X) \longrightarrow \mathfrak{Mod}(\varprojlim_\Phi(\mathcal{O}_X/\mathscr{J}))$$

Composing with the canonical additive functor $\mathfrak{Mod}(\varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})) \longrightarrow \mathfrak{Mod}(\mathcal{O}_{X/Y})$ we have an additive functor

$$(-)_{/Y} : \mathfrak{Mod}(\mathcal{O}_X) \longrightarrow \mathfrak{Mod}(\mathcal{O}_{X/Y})$$

- Applying j^{-1} to the morphism of sheaves of rings $\mathcal{O}_X \longrightarrow \varprojlim_\Phi(\mathcal{O}_X/\mathscr{J})$ we have a morphism of sheaves of rings $j^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{X/Y}$. By adjointness there is a morphism of sheaves of rings $\mathcal{O}_X \longrightarrow j_*\mathcal{O}_{X/Y}$ and therefore a morphism of ringed spaces $(Y, \mathcal{O}_{X/Y}) \longrightarrow (X, \mathcal{O}_X)$.

We denote the ringed space $(Y, \mathcal{O}_{X/Y})$ by $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ and call it the *formal completion of* X *along* Y. If $\mathscr{F}$ is a sheaf of modules on X then we denote the sheaf of modules $\mathscr{F}_{/Y}$ on $\widehat{X}$ by $\widehat{\mathscr{F}}$ and call it the *completion of* $\mathscr{F}$ *along* Y. There is a canonical morphism $i_X : \widehat{X} \longrightarrow X$ of ringed spaces and completion defines an additive functor $\mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(\widehat{X})$.